Inexact cutting planes for two-stage mixed-integer stochastic programs

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We propose a novel way of applying cutting plane techniques to two-stage mixed-integer stochastic programs. Instead of using cutting planes that are always valid, our idea is to apply inexact cutting planes to the second-stage feasible regions that may cut away feasible integer second-stage solutions for some scenarios and may be overly conservative for others. The advantage is that it allows us to use cutting planes that are affine in the first-stage decision variables, so that the approximation is convex, and can be solved efficiently using techniques from convex optimization. We derive performance guarantees for using particular types of inexact cutting planes for simple integer recourse models. Moreover, we show in general that using inexact cutting planes leads to good first-stage solutions if the total variations of the probability density functions of the random variables in the model are small enough.

Key words: stochastic programming, integer programming, cutting plane techniques, convex approximations

1. Introduction

Many practical problems under uncertainty in, e.g., energy, finance, logistics, and healthcare involve integer decision variables. Such problems can be modelled as mixed-integer stochastic programs (MISPs), but are notoriously difficult to solve. In this paper, we do not attempt to solve these problems exactly. Instead, we introduce a novel approach to approximately solve two-stage MISPs, and we derive performance guarantees for the resulting approximating solutions.

Traditional solution methods for MISPs combine solution approaches for continuous stochastic programs and deterministic mixed-integer programs (MIPs). See, e.g., Ahmed et al. [1] for branch-and-bound, Sen and Higle [18] and Ntaimo [13] for disjunctive decomposition, Carøe and Schultz [5] for dual decomposition, Laporte and Louveaux [11] for the integer L-shaped method, and Zhang and Küçükyavuz [23] for cutting plane techniques. All these solution methods aim at finding the exact optimal solution for MISPs, but generally have difficulties scaling up to solve large problem instances. This is not surprising, since contrary to their continuous counterparts, these MISPs are non-convex in general [14]. This means that efficient techniques from convex optimization cannot be used to solve these problems.

Based on this observation and inspired by the success of cutting plane techniques for deterministic MIPs, we propose to use *cutting planes* to solve two-stage MISPs. However, we will use them in a fundamentally different way than in existing methods for both deterministic and stochastic MIPs. Instead of using exact cutting planes that are always valid, we propose to use *inexact* cutting planes for the second-stage feasible regions in such a way that the approximating problem remains convex in the first-stage decision variables, and thus efficient convex optimization techniques can be used to solve the approximation.

The disadvantage of using inexact cutting planes is that they may cut away part of the second-stage feasible region or that they may be overly conservative, so that we significantly over- or underestimate the second-stage costs, respectively. However, for MISPs this may be justified since our aim is not to find the exact and complete characterization of the integer hulls of the second-stage feasible regions, but rather to obtain good first-stage decisions. In fact, one of our main contributions is that we show that it is possible to find good or even near-optimal *first-stage* decisions despite the fact that the integer hulls of the second-stage feasible regions are inexactly approximated.

For simple integer recourse (SIR) models, a special type of MISP, our inexact cutting plane approximation turns out to be equivalent to convex α-approximations, derived by Klein Haneveld et al. [10] from a completely different perspective. By reinterpreting these α-approximations using inexact cutting planes, we connect two existing solution methodologies for MISPs that use convex approximations and exact cutting planes, respectively. Moreover, this reinterpretation allows us to apply existing performance guarantees derived in Romeijnders et al. [16] for α-approximations to inexact cutting plane techniques for SIR models. Furthermore, we use results from Romeijnders et al. [15] to derive conditions for general MISPs under which inexact cutting plane techniques are *asymptotically accurate*. Intuitively, this means that using inexact cutting planes yields good approximations if the variability of the random parameters in the model is large enough. We derive inexact mixed-integer Gomory cuts for general two-stage MISPs and inexact cutting planes for a nurse scheduling problem that are asymptotically accurate.

Summarizing, the main contributions of our paper are as follows.

- We propose a novel solution approach for two-stage MISPs by applying inexact cutting planes to second-stage feasible regions.

- We reinterpret α-approximations for SIR models as inexact cutting plane approximations, connecting two existing solution methodologies for MISPs, and yielding a tight error bound for applying inexact cutting planes to SIR models.

- We derive a performance guarantee for applying inexact cutting planes to MISPs in general, proving that inexact cutting plane techniques are asymptotically accurate.

- We derive inexact mixed-integer Gomory cuts for general MISPs and derive inexact cutting planes for a nurse scheduling problem.

The remainder of this paper is organized as follows. In Section 2 we define MISPs and explain our inexact cutting plane approach. In Section 3, we reintrepret α-approximations for SIR models using inexact cutting planes, and in Section 4 we prove for MISPs in general that inexact cutting plane techniques are asymptotically accurate. In Section 5, we derive inexact mixed-integer Gomory cuts, and apply inexact cutting planes to a nurse scheduling problem. We end with a discussion in Section 6.

2. Problem definition and solution approach

2.1. Problem definition

Two-stage MISPs can be interpreted as hierarchical planning problems. In the first stage, decisions x have to be made before some random parameters ω are known, whereas in the second stage, decisions y are made after the realizations of these random parameters ω are revealed. We assume that the probability distribution of ω is known, with F denoting the cumulative distribution function and Ω the support of ω. The MISPs that we consider are defined as

$$\min_{x,z} \left\{ c^\top x + Q(z) : \ Ax = b, \ z = Tx, \ x \in X \right\}, \tag{1}$$

where $z = Tx \in \mathbb{R}^m$ represent tender variables. Moreover, the *expected value function* Q represents the expected second-stage costs

$$Q(z) := \mathbb{E}_\omega[v(\omega, z)], \quad z \in \mathbb{R}^m, \tag{2}$$

where the *second-stage value function* v is defined as

$$v(\omega, z) := \min_y \left\{ q^\top y : \ Wy = \omega - z, \ y \in Y \right\}, \quad \omega \in \Omega, z \in \mathbb{R}^m. \tag{3}$$

The second-stage decisions y are also called *recourse actions*. Indeed, if $Tx = \omega$ represents random goal constraints, then the second-stage optimization problem v models all possible recourse actions y, and their corresponding costs, to compensate for infeasibilities of these goal constraints. Observe that we only consider randomness in the right-hand side of these goal constraints. Moreover, we assume that at least some of the second-stage decision variables y_i are restricted to be integer. This is captured by the feasible regions $X \subset \mathbb{R}^{n_1}_+$ and $Y \subset \mathbb{R}^{n_2}_+$ that may impose integrality restrictions on the first- and second-stage decision variables, respectively.

Throughout this paper we make the following assumptions. The first is often referred to as the *complete recourse* assumption, meaning that there always exists a feasible recourse action y, ensuring that $v(\omega,z)<+\infty$ for all $\omega\in\Omega$ and $z\in\mathbb{R}^m$. The second is equivalent to the dual feasible region of the LP-relaxation of v being non-empty, implying that $v(\omega,z)>-\infty$ for all $\omega\in\Omega$ and $z\in\mathbb{R}^m$. Together with the third assumption, these assumptions guarantee that $Q(z)$ is finite for every $z\in\mathbb{R}^m$.

Assumption 1. *We assume that*

- *there exists* $y\in Y$ *such that* $Wy=\omega-z$ *for every* $\omega\in\Omega$ *and* $z\in\mathbb{R}^m$,
- *there exists* $\lambda\in\mathbb{R}^m$ *such that* $W^\top\lambda\leq q$, *and*
- $\mathbb{E}_\omega[|\omega_i|]<+\infty$, *for all* $i=1,\dots,m$.

2.2. Novel solution approach: inexact cutting planes

To solve the MISP defined in (1), we propose to relax the integrality restrictions on the second-stage decision variables y and to add *inexact* cutting planes to the second-stage feasible region

$$Y(\omega,z):=\Big\{y\in Y:\ Wy=\omega-z\Big\}.$$

In particular, we assume that the cutting planes are of the form $\hat{W}(\omega)y\geq\hat{h}(\omega)-\hat{T}(\omega)z$, so that they are affine in the tender variables z.

Definition 1. Consider the second-stage value function v defined in (3). Then, we call $\hat{v}$ an *inexact cutting plane approximation* of v if it is of the form

$$\hat{v}(\omega,z)=\min_y\{q^\top y:\ Wy=\omega-z,\ \hat{W}(\omega)y\geq\hat{h}(\omega)-\hat{T}(\omega)z,\ y\in\mathbb{R}^{n_2}_+\},\quad \omega\in\Omega,\, z\in\mathbb{R}^m.$$

Moreover, we define the inexact cutting plane approximation $\hat{Q}$ of the expected value function Q, defined in (2), as $\hat{Q}(z):=\mathbb{E}_\omega[\hat{v}(\omega,z)]$, $z\in\mathbb{R}^m$.

The main reason we use inexact cutting planes that are *affine in* z is that the approximating value function $\hat{v}(\omega,z)$ with feasible region

$$\hat{Y}(\omega,z):=\left\{y\in\mathbb{R}^{n_2}_+:\ \begin{array}{l}Wy=\omega-z\\ \hat{W}(\omega)y\geq\hat{h}(\omega)-\hat{T}(\omega)z\end{array}\right\}$$

is *convex* in z for every fixed $\omega\in\Omega$, and thus the corresponding approximating expected value function $\hat{Q}$ is convex. This means that the MISP in (1) with Q replaced by $\hat{Q}$ can be solved efficiently using techniques from convex optimization.

Lemma 1. *Consider the inexact cutting plane approximations* $\hat{v}$ *and* $\hat{Q}$ *of Definition 1. Then,* $\hat{Q}$ *is convex, and* $\hat{v}(\omega,z)$ *is convex in* z *for every fixed* $\omega\in\Omega$. □

In Section 5 we derive inexact mixed-integer Gomory cuts and inexact cutting planes for a nurse scheduling problem. However, the main focus of this paper is not on how to obtain the inexact cutting plane approximation from Definition 1. Instead, we assume that the inexact cutting planes are given or can be iteratively generated by an algorithm, and we consider the performance of using such cutting planes.

The performance of these inexact cutting planes may be surprisingly good, even if they cut away feasible integer second-stage solutions or admit second-stage solutions outside the integer hull $\bar{Y}(\omega,z)$ of the second-stage feasible region $Y(\omega,z)$; in these cases, $\hat{v}(\omega,z)$ may significantly over- or underestimate $v(\omega,z)$, respectively. However, to obtain good first-stage decisions x, we do not require $\hat{v}(\omega,z)$ to be a good approximation of $v(\omega,z)$ for *every* $\omega\in\Omega$ and $z\in\mathbb{R}^m$, but merely require $\hat{v}(\omega,z)$ to be a good approximation of $v(\omega,z)$ *on average* for every $z\in\mathbb{R}^m$. This explains why applying inexact cutting planes may work for stochastic MIPs but not for deterministic MIPs.

Using a one-dimensional example, we illustrate the type of inexact cutting planes that we have in mind.

Example 1. Consider a special case of the second-stage value function defined in (3), given by

$$\begin{aligned} v(\omega,z)=\min_{y,u_1,u_2} &\quad qy+ru_1+ru_2 \\ \text{s.t.} &\quad y-u_1+u_2=\omega-z \\ &\quad y\in\mathbb{Z}_+,\ u_1,u_2\in\mathbb{R}_+, \end{aligned} \tag{4}$$

where $0<q<r$. By rewriting the equality in (4) as $u_2=\omega-z-y+u_1$, we can eliminate the variable u_2 from the second-stage value function to obtain

$$\begin{aligned} v(\omega,z)=r(\omega-z)+\min_{y,u_1} &\quad (q-r)y+2ru_1 \\ \text{s.t.} &\quad y-u_1\le\omega-z \\ &\quad y\in\mathbb{Z}_+,\, u_1\in\mathbb{R}_+. \end{aligned} \tag{5}$$

Since the minimization problem in (5) only has two decision variables, y and u_1, we can graphically depict its feasible region $Y(\omega,z)$. The left panel in Figure 1 shows this feasible region for $\omega=2.5$ and $z=1$, and also depicts the feasible region of the LP-relaxation of $v(\omega,z)$. Clearly, the latter is larger than the integer hull $\bar{Y}(\omega,z)$ of $Y(\omega,z)$.

It is well known that the integer hull $\bar{Y}(\omega,z)$ can be obtained by adding a mixed-integer rounding (MIR) inequality, so that for every $\omega\in\Omega$ and $z\in\mathbb{R}$, the integer hull $\bar{Y}(\omega,z)$ equals

$$\bar{Y}(\omega,z):=\left\{(y,u_1)\in\mathbb{R}^2_+:\ y-u_1\le\omega-z,\ y-\frac{1}{1-(\omega-z)+\lfloor\omega-z\rfloor}u_1\le\lfloor\omega-z\rfloor\right\}.$$

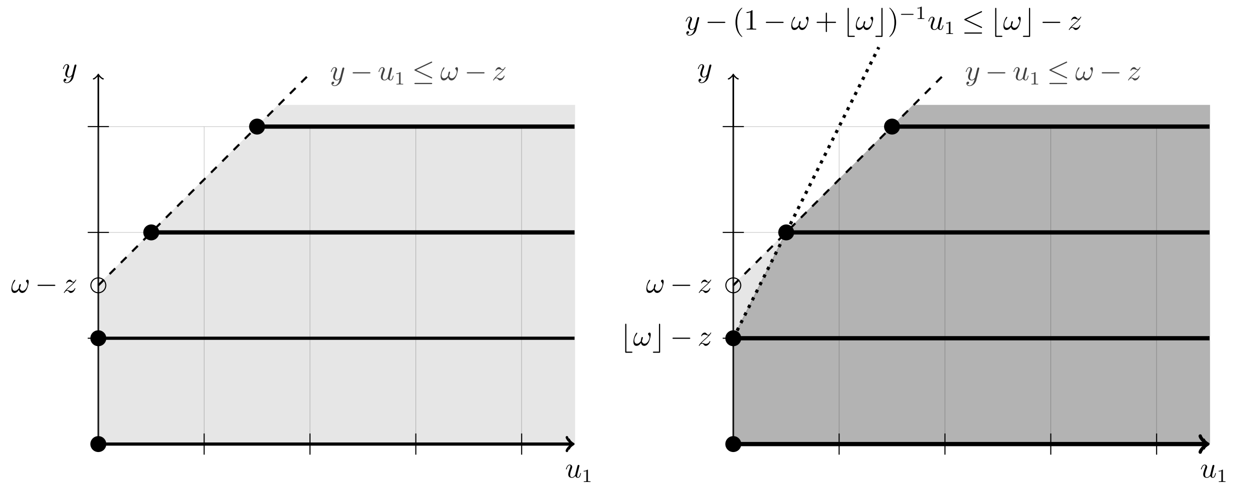

Figure 1 Illustration of the feasible region of $v(\omega, z)$ of Example 1 with $\omega = 2.5$ and $z = 1$. The feasible region $Y(\omega, z)$ is represented by the black dots and the thick black lines. In the left panel the shaded region corresponds to the feasible region of the LP-relaxation of v, whereas in the right panel, the MIR inequality is added, and the dark shaded region represents the integer hull $\bar{Y}(\omega, z)$ of the feasible region $Y(\omega, z)$ of $v(\omega, z)$.

The right panel in Figure 1 shows $\bar{Y}(\omega, z)$ and this MIR inequality.

Observe that the MIR inequality is not affine in z, which means that it will be hard to use for optimization purposes. However, if $z \in \mathbb{Z}$, then it reduces to

$$y - \frac{1}{1-\omega+\lfloor\omega\rfloor}u_1 \leq \lfloor\omega\rfloor - z, \tag{6}$$

which means it is of the form of the inexact cutting planes in Definition 1. Thus, a natural idea is to use the cutting planes in (6), also when $z \notin \mathbb{Z}$. For $z \in \mathbb{Z}$ they will be exact for all $\omega \in \Omega$, and for $z \notin \mathbb{Z}$ they will be inexact. Figure 2 shows the approximating feasible region

$$\hat{Y}(\omega, z) = \left\{(y, u_1) \in \mathbb{R}^2_+ : \ y - u_1 \leq \omega - z, \ y - \frac{1}{1-\omega+\lfloor\omega\rfloor}u_1 \leq \lfloor\omega\rfloor - z\right\},$$

for $z = 0.5$ and $\omega = 1.5, 1.75, 2, 2.25$. We observe that for $\omega = 2$, the approximating MIR inequality coincides with the constraint $y - u \leq \omega - z$, so that $\hat{Y}(\omega, z)$ is equal to the feasible region of the LP-relaxation of $v(\omega, z)$, and thus admits solutions outside the integer hull $\bar{Y}(\omega, z)$. For $\omega = 1.5$, on the other hand, the approximating MIR inequality cuts away feasible integer solutions. For $\omega = 1.75$ and $\omega = 2.25$ we see a combination of both.

In Section 5.2 we will numerically assess the performance of the inexact cutting plane approximation

$$\hat{v}(\omega, z) := r(\omega - z) + \min_{y,u_1}\left\{(q-r)y + 2ru_1 : \ (y, u_1) \in \hat{Y}(\omega, z)\right\}, \quad \omega \in \Omega, \ z \in \mathbb{R}, \tag{7}$$

and show that for a normally distributed random variable $\omega \sim N(\mu, \sigma^2)$, $\hat{Q}$ is a good approximation of Q for medium to large values of the standard deviation σ.

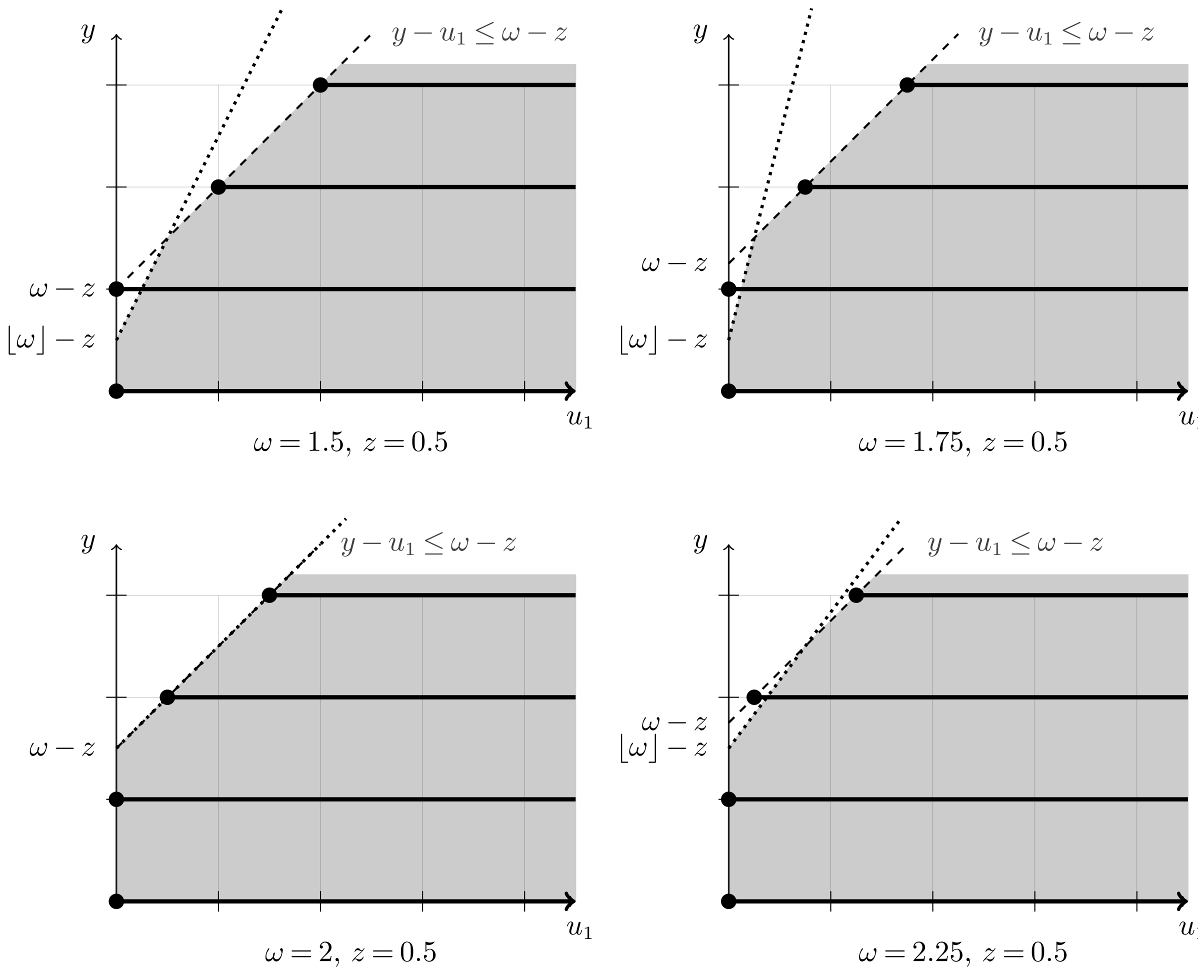

Figure 2 Illustration of the feasible region of $v(\omega, z)$ of Example 1 with $z = 0.5$ and $\omega = 1.5, 1.75, 2,$ and 2.25. The feasible region $Y(\omega, z)$ is represented by the black dots and the thick black lines. The dotted line represents the inexact MIR inequality defined in (6), and the shaded regions the approximating feasible region $\hat{Y}(\omega, z)$.

3. Inexact cutting planes for simple integer recourse models

In this section we show that existing convex approximations for simple integer recourse (SIR) models can be interpreted as inexact cutting plane approximations. SIR models are introduced by Louveaux and Van der Vlerk [12], and can be considered the most simple version of a MISP as defined in (1). For ease of exposition, we consider here the one-sided and one-dimensional version of SIR, where the second-stage value function v is defined as

$$v(\omega, z) = \min_y \left\{ qy : \ y \geq \omega - z, \ y \in \mathbb{Z}_+ \right\}, \quad \omega \in \Omega, z \in \mathbb{R}.$$

Observe that we can derive a closed-form expression for v since for every $\omega \in \Omega$ and $z \in \mathbb{R}$, the optimal solution is $y^* = \lceil \omega - z \rceil^+ := \max\{0, \lceil \omega - z \rceil\}$, and thus $v(\omega, z) = q \lceil \omega - z \rceil^+$. Clearly, $v(\omega, z)$ is a non-convex function of z because of the round-up operator.

We, however, focus on the feasible region $Y(\omega,z)=\{y\in\mathbb{Z}_+ : y\geq \omega - z\}$ and its integer hull

$$\bar{Y}(\omega,z)=\Big\{y\in\mathbb{R}_+ : y\geq \lceil \omega - z\rceil\Big\},\quad \omega\in\Omega,\, z\in\mathbb{R}.$$

Here, the cutting plane $y\geq\lceil\omega-z\rceil$ makes the original constraint $y\geq\omega-z$ redundant. Similar to Example 1, this exact cutting plane is not affine in z and thus not suitable for optimization purposes. However, if $z\in\mathbb{Z}$, then the cutting plane is equivalent to $y\geq\lceil\omega\rceil-z$, which we can use as an inexact cutting plane for $z\notin\mathbb{Z}$. In fact, we define a family of inexact cutting plane approximations $\hat{v}_\alpha$, each of them using the cutting plane $y\geq\lceil\omega-\alpha\rceil+\alpha-z$ that is exact for $z\in\alpha+\mathbb{Z}$.

Definition 2. For every $\alpha\in\mathbb{R}$, define the inexact cutting plane approximation $\hat{v}_\alpha$ for the SIR second-stage value function v as

$$\hat{v}_\alpha(\omega,z)=\min_y\Big\{qy:\ y\geq\lceil\omega-\alpha\rceil+\alpha-z,\ y\in\mathbb{R}_+\Big\}=q\Big(\lceil\omega-\alpha\rceil+\alpha-z\Big)^+,\quad \omega\in\Omega,\, z\in\mathbb{R}.$$

Moreover, define the corresponding inexact cutting plane approximation $\hat{Q}_\alpha$ for the SIR expected value function Q as $\hat{Q}_\alpha(z)=q\mathbb{E}_\omega[(\lceil\omega-\alpha\rceil+\alpha-z)^+]$, $z\in\mathbb{R}$.

Surprisingly, the inexact cutting plane approximation $\hat{Q}_\alpha$ equals the α-approximations of Klein Haneveld et al. [10], derived from a completely different perspective. They first identify all probability distributions of ω for which the expected value function Q is convex. This turns out to be all continuous distributions with probability density function f satisfying $f(s)=G(s+1)-G(s)$, $s\in\mathbb{R}$, for some cumulative distribution function G with finite mean. For all other distributions, they use this condition to generate an approximating density function $\hat{f}$, resulting in a convex approximation $\hat{Q}$ of Q. Selecting $G(s+1)=F(\lceil s-\alpha\rceil+\alpha)$, $s\in\mathbb{R}$, yields the α-approximation $\hat{Q}_\alpha(z):=q\mathbb{E}_\omega[(\lceil\omega-\alpha\rceil+\alpha-z)^+]$, $z\in\mathbb{R}$, equivalent to the inexact cutting plane approximation of Definition 1.

In this paper, we reinterpret $\hat{Q}_\alpha$ as an inexact cutting plane approximation, connecting the convex approximation solution philosophy, introduced by Van der Vlerk [20] and continued by among others [10, 15, 16, 17, 19, 21], with exact cutting plane techniques for MISPs, studied in, e.g., [4, 7, 8, 23]. This is particularly relevant, since performance guarantees are available for using convex approximations that may be used for inexact cutting plane approximations. In fact, for SIR models, Romeijnders et al. [16] derive an upper bound on $\|Q-\hat{Q}_\alpha\|_\infty:=\sup_{z\in\mathbb{R}}|Q(z)-\hat{Q}_\alpha(z)|$ for every $\alpha\in\mathbb{R}$, that depends on the *total variation* of the probability density function f of the random variable ω.

Definition 3. Let $f:\mathbb{R}\to\mathbb{R}$ be a real-valued function and let $I\subset\mathbb{R}$ be an interval. Let $\Pi(I)$ denote the set of all finite ordered sets $P=\{x_1,\dots,x_{N+1}\}$ with $x_1<\cdots<x_{N+1}$ in I. Then, the *total variation* of f on I, denoted $|\Delta|f(I)$, is defined as

$$|\Delta|f(I)=\sup_{P\in\Pi(I)} V_f(P),$$

where $V_f(P)=\sum_{i=1}^N |f(x_{i+1})-f(x_i)|$. We write $|\Delta|f:=|\Delta|f(\mathbb{R})$.

Theorem 1. *Consider the SIR expected value function $Q(z)=q\mathbb{E}_\omega[\lceil\omega-z\rceil^+]$, $z\in\mathbb{R}$, and its inexact cutting plane approximation $\hat{Q}_\alpha(z)=q\mathbb{E}_\omega[(\lceil\omega-\alpha\rceil-\alpha-z)^+]$, $z\in\mathbb{R}$, for $\alpha\in\mathbb{R}$. Then, for every continuous random variable ω with probability density function f, we have*

$$\|Q-\hat{Q}_\alpha\|_\infty\le qh(|\Delta|f),$$

where $h:[0,\infty)\mapsto\mathbb{R}$ is defined as

$$h(|\Delta|f)=\begin{cases} |\Delta|f/8, & |\Delta|f\le 4,\\ 1-2/|\Delta|f, & |\Delta|f\ge 4.\end{cases}$$

Proof. See Romeijnders et al. [16]. □

For unimodal density functions, such as the normal density function in Example 2 below, it holds that the total variation $|\Delta|f$ of the probability density function f of ω decreases as the variance of the random variable ω increases. In general, we conclude from Theorem 1 that the larger the variability in the model the better the inexact cutting plane approximation.

Example 2. Let ω be a normal random variable with mean μ and standard deviation σ. Then, the probability density function f of ω is given by

$$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\Big\{-\frac{(x-\mu)^2}{2\sigma^2}\Big\},\quad x\in\mathbb{R},$$

which is unimodal with mode μ, and thus has total variation $|\Delta|f=2f(\mu)=\sigma^{-1}\sqrt{2/\pi}$. Hence, if the standard deviation σ increases, then the total variation $|\Delta|f$ of f will decrease, and thus the upper bound on $\|Q-\hat{Q}_\alpha\|_\infty$ in Theorem 1 will decrease. In other words, if the standard deviation is large, then $\hat{Q}_\alpha$ is a close approximation of Q, and thus the resulting approximating first-stage decision $\hat{x}_\alpha$ will be good.

4. Inexact cutting plane approximations for general MISPs

Based on Section 3 and Example 1 in Section 2, we observe that it is possible to use exact cutting planes, that are valid for all $\omega\in\Omega$ and for z on a grid of points, as inexact cutting planes for all ω and z. That is why we make the following assumption for inexact cutting plane approximations.

Assumption 2. *There exist $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{Z}^m$ such that for all $z \in \mathbb{R}^m$ with $z \in \alpha + \beta\mathbb{Z}^m$ and for all $\omega \in \Omega$,*

- $\Big\{y \in \mathbb{Z}_+^{p_2} \times \mathbb{R}_+^{n_2-p_2} : Wy = \omega - z\Big\} \subset \Big\{y \in \mathbb{R}_+^{n_2} : Wy = \omega - z, \hat{W}(\omega)y \geq \hat{h}(\omega) - \hat{T}(\omega)z\Big\}$,
- $\hat{v}(\omega, z) = v(\omega, z)$.

Remark 1. With slight abuse of notation we will use $\alpha + \beta\mathbb{Z}^m$ to represent the grid of points

$$\alpha + \beta\mathbb{Z}^m := \Big\{(\alpha_1 + \beta_1 l_1, \ldots, \alpha_m + \beta_m l_m) : \ l \in \mathbb{Z}^m\Big\}.$$

In the remainder of this section we will prove that under Assumption 2, inexact cutting plane approximations are asymptotically accurate. That is, the error of using inexact cutting planes vanishes as the total variations of the one-dimensional conditional pdfs of the random vector ω in the model go to zero. For example, for normally distributed ω this means that the solutions obtained by using inexact cutting planes are good if the variance of ω is large enough. The final result is Theorem 2, which is conveniently stated here below. This result also holds when only the first condition in Assumption 2 holds, but then the inexact cutting plane approximation is an asymptotic lower bound.

Definition 4. For every $i = 1, \ldots, m$ and $t \in \mathbb{R}^m$, we let $t_{-i} \in \mathbb{R}^{m-1}$ denote the vector t without its i-th component.

Definition 5. For every $i = 1, \ldots, m$ and $t_{-i} \in \mathbb{R}^{m-1}$, define the i-th *conditional density function* $f_i(\cdot|t_{-i})$ of the m-dimensional joint pdf f as

$$f_i(t_i|t_{-i}) = \begin{cases} \dfrac{f(t)}{f_{-i}(t_{-i})}, & f_{-i}(t_{-i}) > 0, \\ 0, & f_{-i}(t_{-i}) = 0, \end{cases}$$

where f_{-i} represents the joint density function of ω_{-i}, the random vector obtained by removing the i-th element of ω.

Definition 6. Let $\mathcal{H}^m$ denote the set of all m-dimensional joint pdfs f whose conditional density functions $f_i(\cdot|t_{-i})$ are of bounded variation.

Theorem 2. *Consider the mixed-integer recourse function Q and its inexact cutting plane approximation $\hat{Q}$. Under Assumptions 1 and 2, there exists a constant $C \in \mathbb{R}$ with $C > 0$ such that for all ω with pdf $f \in \mathcal{H}^m$,*

$$\|Q - \hat{Q}\|_\infty \leq C \sum_{i=1}^m \mathbb{E}_{\omega_{-i}}\Big[|\Delta| f_i(\cdot|\omega_{-i})\Big].$$

The proof of Theorem 2 is postponed to Section 4.4. First, however, we discuss preliminary results required for this proof. In particular, in Section 4.1 we discuss properties of the mixed-integer value function $v(\omega, z)$, in Section 4.2 we show that the inexact cutting plane approximation $\hat{v}(\omega, z)$ is affine in z on parts of its domain, and in Section 4.3 we derive bounds on $\hat{v}$. The proofs of our auxiliary lemmas and propositions in these sections are postponed to the Appendix.

4.1. Properties of mixed-integer value functions

Let B be a dual feasible basis matrix of the LP-relaxation v_{LP} of v. Then, we can rewrite v_{LP} as

$$\begin{aligned} v_{LP}(\omega, z) = \min_{y_B, y_N} \quad & q_B^\top y_B + q_N^\top y_N \\ \text{s.t.} \quad & B y_B + N y_N = \omega - z \\ & y_B \in \mathbb{R}_+^m, \ y_N \in \mathbb{R}_+^{n_2 - m}, \end{aligned} \tag{8}$$

where y_B denote the basic variables and y_N the non-basic variables. Using the equality in (8) to solve for the basic variables y_B, we obtain the equivalent representation

$$\begin{aligned} v_{LP}(\omega, z) = q_B^\top B^{-1}(\omega - z) + \min_{y_N} \quad & \bar{q}_N^\top y_N \\ \text{s.t.} \quad & B^{-1}(\omega - z) - B^{-1} N y_N \geq 0 \\ & y_N \in \mathbb{R}_+^{n_2 - m}, \end{aligned} \tag{9}$$

with reduced costs $\bar{q}_N^\top := q_N^\top - q_B^\top B^{-1} N \geq 0$. Obviously, it is optimal to select the non-basic variables y_N equal to zero in the minimization problem in (9) if $B^{-1}(\omega - z) \geq 0$. The latter condition can conveniently be rewritten as $\omega - z \in \Lambda$, where the simplicial cone Λ is defined as $\Lambda := \{t \in \mathbb{R}^m : B^{-1} t \geq 0\}$. Thus, if $\omega - z \in \Lambda$, then

$$v_{LP}(\omega, z) = q_B^\top B^{-1}(\omega - z).$$

This result holds for every dual feasible matrix B. In fact, the basis decomposition theorem of Walkup and Wets [22] shows that there exist basis matrices B_k and corresponding simplicial cones $\Lambda^k := \{t \in \mathbb{R}^m : B_k^{-1} t \geq 0\}$, $k = 1, \ldots, K$, such that these cones Λ^k cover $\mathbb{R}^m$, the interiors of these cones Λ^k are mutually disjoint, and $v_{LP}(\omega, z) = q_{B_k}^\top B_k^{-1}(\omega - z)$ for $\omega - z \in \Lambda^k$ for every $k = 1, \ldots, K$.

Romeijnders et al. [15] prove a similar result for the *mixed-integer* value function v, involving the same basis matrices B_k and simplicial cones Λ^k, $k = 1, \ldots, K$. They show that there exist distances $d_k \geq 0$ such that if $\omega - z \in \Lambda^k(d_k)$, i.e., if $\omega - z \in \Lambda^k$ and $\omega - z$ has at least Euclidean distance d_k to the boundary of Λ^k, then

$$v(\omega, z) = q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - z),$$

where ψ^k is a B_k-periodic function, see Definition 7 below. The first term is the same as the LP-relaxation v_{LP}, and thus the second term can be interpreted as the additional costs of having integer variables instead of continuous ones. Theorem 3 summarizes these results.

DEFINITION 7. Let $B \in \mathbb{Z}^{m\times m}$ be an integer matrix. Then, a function $\psi : \mathbb{R}^m \mapsto \mathbb{R}$ is called B-periodic if and only if $\psi(z) = \psi(z + Bl)$ for every $z \in \mathbb{R}^m$ and $l \in \mathbb{Z}^m$.

THEOREM 3. *Consider the mixed-integer value function*

$$v(\omega, z) = \min\left\{q^\top y : Wy = \omega - z,\, y \in \mathbb{Z}_+^{n_2} \times \mathbb{R}_+^{n_3}\right\}, \quad z \in \mathbb{R}^m,$$

where W is an integer matrix, and $v(\omega,z)$ is finite for all $\omega \in \Omega$ and $z \in \mathbb{R}^m$ by Assumption 1. Then, there exist dual feasible basis matrices B_k of v_{LP}, $k = 1,\dots,K$, simplicial cones $\Lambda^k := \{t \in \mathbb{R}^m : B_k^{-1}t \geq 0\}$, distances $d_k \geq 0$, and bounded B_k-periodic functions ψ^k such that

- $\bigcup_{k=1}^{K} \Lambda^k = \mathbb{R}^m$,
- $(\operatorname{int}\Lambda^k) \cap (\operatorname{int}\Lambda^l) = \emptyset$ *for every $k,l \in \{1,\dots,K\}$ with $k \neq l$, and*
- $v(\omega,z) = q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - z)$ *for every $\omega - z \in \Lambda^k(d_k)$.*

Proof. See [15]. □

4.2. Linearity regions of inexact cutting plane approximations

Let $k = 1,\dots,K$ be given and consider a fixed $\omega \in \Omega$. Theorem 3 shows that for all $z \in \mathbb{R}^m$ with $\omega - z \in \Lambda^k(d_k)$, i.e., for all $z \in \omega - \Lambda^k(d_k)$, the mixed-integer value function v is given by

$$v(\omega,z) = q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - z).$$

Since ψ^k is B_k-periodic there exist values of β for which $\psi^k(\omega - z) = \psi^k(\omega - \alpha)$ for all $z \in \alpha + \beta\mathbb{Z}^m$; see the proof of Proposition 1. For simplicity, however, assume for the moment that β equals such a value. Then, for all $z \in \omega - \Lambda^k(d_k)$ and $z \in \alpha + \beta\mathbb{Z}^m$, we have $\hat{v}(\omega,z) = v(\omega,z)$, and thus the inexact cutting plane approximation $\hat{v}$ equals

$$\hat{v}(\omega,z) = q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - \alpha). \tag{10}$$

Thus, for a fixed $\omega \in \Omega$, the inexact cutting plane approximation $\hat{v}(\omega,z)$ is affine in z over a grid of points in $\omega - \Lambda^k(d_k)$. Since $\hat{v}(\omega,z)$ is convex in z, we intuitively expect $\hat{v}(\omega,z)$ to satisfy (10) for points outside the grid in $\omega - \Lambda^k(d_k)$ as well. Lemma 2 confirms our intuition.

LEMMA 2. *Let $v : \mathbb{R}^m \mapsto \mathbb{R}$ be a convex function and let $C \subset \mathbb{R}^m$ be a closed convex set with extreme points $z^j \in C$, $j = 1,\dots,J$, and interior point $z^0 \in C$. Suppose that there exist $a \in \mathbb{R}^m$ and $b \in \mathbb{R}$ such that $v(z^j) = a^\top z^j + b$ for all $j = 0,\dots,J$. Then, $v(z) = a^\top z + b$ for all $z \in C$.*

To apply Lemma 2 to $\hat{v}(\omega, z)$ we introduce hyperrectangles $C^l(\alpha, \beta)$ that have extreme points on the grid $\alpha + \beta\mathbb{Z}^m$.

DEFINITION 8. Let $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^m$ be given. For every $l \in \mathbb{Z}^m$, we define the hyperrectangle $C^l(\alpha, \beta)$ as

$$C^l(\alpha, \beta) := \prod_{i=1}^{m} \Big[\alpha_i + \beta_i(l_i - 1), \alpha_i + \beta_i(l_i + 1)\Big].$$

For every value of $\alpha, \beta \in \mathbb{R}^m$ and $l \in \mathbb{Z}^m$, the hyperrectangle $C^l(\alpha, \beta) \subset \mathbb{R}^m$ is convex. Moreover, all its extreme points and the interior point $(\alpha_1 + \beta_1 l_1, \ldots, \alpha_m + \beta_m l_m)$ are on the grid $\alpha + \beta\mathbb{Z}^m$. Thus, if $C^l(\alpha, \beta) \subset \omega - \Lambda^k(d_k)$, then we can apply Lemma 2 to $\hat{v}(\omega, \cdot)$ with $C := C^l(\alpha, \beta)$ to conclude that $\hat{v}(\omega, z)$ satisfies (10) for all $z \in C^l(\alpha, \beta)$, and thus $\hat{v}(\omega, z)$ is affine in z over $C^l(\alpha, \beta)$. Applying Lemma 2 for all $C^l(\alpha, \beta)$ that are completely contained in $\omega - \Lambda^k(d_k)$, we can show that $\hat{v}(\omega, z)$ is affine in z over at least $\omega - \Lambda^k(d_k + 2\|\beta\|)$. This is true since the *diameter* of $C^l(\alpha, \beta)$ is $2\|\beta\|$, and $\Lambda^k(d_k + 2\|\beta\|)$ represents all points in Λ^k with at least Euclidean distance $d_k + 2\|\beta\|$ to the boundary of Λ^k. Thus, for every $z \in \omega - \Lambda^k(d_k + 2\|\beta\|)$ there exists a hyperrectangle $C^l(\alpha, \beta) \subset \omega - \Lambda^k(d_k)$ that contains z. Here, the diameter of $C^l(\alpha, \beta)$ is defined as

$$\max_{z_1, z_2} \Big\{ \|z_1 - z_2\| : z_1, z_2 \in C^l(\alpha, \beta) \Big\} = 2\|\beta\|.$$

Proposition 1 shows all *linearity regions* of $\hat{v}(\omega, z)$ for fixed $\omega \in \Omega$. These are subsets of the domain of $\hat{v}(\omega, \cdot)$ on which $\hat{v}(\omega, z)$ is affine in z.

PROPOSITION 1. *Consider an inexact cutting plane approximation $\hat{v}(\omega, z)$ as defined in Definition 1, and let Λ^k, $k = 1, \ldots, K$, denote the simplicial cones from Theorem 3. Then, under Assumptions 1 and 2, for every $k = 1, \ldots, K$, there exists a distance $d'_k \geq 0$ such that if $\omega - z \in \Lambda^k(d'_k)$, then*

$$\hat{v}(\omega, z) = q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - \alpha).$$

4.3. Bounds on the value function of an inexact cutting plane approximation

Proposition 1 defines $\hat{v}(\omega, z)$ on the linearity regions $\Lambda^k(d'_k)$. In fact, on these linearity regions, $v(\omega, z) = q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - z)$ and $\hat{v}(\omega, z) = q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - \alpha)$, so that the difference between the two equals

$$v(\omega, z) - \hat{v}(\omega, z) = \psi^k(\omega - z) - \psi^k(\omega - \alpha), \quad z \in \omega - \Lambda^k(d'_k).$$

This difference is B_k-periodic and bounded, since ψ^k is a bounded B_k-periodic function by Theorem 3. These properties will be exploited to derive an error bound for the inexact cutting plane approximation $\hat{Q}$ in Section 4.4.

Outside the linearity regions, i.e., on $\mathcal{N} := \mathbb{R}^m \setminus \bigcup_{k=1}^K \Lambda^k(d'_k)$, we cannot prove such properties for $v(\omega, z)$ and $\hat{v}(\omega, z)$. However, we can show that the difference between the two is bounded. That is, there exists $R \in \mathbb{R}$ such that

$$\|v - \hat{v}\|_\infty := \sup_{\omega, z} |v(\omega, z) - \hat{v}(\omega, z)| \leq R.$$

To prove this result we use that $\mathcal{N}$ can be covered by finitely many hyperslices H_j, $j \in \mathcal{J}$, see [15].

Definition 9. Let $\delta > 0$ and normal vector $a \in \mathbb{R}^m \setminus \{0\}$ be given. Then, the *hyperslice* $H(a, \delta)$ is defined as

$$H(a, \delta) := \{z \in \mathbb{R}^m : 0 \leq a^\top z \leq \delta\}.$$

However, before we derive an upper bound on $\|v - \hat{v}\|_\infty$, we first derive a lower bound and upper bound on the value function $\hat{v}(\omega, z)$ of the inexact cutting plane approximation. The lower bound follows directly from Proposition 1 and the fact that $\hat{v}(\omega, z)$ is convex in z for every fixed $\omega \in \Omega$.

Lemma 3. *Consider an inexact cutting plane approximation $\hat{v}(\omega, z)$ as defined in Definition 1. Then, under Assumptions 1 and 2, we have for every $\omega \in \Omega$ and $z \in \mathbb{R}^m$ that*

$$\hat{v}(\omega, z) \geq \max_{k=1,\dots,K} \left\{ q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - \alpha) \right\}.$$

The lower bound of $\hat{v}(\omega, z)$ in Lemma 3 is not only valid on the linearity regions of $\hat{v}(\omega, \cdot)$, but also on $\omega - \mathcal{N}$. We will show that the difference between $\hat{v}(\omega, z)$ and this lower bound is bounded. Again, we use the fact that $\hat{v}(\omega, z)$ is convex in z for every fixed $\omega \in \Omega$.

Lemma 4. *Consider an inexact cutting plane approximation $\hat{v}(\omega, z)$ as defined in Definition 1. Then, under Assumptions 1 and 2, there exists $R' \in \mathbb{R}$ such that*

$$\hat{v}(\omega, z) - \max_{k=1,\dots,K} \left\{ q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - \alpha) \right\} \leq R'. \tag{11}$$

Now we are ready to prove an upper bound on $\|v - \hat{v}\|_\infty$. The idea of the proof is that we can use Lemma 3 and 4 to bound $\|\hat{v} - v_{LP}\|_\infty$, where the LP-relaxation $v_{LP}(\omega, z)$ of $v(\omega, z)$ is equal to

$$v_{LP}(\omega, z) = \max_{k=1,\dots,K} \left\{ q_{B_k}^\top B_k^{-1}(\omega - z) \right\}, \tag{12}$$

and the maximum difference between v and v_{LP} is known.

Proposition 2. *Consider an inexact cutting plane approximation $\hat{v}(\omega, z)$ as defined in Definition 1. Then, under Assumptions 1 and 2, there exists $R \in \mathbb{R}$ such that*

$$\|v - \hat{v}\|_\infty \leq R.$$

4.4. Proof of error bound

In this section we give the proof of Theorem 2. Whereas the focus in Sections 4.2 and 4.3 was on $\hat{v}(\omega,z)$ as a function of z for fixed $\omega\in\Omega$, we now consider the difference $v(\omega,z)-\hat{v}(\omega,z)$ as a function of ω for fixed $z\in\mathbb{R}^m$. This is because $Q(z)-\hat{Q}(z)=\mathbb{E}_\omega[v(\omega,z)-\hat{v}(\omega,z)]$, $z\in\mathbb{R}^m$, and thus $v(\omega,z)-\hat{v}(\omega,z)$ can be interpreted as the underlying difference function for fixed $z\in\mathbb{R}^m$. Based on Propositions 1 and 2, we know that for $\omega\in z+\Lambda^k(d'_k)$, $k=1,\dots,K$,

$$v(\omega,z)-\hat{v}(\omega,z)=\psi^k(\omega-z)-\psi^k(\omega-\alpha),$$

and for $\omega\in z+\mathcal{N}$,

$$\Big|v(\omega,z)-\hat{v}(\omega,z)\Big|\le R.$$

We will use these two main properties to derive an upper bound for $\|Q-\hat{Q}\|_\infty$ that depends on the total variations of the one-dimensional conditional probability density functions of the random variables in the model, showing that inexact cutting plane approximations are asymptotically accurate.

Proof of Theorem 2. Combining Theorem 3 and Proposition 1, there exist basis matrices B_k, corresponding simplicial cones Λ^k, distances $d'_k\ge 0$, and bounded B_k-periodic functions ψ^k such that for $\omega-z\in\Lambda^k(d'_k)$,

$$v(\omega,z)-\hat{v}(\omega,z)=\psi^k(\omega-z)-\psi^k(\omega-\alpha).$$

Moreover, by Proposition 2, there exists $R\in\mathbb{R}$ such that $\|v-\hat{v}\|_\infty\le R$.

Fix $z\in\mathbb{R}^m$ and consider the difference $v(\omega,z)-\hat{v}(\omega,z)$ as a function of ω. We will use that for $\omega\in z+\Lambda^k(d'_k)$, this difference is B_k-periodic, and for $\omega\in z+\mathcal{N}$, it is bounded by R. In fact, using trivially adjusted versions of Theorems 4.6 and 4.13 in [15] we can show that there exist constants $D>0$ and $C'_k>0$, $k=1,\dots,K$, such that

$$\mathbb{P}\{\omega\in z+\mathcal{N}\}\le D\sum_{i=1}^m \mathbb{E}_{\omega_{-i}}\Big[|\Delta|f_i(\cdot|\omega_{-i})\Big], \tag{13}$$

and for every $k=1,\dots,K$,

$$\left|\int_{z+\Lambda^k(d'_k)}(\psi^k(t-z)-\psi^k(t-\alpha))f(t)dt\right|\le C'_k\sum_{i=1}^m \mathbb{E}_{\omega_{-i}}\Big[|\Delta|f_i(\cdot|\omega_{-i})\Big]. \tag{14}$$

Then,

$$\begin{aligned}|Q(z)-\hat{Q}(z)| &= \left|\int_{\mathbb{R}^m}(v(t,z)-\hat{v}(t,z))f(t)dt\right|\\ &\le \left|\int_{z+\mathcal{N}}(v(t,z)-\hat{v}(t,z))f(t)dt\right|+\sum_{k=1}^K\left|\int_{z+\Lambda^k(d'_k)}(v(t,z)-\hat{v}(t,z))f(t)dt\right|\\ &\le R\mathbb{P}\{\omega\in z+\mathcal{N}\}+\sum_{k=1}^K\left|\int_{z+\Lambda^k(d'_k)}(v(t,z)-\hat{v}(t,z))f(t)dt\right|.\end{aligned}$$

Applying the bound in (13) to the first term and the bounds in (14) to the second term, we obtain

$$\begin{aligned}|Q(z)-\hat{Q}(z)| &\leq RD\sum_{i=1}^{m}\mathbb{E}_{\omega_{-i}}\Big[|\Delta|f_i(\cdot|\omega_{-i})\Big]+\sum_{k=1}^{K}C_k'\sum_{i=1}^{m}\mathbb{E}_{\omega_{-i}}\Big[|\Delta|f_i(\cdot|\omega_{-i})\Big]\\ &= C\sum_{i=1}^{m}\mathbb{E}_{\omega_{-i}}\Big[|\Delta|f_i(\cdot|\omega_{-i})\Big],\end{aligned}$$

where the constant C is defined as $C := RD + \sum_{k=1}^{K} C_k'$. □

5. Examples of inexact cutting planes

In this section we consider examples of inexact cutting plane approximations that are asymptotically accurate. We derive inexact mixed-integer Gomory cuts in Section 5.1, and an inexact cutting plane approximation for a nurse scheduling problem in Section 5.2.

5.1. Inexact mixed-integer Gomory cuts

In this section we will derive inexact mixed-integer Gomory cuts that satisfy the first condition of Assumption 2. It can be shown, analogously to Theorem 2, that these inexact cuts are asymptotically accurate, or in fact yield an asymptotic lower bound.

Consider the second-stage value function

$$v(\omega,z) := \min_{y_B,y_N}\Big\{q_B^\top y_B + q_N^\top y_N :\ By_B + Ny_N = \omega - z,\ y_B\in Y_N,\, y_N\in Y_N\Big\},\quad \omega\in\Omega,\, z\in\mathbb{R}^m,$$

where similar as in Section 4.1, we let B denote a dual feasible basis matrix of the LP-relaxation of v. Multiplying the equality constraint in $v(\omega,z)$ by $e_i^\top B^{-1}$, where e_i is the i-th unit vector, we obtain

$$y_{B_i} + e_i^\top B^{-1}Ny_N = e_i^\top B^{-1}(\omega - z), \tag{15}$$

where y_{B_i} denotes the i-th basic variable. Let $\bar{w}_{ij}$ denote the j-th component of the vector $e_i^\top B^{-1}N$, let y_{N_j} denote the j-th non-basic variable, and let $r_i(\omega,z) := e_i^\top B^{-1}(\omega-z) - \lfloor e_i^\top B^{-1}(\omega-z)\rfloor$. If the i-th basic variable y_{B_i} is restricted to be integer, then we can derive from (15) the exact mixed-integer Gomory cut

$$\sum_{j\in J_1}\min\left\{\frac{-\bar{w}_{ij}-\lfloor -\bar{w}_{ij}\rfloor}{r_i(\omega,z)},\frac{\bar{w}_{ij}+\lceil -\bar{w}_{ij}\rceil}{1-r_i(\omega,z)}\right\}y_{N_j} + \sum_{j\in J_2}\max\left\{\frac{-\bar{w}_{ij}}{r_i(\omega,z)},\frac{\bar{w}_{ij}}{1-r_i(\omega,z)}\right\}y_{N_j}\geq 1, \tag{16}$$

where J_1 denotes the index set of integer non-basic variables y_{N_j} and J_2 the index set of continuous non-basic variables y_{N_j}; see e.g. [2].

Obviously, the exact mixed-integer Gomory cut in (16) is not affine in z, among others since $r_i(\omega,z)$ is not affine in z. However, if $z\in\beta\mathbb{Z}^m$, where $\beta := |\det(B)|e$ with e the all-one vector, then under the assumption that W is integer, we can show that $r_i(\omega,z) = r_i(\omega,0)$, and thus the mixed-integer Gomory cut in (16) does not depend on z. This is true, since for such z, we have

$$e_i^\top B^{-1} z = e_i^\top \Big(\det(B)^{-1} \operatorname{adj}(B) \Big) z \in \mathbb{Z},$$

and thus

$$r_i(\omega,z) = e_i^\top B^{-1}(\omega - z) - \lfloor e_i^\top B^{-1}(\omega - z) \rfloor = e_i^\top B^{-1}\omega - \lfloor e_i^\top B^{-1}\omega \rfloor = r_i(\omega,0).$$

Similarly, if $z \in \alpha + \beta\mathbb{Z}^m$ with $\beta := |\det(B)|e$, then $r_i(\omega,z) = r_i(\omega,\alpha)$. Thus, replacing $r_i(\omega,z)$ by $r_i(\omega,\alpha)$ in (16) yields an inexact mixed-integer Gomory cut that does not depend on z and is valid for all ω and for all z on a grid of points $\alpha + \beta\mathbb{Z}^m$. Hence, it satisfies the first condition of Assumption 2, and thus analoguously to Theorem 2 the inexact mixed-integer Gomory cut

$$\sum_{j \in J_1} \min\left\{ \frac{-\bar{w}_{ij} - \lfloor -\bar{w}_{ij} \rfloor}{r_i(\omega,\alpha)}, \frac{\bar{w}_{ij} + \lceil -\bar{w}_{ij} \rceil}{1 - r_i(\omega,\alpha)} \right\} y_{N_j} + \sum_{j \in J_2} \max\left\{ \frac{-\bar{w}_{ij}}{r_i(\omega,\alpha)}, \frac{\bar{w}_{ij}}{1 - r_i(\omega,\alpha)} \right\} y_{N_j} \geq 1,$$

is asymptotically valid for all $z \in \mathbb{R}^m$.

5.2. Nurse scheduling problem

In this section we will apply inexact cutting planes to a nurse scheduling problem, introduced by Kim and Mehrotra [9]. In this problem, a regular work schedule for the nurses is determined in the first stage, resulting in an available number z_t of nurses per time period $t = 1, \dots, T$. This regular work schedule is determined before the random demand ω_t for nurses per time period is known. Thus, it may turn out that we have a shortage or surplus of nurses in some of the time periods. In this case, it is possible to add or subtract nurse shifts, consisting of several consecutive time periods, after the demands ω_t are known. Moreover, we penalize any remaining nurse shortages and nurse surpluses using unit penalty costs per time period. The corresponding second-stage value function v is given by

$$\begin{aligned} v(\omega,z) = \min_{y,u_1,u_2} \quad & q^\top y + r_1^\top u_1 + r_2^\top u_2 \\ \text{s.t.} \quad & Wy - u_1 + u_2 = \omega - z \\ & y \in \mathbb{Z}_+^{n_2},\ u_1, u_2 \in \mathbb{R}^T, \end{aligned} \tag{17}$$

where $y \in \mathbb{Z}_+^{n_2}$ represents the possibility to add or subtract nurse shifts, and W is a $\{-1,0,1\}$-matrix, modelling which time periods are contained in which shift. Kim and Mehrotra [9] show that W is a totally unimodular matrix. Moreover, they show that if $z \in \mathbb{Z}^T$, then the cutting planes $Wy - \hat{D}(\omega)u_1 \leq \lfloor \omega \rfloor - z$, with $\hat{D}(\omega)$ a diagonal matrix with t-th diagonal component $\hat{D}_{tt}(\omega)$ equal to

$$\hat{D}_{tt}(\omega) = \frac{1}{1 - \omega_t + \lfloor \omega_t \rfloor}, \quad t = 1, \dots, T,$$

are valid for all $\omega \in \Omega$. In particular, combined with the constraints $Wy - u_1 + u_2 = \omega - z$, they completely define the integer hull $\bar{Y}(\omega, z)$ of the feasible region $Y(\omega, z)$ of $v(\omega, z)$. That is, for every $\omega \in \Omega$ and $z \in \mathbb{Z}^T$,

$$\bar{Y}(\omega, z) = \left\{ (y, u_1, u_2) \in \mathbb{R}_+^{n_2+2T} : \ Wy - u_1 + u_2 = \omega - z, \ Wy - \hat{D}(\omega) u_1 \leq \lfloor \omega \rfloor - z \right\}.$$

If we assume, contrary to [9], that z is not necessarily integral, then we may use the inexact cutting planes $Wy - \hat{D}(\omega)u_1 \leq \lfloor \omega \rfloor - z$ to derive the inexact cutting plane approximation

$$\begin{aligned} \hat{v}(\omega, z) = \min_{y, u_1, u_2} \quad & q^\top y + r_1^\top u_1 + r_2^\top u_2 \\ \text{s.t.} \quad & Wy - u_1 + u_2 = \omega - z \\ & Wy - \hat{D}(\omega) u_1 \leq \lfloor \omega \rfloor - z \\ & y \in \mathbb{R}_+^{n_2}, \ u_1, u_2 \in \mathbb{R}^T. \end{aligned}$$

Observe that $\hat{v}(\omega, z)$ satisfies Assumption 2, so that by Theorem 2, the corresponding inexact cutting plane approximation $\hat{Q}$ is asymptotically accurate. In Example 3 below, we numerically show the actual performance of this inexact cutting plane approximation for the one-dimensional second-stage value function of Example 1 in Section 2, which can be considered a special case of (17).

Example 3. Consider the second-stage value function $v(\omega, z)$ of Example 1,

$$\begin{aligned} v(\omega, z) = r(\omega - z) + \min_{y, u_1} \quad & (q - r)y + 2ru_1 \\ \text{s.t.} \quad & y - u_1 \leq \omega - z \\ & y \in \mathbb{Z}_+, \, u_1 \in \mathbb{R}_+, \end{aligned}$$

and its inexact cutting plane approximation defined in (7). Let ω be a normal random variable with mean μ and standard deviation σ. Then, as shown in Example 2, the total variation $|\Delta| f$ of the probability density function f of ω equals $|\Delta| f = \sigma^{-1}\sqrt{2/\pi}$. Figure 3 shows $\|Q - \hat{Q}\|_\infty$, the maximum difference between the expected value function Q and its inexact cutting plane approximation $\hat{Q}$, as a function of the standard deviation σ for $q = 1$ and $r = 2$. We observe that this difference decreases if σ increases. This is in line with Theorem 2, since the total variation $|\Delta| f$ of a normal probability density function f decreases if the standard deviation σ increases.

6. Discussion

We consider a new solution method for solving two-stage mixed-integer stochastic programs (MISPs). Instead of applying exact cuts to the second-stage feasible regions that are always valid,

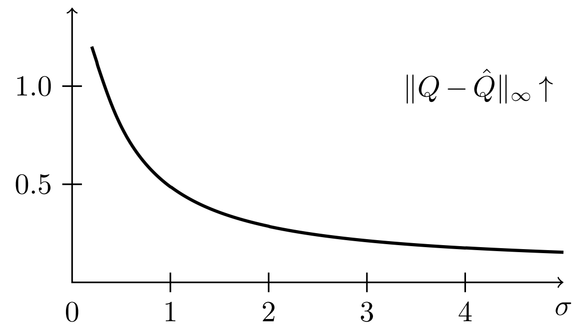

Figure 3 The maximum difference between Q and its inexact cutting plane approximation $\hat{Q}$ of Example 3, with $q=1$ and $r=2$, as a function of the standard deviation σ of a normal random variable ω.

we propose to use *inexact* cutting planes that are affine in the first-stage decision variables. The advantage is that the approximating problem, that uses these inexact cuts, is convex, and can thus be solved efficiently using techniques from convex optimization.

For simple integer recourse models, we show that we can obtain the α-approximations of Klein Haneveld et al. [10] using inexact cutting planes. A direct consequence of this result is that we obtain an error bound on the quality of the solution obtained using inexact cutting planes. This bound is small if the total variation of the probability density function of the random variable in the model is small. For general MISPs we show that under mild assumptions inexact cutting plane approximations are asymptotically accurate. For general MISPs we also derive inexact mixed-integer Gomory cuts, and we derive asymptotically accurate inexact cutting planes for a nurse scheduling problem. Numerical experiments show that the error of using the inexact cutting planes indeed converges to zero if the total variations of the random variables in the model go to zero.

A direction for future research is to derive problem-specific inexact cutting planes for specific applications of two-stage MISPs. Moreover, tighter error bounds may be derived for these problem-specific inexact cutting plane approximations using the special structure of the problems, similar as for simple integer recourse models. Another future research direction is to combine exact and inexact cutting planes to obtain more accurate approximations at the expense of increasing the computational effort of solving the approximation.

Appendix

Proof of Lemma 2. Since C is convex, every $z \in C$ can be written as a convex combination of its extreme points:

$$z = \sum_{j=1}^{J} \mu_j z^j,$$

with $\sum_{j=1}^J \mu_j = 1$, and $\mu_j \geq 0, j = 1, \dots, J$. Since v is convex, this implies that for all $z \in C$

$$v(z) = v\Big(\sum_{j=1}^J \mu_j z^j\Big) \leq \sum_{j=1}^J \mu_j v(z^j) = \sum_{j=1}^J \mu_j (a^\top z^j + b) = a^\top z + b. \tag{18}$$

To prove that also $v(z) \geq a^\top z + b$ for all $z \in C$, assume for contradiction that there exists $\bar{z} \in C$ such that $v(\bar{z}) < a^\top \bar{z} + b$. Since C is convex and z^0 is an interior point of C there exists $\epsilon > 0$ such that $\hat{z} := z^0 + \epsilon(z^0 - \bar{z}) \in C$. This point $\hat{z}$ is defined in such a way that z^0 can be written as a convex combination of $\bar{z}$ and $\hat{z}$:

$$z^0 = \frac{1}{1+\epsilon}\hat{z} + \frac{\epsilon}{1+\epsilon}\bar{z}.$$

Since v is convex, this implies that

$$v(z^0) \leq \frac{1}{1+\epsilon} v(\hat{z}) + \frac{\epsilon}{1+\epsilon} v(\bar{z}) < \frac{1}{1+\epsilon}(a^\top \hat{z} + b) + \frac{\epsilon}{1+\epsilon}(a^\top \bar{z} + b) = a^\top z^0 + b, \tag{19}$$

where we use that $v(\bar{z}) < a^\top \bar{z} + b$ by assumption and $v(\hat{z}) \leq a^\top \hat{z} + b$ by (18). Since (19) contradicts the assumption that $v(z^0) = a^\top z^0 + b$, we conclude that $v(z) = a^\top z + b$ for all $z \in C$. □

Proof of Proposition 1. Since $\hat{v}(\omega, z) = v(\omega, z)$ for all $\omega \in \Omega$ and $z \in \alpha + \beta\mathbb{Z}^m$, it follows from Theorem 3 that for every $k = 1, \dots, K$, there exists d_k such that for all $\omega \in \Omega$ and $z \in \alpha + \beta\mathbb{Z}^m$ with $\omega - z \in \Lambda^k(d_k)$,

$$\hat{v}(\omega, z) = q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - z).$$

Fix $\omega \in \Omega$. Then, $\psi^k(\omega - z)$ is B_k-periodic in z and thus $\psi^k(\omega - z) = \psi^k(\omega - z - \det(B_k)l)$ for every $l \in \mathbb{Z}^m$ by Lemma 4.8 in [15]. Define $\delta^k := \det(B_k)\beta \in \mathbb{Z}^m$ and let $l \in \mathbb{Z}^m$ be given, and consider the hyperrectangle $C^l(\alpha, \delta^k)$. Let z^j, $j = 1, \dots, J$, denote its extreme points and let $z^0 := (\alpha_1 + \delta_1^k l_1, \dots, \alpha_m + \delta_m^k l_m)$ be an interior point. If $C^l(\alpha, \delta^k) \subset \omega - \Lambda^k(d_k)$, then we can apply Lemma 2 with $a := -q_{B_k}^\top B_k^{-1}$, $b := q_{B_k}^\top B_k^{-1}\omega + \psi^k(\omega - \alpha)$, and $C := C^l(\alpha, \delta^k)$ to conclude that $\hat{v}(\omega, z) = q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - \alpha)$ for all $z \in C^l(\alpha, \delta^k)$. Since the diameter of $C^l(\alpha, \delta^k)$ is $2\|\delta^k\|$, we conclude that the result holds for all $\omega - z \in \Lambda^k(d_k + 2\|\delta^k\|)$. Indeed, $\omega - z$ will be in $\omega - C^l(\alpha, \delta^k)$ for some $l \in \mathbb{Z}^m$. The claim now follows by defining $d_k' := d_k + 2\|\delta^k\|$. □

Proof of Lemma 3. Fix $\omega \in \Omega$. Then, by Proposition 1 it follows that for every $k = 1, \dots, K$,

$$\hat{v}(\omega, z) = q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - \alpha), \quad z \in \omega - \Lambda^k(d_k').$$

Since $\hat{v}(\omega, z)$ is convex in z, and affine on $\omega - \Lambda^k(d_k')$, we can derive a subgradient inequality for each $k = 1, \dots, K$:

$$\hat{v}(\omega, z) \geq q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - \alpha), \quad z \in \mathbb{R}^m.$$

Combining these inequalities over all $k = 1, \dots, K$, yields the desired result. □

Proof of Lemma 4. Fix $\omega \in \Omega$. By Proposition 1, there exist distances $d'_k \geq 0$ such that for every $k = 1, \dots, K$,

$$\hat{v}(\omega, z) = q_{B_k}^\top B_k^{-1}(\omega - z) + \psi^k(\omega - \alpha), \quad z \in \omega - \Lambda^k(d'_k).$$

Thus, on the linearity regions $\omega - \Lambda^k(d'_k)$, $\hat{v}(\omega, z)$ equals its lower bound from Lemma 3. Therefore, we only have to show (11) for $z \in \omega - \mathcal{N}$. To this end, let $z \in \omega - \mathcal{N}$ be given. Since $\mathcal{N}$ can be covered by finitely many hyperslices, there exist $a_j \in \mathbb{R}^m \backslash \{0\}$ and $\delta_j > 0$, $j \in \mathcal{J}$, such that

$$\mathcal{N} \subset \bigcup_{j \in \mathcal{J}} H_j,$$

where $H_j := H(a_j, \delta_j)$, $j \in \mathcal{J}$. We will construct points z_1 and z_2 in the linearity regions of $\hat{v}(\omega, \cdot)$, so that z is a convex combination of z_1 and z_2. Then, we can use that $\hat{v}(\omega, z)$ is convex in z to derive an upper bound on $\hat{v}(\omega, z)$ in terms of $\hat{v}(\omega, z_1)$ and $\hat{v}(\omega, z_2)$. Since z_1 and z_2 are in the linearity regions, these values are known.

To construct such z_1 and z_2, let $d \in \mathbb{R}^m \backslash \{0\}$ be a direction of unit length not parellel to any of the hyperslices H_j, and thus not orthogonal to any of the normal vectors a_j, $j \in \mathcal{J}$. Then, $a_j^\top d \neq 0$, $j \in \mathcal{J}$, and $\|d\| = 1$. We consider the line through z with direction d and define the halflines L_1 and L_2 as

$$L_1 := \{z + \mu d : \mu \in \mathbb{R}_+\} \quad \text{and} \quad L_2 := \{z - \mu d : \mu \in \mathbb{R}_+\}.$$

Since the direction d is not parallel to any of the hyperslices, we have $L_1 \not\subset \bigcup_{j \in \mathcal{J}} H_j$ and $L_2 \not\subset \bigcup_{j \in \mathcal{J}} H_j$, and thus $L_i \cap (\omega - \bigcup_{k=1}^K \Lambda^k(d'_k)) \neq \emptyset$, $i = 1, 2$. This means that it is possible to select $z^1, z^2 \in \omega - \bigcup_{k=1}^K \Lambda^k(d'_k)$ on L_1 and L_2, respectively, with minimal distance to z:

$$z^i := \arg\min_{z'} \left\{ \|z - z'\| : z' \in L_i \cap \left(\omega - \bigcup_{k=1}^K \Lambda^k(d'_k) \right) \right\}, \quad i = 1, 2.$$

Since z is on the line segment between z^1 to z^2, we can write z as a convex combination $z = \mu z^1 + (1 - \mu) z^2$ of z^1 and z^2 with $\mu \in [0, 1]$. We will use the convexity of $\hat{v}(\omega, \cdot)$ to derive an upper bound on $\hat{v}(\omega, z)$. Here, we will assume without loss of generality that $z^1 \in \omega - \Lambda^{k_1}(d'_{k_1})$ and $z^2 \in \omega - \Lambda^{k_2}(d'_{k_2})$ with $k_1, k_2 \in \{1, \dots, K\}$. We obtain

$$\begin{aligned} \hat{v}(\omega, z) &\leq \mu \hat{v}(\omega, z^1) + (1 - \mu) \hat{v}(\omega, z^2) \\ &= \mu \Big(q_{B_{k_1}}^\top B_{k_1}^{-1}(\omega - z^1) + \psi^{k_1}(\omega - \alpha) \Big) + (1 - \mu) \Big(q_{B_{k_2}}^\top B_{k_2}^{-1}(\omega - z^2) + \psi^{k_2}(\omega - \alpha) \Big). \end{aligned}$$

To obtain the bound in (11) on the difference between $\hat{v}(\omega, z)$ and its lower bound, we subtract this lower bound from both the left- and right-hand side of the inequality above. Defining $k^* :=$

$\arg\max_{k=1,\ldots,K}\{q_{B_k}^\top B_k^{-1}(\omega-z)+\psi^k(\omega-\alpha)\}$, the difference between $\hat{v}(\omega,z)$ and its lower bound can then be bounded by

$$\mu\Big(q_{B_{k_1}}^\top B_{k_1}^{-1}(\omega-z^1)+\psi^{k_1}(\omega-\alpha)-q_{B_{k^*}}^\top B_{k^*}^{-1}(\omega-z)-\psi^{k^*}(\omega-\alpha)\Big)$$
$$+(1-\mu)\Big(q_{B_{k_2}}^\top B_{k_2}^{-1}(\omega-z^2)+\psi^{k_2}(\omega-\alpha)-q_{B_{k^*}}^\top B_{k^*}^{-1}(\omega-z)-\psi^{k^*}(\omega-\alpha)\Big).$$

Since k_1 and k_2 are not necessarily the maximizing index for $\max_{k=1,\ldots,K}\{q_{B_k}^\top B_k^{-1}(\omega-z)+\psi^k(\omega-\alpha)\}$, we may replace k^* by k_1 and k_2, respectively, to obtain after straightforward simplifications,

$$\hat{v}(\omega,z)-\max_{k=1,\ldots,K}\Big\{q_{B_k}^\top B_k^{-1}(\omega-z)+\psi^k(\omega-\alpha)\Big\}\le \mu q_{B_{k_1}}^\top B_{k_1}^{-1}(z^1-z)+(1-\mu)q_{B_{k_2}}^\top B_{k_2}^{-1}(z^2-z).$$

We will bound the right-hand side in terms of the distance $\|z^1-z^2\|$ between z^1 and z^2. Here, we use $\lambda_i^*:=\max_{k=1,\ldots,K}|q_{B_k}^\top(B_k)^{-1}e_i|$, for $i=1,\ldots,m$, where e_i is the i-th unit vector. We have

$$\begin{aligned}\hat{v}(\omega,z)-\max_{k=1,\ldots,K}\Big\{q_{B_k}^\top B_k^{-1}(\omega-z)+\psi^k(\omega-\alpha)\Big\}&\le \mu\sum_{i=1}^m\lambda_i^*|z_i^1-z_i|+(1-\mu)\sum_{i=1}^m\lambda_i^*|z_i^2-z_i|\\&\le \mu\sum_{i=1}^m\lambda_i^*\|z^1-z\|+(1-\mu)\sum_{i=1}^m\lambda_i^*\|z^2-z\|\\&\le \|z^1-z^2\|\sum_{i=1}^m\lambda_i^*,\end{aligned}\tag{20}$$

where the last inequality holds since z is on the line segment between z_1 to z_2, and thus $\|z^1-z\|\le\|z^1-z^2\|$ and $\|z^2-z\|\le\|z^1-z^2\|$.

It remains to derive an upper bound on $\|z^1-z^2\|$. To do so, observe that $\omega-z$ is on the line segment $\omega-L$ between $\omega-z^1$ and $\omega-z^2$. Moreover, in the worst-case this line segment is completely contained in the union of the hyperslices H_j, $j\in\mathcal{J}$. Hence,

$$\|z^1-z^2\|=\|(\omega-z^1)-(\omega-z^2)\|\le\|(\omega-L)\cap(\bigcup_{j\in\mathcal{J}}H_j)\|=\|\bigcup_{j\in\mathcal{J}}((\omega-L)\cap H_j)\|\le\sum_{j\in\mathcal{J}}\|(\omega-L)\cap H_j\|,$$

where $\|L\|$ denotes the total length of the line segments in L. To find $\|(\omega-L)\cap H_j\|$, observe that $\hat{z}\in L$ satisfies $\hat{z}=z-\hat{\mu}d$ for some $\hat{\mu}\in\mathbb{R}$. Moreover, $\omega-\hat{z}\in H_j:=H(a_j,\delta_j)$ if $0\le a_j^\top(\omega-z+\hat{\mu}d)\le\delta_j$, or equivalently if

$$\begin{cases}\dfrac{-a_j^\top(\omega-z)}{a_j^\top d}=:\underline{\mu}\le\hat{\mu}\le\overline{\mu}:=\dfrac{\delta_j-a_j^\top(\omega-z)}{a_j^\top d}, & \text{if } a_j^\top d>0,\\[2ex] \dfrac{\delta_j-a_j^\top(\omega-z)}{a_j^\top d}=:\underline{\mu}\le\hat{\mu}\le\overline{\mu}:=\dfrac{-a_j^\top(\omega-z)}{a_j^\top d}, & \text{if } a_j^\top d<0.\end{cases}$$

Then, $\|(\omega-L)\cap H_j\|=(\overline{\mu}-\underline{\mu})\|d\|=\frac{\delta_j}{|a_j^\top d|}$, where we use that $\|d\|=1$. Thus, by defining

$$R':=\Big(\sum_{i=1}^m\lambda_i^*\Big)\Big(\sum_{j\in\mathcal{J}}\frac{\delta_j}{|a_j^\top d|}\Big),$$

the claim follows from combining $\|z^1-z^2\|\le\sum_{j\in\mathcal{J}}\frac{\delta_j}{|a_j^\top d|}$ and (20). □

Proof of Proposition 2. Consider the LP-relaxation $v_{LP}(\omega, z)$ of $v(\omega, z)$ as defined in (12). Then, by, e.g., [3] and [6], there exists R'' such that $\|v - v_{LP}\|_\infty \leq R''$. Moreover, by combining Lemma 3 and 4, we conclude that $\|v_{LP} - \hat{v}\|_\infty \leq R' + \max_{k=1,\dots,K} \sup_{s\in\mathbb{R}^m} |\psi^k(s)|$. If we define $R := R'' + R' + \max_{k=1,\dots,K} \sup_{s\in\mathbb{R}^m} |\psi^k(s)|$, then

$$\|v - \hat{v}\| \leq \|v - v_{LP}\| + \|v_{LP} - \hat{v}\| \leq R'' + R' + \max_{k=1,\dots,K} \sup_{s\in\mathbb{R}^m} |\psi^k(s)| =: R,$$

where the first inequality follows from the triangle inequality. □

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