Bounds for Probabilistic Programming with Application to a Blend Planning Problem

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Abstract

In this paper, we derive deterministic inner approximations for single and joint probabilistic constraints based on classical inequalities from probability theory such as the one-sided Chebyshev inequality, Bernstein inequality, Chernoff inequality and Hoeffding inequality (see [25]). New assumptions under which the bounds based approximations are convex allowing to solve the problem efficiently are derived. When the convexity condition can not hold, an efficient sequential convex approximation approach is further proposed to solve the approximated problem. Piecewise linear and tangent approximations are also provided for Chernoff and Hoeffding inequalities allowing to reduce the computational complexity of the associated optimization problem. Extensive numerical results on a blend planning problem under uncertainty are finally provided allowing to compare the proposed bounds with the Second Order Cone (SOCP) formulation and Sample Average Approximation (SAA).

Keywords: stochastic programming, single chance-constraint, joint chance-constraints, bounds, blending problem.

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1. Introduction

Chance constrained optimization problems are an important class of optimization problems under uncertainty which involve constraints that are required to hold with specified probabilities. The reader is referred to [28, 32] for in-deep results and extensive reviews on the theory and applications of chance constrained optimization problems.

Several applications of chance constraints are considered in economics and finance [3], water reservoir management [2, 23], system optimization [11], the electrical industry [36], optimal power flow [35] and many others.

The main difficulty of this class of problems is that their feasible sets are generally non-convex. On this purpose [26] investigated a wide family of *logarithmically concave distributions*, showing that under this assumption the feasible set is convex. For the case of symmetric elliptical probability distributions, convex relaxations or reformulations via Second Order Cone Programming (SOCP) were proposed in [13, 6, 7, 5, 18].

In order to solve chance constrained problems efficiently, we need both the convexity of the corresponding feasible set and efficient computability of the considered probability [22]. This combination is rare, and very few are the cases in which chance constraints can be processed efficiently (see [8, 15, 29]). Moreover, when the random variables are not elliptically distributed, e.g., truncated distributions, SOCP cannot be used. Whenever this is the case, bounds and tractable approximations of chance constraints can be very useful.

A computationally tractable approximation of chance constrained problems could also be given by scenario approaches, based on Monte Carlo sampling techniques [19, 22, 24], where the probabilistic constraint is replaced by a sampled set of constraints. The sample size is chosen to guarantee that a solution to the sampled problem is feasible to the probabilistic constrained one with a high probability. See [21] for a survey of safe and scenario approximations of chance constraints.

An alternative to scenario approaches consists in providing bounds based on using deterministic analytical approximations of chance constraints. For the case of individual chance constraint, the bounds are mainly based on extensions of Chebyshev inequality which requires the knowledge only of the first two moments of the distribution [14, 25]. For joint chance constraints, deterministic equivalent approximations have been discussed in [6, 7, 5, 18] and for special distributions, such as the multivariate gamma, in [33]. In

[4] a new formulation for approximating joint chance constrained problems that improves upon the standard approach using Bonferroni inequality is proposed. The approach decomposes the joint chance constraint into a problem with individual chance constraints, and then applies safe robust optimization approximation on each one of them. Connections with bounds on the conditional-value-at-risk (CVaR) measure are also provided. Besides, in [22] a class of analytical approximations of single and joint independent chance constraints are developed and referred to as Bernstein approximations. Relaxations and approximations of linear chance constraints in the setting of a finite distribution of the stochastic parameters has been studied in [31, 27, 30, 8, 1, 9]. The case of integer programs with probabilistic constraints has been addressed in [10] and bounds via binomial moments proposed.

Relaxations for probabilistically constrained stochastic programming problems in which the random variables are in the right-hand sides of the stochastic inequalities defining the joint chance constraints are reviewed and provided in [16].

In this paper, we study deterministic inner approximations (restrictions) for single and joint probabilistic constraints. The derived upper bounds are based on classical inequalities from probability theory such as the one-sided Chebyshev inequality, Bernstein inequality, Chernoff inequality and Hoeffding inequality (see [25]). Notice that our proposed bounds do not require any particular assumption on probability distributions. We derive new assumptions under which the bounds based approximations of joint chance constrained problems are convex and the approximated problem can be optimized efficiently. When the convexity condition can not hold, an efficient sequential convex approximation approach is further proposed to solve the approximated problem. Piecewise linear and tangent approximations are also provided for Chernoff and Hoeffding inequalities allowing to reduce the computational complexity of the associated optimization problem. To the best of our knowledge, these results are new in the literature since the majority of the abovementioned contributions deals with symmetric elliptical distributions.

The bounds are tested on a refinery multiproduct blend planning problem with uncertain raw materials qualities, where each product is required to be on specification with high probability (see [34]). The associated optimization model is a joint probabilistic constrained optimization model in which the number of joint independent chance constraints equals the number of final products. In each joint chance constraint the uncertainty appears in the left-hand side coefficient matrix having independent matrix vector rows, and it ensures that all the qualities are on specification with high probability. Extensive numerical results on this problem are provided allowing to compare the proposed bounds with the Second Order Cone (SOCP) formulation for individual chance constraints and Sample Average Approximation (SAA) for joint chance constraints.

The main contributions and research questions of the paper can be summarized as follows:

- to review inner approximations for individual chance constraints based on classical probabilistic inequalities such as the one-sided Chebyshev, Chernoff, Bernstein and Hoeffding inequalities (see [25]);
- to extend such approximations to the joint independent case;
- to derive new sufficient conditions under which the aforementioned approximations are tractable via logarithmic transformation;
- to propose a sequential convex approximation method for the cases in which a logarithmic transformation cannot be applied;
- to provide an extensive numerical campaign based on a blending problem, with the aim of:
  - understanding the performance of the considered inner approximations in terms of percentage GAP, CPU time and optimal blending recipes with respect to the exact SOCP reformulation for the single chance constraint case and with respect to the Sample Average Approximations (SAA) method for the joint chance constraint case;
  - reducing the computational complexity of the derived problems via a piecewise linear approximation based on tangent and segment approximation;
  - analyzing the sensitivity of the solution for increasing values of considered products qualities;
  - analyzing the performance of the solutions obtained using the aforementioned bounds over the realization of different probability distributions including truncated distributions;

– providing managerial insights on the usage of bounds.

The paper is organized as follows: Section 2 investigates how to derive Chebyshev, Chernoff, Bernstein and Hoeffding bounds both for individual and joint probabilistic constrained problems. A refinery blending problem under uncertainty is described in Section 3 and a probabilistic constrained model for this problem presented. Numerical results on the refinery blend planning problem are in Section 4. Conclusions follow.

2. Bounds for probabilistic constrained problems

We consider the following joint chance constrained linear program:

$$\begin{aligned} \min_{x} \quad & c^T x \\ s.t. \quad & \mathbb{P}\left\{\Xi x \le H\right\} \ge \alpha, \\ & x \in X, \end{aligned} \tag{1}$$

where $H = (h_1, \dots, h_K) \in \mathbb{R}^K$, $\Xi = [\xi_1, \dots, \xi_K]^T$ is a $K \times n$ random matrix, with ξ_k, $k = 1, \dots, K$ a random vector in $\mathbb{R}^n$. We denote with $\mathbb{P}$ a probability measure, x a decision vector with feasible set $X \subseteq \mathbb{R}^n_+$, $c \in \mathbb{R}^n$ and $0 < \alpha < 1$ a prespecified confidence parameter. Notice that the objective function parameters c can also be considered as random variables. For the sake of simplicity we replace them by their means. Our goal is to come up with a deterministic equivalent problem of (1) such that the feasible set $S(\alpha) := \{x \in X : \mathbb{P}\left\{\Xi^T x \le H\right\} \ge \alpha\}$ of (1) is convex.

In particular, an individual chance constrained problem can be written as follows:

$$\begin{aligned} \min_{x} \quad & c^T x \\ s.t. \quad & \mathbb{P}\left\{\xi^T x \le h\right\} \ge \alpha, \\ & x \in X. \end{aligned} \tag{2}$$

If we consider the case of a multivariate normally distributed vectors ξ with mean $\bar{\xi} = \mathbb{E}(\xi)$ and positive definite variance-covariance matrix Σ, the following relations hold true:

$$\mathbb{P}(\xi^T x \le h) \ \ge \ \alpha, \tag{3}$$

$$\Updownarrow$$

$$\bar{\xi}^T x + F^{-1}(\alpha) \left\|\Sigma^{1/2} x\right\| \ \le \ h, \tag{4}$$

where $F^{-1}(\cdot)$ is the inverse of F, the standard normal cumulative distribution function. The same scheme can be applied to elliptical distributions, e.g., normal distribution, Laplace distribution, t-Student distribution, Cauchy distribution, Logistic distribution [6, 18].

When the probability distributions are not elliptical or not known in advance, lower and upper bounds on the individual or joint chance constraint, can be very useful. We will investigate them in the following sections.

2.1. Chebyshev and Chernoff Bounds

In the following, we provide bounds for problems (1) and (2) based on deterministic approximations of probabilistic inequalities such as the one-sided Chebyshev and Chernoff inequalities.

2.1.1. Chebychev bounds

We consider the *one-sided Chebyshev* inequality [25, 17]. We assume that ξ has finite second moments and denote $\sigma_\xi^2 = Var(\xi)$ and $\bar{\xi} = \mathbb{E}(\xi)$ its mean. The one-sided Chebyshev inequality is given by

$$\mathbb{P}(\xi - \bar{\xi} \geq h) \leq \frac{\sigma_\xi^2}{\sigma_\xi^2 + h^2}. \tag{5}$$

For the individual chance constraint problem (2), we have the following result:

Proposition 1. Assume that ξ has finite first and second moments with variance-covariance matrix Σ and mean $\bar{\xi}$. Under one-sided Chebyshev inequality (5), an inner approximation of Problem (2) is obtained as follows

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & \bar{\xi}^T x + \sqrt{\frac{\alpha}{1-\alpha}} \left\| \Sigma^{1/2} x \right\| \leq h, \\ & x \in X. \end{aligned} \tag{6}$$

Moreover, (6) is a convex problem.

Proof. First, we note that, assuming $\mathbb{P}(\xi^T x = h) = 0$ a.s., then

$$\begin{aligned} \mathbb{P}\left(\xi^T x \leq h\right) &\geq \alpha, \quad (7)\\ &\Updownarrow \\ \mathbb{P}\left(\xi^T x \geq h\right) &\leq 1-\alpha, \quad (8)\\ &\Updownarrow \\ \mathbb{P}\left(\xi^T x - \bar{\xi}^T x \geq h - \bar{\xi}^T x\right) &\leq 1-\alpha. \quad (9)\end{aligned}$$

Then, we apply (5) to (9):

$$\mathbb{P}\left(\xi^T x - \bar{\xi}^T x \geq h - \bar{\xi}^T x\right) \leq \frac{\sigma_\xi^2}{\sigma_\xi^2 + (h - \bar{\xi}^T x)^2}, \quad (10)$$

where $\sigma_\xi^2 = x^T \Sigma x$ with variance-covariance matrix Σ. If $\frac{\sigma_\xi^2}{\sigma_\xi^2 + (h - \bar{\xi}^T x)^2} \leq 1-\alpha$, then (7) will be satisfied. Therefore,

$$\frac{x^T \Sigma x}{x^T \Sigma x + (h - \bar{\xi}^T x)^2} \leq 1-\alpha, \Longleftrightarrow \frac{\alpha}{1-\alpha} x^T \Sigma x \leq (h - \bar{\xi}^T x)^2,$$

which is equivalent to

$$\sqrt{\frac{\alpha}{1-\alpha}} \left\| \Sigma^{1/2} x \right\| \leq h - \bar{\xi}^T x. \quad (11)$$

□

In the following, we extend our results to the case of independent joint chance constraints. In particular, if ξ_k, $k = 1, \ldots, K$ are independent row vectors, $\mathbb{P}\left\{\Xi x \leq H\right\} \geq \alpha$ is equivalent to

$$\prod_{k=1}^{K} \mathbb{P}\left\{\xi_k^T x \leq h_k\right\} \geq \alpha = \prod_{k=1}^{K} \alpha^{y_k}, \quad (12)$$

with $\sum_{k=1}^{K} y^k = 1$, $y_k \geq 0$, $k = 1, \ldots, K$ and $y = (y_1, \ldots, y_K)^T$.

We now provide an upper bound to problem (1) based on the *one-sided Chebyshev* inequality. We assume that ξ_k, $k = 1, \ldots, K$ has finite second moments. Let $\sigma_{\xi_k} = Var(\xi_k)$ be the variance of ξ_k with variance-covariance matrix Σ_k and $\bar{\xi}_k = \mathbb{E}(\xi_k)$ its mean. The following result holds true:

Proposition 2. Based on one-sided Chebyshev inequality, an inner approximation of problem (1) can be obtained by solving the following non linear optimization problem

$$\begin{aligned} \min_{x,y} \quad & c^T x \\ \text{s.t.} \quad & \bar{\xi}_k^T x + \sqrt{\frac{\alpha^{y_k}}{1-\alpha^{y_k}}} \left\| \Sigma_k^{1/2} x \right\| \leq h_k, \ k = 1, \dots, K, \\ & \sum_{k=1}^{K} y_k = 1, \ y_k \geq 0, \ k = 1, \dots, K, \ x \in X. \end{aligned} \tag{13}$$

Proof. From the reformulation (12) and the proof of Proposition 1, we can immediately obtain the conclusion. □

Assumption 1. $X = \mathbb{R}_+^n \cap \mathbf{L}$, $\mathbf{L}$ is selected such that $Z = \{z \in \mathbb{R}^n : z_j = \ln(x_j), j = 1, \dots, n, x \in \mathbf{L}\}$ is convex.

Problem (13) is not convex but biconvex (see [12] for the definition) because of the first group of constraints. To come-up with a tractable convex reformulation, with Assumption 1, we use the following logarithmic transformation $z = \ln x$. In this case, Problem (13) can be reformulated as follows

$$\begin{aligned} \min_{z,y} \quad & c^T e^z \\ \text{s.t.} \quad & \bar{\xi}_k^T e^z + \left\| \Sigma_k^{1/2} e^{\ln\left(\sqrt{\frac{\alpha^{y_k}}{1-\alpha^{y_k}}}\right) \cdot e_n + z} \right\| \leq h_k \ k = 1, \dots, K, \\ & \sum_{k=1}^{K} y_k = 1, \ y_k \geq 0, \ k = 1, \dots, K, \ z \in Z, \end{aligned} \tag{14}$$

where e_n is an $n \times 1$ vector of ones.
We now prove that problem (14) is convex for all $\alpha \in [0, 1]$.

Lemma 2. *Given the sets X, Y, Z with X, Y convex sets; let $f : X \to Y$ be a convex function in C^2 and $g : Y \to Z$ be a nonincreasing concave function in C^2. Then, $g \circ f : X \to Z$ is a concave function.*

The proof is given in Appendix A.

Assumption 3. $c \geq 0$. For each $k = 1, \dots, K$, all the components of $\bar{\xi}_k$ and $\Sigma_k^{1/2}$ are non-negative.

Proposition 3. If Assumption 3 holds, then problem (14) is convex for all $\alpha \in [0, 1]$.

Proof. To show the convexity of problem (14), we firstly need to show the convexity of $\ln\left(\sqrt{\frac{\alpha^{y_k}}{1-\alpha^{y_k}}}\right)$ in y_k, when $\sqrt{\frac{\alpha^{y_k}}{1-\alpha^{y_k}}} \geq 0$. As $\ln\left(\sqrt{\frac{\alpha^{y_k}}{1-\alpha^{y_k}}}\right) = \frac{1}{2}\left(y_k \ln \alpha - \ln\left(1 - \alpha^{y_k}\right)\right)$, we can deduce the convexity of function $\ln\left(\sqrt{\frac{\alpha^{y_k}}{1-\alpha^{y_k}}}\right)$ if the term $\ln\left(1 - \alpha^{y_k}\right)$ is concave.

Since $p \mapsto \ln(1-p)$ is decreasing and concave with respect to p and $y_k \mapsto \alpha^{y_k}$ is convex with respect to y_k, we have that $\ln\left(1 - \alpha^{y_k}\right)$ is concave with respect to y_k as shown by Lemma 2.

Since the norm is a convex function and it is also a nondecreasing function on nonnegative space, then the composition function $\left\|\Sigma_k^{1/2} e^{\ln\left(\sqrt{\frac{\alpha^{y_k}}{1-\alpha^{y_k}}}\right)\cdot e_n + z}\right\|$ is a convex function. Moreover, the function $z \mapsto \bar{\xi}_k^T e^z$ in problem (14) is convex because $\bar{\xi}_k \geq 0$. Hence, the problem (14) is convex for all $\alpha \in [0, 1]$. □

When Assumption 1 does not hold, a natural approach is to apply the sequential convex approximation by adjusting $y_k, k = 1, \ldots, K$ with respect to x. Following [18], given y_k^* such that $\sum_{k=1}^K y_k^* = 1,\ y_k^* \geq 0$, we first fix $y_k = y_k^*$ and obtain x^* by solving

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & \bar{\xi}_k^T x + \sqrt{\frac{\alpha^{y_k^*}}{1-\alpha^{y_k^*}}} \left\|\Sigma_k^{1/2} x\right\| \leq h_k,\ k = 1, \ldots, K, \\ & x \in X, \end{aligned} \tag{15}$$

and then fix $x = x^*$ and update y_k by solving

$$\begin{aligned} \min_y \quad & \psi^T y \\ \text{s.t.} \quad & \sqrt{\frac{\alpha^{y_k}}{1-\alpha^{y_k}}} \leq \frac{h_k - \bar{\xi}_k^T x^*}{\left\|\Sigma_k^{1/2} x^*\right\|},\ k = 1, \ldots, K, \\ & \sum_{k=1}^K y_k = 1,\ y_k \geq 0,\ k = 1, \ldots, K. \end{aligned} \tag{16}$$

Here, ψ is a chosen searching direction. The constraint

$$\sqrt{\frac{\alpha^{y_k}}{1-\alpha^{y_k}}} \leq \frac{h_k - \bar{\xi}_k^T x^*}{\left\|\Sigma_k^{1/2} x^*\right\|},$$

can be reformulated as

$$y_k \geq \log_\alpha \frac{(\tau_k^n)^2}{1+(\tau_k^n)^2},$$

where $\tau_k^n = \frac{h_k - \bar{\xi}_k^T x^*}{\left\|\Sigma_k^{1/2} x^*\right\|}$.

Theorem 2 in [18] shows the convergence of this algorithm, which provides an upper bound for problem (13).

2.1.2. Chernoff bounds

We consider now the *Chernoff bound*:

$$\mathbb{P}(\xi \geq h) \leq \frac{\mathbb{E}(e^{t\xi})}{e^{th}}, \tag{17}$$

where $\mathbb{E}(e^{t\xi})$ is the moment generating function of the random variable ξ and $t > 0$. We denote with $\bar{\xi}$ the mean of ξ and with $\sigma^2 = Var(\xi)$ its variance.

First, we proof the convexity of $\mathbb{E}(e^{t\xi^T x})$.

Lemma 4. *For any $t > 0$, $x \mapsto \mathbb{E}(e^{t\xi^T x})$ is a convex function.*

The proof is given in Appendix A.

Proposition 4. If ξ follows a normal distribution with mean vector $\bar{\xi}$ and variance-covariance matrix Σ, under Chernoff bound (17), an inner approximation of Problem (2) is obtained as follows

$$\begin{aligned}
\min_x \quad & c^T x \\
\text{s.t.} \quad & \bar{\xi}^T x + \sqrt{2 \ln \frac{1}{(1-\alpha)}} \left\|\Sigma^{1/2} x\right\| \leq h, \\
& x \in X.
\end{aligned} \tag{18}$$

Moreover, Problem (18) is a convex optimization problem.

Proof. First, we have from (8)

$$\mathbb{P}(\xi^T x \leq h) \geq \alpha \Longleftrightarrow \mathbb{P}(\xi^T x \geq h) \leq 1-\alpha.$$

This implies

$$\mathbb{P}(\xi^T x \geq h) \leq \frac{\mathbb{E}(e^{t\xi^T x})}{e^{th}}. \tag{19}$$

Given $t > 0$, if we choose $\frac{\mathbb{E}(e^{t\xi^T x})}{e^{th}} \leq 1-\alpha$, then we get an upper bound to problem (2) with feasible region

$$\bar{S}(\alpha) = \left\{x \in X \subseteq \mathbb{R}^n_+ | \mathbb{E}(e^{t\xi^T x}) \leq (1-\alpha)e^{th}\right\}, \tag{20}$$

which is convex as $x \mapsto \mathbb{E}(e^{t\xi^T x})$ is convex, as shown by Lemma 4.

If ξ is a normal distribution with mean $\bar{\xi}$ and variance-covariance Σ, i.e. $\xi \sim N(\bar{\xi}, \Sigma)$ then in (20) we have $\mathbb{E}(e^{t\xi^T x}) = e^{t\bar{\xi}^T x} \cdot e^{\frac{1}{2}x^T \Sigma x t^2}$. The feasible region $\bar{S}(\alpha)$ can be written as:

$$\bar{S}(\alpha) = \left\{x \in X \subseteq \mathbb{R}^n_+ | \exists\, t > 0 : \frac{1}{2}x^T \Sigma x t^2 + t\bar{\xi}^T x - th \leq \ln(1-\alpha)\right\}. \tag{21}$$

The set (21) is equivalent to:

$$\inf_{t>0} \left\{\frac{1}{2}x^T \Sigma x t^2 + t\bar{\xi}^T x - th\right\} \leq \ln(1-\alpha). \tag{22}$$

From $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}x^T \Sigma x t^2 + t\bar{\xi}^T x - th) = 0$, we get $t = \frac{h-\bar{\xi}^T x}{x^T \Sigma x}$. Since $t > 0$ we require $h - \bar{\xi}^T x > 0$.

Therefore (21) is equivalent to

$$\bar{S}(\alpha) = \left\{x \in X \subseteq \mathbb{R}^n_+ | -(h-\bar{\xi}^T x)^2 \leq 2\ln(1-\alpha) x^T \Sigma x\right\}, \tag{23}$$

which is equivalent to the following convex set:

$$\bar{S}(\alpha) = \left\{x \in X \subseteq \mathbb{R}^n_+ | h - \bar{\xi}^T x \geq \sqrt{2\ln\frac{1}{(1-\alpha)}} \|\Sigma^{1/2} x\|\right\}. \tag{24}$$

□

We extend our results to the case of independent joint chance constraints.

If we assume that ξ_k, $k = 1, \ldots, K$ are multivariate normally distributed independent row vectors with mean vector $\bar{\xi}_k = (\bar{\xi}_{k1}, \ldots, \bar{\xi}_{kn})^T$ and covariance matrix Σ_k, we can derive a deterministic reformulation of problem (1) based on (12). We consider now an upper bound to problem (1) based on *Chernoff bound.*

Proposition 5. If ξ_k, $k = 1, \ldots, K$ are pairwise independent and normally distributed with mean vector $\bar{\xi}_k$ and covariance matrix Σ_k, based on Chernoff bound, an inner approximation of Problem (1) is obtained by solving the following problem

$$\begin{aligned} \min_{z,y} \quad & c^T x \\ \text{s.t.} \quad & \bar{\xi}_k^T x + \sqrt{2 \ln \left(\frac{1}{1 - \alpha^{y_k}} \right)} \|\Sigma_k^{1/2} x\| \leq h_k, \ k = 1, \ldots, K, \\ & \sum_{k=1}^{K} y_k = 1, \ y_k \geq 0, \ k = 1, \ldots, K, \ x \in X. \end{aligned} \tag{25}$$

Proof. First, we note that

$$\mathbb{P}(\xi_k^T x \leq h_k) \geq \alpha^{y_k} \Longleftrightarrow \mathbb{P}(\xi_k^T x \geq h_k) \leq 1 - \alpha^{y_k}, \ k = 1, \ldots, K.$$

Chernoff bound leads to

$$\mathbb{P}(\xi_k^T x \geq h_k) \leq \frac{\mathbb{E}(e^{t\xi_k^T x})}{e^{t h_k}}, \ k = 1, \ldots, K, \tag{26}$$

given $t > 0$. An upper bound to problem (1) is then obtained by solving the following problem for a given $t > 0$:

$$\begin{aligned} \min_{z,y} \quad & c^T x \\ \text{s.t.} \quad & \mathbb{E}(e^{t\xi_k^T x}) \leq (1 - \alpha^{y_k}) e^{t h_k}, \ k = 1, \ldots, K, \\ & \sum_{k=1}^{K} y_k = 1, \ y_k \geq 0, \ k = 1, \ldots, K, \ x \in X. \end{aligned} \tag{27}$$

However, if the probability distributions of ξ_k, $k = 1, \ldots, K$ are not known, the main difficulty of the model (27) is given by the computation of $\mathbb{E}(e^{t\xi_k^T x})$. On the other hand, if we assume ξ_k, $k = 1, \ldots, K$ are normal distributions with mean $\bar{\xi}_k$ and variance-covariance Σ_k, i.e. $\xi_k \sim N(\bar{\xi}_k, \Sigma_k)$, then we have that $\mathbb{E}(e^{t\xi_k^T x}) = e^{t\bar{\xi}_k^T x} \cdot e^{\frac{1}{2}x^T \Sigma_k x t^2}$, $k = 1, \ldots, K$. Consequently problem (27) can be written as

$$
\begin{aligned}
\min_{z,y} \quad & c^T x \\
\text{s.t.} \quad & \frac{1}{2} x^T \Sigma_k x t^2 + t\bar{\xi}_k^T x - t h_k \leq \ln(1 - \alpha^{y_k}), \ k = 1, \ldots, K, \\
& \sum_{k=1}^{K} y_k = 1, \ y_k \geq 0, \ k = 1, \ldots, K, \ x \in X.
\end{aligned} \tag{28}
$$

Similarly to the individual chance constraint case, we have:

$$
\begin{aligned}
\min_{z,y} \quad & c^T x \\
\text{s.t.} \quad & h_k - \bar{\xi}_k^T x \geq \sqrt{2 \ln \left(\frac{1}{1 - \alpha^{y_k}} \right)} \|\Sigma_k^{1/2} x\|, \ k = 1, \ldots, K, \\
& \sum_{k=1}^{K} y_k = 1, \ y_k \geq 0, \ k = 1, \ldots, K, \ x \in X.
\end{aligned}
$$

□

Problem (25) is not a convex optimization problem. Therefore, with Assumption 1, we apply the transformation $z = \ln x$ and get:

$$
\begin{aligned}
\min_{z,y} \quad & c^T e^z \\
\text{s.t.} \quad & \bar{\xi}_k^T e^z + \left\| \Sigma_k^{1/2} e^{\ln\left(\sqrt{2\ln\left(\frac{1}{1-\alpha^{y_k}}\right)}\right) + z} \right\| \leq h_k, \ k = 1, \ldots, K, \\
& \sum_{k=1}^{K} y_k = 1, \ y_k \geq 0, \ k = 1, \ldots, K, \ z \in Z.
\end{aligned} \tag{29}
$$

Moreover, if $\bar{\xi}_k \geq 0, \ k = 1, 2, \ldots, K$, and the function $\ln\left(\sqrt{2\ln(\frac{1}{1-\alpha^{y_k}})}\right)$ is convex, then Problem (29) is convex. The following lemma shows the convexity of $\ln\left(\sqrt{2\ln(\frac{1}{1-\alpha^{y_k}})}\right)$.

Lemma 5. *If* $\alpha \geq 1-e^{-1} \approx 0.6321$*, then the function* $y_k \mapsto \ln\left(\sqrt{2\ln(\frac{1}{1-\alpha^{y_k}})}\right)$ *is convex.*

The proof is given in Appendix A. Therefore, when $c \geq 0$, $\alpha \geq 1-e^{-1}$, $\bar{\xi}_k \geq 0,\ k = 1, 2, \ldots, K$, problem (29) is convex.

When Assumption 1 and the convexity condition for problem (29) do not hold, a sequential convex approximation algorithm can be applied to problem (25) in a similar way of what proposed before for Chebyshev bounds.

Notice that SOCP requires the inverse of a CDF which is typically a difficult function to deal with in the case of joint chance constraints. However, Chernoff bound doesn't require any CDF function, and its main advantage is then its easy-use especially from implementation point of view (the CDF is not implemented in current software platform developments). Moreover, the Chernoff bound can be applied with any distribution if the corresponding generated moment function can be reformulated explicitly.

2.2. Bernstein and Hoeffding Bounds

Bernstein and Hoeffding bounds are considered as exponential type estimates of probabilities. These inequalities are frequently used for investigating for instance the law of large numbers. They are also often used in statistics and probability theory. In this section, we investigate these bounds for the case of individual (2) and joint chance constraints (1) and we apply them to the blend planning problem.

2.2.1. Bernstein bounds

In this section, we consider *Bernstein bound* [25]. We assume that the mean and the range parameters for all independent components ξ_i of the random vector ξ are known, i.e. $l_i \leq \xi_i \leq u_i$, and $\mathbb{E}(\xi_i) = \bar{\xi}_i$, for $i = 1, \ldots, n$. The Bernstein-type exponential estimate, given by

$$e^{-g^* h} \prod_{i=1}^{n} \left\{ \frac{u_i - \bar{\xi}_i}{u_i - l_i} e^{g^* l_i} + \frac{\bar{\xi}_i - l_i}{u_i - l_i} e^{g^* u_i} \right\} \leq \alpha, \tag{30}$$

with arbitrary $g^* > 0$, implies $\mathbb{P}(\sum_{i=0}^{n} \xi_i \geq h) \leq \alpha$.

Proposition 6. Given a random vector ξ with independent components ξ_i such that $l_i \leq \xi_i \leq u_i$, and $\mathbb{E}(\xi_i) = \bar{\xi}_i$, for $i = 1, \ldots, n$, an upper bound for problem (2) can be obtained by solving the following problem

$$\begin{aligned}
\min_{x} \quad & c^T x \\
\text{s.t.} \quad & \sum_{i=1}^{n} \ln \left\{ \frac{u_i - \bar{\xi}_i}{u_i - l_i} e^{g^* l_i x_i} + \frac{\bar{\xi}_i - l_i}{u_i - l_i} e^{g^* u_i x_i} \right\} \leq \ln(1-\alpha) + g^* h, \\
& x \in X
\end{aligned} \tag{31}$$

with arbitrary constant $g^* > 0$.

Proof. Applying Bernstein inequality to the chance constraint in (2), and passing to the logarithm both sides, the proof follows. □

We provide now an upper bound to Problem (1) based on the Bernstein inequality.

Proposition 7. Given the random vectors ξ_k for $k = 1, \dots, K$, with independent components $(\xi_k)_i$ such that $(l_k)_i \leq (\xi_k)_i \leq (u_k)_i$, and $\mathbb{E}[(\xi_k)_i] = (\bar{\xi}_k)_i$, for $k = 1, \dots, K$ and $i = 1, \dots, n$, an inner approximation of Problem (2) can be obtained by solving the following problem

$$\begin{aligned}
\min_{x,y} \quad & c^T x \\
\text{s.t.} \quad & \sum_{i=1}^{n} \ln \left\{ \frac{(u_k)_i - (\bar{\xi}_k)_i}{(u_k)_i - (l_k)_i} e^{g_k^* (l_k)_i x_i} + \frac{(\bar{\xi}_k)_i - (l_k)_i}{(u_k)_i - (l_k)_i} e^{g_i^* (u_k)_i x_i} \right\} \leq g_k^* h_k + \\
& \ln(1 - \alpha^{y_k}),\ k = 1, \dots, K, \\
& \sum_{k=1}^{K} y_k = 1,\ y_k \geq 0,\ k = 1, \dots, K,\ x \in X,
\end{aligned} \tag{32}$$

with arbitrary constants $g_k^* > 0$, $k = 1, \dots, K$.

Proof. According to Bernstein-type exponential estimate, the condition

$$e^{-g_k^* h_k} \prod_{i=1}^{n} \left\{ \frac{(u_k)_i - (\bar{\xi}_k)_i}{(u_k)_i - (l_k)_i} e^{g_k^* (l_k)_i x_i} + \frac{(\bar{\xi}_k)_i - (l_k)_i}{(u_k)_i - (l_k)_i} e^{g_k^* (u_k)_i x_i} \right\} \leq \alpha^{y_k}, \tag{33}$$

with arbitrary $g_k^* > 0$, implies $\mathbb{P}(\sum_{i=0}^{n} (\xi_k)_i x_i \geq h_k) \leq \alpha^{y_k}$, $k = 1, \dots, K$. We note that

$$\mathbb{P}\left(\xi_k^T x \leq h_k\right) \geq \alpha^{y_k},\ k = 1, \dots, K, \tag{34}$$

$$\Updownarrow$$

$$\mathbb{P}\left(\sum_{i=1}^{n} (\xi_k)_i x_i \geq h_k\right) \leq 1 - \alpha^{y_k},\ k = 1, \dots, K. \tag{35}$$

Problem (35) can be approximated as

$$\sum_{i=1}^{n} \ln \left\{ \frac{(u_k)_i - (\bar{\xi}_k)_i}{(u_k)_i - (l_k)_i} e^{g_k^*(l_k)_i x_i} + \frac{(\bar{\xi}_k)_i - (l_k)_i}{(u_k)_i - (l_k)_i} e^{g_k^*(u_k)_i x_i} \right\} \leq \ln(1 - \alpha^{y_k}) + g_k^* h_k,$$

for any $g_k^* > 0$, $k = 1, \ldots, K$. □

From Proposition 4.1 in [25] and the concavity of function $\ln(1 - \alpha^{y_k})$, problem (32) is convex.

2.2.2. Hoeffding bounds

We consider now an approximation based on *Hoeffding inequality* ([14]) given as follows:

$$\mathbb{P}(\frac{\xi^T e_n}{n} - \frac{\bar{\xi}^T e_n}{n} \geq h) \leq e^{\frac{-2n^2h^2}{\sum_{i=1}^{n}(u_i - l_i)^2}}, \tag{36}$$

with l_i, u_i the range parameters of the independent components ξ_i of the random vector ξ, i.e. $l_i \leq \xi_i \leq u_i$, $i = 1, \ldots, n$, $\bar{\xi} = \mathbb{E}(\xi)$ and $e_n \in \mathbb{R}^n$ is a vector with all elements equal to 1.

Proposition 8. With the assumption of ξ mentioned above, an inner approximation of Problem (2) can be obtained by solving the following convex problem

$$\begin{aligned} \min_{x} \quad & c^T x \\ \text{s.t.} \quad & \bar{\xi}^T x + \frac{\sqrt{2}}{2}\sqrt{-\ln(1-\alpha)}\|Mx\| \leq h, \\ & x \in X, \end{aligned} \tag{37}$$

where $M = diag(u - l)$, $u = (u_1, \ldots, u_n)^T$, $l = (l_1, \ldots, l_n)^T$.

Proof. We note that

$$\mathbb{P}\left(\xi^T x \leq h\right) \geq \alpha, \tag{38}$$

$$\Updownarrow$$

$$\mathbb{P}\left(\xi^T x - \bar{\xi}^T x \geq h - \bar{\xi}^T x\right) \leq 1 - \alpha. \tag{39}$$

Then, we apply (36) to (39) and get:

$$\mathbb{P}\left(\xi^T x - \bar{\xi}^T x \geq h - \bar{\xi}^T x\right) \leq e^{\frac{-2(h-\bar{\xi}^T x)^2}{\sum_{i=1}^n (u_i - l_i)^2 x_i^2}}. \tag{40}$$

If

$$e^{\frac{-2(h-\bar{\xi}^T x)^2}{\sum_{i=1}^n (u_i - l_i)^2 x_i^2}} \leq 1-\alpha, \tag{41}$$

then (38) will be satisfied. Logarithmic transformation of (41) leads to

$$\frac{-2(h-\bar{\xi}^T x)^2}{\sum_{i=1}^n (u_i - l_i)^2 x_i^2} \leq \ln(1-\alpha), \tag{42}$$

which can be written as

$$h - \bar{\xi}^T x \geq \frac{\sqrt{2}}{2}\sqrt{-\ln(1-\alpha)}\|Mx\|, \tag{43}$$

where $M = diag(u-l)$ and then (43) is a convex inequality. □

Proposition 9. Assume that the mean and the range parameters for all independent components $(\xi_k)_i$ of the random vectors ξ_k are known, i.e. $(l_k)_i \leq (\xi_k)_i \leq (u_k)_i$, for $k = 1, \ldots, K$ and $i = 1, \ldots, n$. An inner approximation of problem (1) based on Hoeffding inequality can be given by

$$\begin{aligned}
\min_{x,y} \quad & c^T x \\
\text{s.t.} \quad & \bar{\xi}_k^T x + \frac{\sqrt{2}}{2}\sqrt{\ln\left(\frac{1}{1-\alpha^{y_k}}\right)}\|M_k x\| \leq h_k, \; k = 1, \ldots, K, \\
& \sum_{k=1}^K y_k = 1, \; y_k \geq 0, \; k = 1, \ldots, K, \; x \in X,
\end{aligned} \tag{44}$$

where $M_k = diag(u_k - l_k), u_k = ((u_k)_1, \ldots, (u_k)_n)^T, \; l_k = ((l_k)_1, \ldots, (l_k)_n)^T$, $k = 1, \ldots, K$.

Proof. With almost the same proof as Proposition 8, the conclusion can be obtained. □

Additionally, with Assumption 1, an equivalent inner approximation of problem (1) based on Hoeffding inequality can be obtained by applying the following transformation $z = \ln x$:

$$\begin{aligned} \min_{z,y} \quad & c^T e^z \\ \text{s.t.} \quad & \bar{\xi}_k^T e^z + \frac{1}{2}\|M_k e^{\ln\left(\sqrt{2\ln\left(\frac{1}{1-\alpha^{y_k}}\right)}\right)+z}\| \le h_k,\ k = 1,\dots,K, \\ & \sum_{k=1}^{K} y_k = 1,\ y_k \ge 0,\ k = 1,\dots,K,\ z \in Z. \end{aligned} \tag{45}$$

From Lemma 5, function $\ln\left(\sqrt{2\ln(\frac{1}{1-\alpha^{y_k}})}\right)$ is convex, when $\alpha \ge 1 - e^{-1}$. Hence, if $c \ge 0$, $\alpha \ge 1 - e^{-1}$, Problem (45) is convex.

When Assumption 1 and the convexity condition for problem (45) do not hold, we should apply a sequential convex approximation algorithm similar to what proposed before for the Chebyshev bound.

3. A Refinery Blend Planning Problem under Uncertainty

In this section, we describe a problem of refinery operations which was originally proposed in [34]. An important blend planning problem for a refinery consists of determining the optimal quantities $x_{ip} \in \mathbb{R}_+$ of material $i \in \mathcal{B}$ to blend together to obtain final products $p \in \mathcal{P}$ to sell on the market. The refinery operation incurs a cost $v_i \in \mathbb{R}_+$ for producing or acquiring one unit of blendstock $i \in \mathcal{B}$. On the other hand, selling the end product on the market yields a revenue $f_p \in \mathbb{R}_+$, $p \in \mathcal{P}$ per unit. The objective of the refinery is to maximize the profit defined as the difference between revenues made from selling the end product and the cost of acquiring all blendstocks taking into account of the maximum production capacity mp of each product $p \in \mathcal{P}$, a maximum flow rate mf_i, a density d_i of each blendstock $i \in \mathcal{B}$, and a maximum target density $\bar{d}_p$ of each product $p \in \mathcal{P}$.

Typically, quality targets are imposed by law to prevent companies from selling substandard products which may damage engines and/or pollute the environment. We denote the target for quality $k \in \mathcal{K}$ in product p by t_{kp} and let ζ_{ki} the random value of quality k in blendstock $i \in \mathcal{B}$. We assume that quality $k \in \mathcal{K}$ of a product $p \in \mathcal{P}$ is a linear function of the fraction of each blendstock used to produce it. Notice that the true qualities of blendstocks are actually unknown in real life applications. This means that if the operator considers ζ_{ki} fixed to a nominal value $\bar{\zeta}_{ki}$, the blending plan will lead to a significant loss since it will often yield off specification products which cannot be sold in the market. This issue implies that the refinery operator should explicitly account for uncertainties. In the following we consider a model which achieves this goal.

3.1. A probabilistic constrained programming formulation

In this section, we present a probabilistic constrained programming formulation of the refinery blend planning problem under uncertainty described above. We assume that the uncertain qualities ζ_{ki} are modeled as random variables defined in the probability space $(\mathbb{R}^{K\times n}, \mathcal{F}, \mathbb{P})$ which consists of the sample space $\mathbb{R}^{K\times n}$, $\sigma-$algebra $\mathcal{F}$ and probability measure $\mathbb{P}$.
Let us now define the following notation:
Sets:

- $\mathcal{B} := \{1, \ldots, n\}$: set of materials to blend (blendstocks);
- $\mathcal{P} := \{1, \ldots, P\}$: set of final products to produce;
- $\mathcal{K} := \{1, \ldots, K\}$: set of qualities;

Deterministic parameters:

- v_i: cost for producing or acquiring one unit of blendstock $i \in \mathcal{B}$;
- f_p: revenue for selling one unit of product $p \in \mathcal{P}$;
- t_{kp}: target for quality $k \in \mathcal{K}$ in product $p \in \mathcal{P}$;
- mp: maximum production of each type of gasoline;
- mf_i: maximum flow of each blendstock $i \in \mathcal{B}$;
- d_i: density of each blendstock $i \in \mathcal{B}$;
- $\bar{d}_p$: maximum target density for product $p \in \mathcal{P}$;
- $\bar{\zeta}_{ki}$: expected value of quality $k \in \mathcal{K}$ in blendstock $i \in \mathcal{B}$;
- α: target probability;

Uncertain parameters:

- ζ_{ki}: value of quality $k \in \mathcal{K}$ in blendstock $i \in \mathcal{B}$;

Decision variables:

- x_{ip}: quantity of blendstock $i \in \mathcal{B}$ to blend to obtain product $p \in \mathcal{P}$.

The probabilistic constrained programming model for the blend planning problem is formulated as follows:

$$\min \quad \sum_{p\in\mathcal{P}} \sum_{i\in\mathcal{B}} (v_i - f_p) x_{ip} \tag{46}$$

$$s.t. \quad \mathbb{P}\left\{ \sum_{i\in\mathcal{B}} (\zeta_{ki} x_{ip}) \leq t_{kp} \sum_{i\in\mathcal{B}} x_{ip},\ \forall k \in \mathcal{K} \right\} \geq \alpha,\ \forall p \in \mathcal{P}, \tag{47}$$

$$\sum_{i\in\mathcal{B}} x_{ip} \leq mp,\ \forall p \in \mathcal{P}, \tag{48}$$

$$\sum_{p\in\mathcal{P}} x_{ip} \leq mf_i,\ \forall i \in \mathcal{B}, \tag{49}$$

$$\sum_{i\in\mathcal{B}} d_i x_{ip} \leq \bar{d}_p \sum_{i\in\mathcal{B}} x_{ip},\ \forall p \in \mathcal{P}, \tag{50}$$

$$x_{ip} \geq 0, \forall i \in \mathcal{B},\ \forall p \in \mathcal{P}. \tag{51}$$

The objective function (46) expresses a minimization of the refinery costs defined as the difference between cost of acquiring all blendstocks and revenues made from selling. Constraints (47) guarantee that each product $p \in \mathcal{P}$ is on-specification with probability greater than α only when all qualities $k \in \mathcal{K}$ independently and jointly meet their targets. This generates the so-called joint independent probabilistic constraints. The deterministic resource constraints (48)-(49)-(50) respectively impose a maximum production of each product, a maximum flow rate of each blendstock and a maximum target density of each product. Finally constraints (51) define the decision variables of the problem.

Problem (46)-(51) is typically noncovex and thus difficult to solve. A simple approach to deal with it, is to decompose the joint chance constraint (47), for each product $p \in \mathcal{P}$ into K individual chance constraints as follows:

$$\mathbb{P}\left\{ \sum_{i\in\mathcal{B}} (\zeta_{ki} x_{ip}) \leq t_{kp} \sum_{i\in\mathcal{B}} x_{ip} \right\} \geq \alpha,\ \forall p \in \mathcal{P},\ \forall k \in \mathcal{K}, \tag{52}$$

which corresponds to an inner approximation to constraint (47) according to Bonferroni inequality [6]. In the following we will solve problem (46)-(51) based on probabilistic inequalities derived in Section 2 which will enable us to obtain much less conservative approximations.

4. Numerical results

In this section, we provide an extensive numerical campaign to investigate performance of the bounds proposed in Section 2 on the refinery blend planning problem under uncertainty (46)-(51), with the aim of:

- understanding the performance of the considered inner approximations in terms of percentage GAP, CPU time and optimal blending recipes with respect to the exact SOCP reformulation for the single chance constraint case (see Subsection 4.1) and with respect to the Sample Average Approximations (SAA) method for the joint chance constraint case (see Subsection 4.2);

- reducing the computational complexity of the derived problems via a piecewise linear approximation based on tangent and segment approximation (see Appendix B);

- analyzing the sensitivity of the solution for increasing values of considered products qualities;

- analyzing the performance of the solutions obtained using the aforementioned bounds over the realization of different probability distributions (see Subsection 4.3).

The bounds have been implemented under Matlab R2018b environment using the CVX software, a modeling system for constructing and solving convex programs and SeDuMi solver. The computations have been performed on a 64-bit machine with 8 GB of RAM and a 1.8 GHz Intel i7 processor.

We first considered benchmark instances available in the literature (see [34]) with some slight modifications as follows: by blending 10 types of intermediate flows $\mathcal{B} = \{1, \dots, 10\}$, three types of gasoline $\mathcal{P} = \{\text{Type-1, Type-2, Type-3}\}$ are produced. The maximum production of each type of gasoline is $mp = 50$ and the set of qualities to be met is $\mathcal{K} = \{\text{RVP, (RON+MON)/2, Sulfur, Benzene}\}$. We assume that the random variable ζ_{ki} is distributed according to a normal distribution with mean $\bar{\zeta}_{ki}$ and standard deviation σ_{ki} reported in tables 6-7 available in the Supplementary Material, respectively. Under this assumption, chance constraints can be equivalently reformulated as Second Order Cone Programming (SOCP) constraints as in (4). This will allow us to make a fair comparison of the bounds with the exact SOCP reformulation. Tables 8-9 in the Supplementary Material report the deterministic parameters mf_i, v_i, d_i, with $i \in \mathcal{B}$ and $\bar{d}_p$, f_p, with $p \in \mathcal{P}$. Finally the targets t_{kp} for quality $k \in \mathcal{K}$ in product $p \in \mathcal{P}$ are reported in Table 10. For both the individual and joint chance constraints cases, we choose a target probability $\alpha = 0.95$. Typical values for an oil company are in the range $\alpha \in [95\%; 99\%]$ (see [34]).

Secondly, we generate a set of instances extending the number of qualities up to $|\mathcal{K}| = 100$. The new data (expected values $\bar{\zeta}_{ki}$ and standard deviations σ_{ki}) are uniformly generated respectively in the intervals $[1, 125]$ and $(0, 1)$ and reported in Supplementary Material (see Tables 11, 12 and 13). Although the problems which give upper bounds obtained by Chebyshev, Chernoff and Hoeffding inequalities for problem (1) are convex under some conditions, they are still hard to solve directly by current tools because of the following term: $\ln\left(\sqrt{2\ln(\frac{1}{1-\alpha^{y_k}})}\right)$. In Appendix B we propose piecewise linear approximations for this function based on tangent and segment approximations as in [6].

4.1. The individual chance constrained case

We first consider model (46)-(51) for the blend planning problem where for any product $p \in \mathcal{P}$ the probabilistic constraints (47) are replaced by K individual chance constraints as in equation (52). Table 1 exhibits a comparison of the objective function optimal values and percentage gaps of the four bounds (Chebyshev, Chernoff, Hoeffding, and Bernstein) versus SOCP formulation assuming a normal distribution, as well as the CPU time (given in seconds) considering the data reported in Tables 6-10. As upper and lower bounds (u_{ki} and l_{ki}) for random vector ζ_{ki} are needed in Bernstein and Hoeffding bounds, we generate 3000 samples following normal distributions with means $\bar{\zeta}_{ki}$ and standard deviations σ_{ki} specified before. The maximal values of these 3000 samples are selected as upper bounds while the minimal values as lower bounds.

We first refer to the case of a collection of $|\mathcal{K}| \times |\mathcal{P}| = 12$ individual chance constraints. Results show that the best upper bound is obtained by Chernoff reformulation with a percentage gap of only 0.74%, followed respectively by

Chebyshev, Bernstein and Hoeffding having a percentage gap of 14.05%. In terms of CPU time, the most expensive is the Bernstein bound while the others are relatively comparable with SOCP.

	SOCP	*Chebyshev*	*Chernoff*	*Hoeffding*	*Bernstein*
Optimal value	-775.86	-747.16	-770.05	-666.85	-717.09
GAP (%)	-	3.69%	0.74%	14.05%	7.57%
CPU time	1.3125	0.9531	1.0781	0.8281	2.7344

Table 1: Comparison of SOCP formulation versus bounds for the refinery blending planning problem with $|\mathcal{K}| \times |\mathcal{P}| = 12$ individual chance constraints as in (52).

The blending recipes x_{ip} for the three types of gasolines (Type-1, Type-2, Type-3) over the ten blendstocks, are shown in Figures 1-2 for the SOCP formulation and all the considered bounds respectively.

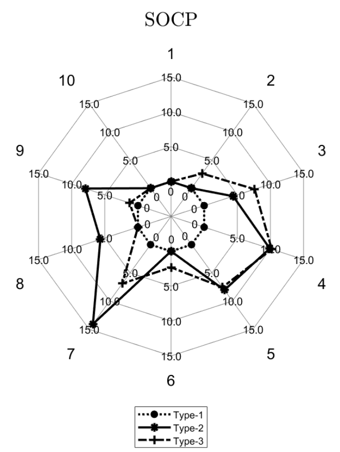

Figure 1: Optimal blending recipes x_{ip} for the three types of gasolines for SOCP formulation for the refinery blending planning problem with $|\mathcal{K}| \times |\mathcal{P}| = 12$ individual chance constraints as in (52).

Results show that the most similar compositions to SOCP solution (0 for Type-1, 50 for Type-2 and 38.43 for Type-3) are given by Chernoff followed by Chebyshev ones, suggesting not to produce Type-1 and to saturate the production of Type-2, while they slightly underestimate the production of Type-3 with only 36.88 and 27.38 units respectively. Different is the composition obtained with Bernstein (0.20 for Type-1, 50 for Type-2 and 14.32 for

Type-3) and Hoeffding solutions (5.55 for Type-1, 50 for Type-2 and 12.68 for Type-3) where Type-1 is suggested to be produced and the production of Type-3 is strongly underestimated.

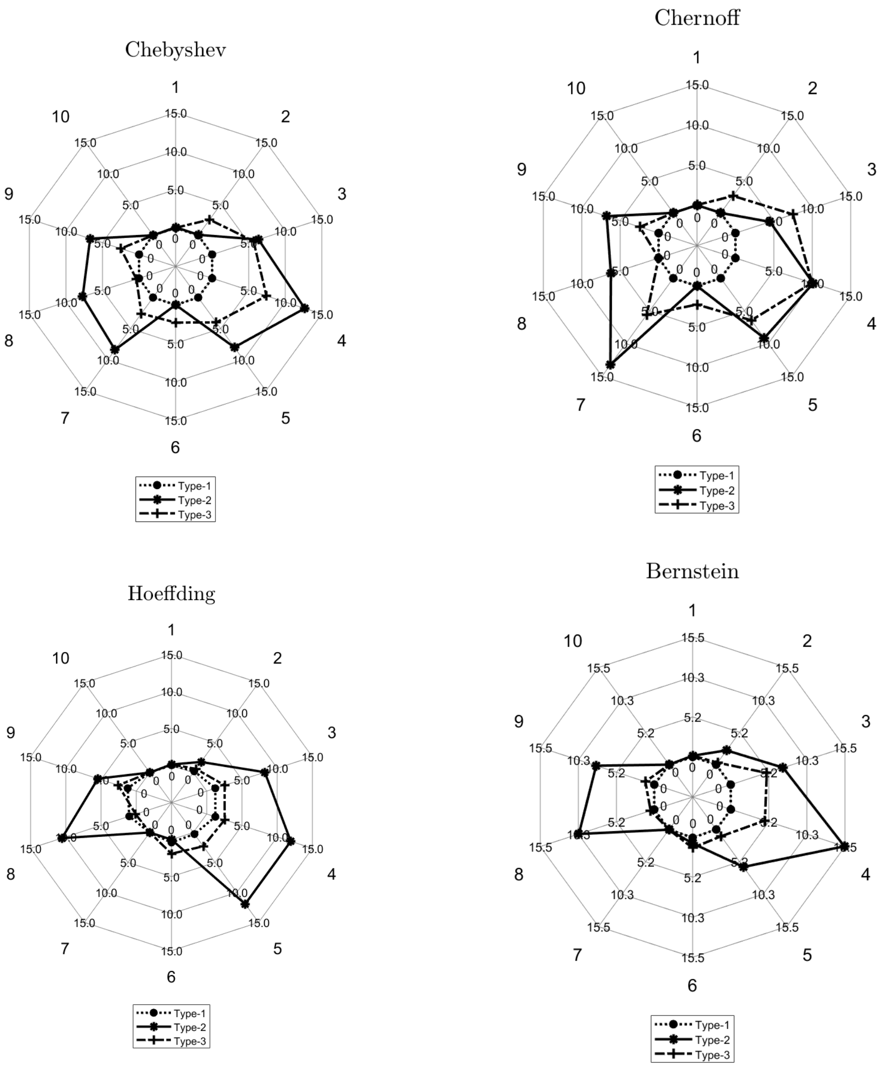

Figure 2: Optimal blending recipes x_{ip} for the three types of gasolines for SOCP formulation and all the considered bounds for the refinery blending planning problem with $|\mathcal{K}| \times |\mathcal{P}| = 12$ individual chance constraints as in (52).

We secondly refer to the case of a collection of $|\mathcal{K}| \times |\mathcal{P}| = 3K$ individual chance constraints with $|\mathcal{K}| = 1, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100$ qualities

	SOCP	*Chebyshev*	*Chernoff*	*Hoeffding*	*Bernstein*
$K = 1$					
Optimal value	-965.34	-957.36	-962.97	-950.57	-941.13
GAP (%)	-	0.83%	0.25%	1.53%	2.51%
CPU time	0.72	0.66	0.67	0.50	2.03
$K = 10$					
Optimal value	-481.23	-387.48	-453.56	-266.16	-
GAP (%)	-	19.48%	5.75%	44.69%	-
CPU time	1.17	1.08	1.39	1.22	68.50
$K = 20$					
Optimal value	-432.45	-340.23	-407.42	-245.08	-
GAP (%)	-	21.32%	5.79%	43.33%	-
CPU time	2.78	2.30	2.08	0.88	198.52
$K = 30$					
Optimal value	-431.59	-302.97	-400.58	-250.29	-
GAP (%)	-	29.80%	7.18%	42.01%	-
CPU time	2.89	2.67	2.56	0.95	176.53
$K = 40$					
Optimal value	-422.79	-298.04	-396.14	-259.23	-
GAP (%)	-	29.51%	6.30%	38.69%	-
CPU time	2.73	2.44	2.36	1.11	217.66
$K = 50$					
Optimal value	-422.79	-298.04	-396.14	-257.58	-
GAP (%)	-	29.51%	6.30%	39.08%	-
CPU time	3.06	3.03	2.88	0.95	309.38
$K = 60$					
Optimal value	-422.79	-297.80	-396.14	-250.94	-
GAP (%)	-	29.56%	6.30%	40.65%	-
CPU time	3.75	3.55	3.94	1.22	313.34
$K = 70$					
Optimal value	-421.56	-280.52	-383.10	-254.58	-
GAP (%)	-	33.46%	9.12%	39.61%	-
CPU time	4.36	4.14	4.69	1.67	327.58
$K = 80$					
Optimal value	-401.91	-261.34	-360.53	-237.24	-
GAP (%)	-	34.98%	10.30%	40.97%	-
CPU time	6.78	6.22	5.27	1.56	410.38
$K = 90$					
Optimal value	-401.91	-261.34	-360.53	-247.43	-
GAP (%)	-	34.98%	10.30%	38.44%	-
CPU time	5.41	5.19	5.33	1.67	572.45
$K = 100$					
Optimal value	-401.91	-261.34	-360.53	-249.29	-
GAP (%)	-	34.98%	10.30%	37.97%	-
CPU time	6.03	7.23	6.17	1.81	476.98

Table 2: Comparison of SOCP formulation versus bounds for the refinery blending planning problem with $3K$ individual chance constraints as in (52) with $K = 1, 10, \ldots, 100$.

and $|\mathcal{P}| = 3$ products. A comparison between the results of the exact SOCP formulation and its approximations (Chebyshev, Chernoff, Hoeffding, and Bernstein) is reported in Table 2. Specifically, these results are obtained by considering the data corresponding to the first K rows of Tables 11, 12 and 13 (see Supplementary Material).

As in the case of $|\mathcal{K}| = 4$, numerical results confirm that the Chernoff approximation outperforms the others, as its percentage gaps vary in the interval 0.25%–10.30% increasing with an increasing number of qualities $|\mathcal{K}|$. On the other hand, gaps for Chebyshev approximations range between 0.83%–34.98%, while gaps for Hoeffding approximations span between 1.53%–44.69%. A comparison of the percentage gap values with respect to

SOCP formulation for increasing number of qualities $|\mathcal{K}| = 5, \dots, 100$ is shown in Figure 5 in Supplementary Material. Results confirm the dominance of Chernoff bound, followed by Chebyshev and then by Hoeffding and the increasing behavior for considering a lager number of qualities $|\mathcal{K}|$. The optimal total costs and corresponding percentage gaps of Bernstein bound have been obtained only in the case of one and five qualities with percentage gaps of respectively 2.51% ($|\mathcal{K}| = 1$) and 30.50% ($|\mathcal{K}| = 5$). For larger values ($|\mathcal{K}| \geq 10$) the approximated problem via Bernstein becomes infeasible via CVX software. Albeit Bernstein approximation seems performing better that the Hoeffding approximation, there is not enough empirical evidence to support this conclusion because of lack of numerical results for larger number of qualities.

Finally Table 2 and Figure 6 in Supplementary Material show the CPU time of Chebyshev,Chernoff and Hoeffding bounds for increasing number of qualities $|\mathcal{K}| = 1, \dots, 100$. Bernstein approximation CPU time is omitted since considerably larger than the one required by the others (and as said before we cannot compute the bound for more than $|\mathcal{K}| = 5$). CPU time for both Chebyshev and Chernoff approximations increases with the number of qualities in a similar way reaching a maximum of 7 CPU seconds, while Hoeffding approximation takes a maximum of only 1.81 CPU seconds with $|\mathcal{K}| = 100$. Results suggest to use Hoeffding bound only when we need a rough solution in short time at expenses of low precision.

4.2. Joint chance constraint case

We investigate now the performance of the bounds for the blend planning problem with joint probabilistic constraints based on model (46)-(51).

Notice that, the blending problem cannot be solved according to the proposed logarithmic transformation for the following reasons: because of equation (47), after applying the logarithmic transformation on x_{ip}, the set Z for the new variable z is not convex. This is in contradiction with Assumption 1. Besides, the convexity condition for problems (14)-(29)-(45) does not hold, being $c < 0$. Therefore, the Chebyshev, Chernoff and Hoeffding bounds will be computed via a sequential convex approximation algorithm described above.

Besides, a *Sample Average Approximation* (SAA) method (see [24]) to solve the original joint chance constrained model (46)-(51), has been adopted. We now study the performance of the four bounds versus SAA assuming a normal distribution for ζ_{ki}, $k \in \mathcal{K}$, $i \in \mathcal{B}$. As for the individual chance constraints case, as upper and lower bounds (u_{ki} and l_{ki}) for random vector ζ_{ki} are needed in Bernstein and Hoeffding bounds, we generate 3000 samples following normal distributions with means $\bar{\zeta}_{ki}$ and standard deviations σ_{ki} specified before. The maximal values of these 3000 samples are selected as upper bounds while the minimal values as lower bounds. Besides, 1000 samples are generated according to the corresponding normal distribution of ζ_{ki} for SAA method.

As before, we first refer to the case of joint chance constraints with $|\mathcal{K}| = 4$ and $|\mathcal{P}| = 3$ considering the data reported in Tables 6-10. Results in Table 3 show that the best upper bound is obtained by Chernoff reformulation with a percentage gap of only 0.24%, followed respectively by Bernstein, Chebyshev and Hoeffding having a percentage gap of 14.48%. The CPU time required by all the four bounds is considerably lower than the one required by SAA. This is due to the fact that the good quality of SAA solution requires a large size sample with an important number of binary variables and a high computational effort as consequence.

	SAA	*Chebyshev*	*Chernoff*	*Hoeffding*	*Bernstein*
Optimal value	-767.56	-682.80	-765.69	-656.36	-714.57
GAP (%)	-	11.04%	0.24%	14.48%	6.90%
CPU time	152023	2.9	4.5	3.4	2

Table 3: Comparison of SAA method versus bounds for the refinery blending planning problem with joint chance constraints with $|\mathcal{K}| = 4$ and $|\mathcal{P}| = 3$.

The corresponding blending recipes x_{ip} for the three types of gasolines (Type-1, Type-2, Type-3) over the ten blendstocks, are shown in Figures 3-4 for the SAA method and all the considered bounds respectively. Results show that the most similar compositions to SOCP solution (0 for Type-1, 50 for Type-2 and 36.27 for Type-3) are given by Chernoff one, suggesting not to produce Type-1 and to saturate the production of Type-2, while it slightly underestimates the production of Type-3 with only 35.39. Different is the composition obtained with Bernstein (0.22 for Type-1, 50 for Type-2 and 12.02 for Type-3) where a small quantity of Type-1 is suggested to be produced and the production of Type-3 is underestimated. Finally both Chebyshev solution (0 for Type-1, 50 for Type-2 and 7.75 for Type-3) and Hoeffding solution (0 for Type-1, 50 for Type-2 and 4.13 for Type-3) match the correct quantities of Type-1 and Type-2 but strongly underestimated the one of Type-3.

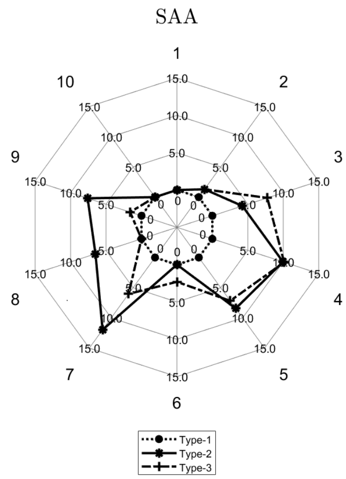

Figure 3: Optimal blending recipes x_{ip} for the three types of gasolines for SAA method for the refinery blending planning problem with joint chance constraints ($|\mathcal{P}| = 3$ and $|\mathcal{K}| = 4$).

As done for the single chance constrained case, we extend now the investigation considering a larger number of qualities up to $|\mathcal{K}| = 100$ with 3 products i.e., $|\mathcal{P}| = 3$. Table 4 shows the comparison between the results of the SAA and its approximations (Chebyshev, Chernoff, Hoeffding, and Bernstein) for, respectively, $|\mathcal{K}| = 5$, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100. Specifically, the results are obtained by considering the data corresponding to the first K rows of Tables 11, 12 and 13 (see Supplementary Material).

Similarly to the individual case, the results in Table 4 show that the Chernoff approximation outperforms the others, as its percentage gaps vary in the interval 0.25%–11.02% and it increases with an increasing number of qualities. On the other hand, gaps for Chebyshev and Hoeffding bounds reach a percentage gap of 100% when $|\mathcal{K}| \geq 20$. The optimal total costs and corresponding percentage gaps for Bernstein bound can be obtained only in the case of five qualities with percentage gaps of 25.63%. This is because, when $|\mathcal{K}| \geq 10$, the approximated problem based on Bernstein bound becomes infeasible via CVX software. The gaps for Hoeffding approximations span between 42.98%–66.26% when $|\mathcal{K}| = 5, 10$ while it becomes 100% when $|\mathcal{K}| \geq 20$.

A comparison of the percentage gap values with respect to SAA for increasing number of qualities $|\mathcal{K}| = 5, \ldots, 100$ is shown in Figure 7 in Sup-

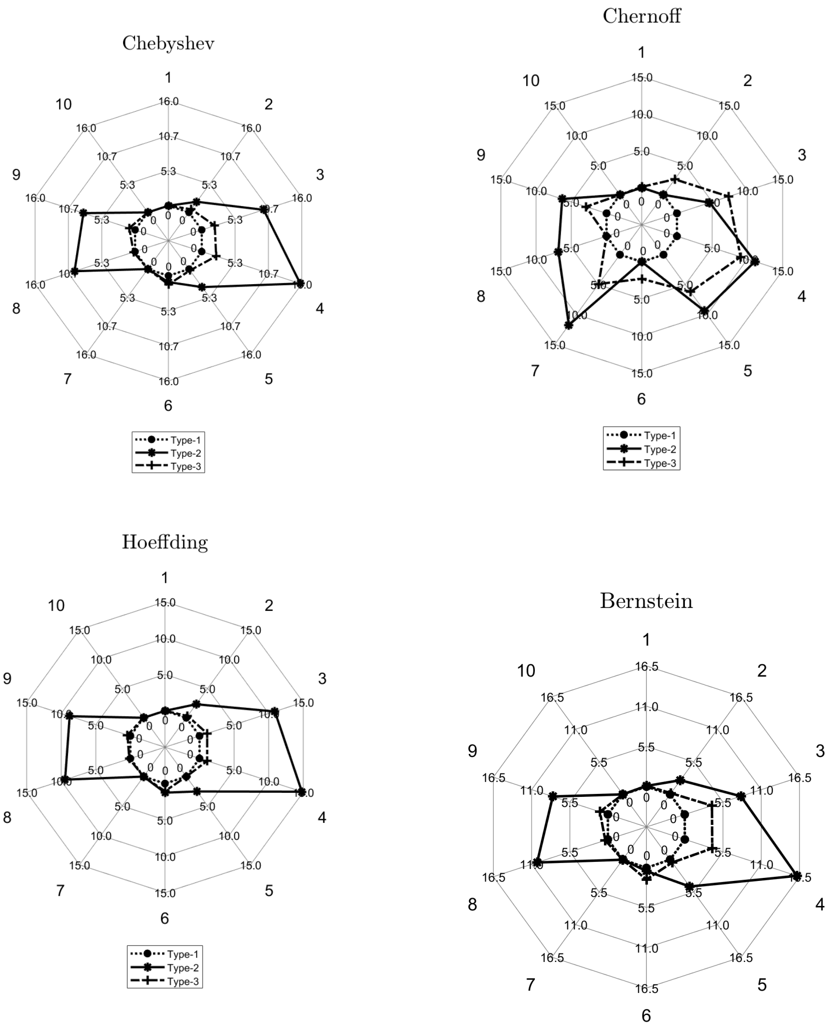

Figure 4: Optimal blending recipes x_{ip} for the three types of gasolines for all the considered bounds for the refinery blend planning problem with joint chance constraints ($|\mathcal{P}| = 3$ and $|\mathcal{K}| = 4$).

plementary Material. Results confirm the dominance of Chernoff bound, followed by Bernstein for $K = 5$, and then by Hoeffding and Chebyshev. The increasing behavior for increasing numbers of qualities is also confirmed. Results show that for large number of qualities, only Chernoff bound is appropriated in a joint chance constraint context, while the other approximations provide very loose bounds for large number of joint probabilistic conditions.

	SAA	*Chebyshev*	*Chernoff*	*Hoeffding*	*Bernstein*
$K = 5$					
Optimal value	-454.53	-260.52	-444.14	-259.19	-338.02
GAP (%)	-	42.68%	2.2851%	42.98%	25.63%
CPU time	158638.59	3.00	5.67	4.06	26.50
$K = 10$					
Optimal value	-447.46	-83.50	-437.24	-150.97	-
GAP (%)	-	81.34%	2.28%	66.26%	-
CPU time	$> 10^6$	3.81	13.11	5.75	-
$K = 20$					
Optimal value	-409.36	0.00	-389.36	0.00	-
GAP (%)	-	100.00%	4.89%	100.00%	-
CPU time	$> 10^6$	2.50	5.13	1.98	-
$K = 30$					
Optimal value	-364.98	0.00	-345.63	0.00	-
GAP (%)	-	100.00%	5.30%	100.00%	-
CPU time	$> 10^6$	3.02	2.69	2.42	-
$K = 40$					
Optimal value	-359.67	0.00	-329.71	0.00	-
GAP (%)	-	100.00%	8.33%	100.00%	-
CPU time	$> 10^6$	3.70	3.55	3.25	-
$K = 50$					
Optimal value	-357.96	0.00	-326.67	0.00	-
GAP (%)	-	100.00%	8.74%	100.00%	-
CPU time	$> 10^6$	4.52	4.05	3.52	-
$K = 60$					
Optimal value	-350.83	0.00	-323.55	0.00	-
GAP (%)	-	100.00%	8.83%	100.00%	-
CPU time	$> 10^6$	4.64	4.17	4.19	-
$K = 70$					
Optimal value	-338.56	0.00	-304.23	0.00	-
GAP (%)	-	100.00%	10.14%	100.00%	-
CPU time	$> 10^6$	6.34	5.53	5.02	-
$K = 80$					
Optimal value	-316.28	0.00	-282.71	0.00	-
GAP (%)	-	100.00%	10.62%	100.00%	-
CPU time	$> 10^6$	6.63	6.48	6.03	-
$K = 90$					
Optimal value	-315.28	0.00	-281.37	0.00	-
GAP (%)	-	100.00%	10.76%	100.00%	-
CPU time	$> 10^6$	6.47	6.41	6.19	-
$K = 100$					
Optimal value	-314.89	0.00	-280.19	0.00	-
GAP (%)	-	100.00%	11.02%	100.00%	-
CPU time	$> 10^6$	7.72	7.48	7.08	-

Table 4: Comparison of SAA method versus bounds for the refinery blending planning problem with joint chance constraints with $K = 5, \dots, 100$ qualities and 3 products.

Finally Table 4 and Figure 8 in Supplementary Material show the CPU time of Chebyshev,Chernoff and Hoeffding bounds for increasing number of qualities $|\mathcal{K}| = 5, \dots, 100$. Bernstein approximation CPU time is omitted since the approximated problem based on Bernstein bound is infeasible when $K \geq 10$. CPU time for both Chebyshev and Hoeffding approximations increases with the number of qualities in a similar way reaching a maximum of 7.72 CPU seconds for 100 qualities, while Chernoff approximation takes a maximum of 13.11 CPU seconds with 10 qualities. The most remarkable result is that for $|\mathcal{K}| \geq 5$, the CPU time required by all the four bounds is considerably lower than the one required by the SAA (see Table 4). In case of large number of joint probabilistic constraints, results suggest to use Chernoff bound while the other ones only when we need a rough solution in short time at expenses of low precision.

4.3. The error of the bounds

In this section we analyze the performance of the solutions obtained using the aforementioned bounds over the realization of different truncated probability distributions (see [20]).

Let x_ξ be the solution obtained by solving problem (2) using the probability distribution ξ and $z_\xi(x_\xi)$ be its optimal objective function value.

Let x_{bound} be the solution obtained by solving problem (2) using one of the considered bounds (Chebyshev, Chernoff, Hoeffding and Bernstein) and let $z_\xi(x_{bound})$ the corresponding objective function value. Notice that $z_\xi(x_{bound})$ provides an upper bound of $z_\xi(x_\xi)$. We define the *Percentage Error of the Bound* $\%EB$ as

$$\%EB := \frac{z_\xi(x_{bound}) - z_\xi(x_\xi)}{z_\xi(x_\xi)} \cdot 100. \tag{53}$$

The greater is $\%EB$, the greater is the objective function value increase of using the solution x_{bound} when ξ occurs. When x_{bound} is not feasible under distribution ξ we set $\%EB = \infty$.

In Table 5 we compute $\%EB$ using one of the bounds (Chebyshev, Chernoff, Hoeffding and Bernstein via SAA) while the distribution ξ can be Cauchy, Logistic, Laplace and T-Student (with degree of freedom respectively equal to 2 and 3). Notice that the aforementioned distributions have been truncated in the same interval $[l_{ki}, u_{ki}]$ specified for Bernstein and Hoeffding bounds. Results refer to the case of a collection of $|\mathcal{K}| \times |\mathcal{P}| = 12$ individual chance constraints. Chebyshev bound performs very well having $\%EB \in [3.07\%, 3.32\%]$. Excellent is the performance of the Chernoff bound when ξ follows a Laplace or a T-Student (with degree of freedom equal to 3) with $\%EB \in [0.16\%, 0.34\%]$ while is infeasible when a Cauchy, a Logistic or T-Student (with degree of freedom equal to 2) occur. Larger are the errors using Bernstein bound solution having $\%EB \in [7.30\%, 7.55\%]$. Worse is the performance of Hoeffding bound solution having $\%EB \in [15.83\%, 16.05\%]$.

5. Conclusions

In this paper, we study deterministic inner approximations for single and joint probabilistic constraints. The derived upper bounds are based on clas-

		Bounds			
	$z_{bound}(x_{bound})$ $z_\xi(x_\xi)$ $z_\xi(x_{bound})$ $\%EB$	*Chebyshev*	*Chernoff*	*Hoeffding*	*Bernstein*
Distribution ξ	*Truncated Cauchy*	-747.16 -772.89 -747.16 3.32	-770.05 -772.89 ∞ ∞	-648.79 -772.89 -648.79 16.05	-714.51 -772.89 -714.51 7.55
	Truncated Logistic	-747.16 -770.85 -747.16 3.07	-770.05 -770.85 ∞ ∞	-648.79 -770.85 -648.79 15.83	-714.51 -770.85 -714.51 7.30
	Truncated Laplace	-747.16 -771.30 -747.16 3.12	-770.05 -771.30 -770.05 0.16	-648.79 -771.30 -648.79 15.88	-714.51 -771.30 -714.51 7.36
	Truncated T-Student (2)	-747.16 -771.70 -747.16 3.17	-770.05 -771.70 ∞ ∞	-648.79 -771.70 -648.79 15.92	-714.51 -771.70 -714.51 7.41
	Truncated T-Student (3)	-747.16 -772.69 -747.16 3.30	-770.05 -772.69 -770.05 0.34	-648.79 -772.69 -648.79 16.03	-714.51 -772.69 -714.51 7.52

Table 5: Percentage Error of the Bound ($\%EB$) obtained solving problem (2) using one of the bounds (Chebyshev, Chernoff, Hoeffding and Bernstein) while the distribution ξ (among truncated Cauchy, truncated Logistic, truncated Laplace and truncated T-Student with degree of freedom respectively equal to 2 and 3) occurs.

sical inequalities from probability theory such as the one-sided Chebyshev inequality, Bernstein inequality, Chernoff inequality and Hoeffding inequality. We show that under mild assumptions, the bounds based approximations of joint chance constrained problems are convex and the approximated problem can be solved efficiently. When the convexity condition can not hold, we show how we to apply an efficient sequential convex approximation approach to solve the approximated problem. Piecewise linear and tangent approximations are also provided for Chernoff and Hoeffding inequalities allowing to reduce the computational complexity of the associated optimization problem. To the best of our knowledge, these results are new in the literature since the majority of the contributions deals with symmetric elliptical distributions while we do not require any particular assumption on probability distributions.

Interesting results were also obtained by the computational experiments we carried out on a refinery blend planning problem under uncertainty. Comparing the bounds in terms of objective function, blending receipts and CPU time with respect to an exact SOCP formulation or SAA method, we found that the Chernoff approximation outperforms the others, both in case of individual and joint probabilistic constraints. In the individual case, results suggest to use Bernstein approximation only for a limited number of probabilistic constraints while to use Hoeffding bound when we need a rough solution in short time at expenses of low precision. A remarkable result in the joint case is that the CPU time required by all the four bounds is considerably lower than the one needed by SAA which require to solve a difficult mixed integer program with binary variables. Finally a comparison of the error of using the solutions obtained by the bounds over different truncated probability distributions, has confirmed the good performance of Chebyshev bound followed by Bernstein and finally by Hoeffding. Chernoff bound should be considered more carefully having excellent error bounds with Laplace and T-Student (with degree of freedom equal to 3) while is infeasible for all the other considered probability distributions.

Future developments include the investigation of bounds for joint dependent chance constraints.

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Appendix A

Proof of Lemma 2

Proof. Since f is convex, we have that for $\lambda \in [0,1]$ and $x_1, x_2 \in X$, $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$. Therefore, as $g : Y \to Z$ be a nonincreasing concave function, we have

$$g \circ f(\lambda x_1 + (1-\lambda)x_2) \geq g(\lambda f(x_1) + (1-\lambda) f(x_2)) \geq \lambda g \circ f(x_1) + (1-\lambda) g \circ f(x_2),$$

which proves the thesis. □

Proof of Lemma 4

Proof. Since $e^{t\xi^T x}$ is convex with respect to $x \in X$, we have that for $\lambda \in [0,1]$ and $x_1, x_2 \in X$, $e^{t\xi^T(\lambda x_1 + (1-\lambda)x_2)} \leq \lambda e^{t\xi^T x_1} + (1-\lambda) e^{t\xi^T x_2}$. Therefore,

$$\mathbb{E}(e^{t\xi^T(\lambda x_1 + (1-\lambda)x_2)}) \leq \mathbb{E}(\lambda e^{t\xi^T x_1} + (1-\lambda) e^{t\xi^T x_2}) = \lambda \mathbb{E}(e^{t\xi^T x_1}) + (1-\lambda)\mathbb{E}(e^{t\xi^T x_2}),$$

which proves the thesis. □

Proof of Lemma 5

Proof. We only need to prove the convexity of the function $b \mapsto \ln\left(\sqrt{2\ln(\frac{1}{1-b})}\right)$ since the convexity of composite function $\ln\left(\sqrt{2\ln(\frac{1}{1-\alpha^{y_k}})}\right)$ is implied when $b \mapsto \ln\left(\sqrt{2\ln(\frac{1}{1-b})}\right)$ is nondecreasing and convex. As $\ln\left(\sqrt{2\ln(\frac{1}{1-b})}\right)$ is monotone (since $\frac{\mathrm{d}}{db}\ln\left(\sqrt{2\ln(\frac{1}{1-b})}\right) = \frac{1}{2(1-b)\log\left(\frac{1}{1-b}\right)} < 0$ with $b < 1$), we need to show the convexity of $b \mapsto \ln\left(\sqrt{2\ln(\frac{1}{1-b})}\right)$. We can notice that $\ln\left(\sqrt{2\ln(\frac{1}{1-b})}\right) = \dfrac{1}{2}\ln\left(2\ln(\frac{1}{1-b})\right)$. Therefore, we only need to focus on the convexity of $b \mapsto \ln\left(2\ln(\frac{1}{1-b})\right)$.

We have

$$\frac{\mathrm{d}^2}{\mathrm{d}^2 b}\ln\left(2\ln(\frac{1}{1-b})\right) = \frac{\ln\left(\frac{1}{1-b}\right) - 1}{(b-1)^2 \ln^2\left(\frac{1}{1-b}\right)}.$$

Then, $b \mapsto \ln\left(2\ln(\frac{1}{1-b})\right)$ is convex if and only if $\ln(1-b) + 1 \leq 0$, i.e. $b \geq 1 - e^{-1}$.

As $y_k \mapsto \alpha^{y_k}$ is convex and $\alpha^{y_k} \geq \alpha$ for any $0 \leq y_k \leq 1$, if $\alpha^{y_k} \geq \alpha \geq 1 - e^{-1}$, then the function $y_k \mapsto \ln\sqrt{2\ln(\frac{1}{1-\alpha^{y_k}})}$ is convex. □

Appendix B

Tangent approximation

We choose S different linear functions:

$$l_s(y_k) = a_s y_k + b_s, \quad s = 1, \ldots, S,$$

such that

$$l_s(y_k) \leq \Upsilon(y_k), \ \forall y_k \in [\rho, 1), \ k = 1, \ldots, K.$$

Here $\rho \geq 0$ is a constant such that $\Upsilon(y_k)$ is convex on $[\rho, 1)$. Then, $\Upsilon(y_k)$ can be approximated by the following piecewise linear function

$$l(y_k) = \max_{s=1,\ldots,S} l_s(y_k),$$

which provides a lower approximation for $\Upsilon(y_k)$.

In order to achieve the expected precision, we set $l_s(y_k)$ as the tangent line of $\Upsilon(y_k)$ at S points $\tau_1, \ldots, \tau_S$ with $\tau_s \in [\rho, 1)$, $s = 1, \ldots, S$. Then, we have

$$a_s = \left.\frac{\mathrm{d}\Upsilon(y_k)}{\mathrm{d}y_k}\right|_{y_k=\tau_s}, \quad b_s = \Upsilon(\tau_s) - a_s \tau_s.$$

Thanks to these piecewise linear approximations for $\Upsilon(y_k)$, we have the following results:

Proposition 10. Under the aforementioned convex conditions, if we replace in problems (14), (29), and (45) $\Upsilon(y_k)$ by $l(y_k)$, we obtain their convex approximations. The optimum values of the approximation problems are lower bounds for problems (14), (29), (45), respectively. Moreover, the approximation problems become asymptotically an equivalent reformulation of problems (14), (29), and (45) when S goes to infinity.

Proof. As the approximation problems are obtained by relaxing some constraints in problems (14), (29), (45), it is easy to see that the optimal values of the approximation problems are lower bounds for problems (14), (29), (45), respectively.

We know under convex conditions for problems (14), (29), and (45), $\Upsilon(y_k)$ is convex for each problem. As the S tangent functions are selected differently, when S goes to infinity, the constraints in the approximation problems are asymptotically equivalent to the constraints in problems (14), (29), and (45), respectively. □

Segment approximation

In order to come up with conservative bounds for the optimum values of problems (14), (29), and (45), we use the linear segments $\bar{a}_s y_k + \bar{b}_s,\ s = 1, \ldots, S$, between $\tau_1, \tau_2, \ldots, \tau_{S+1} \in [\rho, 1)$ to construct a piecewise linear function

$$\bar{l}(y_k) = \max_{s=1,\ldots,S} \left\{ \bar{a}_s y_k + \bar{b}_s \right\}, \tag{54}$$

where

$$\bar{a}_s = \frac{\Upsilon(\tau_{s+1}) - \Upsilon(\tau_s)}{\tau_{s+1} - \tau_s}, \quad \bar{b}_s = \Upsilon(\tau_s) - \bar{a}_s \tau_s,\ s = 1, \ldots, S.$$

Using the piecewise linear function $\bar{l}(y_k)$ to replace $\Upsilon(y_k)$ in problems (14), (29), and (45), gives the corresponding approximation problems.

Similar to Proposition 10, we can derive the following result for the linear approximation:

Proposition 11. Under the aforementioned convex conditions, if we replace in problems (14), (29), and (45) $\Upsilon(y_k)$ by $\bar{l}(y_k)$, we obtain the convex approximations of these problems.

The optimum values of the approximation problems are an upper bound for problems (14), (29), and (45), respectively. Moreover, the approximation problems become asymptotically an equivalent reformulation of problems (14), (29), and (45), respectively, when S goes to infinity.

The proof of this Proposition follows the same pattern as the proof of Proposition 10.

Supplementary Material

$\bar{\zeta}_{ki}$	1	2	3	4	5	6	7	8	9	10
RVP	60	10.4	1.6	11.2	2.7	4.6	10.6	4.4	3.3	2.1
(RON+MON)/2	84.8	69.8	71.9	82.1	99	68	91.6	86.7	85.1	93.7
Sulfur	10	0	6	1	0	0	41	111	30	20
Benzene	0	3.8	0.4	0	0	0	0.6	0	0	0

Table 6: Expected value $\bar{\zeta}_{ki}$ of quality $k \in \mathcal{K}$ in blendstock $i \in \mathcal{B}$.

σ_{ki}	1	2	3	4	5	6	7	8	9	10
RVP	0.7746	0.3225	0.1265	0.3347	0.1643	0.2145	0.3256	0.2098	0.1817	0.1449
(RON+MON)/2	2.0591	1.8682	1.8960	2.0261	2.2249	1.8439	2.1401	2.0821	2.0628	2.1645
Sulfur	0.3162	1.e-10	0.2449	0.1	1.e-10	1.e-10	0.6403	1.0536	0.5477	0.4472
Benzene	1.e-10	0.1949	0.0632	1.e-10	1.e-10	1.e-10	0.0775	1.e-10	1.e-10	1.e-10

Table 7: Standard deviation σ_{ki} of quality $k \in \mathcal{K}$ in blendstock $i \in \mathcal{B}$.

	1	2	3	4	5	6	7	8	9	10
mf_i	8	2.6	12	20.1	15.6	2.3	26.3	17.5	9.2	18
d_i	0.565	0.656	0.772	0.618	0.855	0.693	0.679	0.757	0.803	0.713
v_i	36	40	39	41	52	42	47	44	45	55

Table 8: Deterministic parameters mf_i, d_i and v_i in blendstock $i \in \mathcal{B}$.

	Type-1	Type-2	Type-3
$\bar{d}_p$	0.79	0.79	0.79
f_p	49.7	54.6	52

Table 9: Deterministic parameters $\bar{d}_p$ and f_p in product $p \in \mathcal{P}$.

t_{kp}	Type-1	Type-2	Type-3
RVP (max)	7	7	7
(RON+MON)/2 (max)	85	90	85
Sulfur (max)	30	30	10
Benzene (max)	0.6	0.6	0.6

Table 10: Target t_{kp} for quality $k \in \mathcal{K}$ in product $p \in \mathcal{P}$.

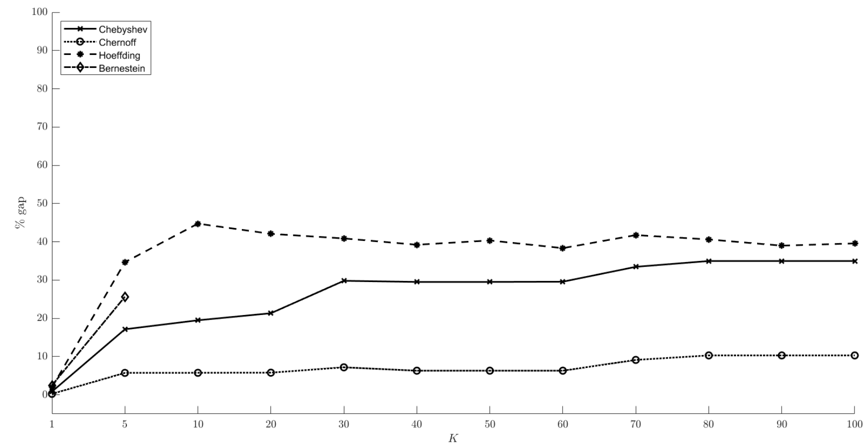

Figure 5: % GAP of the bounds with respect to SOCP formulation for the refinery blending planning problem with $3K$ individual chance constraints with $K = 5, \ldots, 100$.

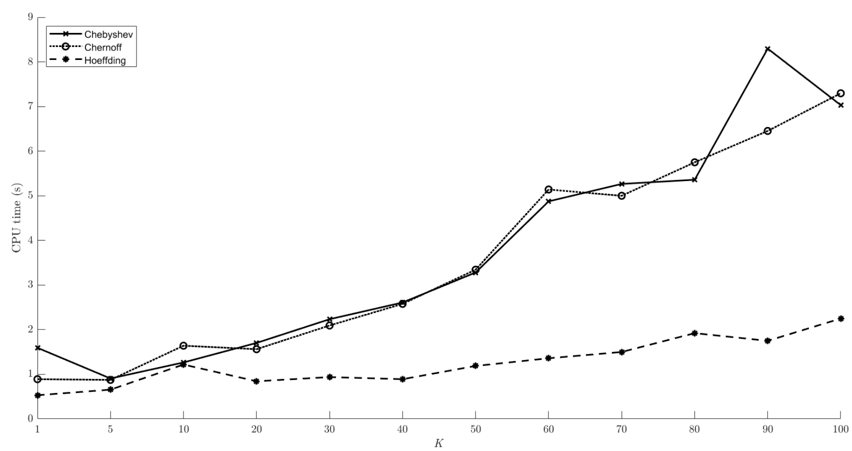

Figure 6: CPU time required by the different bounds for the refinery blending planning problem with $3K$ individual chance constraints with $K = 5, \ldots, 100$.

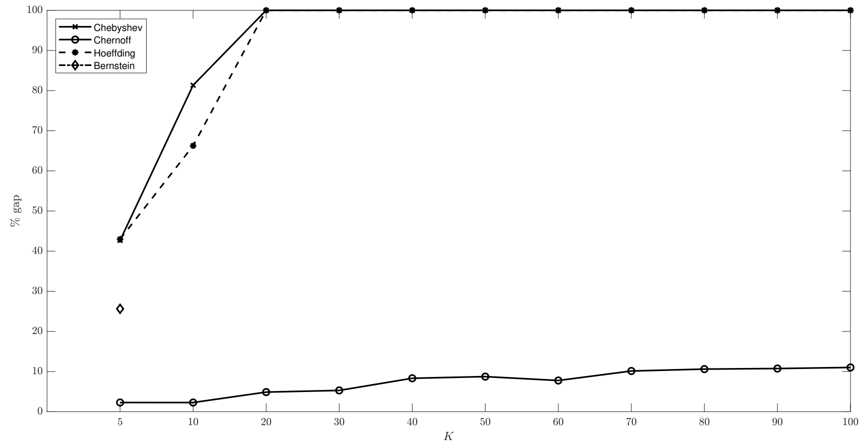

Figure 7: % GAP of the bounds with respect to SAA method for the refinery blend planning problem with joint chance constraints with $K = 5, \ldots, 100$ and 3 products.

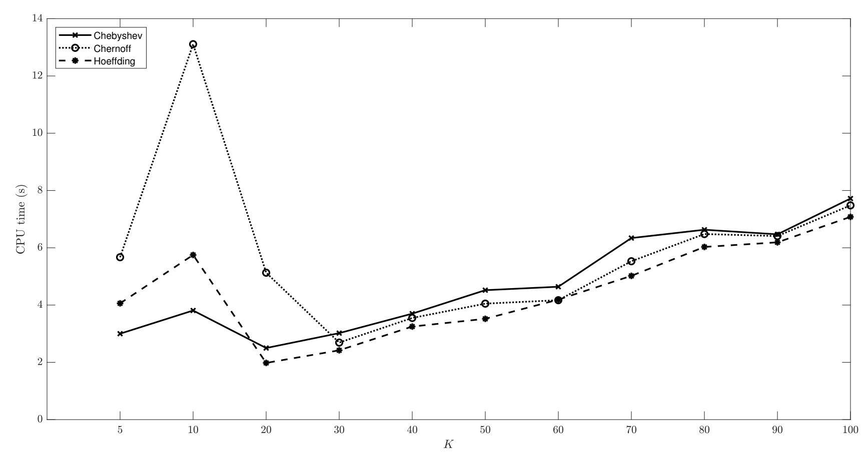

Figure 8: CPU time required by the different bounds for the refinery blending planning problem with joint chance constraints with $K = 5, \ldots, 100$ and 3 products.

ζ_{ki}	1	2	3	4	5	6	7	8	9	10
Quality 1	4	20	50	8	10	24	40	20	14	12
Quality 2	20	8	16	23	9	5	9	7	13	13
Quality 3	42	45	72	9	33	21	75	36	63	45
Quality 4	100	10	120	10	105	5	25	25	20	75
Quality 5	9	2	11	19	6	4	21	20	9	8
Quality 6	68	68	76	76	28	80	56	36	4	68
Quality 7	1	11	5	23	5	20	12	5	21	10
Quality 8	25	1	11	6	19	21	18	25	6	21
Quality 9	96	48	8	52	56	96	96	36	36	44
Quality 10	60	20	80	75	115	90	60	70	10	80
Quality 11	52	52	40	100	100	24	20	12	12	28
Quality 12	12	3	54	12	21	33	9	39	12	21
Quality 13	38	36	30	12	32	4	38	4	18	26
Quality 14	110	35	80	65	85	30	10	75	25	65
Quality 15	22	24	21	12	14	6	13	6	11	13
Quality 16	20	12	18	23	20	10	22	3	19	10
Quality 17	125	95	30	90	55	115	10	10	15	85
Quality 18	60	52	52	8	88	40	44	48	80	40
Quality 19	115	60	55	5	85	120	85	110	15	20
Quality 20	48	26	20	36	46	22	16	44	46	22
Quality 21	44	8	56	44	60	88	80	96	52	72
Quality 22	6	48	16	4	10	28	28	50	8	38
Quality 23	36	40	32	4	32	40	8	46	28	6
Quality 24	54	24	30	69	3	42	24	48	27	72
Quality 25	24	30	20	2	30	16	28	28	48	42
Quality 26	56	48	4	68	12	88	64	64	88	28
Quality 27	5	110	35	95	10	40	15	80	25	105
Quality 28	12	30	12	6	40	30	36	14	42	32
Quality 29	65	105	80	70	95	80	115	125	45	95
Quality 30	18	42	12	33	51	15	57	54	33	21
Quality 31	38	2	14	42	22	34	6	26	28	44
Quality 32	5	2	17	17	3	20	24	20	6	11
Quality 33	34	34	50	40	38	2	4	14	50	28
Quality 34	3	8	25	8	23	11	22	21	19	15
Quality 35	17	18	15	5	7	14	19	11	25	13
Quality 36	70	110	60	30	30	115	80	25	100	55
Quality 37	125	65	30	100	40	110	115	35	40	50
Quality 38	75	63	42	39	3	12	72	51	69	69
Quality 39	42	38	4	10	50	40	48	4	30	32
Quality 40	28	16	56	72	8	96	32	60	16	20
Quality 41	18	28	26	8	40	48	2	32	16	34
Quality 42	72	100	92	36	4	36	8	92	12	16
Quality 43	16	68	92	4	48	32	84	60	80	8
Quality 44	75	85	35	100	115	45	55	5	25	70
Quality 45	75	63	51	54	75	15	42	48	33	21
Quality 46	40	44	40	64	28	76	76	56	24	72
Quality 47	32	60	36	4	76	88	80	60	24	100
Quality 48	75	120	100	65	20	40	125	115	90	120
Quality 49	66	21	75	9	24	27	15	30	27	72
Quality 50	8	32	60	92	60	52	80	48	60	84
Quality 51	56	96	84	28	100	92	64	88	80	44
Quality 52	6	27	75	30	6	51	18	54	6	12
Quality 53	63	39	48	24	66	60	36	45	18	42
Quality 54	54	9	42	15	69	3	51	18	9	36
Quality 55	27	75	12	66	75	6	72	39	15	75
Quality 56	70	95	30	45	75	125	95	60	25	85
Quality 57	42	63	72	6	57	63	15	3	3	72
Quality 58	44	36	6	20	50	44	46	44	46	24
Quality 59	34	30	34	40	48	20	6	2	26	12
Quality 60	88	24	20	56	76	84	12	64	36	28
Quality 61	14	2	8	15	24	9	4	5	15	14
Quality 62	14	30	48	42	36	12	38	14	48	30
Quality 63	21	18	21	3	6	6	21	22	19	16
Quality 64	51	57	3	39	69	3	60	60	30	30
Quality 65	11	17	11	3	18	14	8	12	17	8
Quality 66	24	24	15	16	4	10	11	19	3	19
Quality 67	12	16	25	8	18	13	12	24	8	24
Quality 68	17	11	6	2	12	15	14	17	8	21
Quality 69	10	10	125	10	100	20	55	105	75	75
Quality 70	3	75	12	9	57	66	6	30	51	3
Quality 71	16	3	2	3	2	17	9	16	17	9
Quality 72	36	72	27	39	27	3	66	48	63	60
Quality 73	12	63	21	48	30	63	3	45	54	6
Quality 74	28	56	60	96	40	48	8	92	36	4
Quality 75	42	20	8	48	18	22	14	8	8	24

ζ_{ki}	1	2	3	4	5	6	7	8	9	10
Quality 76	10	5	115	45	10	20	30	75	60	70
Quality 77	20	15	20	20	5	6	5	3	4	15
Quality 78	42	44	14	22	8	8	4	2	32	44
Quality 79	12	8	14	8	12	12	13	10	23	4
Quality 80	76	20	16	96	44	32	88	32	80	36
Quality 81	16	15	10	17	3	22	25	24	15	19
Quality 82	7	8	3	25	4	20	22	22	20	18
Quality 83	60	72	12	20	76	92	40	60	12	72
Quality 84	34	46	50	20	2	48	8	48	28	2
Quality 85	20	42	14	44	28	24	14	48	2	50
Quality 86	15	20	20	70	50	30	45	110	95	70
Quality 87	88	8	28	28	56	96	40	76	32	36
Quality 88	42	12	66	57	54	24	69	24	3	63
Quality 89	30	26	14	8	34	8	46	48	2	46
Quality 90	68	28	100	52	92	84	24	40	56	12
Quality 91	45	125	55	45	95	45	110	5	120	95
Quality 92	16	6	14	8	21	21	20	5	14	1
Quality 93	22	12	19	24	23	3	1	19	15	23
Quality 94	42	24	3	42	48	24	45	69	36	72
Quality 95	44	44	16	44	14	50	40	34	32	50
Quality 96	2	18	22	50	46	28	18	46	24	28
Quality 97	16	96	68	80	40	96	72	88	24	40
Quality 98	65	80	30	120	10	40	35	70	90	5
Quality 99	24	60	16	44	16	56	88	92	64	32
Quality 100	8	23	12	22	3	9	1	20	23	24

Table 11: Expected value $\bar{\zeta}_{ki}$ of quality $k \in \mathcal{K}$ in blendstock $i \in \mathcal{B}$.

σ_{ki}	1	2	3	4	5	6	7	8	9	10
Quality 1	0.0333	0.6569	0.3487	0.5053	0.3533	0.2854	0.4984	0.5990	0.3368	0.1889
Quality 2	0.9428	0.0930	0.9699	0.5173	0.1267	0.4141	0.6606	0.6161	0.3154	0.4452
Quality 3	0.6615	0.3144	0.6641	0.6492	0.3878	0.2814	0.1650	0.0289	0.6019	0.3753
Quality 4	0.4221	0.5639	0.8367	0.4091	0.9673	0.7182	0.4576	0.7663	0.9683	0.5440
Quality 5	0.0460	0.7361	0.8123	0.9214	0.4598	0.0405	0.5800	0.4473	0.7305	0.8700
Quality 6	0.2737	0.5691	0.2033	0.0824	0.9303	0.0114	0.0165	0.1831	0.8422	0.5907
Quality 7	0.5929	0.7701	0.1963	0.4749	0.0902	0.9051	0.6335	0.2437	0.8264	0.5979
Quality 8	0.9956	0.5674	0.3659	0.3257	0.4931	0.9409	0.2838	0.3709	0.9550	0.1287
Quality 9	0.7363	0.4086	0.8641	0.7689	0.4371	0.5115	0.2985	0.4879	0.2445	0.6851
Quality 10	0.0377	0.2399	0.6854	0.7733	0.2496	0.0227	0.8833	0.2893	0.6451	0.0660
Quality 11	0.8024	0.4930	0.7012	0.7372	0.5751	0.8220	0.8201	0.2489	0.1335	0.7319
Quality 12	0.6124	0.2711	0.6206	0.2371	0.2535	0.9411	0.9240	0.8942	0.4910	0.3811
Quality 13	0.0325	0.9004	0.5909	0.8833	0.1435	0.4177	0.4116	0.5720	0.2792	0.0980
Quality 14	0.7429	0.7671	0.9523	0.1472	0.9280	0.8428	0.8806	0.0203	0.0660	0.8309
Quality 15	0.7861	0.3307	0.3714	0.9569	0.7326	0.2048	0.4278	0.8758	0.2420	0.7396
Quality 16	0.0041	0.9788	0.7330	0.4686	0.2052	0.1111	0.4831	0.5717	0.2355	0.1559
Quality 17	0.5462	0.8554	0.8626	0.4640	0.1178	0.9143	0.4319	0.2333	0.1506	0.1369
Quality 18	0.4079	0.0298	0.5066	0.9365	0.6543	0.2693	0.2475	0.2605	0.0076	0.3115
Quality 19	0.1087	0.9489	0.4444	0.5722	0.6732	0.1606	0.3227	0.8662	0.7111	0.5713
Quality 20	0.8438	0.0500	0.3592	0.0362	0.9408	0.2178	0.5940	0.4764	0.5453	0.9064
Quality 21	0.0340	0.3293	0.0041	0.2338	0.8450	0.2268	0.0464	0.6244	0.3712	0.2674
Quality 22	0.0103	0.5189	0.2806	0.3052	0.3952	0.6677	0.8258	0.3017	0.4867	0.5713
Quality 23	0.0329	0.6717	0.2023	0.2635	0.8311	0.8382	0.7809	0.3080	0.1408	0.9272
Quality 24	0.5984	0.4712	0.2057	0.4445	0.9471	0.2909	0.1996	0.3804	0.6291	0.7125
Quality 25	0.5461	0.0937	0.7449	0.3835	0.9020	0.0249	0.8419	0.6725	0.7168	0.3197
Quality 26	0.3738	0.8215	0.9610	0.1671	0.3049	0.0625	0.9581	0.2106	0.7718	0.4092
Quality 27	0.3019	0.2386	0.1475	0.3455	0.4950	0.5201	0.8820	0.4415	0.1587	0.2958
Quality 28	0.0243	0.2430	0.4572	0.4646	0.1962	0.0273	0.3150	0.7659	0.8530	0.8030
Quality 29	0.4201	0.4001	0.1417	0.2019	0.9770	0.9072	0.4916	0.7111	0.3220	0.7300
Quality 30	0.0954	0.1283	0.5917	0.4021	0.1455	0.9161	0.6477	0.3540	0.4172	0.0184
Quality 31	0.4211	0.9161	0.4367	0.3937	0.0828	0.7003	0.9229	0.6666	0.4174	0.6039
Quality 32	0.2477	0.1341	0.6600	0.2643	0.5551	0.1339	0.9350	0.3984	0.4407	0.0842
Quality 33	0.4021	0.2000	0.6598	0.2055	0.2038	0.3188	0.8659	0.1788	0.1854	0.0261
Quality 34	0.4148	0.1389	0.9436	0.0096	0.7686	0.2046	0.7646	0.6890	0.6017	0.1839
Quality 35	0.7036	0.6499	0.9894	0.0255	0.7185	0.4853	0.1215	0.7202	0.5509	0.4088
Quality 36	0.5006	0.7891	0.3007	0.7905	0.4175	0.8432	0.1352	0.3075	0.5155	0.6112
Quality 37	0.5010	0.4581	0.3033	0.9125	0.0884	0.6201	0.4941	0.5239	0.6900	0.7752
Quality 38	0.2255	0.4625	0.1650	0.6039	0.1613	0.2811	0.9490	0.6819	0.6354	0.7922
Quality 39	0.9156	0.8472	0.3825	0.1977	0.2701	0.4032	0.7736	0.7733	0.2920	0.1409
Quality 40	0.4384	0.9931	0.0520	0.4561	0.8753	0.6049	0.0146	0.9235	0.7517	0.7000
Quality 41	0.9179	0.5798	0.0125	0.4138	0.9256	0.7911	0.6629	0.6507	0.4778	0.4015
Quality 42	0.9982	0.8910	0.8406	0.6447	0.1977	0.4663	0.9613	0.8388	0.2450	0.9860
Quality 43	0.2504	0.3048	0.1599	0.4070	0.2567	0.9257	0.6421	0.1954	0.0662	0.9646
Quality 44	0.1541	0.3685	0.7339	0.9454	0.6394	0.4436	0.3093	0.0407	0.5600	0.5232
Quality 45	0.6458	0.7752	0.5346	0.6233	0.9799	0.6155	0.2815	0.3584	0.6043	0.1654
Quality 46	0.6043	0.7481	0.4322	0.0764	0.1137	0.7924	0.3126	0.3524	0.2595	0.3995
Quality 47	0.4753	0.9274	0.1562	0.5070	0.9312	0.4318	0.4909	0.6604	0.1478	0.5071
Quality 48	0.1246	0.9934	0.6797	0.0048	0.4979	0.0758	0.3949	0.8198	0.7284	0.0603
Quality 49	0.9630	0.4216	0.6314	0.2995	0.4942	0.6002	0.2119	0.5924	0.6292	0.4539
Quality 50	0.9361	0.0581	0.7038	0.8725	0.0226	0.6633	0.0173	0.2646	0.8423	0.4105
Quality 51	0.2441	0.6092	0.6722	0.9156	0.3682	0.9263	0.0776	0.2513	0.9081	0.5468
Quality 52	0.7519	0.2599	0.7745	0.5755	0.5236	0.4304	0.4321	0.2519	0.0249	0.2250
Quality 53	0.4887	0.9625	0.4735	0.6786	0.7118	0.8561	0.3076	0.3286	0.2978	0.8358
Quality 54	0.0456	0.2964	0.4966	0.3688	0.3375	0.9977	0.2508	0.2223	0.1666	0.9787
Quality 55	0.8683	0.6455	0.8483	0.2994	0.0324	0.1010	0.3874	0.8631	0.8256	0.2095
Quality 56	0.1784	0.3359	0.1689	0.3926	0.3946	0.2085	0.0411	0.7044	0.3232	0.2578
Quality 57	0.3452	0.5581	0.6045	0.4286	0.5995	0.1252	0.4089	0.0295	0.6652	0.7693
Quality 58	0.8231	0.5737	0.2325	0.8416	0.6836	0.7604	0.7338	0.0982	0.8486	0.2388
Quality 59	0.5595	0.3472	0.2648	0.9428	0.0714	0.4674	0.0545	0.8824	0.9779	0.9224
Quality 60	0.4044	0.9961	0.3863	0.4102	0.9954	0.4215	0.7425	0.5270	0.4323	0.5217

σ_{ki}	1	2	3	4	5	6	7	8	9	10
Quality 61	0.4772	0.2349	0.1612	0.8853	0.1178	0.7410	0.2448	0.7537	0.3234	0.0316
Quality 62	0.2859	0.0555	0.9350	0.7332	0.3453	0.3676	0.4757	0.2494	0.4846	0.7884
Quality 63	0.1150	0.2930	0.6542	0.2027	0.5455	0.4986	0.5401	0.7224	0.1938	0.9855
Quality 64	0.3184	0.3442	0.6474	0.5931	0.0659	0.0550	0.3472	0.7890	0.5955	0.1383
Quality 65	0.3634	0.3272	0.3759	0.3989	0.1059	0.0697	0.8975	0.0708	0.5258	0.0707
Quality 66	0.9754	0.1441	0.5559	0.5741	0.1261	0.3228	0.5012	0.6042	0.1499	0.3715
Quality 67	0.6903	0.5296	0.4597	0.4005	0.6662	0.2757	0.2474	0.9301	0.8762	0.8995
Quality 68	0.4732	0.5128	0.5136	0.9689	0.0954	0.0690	0.5824	0.0755	0.8055	0.0726
Quality 69	0.3546	0.0449	0.8344	0.9457	0.4365	0.1005	0.4539	0.7725	0.6687	0.3694
Quality 70	0.5312	0.3220	0.1076	0.3974	0.4143	0.7372	0.2158	0.0329	0.1459	0.2827
Quality 71	0.6488	0.4392	0.1194	0.7673	0.7301	0.3245	0.6164	0.6897	0.5144	0.3299
Quality 72	0.7474	0.9928	0.2678	0.8347	0.1690	0.2861	0.8324	0.1817	0.3395	0.1163
Quality 73	0.1885	0.9346	0.7169	0.5234	0.4094	0.8810	0.7645	0.1934	0.8182	0.5723
Quality 74	0.6801	0.6285	0.0468	0.0289	0.2278	0.5674	0.5385	0.6798	0.6102	0.9962
Quality 75	0.7679	0.6824	0.4162	0.6986	0.7982	0.2873	0.8079	0.0826	0.8347	0.2774
Quality 76	0.0808	0.6009	0.6733	0.1482	0.2535	0.4834	0.0221	0.0083	0.9694	0.2680
Quality 77	0.8624	0.0811	0.1077	0.3286	0.1655	0.8978	0.6048	0.4550	0.3411	0.8257
Quality 78	0.4140	0.4418	0.4412	0.4724	0.4233	0.5533	0.8821	0.9131	0.8507	0.2074
Quality 79	0.0137	0.8277	0.9997	0.6230	0.1668	0.5101	0.4471	0.8022	0.3481	0.5956
Quality 80	0.5094	0.5550	0.9763	0.2599	0.8101	0.9689	0.7248	0.2863	0.3862	0.2952
Quality 81	0.8926	0.7215	0.4757	0.3866	0.6835	0.2583	0.3131	0.1109	0.9769	0.4708
Quality 82	0.5500	0.0301	0.4785	0.7279	0.1565	0.4709	0.7704	0.6717	0.8884	0.5115
Quality 83	0.5476	0.0627	0.5923	0.0166	0.0956	0.7241	0.7572	0.0240	0.6161	0.7284
Quality 84	0.0016	0.9905	0.8405	0.9980	0.3402	0.1696	0.5877	0.4417	0.1482	0.5063
Quality 85	0.1194	0.6276	0.4935	0.4713	0.2681	0.5250	0.9057	0.1555	0.1568	0.1306
Quality 86	0.4113	0.0597	0.0326	0.6476	0.1364	0.2948	0.2111	0.6064	0.7942	0.6763
Quality 87	0.5296	0.0711	0.4109	0.9121	0.2321	0.4567	0.2112	0.2378	0.0808	0.5172
Quality 88	0.6646	0.1797	0.7424	0.2337	0.5717	0.2205	0.2700	0.7186	0.4824	0.3903
Quality 89	0.1152	0.5531	0.7315	0.7815	0.0914	0.2611	0.2702	0.6503	0.1632	0.7566
Quality 90	0.5778	0.4326	0.0878	0.7706	0.3139	0.6448	0.2168	0.9748	0.6524	0.5125
Quality 91	0.5093	0.5714	0.6468	0.7977	0.9428	0.0246	0.5088	0.0503	0.8699	0.8842
Quality 92	0.0149	0.6163	0.4274	0.7112	0.9032	0.3244	0.0923	0.4535	0.9566	0.1502
Quality 93	0.6761	0.1179	0.7552	0.6251	0.0655	0.2866	0.2072	0.6199	0.8180	0.0758
Quality 94	0.9731	0.5893	0.0149	0.6509	0.3878	0.2742	0.1344	0.0940	0.7385	0.5840
Quality 95	0.7359	0.6271	0.4554	0.3959	0.9723	0.3735	0.0211	0.0577	0.5339	0.0862
Quality 96	0.4766	0.0651	0.1636	0.9158	0.9592	0.2341	0.5829	0.7824	0.7226	0.0569
Quality 97	0.2723	0.5836	0.6512	0.0617	0.4077	0.7324	0.2464	0.7394	0.4949	0.2204
Quality 98	0.1035	0.1020	0.9511	0.1021	0.2995	0.9621	0.1123	0.5068	0.8616	0.8200
Quality 99	0.9802	0.9545	0.8294	0.6468	0.5163	0.5738	0.5102	0.4837	0.7042	0.3413
Quality 100	0.0986	0.4981	0.3760	0.5903	0.4712	0.3599	0.4393	0.8626	0.4312	0.0811

Table 12: Standard deviation σ_{ki} of quality $k \in \mathcal{K}$ in blendstock $i \in \mathcal{B}$.

t_{kp}	Type-1	Type-2	Type-3
Quality 1	18.18	20.20	22.22
Quality 2	11.07	12.30	13.53
Quality 3	39.69	44.10	48.51
Quality 4	44.55	49.50	54.45
Quality 5	9.81	10.90	11.99
Quality 6	50.40	56.00	61.60
Quality 7	10.17	11.30	12.43
Quality 8	13.77	15.30	16.83
Quality 9	51.12	56.80	62.48
Quality 10	59.40	66.00	72.60
Quality 11	39.60	44.00	48.40
Quality 12	19.44	21.60	23.76
Quality 13	21.42	23.80	26.18
Quality 14	52.20	58.00	63.80
Quality 15	12.78	14.20	15.62
Quality 16	14.13	15.70	17.27
Quality 17	56.70	63.00	69.30
Quality 18	46.08	51.20	56.32
Quality 19	60.30	67.00	73.70
Quality 20	29.34	32.60	35.86
Quality 21	54.00	60.00	66.00
Quality 22	21.24	23.60	25.96
Quality 23	24.48	27.20	29.92
Quality 24	35.37	39.30	43.23
Quality 25	24.12	26.80	29.48

t_{kp}	Type-1	Type-2	Type-3
Quality 26	46.80	52.00	57.20
Quality 27	46.80	52.00	57.20
Quality 28	22.86	25.40	27.94
Quality 29	78.75	87.50	96.25
Quality 30	30.24	33.60	36.96
Quality 31	23.04	25.60	28.16
Quality 32	11.25	12.50	13.75
Quality 33	26.46	29.40	32.34
Quality 34	13.95	15.50	17.05
Quality 35	12.96	14.40	15.84
Quality 36	60.75	67.50	74.25
Quality 37	63.90	71.00	78.10
Quality 38	44.55	49.50	54.45
Quality 39	26.82	29.80	32.78
Quality 40	36.36	40.40	44.44
Quality 41	22.68	25.20	27.72
Quality 42	42.12	46.80	51.48
Quality 43	44.28	49.20	54.12
Quality 44	54.90	61.00	67.10
Quality 45	42.93	47.70	52.47
Quality 46	46.80	52.00	57.20
Quality 47	50.40	56.00	61.60
Quality 48	78.30	87.00	95.70
Quality 49	32.94	36.60	40.26
Quality 50	51.84	57.60	63.36
Quality 51	65.88	73.20	80.52
Quality 52	25.65	28.50	31.35
Quality 53	39.69	44.10	48.51
Quality 54	27.54	30.60	33.66
Quality 55	41.58	46.20	50.82
Quality 56	63.45	70.50	77.55
Quality 57	35.64	39.60	43.56
Quality 58	32.40	36.00	39.60
Quality 59	22.68	25.20	27.72
Quality 60	43.92	48.80	53.68
Quality 61	9.90	11.00	12.10
Quality 62	28.08	31.20	34.32
Quality 63	13.77	15.30	16.83
Quality 64	36.18	40.20	44.22
Quality 65	10.71	11.90	13.09
Quality 66	13.05	14.50	15.95
Quality 67	14.40	16.00	17.60
Quality 68	11.07	12.30	13.53
Quality 69	52.65	58.50	64.35
Quality 70	28.08	31.20	34.32
Quality 71	8.46	9.40	10.34
Quality 72	39.69	44.10	48.51
Quality 73	31.05	34.50	37.95
Quality 74	42.12	46.80	51.48
Quality 75	19.08	21.20	23.32
Quality 76	39.60	44.00	48.40
Quality 77	10.17	11.30	12.43
Quality 78	19.80	22.00	24.20
Quality 79	10.44	11.60	12.76
Quality 80	46.80	52.00	57.20
Quality 81	14.94	16.60	18.26
Quality 82	13.41	14.90	16.39
Quality 83	46.44	51.60	56.76
Quality 84	25.74	28.60	31.46
Quality 85	25.74	28.60	31.46
Quality 86	47.25	52.50	57.75
Quality 87	43.92	48.80	53.68
Quality 88	37.26	41.40	45.54
Quality 89	23.58	26.20	28.82
Quality 90	50.04	55.60	61.16
Quality 91	66.60	74.00	81.40
Quality 92	11.34	12.60	13.86
Quality 93	14.49	16.10	17.71
Quality 94	36.45	40.50	44.55
Quality 95	33.12	36.80	40.48
Quality 96	25.38	28.20	31.02
Quality 97	55.80	62.00	68.20
Quality 98	49.05	54.50	59.95
Quality 99	44.28	49.20	54.12
Quality 100	13.05	14.50	15.95

Table 13: Target t_{kp} for quality $k \in \mathcal{K}$ in product $p \in \mathcal{P}$.