Robust Multi-product Newsvendor Model with Uncertain Demand and Substitution

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Abstract

This paper studies Robust Multi-product Newsvendor Model with Substitution (R-MNMS), where the demand and the substitution rates are stochastic and are subject to cardinality-constrained uncertainty sets. The goal of this work is to determine the optimal order quantities of multiple products to maximize the worst-case total profit. To achieve this, we first show that for given order quantities, computing the worst-case total profit, in general, is NP-hard. Therefore, we derive the closed-form optimal solutions for the following three special cases: (1) if there are only two products, (2) if there is no substitution among different products, and (3) if the budget of demand uncertainty is equal to the number of products. For a general R-MNMS, we formulate it as a mixed integer linear program with an exponential number of constraints, and develop a branch and cut algorithm to solve it. For large-scale problem instances, we further propose a conservative approximation of R-MNMS and prove that under some certain conditions, this conservative approximation yields an exact optimal solution to R-MNMS. The numerical study demonstrates the effectiveness of the proposed approaches and the robustness of our model.

Keywords: Newsvendor model, robust, cardinality-constrained uncertainty set, mixed integer program, branch and cut algorithm

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1. Introduction

This paper studies Multi-product Newsvendor Model with Substitution (MNMS) under demand and substitution rate uncertainty, in which a retailer determines the optimal order quantity for each product to maximize its total profit. Due to similarity among different products and their occasional unavailability, the phenomenon of substitution among different products is quite common and has been observed in many studies (cf., [3, 37, 13, 44, 47, 12, 51]). For instance, when shopping at Amazon.com, a customer might turn to the blue hat if his or her first-choice green hat were

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currently unavailable. The existence of substitution somehow increases the profit of the retailer (cf. [37]); however, on the other hand, significantly complicates the problem and makes the problem very challenging to handle. Besides, due to the stochasticity of customers' demand and substitution rates, it might be hard to forecast the demand and substitution rates accurately. Therefore, many works (cf., [15, 42, 37, 38, 52]) proposed stochastic programming models to tackle the demand uncertainty by assuming that the probability distribution of the demand is known. However, in many cases, a good estimation of probability distribution might be very challenging. In particular, many nowadays, technology companies and original equipment manufacturers frequently release their new products. For example, every year, Apple Inc. releases its new-generation iPhones and MacBooks. Without enough historical sales data, it is almost impossible to have an accurate prediction of these new products' demand and substitution rates and inaccurate estimations can cause misleading decisions (cf., [49]). Therefore, to foster a more reliable decision, instead, we study the "Robust" Multi-product Newsvendor Model with Substitution (R-MNMS) subject to cardinality-constrained uncertainty set.

An R-MNMS encounters the following technical features. First of all, due to the substitution effect, it has been shown in [52], even when the demand is deterministic, MNMS can be NP-hard. Second, most of the existing works assumed that the customers' demand follows a given probability distribution, which, however, might result in a loss of sales due to inaccurate demand forecasting. Third, although many existing works illustrated interesting properties of MNMS, the closed-form optimal solutions are rarely known; therefore, very limited managerial insights have been discovered so far. In this paper, we will show that under some conditions, all of these features can be appropriately addressed.

1.1. Relevant Literature

There are two types of relevant literature on how to handle the uncertainty of MNMS: (1) using stochastic programming, and (2) using robust optimization. In this section, we will review both approaches.

Many works on MNMS assumed that the probability distribution of the demand is known, for example, [18, 32, 52]. [18] analyzed the decentralized MNMS, where each retailer owns one product and competes with each other, provided the conditions in which the Nash equilibrium exists and provided an iterative algorithm to solve the model. However, its centralized counterpart, where a retailer owns all the products, becomes highly non-convex, which will be studied in this paper. [32] demonstrated that the profit function could be quasi-concave or bi-modal when the demand is deterministic. Recently, [52] formulated stochastic MNMS as a mixed integer linear program and developed polynomial-time approximation algorithms with performance guarantee to solve it. Different from these works, this paper studies centralized R-MNMS under cardinality-constrained uncertainty set.

However, in practice, it might not be easy to learn the distribution of the random demand completely, in particular, when the random demand is not stationary, i.e., the probability distribution of the random demand is subject to change from time to time. In addition, the inaccurate probability distribution might result in unreliable or misleading decisions. Under these circumstances, alternatively, one can choose the robust approach to formulate the model with partial information of the demand, which can be easily characterized or will stay the same at a relatively long period (i.e., mean, variance, or support). Therefore, some works applied the robust optimization to the newsvendor problems [40, 48, 34, 24, 39, 16, 9, 11, 2]. Especially, [40] was the pioneer to introduce the robust idea to analyze single-product newsvendor problem with known mean and variance of the demand. [48] studied several minimax regret formulations for robust multi-item newsvendor models with a budget constraint when the support of demand is known. They developed efficient algorithms to solve the proposed robust models. Similarly, when the demand is subject to a given interval, [24] determined the optimal order quantity as well as the market selection for a minimax regret multi-market newsvendor model. They further developed an approximation algorithm for solving the large-sized problem instances. With known first and second moments, and the shape of the demand distribution, [36] derived the optimal order policy by minimizing the maximum regret of the newsvendor problem. [2] studied the robust optimization with sum of piecewise linear functions and polyhedral uncertainty set, which can be applied to solve the robust multi-product newsvendor problem under budget uncertainty set. However, all of these works either studied robust single-product newsvendor problem or multi-product newsvendor problem without substitution, while different from these existing works, this paper will study robust multi-product newsvendor problem with substitution, i.e., R-MNMS.

There are very limited works on R-MNMS. For decentralized R-MNMS, [19] used the absolute regret criterion to obtain the unique Nash equilibrium. In their work, only the support of the demand is known, and they also showed that the robust model tended to be more tractable than the stochastic counterpart. Recent work in [23] studied a robust two-product newsvendor model with substitution when the first two moments of demand are known. However, the authors were only able to provide the optimal solution for the following two extreme cases: (1) if there exists no substitution, or (2) if there is a perfect substitution between products. Different from these works, this paper studies centralized R-MNMS, and is not only restricted to the two-product cases.

Many robust optimization problems become NP-hard, although their stochastic counterparts can be solved relatively easily (cf., [4]). Therefore, as described in [10, 50, 33], constructing a suitable uncertainty set is an effective way to address the issue of tractability and over-conservatism. In this paper, we study R-MNMS by using the cardinality-constrained uncertainty set to characterize the random demand and substitution rates. The cardinality-constrained uncertainty set was first introduced by [5] into robust optimization to reduce over-conservatism while at the same time, still achieve the robustness. This framework has been successfully applied to many different areas, for example, healthcare (cf., [21, 8, 1]), manufacturing (cf., [26, 27]), inventory management (cf.

[6, 46]), portfolio optimization (cf. [29]), scheduling (cf., [17, 25, 30]), etc. Since no much work has been done on R-MNMS, this paper will fill this gap and apply cardinality uncertainty set into it. We show that under certain conditions (for example, for the two-product case), we can derive closed-form optimal solutions, which allow us to draw interesting managerial insights.

1.2. Summary of Main Contributions

The objective of this paper is to help a retailer determine optimal order quantities of a single-period multi-product newsvendor model with substitution, which optimizes the worst-case total profit under the cardinality-constrained uncertainty set. The main contributions of this paper are summarized as below:

(i) We develop an equivalent reformulation of R-MNMS and prove that computing the worst-case total profit, in general, is NP-hard for given order quantities.

(ii) We derive closed-form solutions for the following three special cases of R-MNMS: (1) if there are only two products; (2) if there is no substitution among different products; or (3) if the budget of demand uncertainty is equal to the number of products.

(iii) We further reformulate R-MNMS as a mixed-integer linear program (MILP) with an exponential number of constraints, and develop branch and cut algorithm to solve it.

(iv) We provide a conservative approximation of R-MNMS, which can be solved more efficiently, and also prove that under certain conditions, the proposed conservative approximation is equivalent to R-MNMS.

The remainder of the paper is organized as follows. Section 2 introduces the problem setting and the model. Section 3 presents the properties of the model and proves the complexity of computing the worst-case total profit. In Section 4, we derive the optimal order quantities for three special cases of the model. Section 5 reformulates the R-MNMS as an MILP, and develops a branch and cut algorithm and a conservative approximation to solve it. Section 6 presents the results of our numerical investigation on the proposed algorithms. Section 7 concludes the paper.

Notation: The following notation is used throughout the paper. We use bold-letters (e.g., $\boldsymbol{x}, \boldsymbol{A}$) to denote vectors and matrices, and use corresponding non-bold letters to denote their components. Given a vector or matrix $\boldsymbol{x}$, its zero norm $\|\boldsymbol{x}\|_0$ denotes the number of its nonzero elements. We let $\boldsymbol{e}$ be the vector or matrix of all ones, and let $\boldsymbol{e}_i$ be the ith standard basis vector. Given an integer n, we let $[n] := \{1, 2, \ldots, n\}$, and use $\mathbb{R}^n_+ := \{\boldsymbol{x} \in \mathbb{R}^n : x_i \geq 0, \forall i \in [n]\}$. Given a real number t, we let $(t)_+ := \max\{t, 0\}$. Given a finite set I, we let $|I|$ denote its cardinality. We let $\tilde{\boldsymbol{\xi}}$ denote a random vector and denote its realizations by $\boldsymbol{\xi}$. Additional notation will be introduced as needed.

2. Model Formulation

In this section, we present the model formulation for R-MNMS.

To begin with, suppose that there is a retailer selling n similar products in the market indexed by $[n] := \{1, \cdots, n\}$ at a given time period. For each product $i \in [n]$, its cost is c_i, price is p_i, and salvage value is s_i, where by convention, we assume that $p_i \geq c_i \geq s_i$. Each product also bears with a random demand $\tilde{D}_i$ for each $i \in [n]$. Ideally, the retailer would like to determine the optimal order quantity for each product $i \in [n]$, denoted as Q_i. Due to the substitution effect, the effective demand of each product will be affected by its realized demand, its order quantity as well as other products' conditions (i.e., whether out-of-stock or not). To formulate this effect, we suppose that the demand of product $j \in [n]$ can be proportionally substituted by another product $i \in [n]$ and $i \neq j$, once the part of the demand of product j cannot be satisfied by its order quantity Q_j. In particular, we let $\tilde{\alpha}_{ji}$ be the substitution rate, which is the proportion of the unmet demand of product j substituted by product i. Note that $\tilde{\alpha}_{ji}$ might not be equal to $\tilde{\alpha}_{ji}$. In this paper, we assume that all the products have the same unit of measurement, and therefore, for each pair of products $i, j \in [n]$, substitution rate satisfies $\tilde{\alpha}_{ji} \in [0, 1]$. Also, by default, we let $\tilde{\alpha}_{ii} = 0$ for each product $i \in [n]$. We let $\tilde{D}_i^s(\boldsymbol{Q})$ denote the effective demand function of product $i \in [n]$ as below:

$$\tilde{D}_i^s(\boldsymbol{Q}) = \tilde{D}_i + \sum_{j \in [n]} \tilde{\alpha}_{ji}(\tilde{D}_j - Q_j)_+, \forall i \in [n], \tag{1}$$

where the second term in the sum is due to its substitution to the unavailable products.

As shown in [52], the retailer's total profit for given order quantities $\boldsymbol{Q}$, substitution rates $\tilde{\boldsymbol{\alpha}}$, and demand $\tilde{\boldsymbol{D}}$ can be formulated as:

$$\widehat{\Pi}(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \tilde{\boldsymbol{\alpha}}) := \sum_{i \in [n]} \left(p_i \min\left(Q_i, \tilde{D}_i^s(\boldsymbol{Q})\right) - c_i Q_i + s_i \left(Q_i - \tilde{D}_i^s(\boldsymbol{Q})\right)_+ \right). \tag{2}$$

2.1. Constructing Uncertainty Sets of Demand and Substitution Rates

Oftentimes, the substitution rates ($\tilde{\boldsymbol{\alpha}}$) and the demand ($\tilde{\boldsymbol{D}}$) of products in (1) are stochastic and their probability distributions are difficult to characterize. To well address the uncertainties of substitution rates and the demand, we will use robust optimization. In particular, we will study R-MNMS under cardinality-constrained uncertainty sets.

First of all, in the demand uncertainty set, suppose that the demand of the n products (i.e., $\tilde{\boldsymbol{D}}$) is within a box, e.g., $\tilde{\boldsymbol{D}} \in [\boldsymbol{D} - \boldsymbol{l}, \boldsymbol{D} + \boldsymbol{u}]$, where $\boldsymbol{D}$ denotes the nominal demand, $\boldsymbol{l}, \boldsymbol{u}$ denote the lower and upper deviations of the demand respectively satisfying $\boldsymbol{l} \in [\boldsymbol{0}, \boldsymbol{D}]$ and $\boldsymbol{u} \geq \boldsymbol{0}$. We also assume that at most $k \in [n] \cup \{0\}$ products are allowed to deviate from their nominal demand $\boldsymbol{D}$. We will discuss the impact of the budget of uncertainty k on optimal order quantities. Therefore, the uncertainty set of the demand can be written as

$$\mathcal{U}_0 = \left\{ \tilde{\boldsymbol{D}} : \tilde{D}_i = D_i + \Delta_i, -l_i \le \Delta_i \le u_i, \forall i \in [n], \|\Delta\|_0 \le k \right\}, \tag{3}$$

Similarly, let us denote the uncertainty set of substitution rate as below

$$\mathcal{U}_\alpha = \left\{ \tilde{\boldsymbol{\alpha}} : \tilde{\alpha}_{ji} = \alpha_{ji} + \Delta^\alpha_{ji}, -l^\alpha_{ji} \le \Delta^\alpha_{ji} \le u^\alpha_{ji}, \forall\, i, j \in [n], \|\boldsymbol{\Delta}^\alpha\|_0 \le k^\alpha \right\}, \tag{4}$$

where $\|\cdot\|_0$ denotes the zero-norm, and k^α is the budget of uncertainty. We suppose that the substitution rates are within a box, e.g., $\tilde{\boldsymbol{\alpha}} \in [\boldsymbol{\alpha} - \boldsymbol{l}^{\boldsymbol{\alpha}}, \boldsymbol{\alpha} + \boldsymbol{u}^{\boldsymbol{\alpha}}]$, where $\boldsymbol{\alpha}$ denotes the nominal substitution rates, $\boldsymbol{l}^{\boldsymbol{\alpha}}, \boldsymbol{u}^{\boldsymbol{\alpha}}$ denote the lower and upper deviations of the substitution rates respectively satisfying $\boldsymbol{l}^{\boldsymbol{\alpha}} \in [\boldsymbol{0}, \boldsymbol{\alpha}]$, $\boldsymbol{u}^{\boldsymbol{\alpha}} \in [0, \boldsymbol{e} - \boldsymbol{\alpha}]$. For notational convenience, we let $\alpha^\alpha_{ii} = l^\alpha_{ii} = u^\alpha_{ii} = 0$ for each $i \in [n]$.

With the notation introduced above, R-MNMS can be formulated as:

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}^n_+} \min_{\tilde{\boldsymbol{D}} \in \mathcal{U}_0, \tilde{\boldsymbol{\alpha}} \in \mathcal{U}_\alpha} \left\{ \widehat{\Pi}(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \tilde{\boldsymbol{\alpha}}) := \sum_{i \in [n]} \left(p_i \min\left(Q_i, \tilde{D}^s_i(\boldsymbol{Q})\right) - c_i Q_i + s_i \left(Q_i - \tilde{D}^s_i(\boldsymbol{Q})\right)_+ \right) \right\}. \tag{5}$$

In Model (5), the objective is to find optimal order quantities to maximize the worst-case total profit over the uncertainty sets $\mathcal{U}_0, \mathcal{U}_\alpha$. For each product $i \in [n]$, we let $\overline{P}_i = p_i - c_i \ge 0$ and $\overline{S}_i = p_i - s_i \ge 0$. Note that $\overline{P}_i$ can be interpreted as the marginal profit or underage cost of product $i \in [n]$, while $\overline{S}_i$ is the sum of the underage cost $(p_i - c_i)$ and overage cost $(c_i - s_i)$ of product $i \in [n]$, where their ratio $\frac{\overline{P}_i}{\overline{S}_i}$ is known as the critical ratio of newsvendor model (c.f., [31]). Since $\min(Q_i, \tilde{D}^s_i(\boldsymbol{Q})) = Q_i - (Q_i - \tilde{D}^s_i(\boldsymbol{Q}))_+$ for each $i \in [n]$, the above Model (5) is equivalent to

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}^n_+} \min_{\tilde{\boldsymbol{D}} \in \mathcal{U}_0, \tilde{\boldsymbol{\alpha}} \in \mathcal{U}_\alpha} \left\{ \widehat{\Pi}(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \tilde{\boldsymbol{\alpha}}) := \sum_{i \in [n]} \overline{P}_i Q_i - \sum_{i \in [n]} \overline{S}_i \left(Q_i - \tilde{D}^s_i(\boldsymbol{Q})\right)_+ \right\}. \tag{6}$$

We treat the uncertainties of demand and substitution rates separately because (i) in practice, the demand estimation and substitution rates estimation follow different procedure [20] and (ii) the separable uncertainty sets allow us to reformulate the model as a mixed integer linear program and to obtain closed-form solutions. For notational convenience, throughout this paper, we will let $\boldsymbol{Q}^*$ denote an optimal solution to R-MNMS (6).

2.2. Discussion about How to Estimate the Uncertainty Sets

The budgets of uncertainty (i.e., k, k^α) in Model 6 plays an important role, and a good choice of these values can achieve both least-conservatism and robustness. The following steps show how to find the optimal budgets of uncertainty $k^*, k^{\alpha*}$ using possibly limited historical data:

Step 0: We split the historical data into two groups Υ_i, $i \in [2]$, and select a candidate set $\mathcal{K} \subseteq \{0\} \cup [n] \times \{0\} \cup [n^2 - n]$ to choose the best $(k^*, k^{\alpha*})$.

Step 1.1: **Determine nominal demand $\widehat{\boldsymbol{\mu}}$, the lower deviation $\boldsymbol{l}$, and the upper deviation $\boldsymbol{u}$.** To do so, we compute the sample mean $\widehat{\mu}_i$ and standard deviation $\widehat{\sigma}_i$ of the first group of historical demand data Υ_1 for each product $i \in [n]$. Then we set the nominal demand $D_i = \widehat{\mu}_i$, and $u_i = l_i = 1.96\widehat{\sigma}_i$ for each product $i \in [n]$.

Step 1.2: **Determine nominal substitution rate $\widehat{\boldsymbol{\mu}}^{\alpha}$, the lower deviation $\boldsymbol{l}^{\alpha}$, and the upper deviation $\boldsymbol{u}^{\alpha}$.** Similarly, we compute the sample mean $\widehat{\mu}^{\alpha}_{ji}$ and standard deviation $\widehat{\sigma}^{\alpha}_{ji}$ of the first group of historical substitution rate data Υ_1 for each pair of products $i, j \in [n]$. Then we set the nominal substitution rate $\alpha_{ji} = \widehat{\mu}^{\alpha}_{ji}$, and $u^{\alpha}_{ji} = l^{\alpha}_{ji} = 1.96\widehat{\sigma}^{\alpha}_{ji}$ for each pair of products $i, j \in [n]$.

Step 2: **Calculate the optimal order quantities $\boldsymbol{Q}^*(k, k^{\alpha})$ and objective value $v^*(k, k^{\alpha})$ for each $(k, k^{\alpha}) \in \mathcal{K}$ by solving Model (6).**

Step 3: **Compute the objective value $\widehat{\Pi}(\boldsymbol{Q}^*(k, k^{\alpha}), \boldsymbol{D}, \boldsymbol{\alpha})$ of Model (2) for each $(k, k^{\alpha}) \in \mathcal{K}$ and each pair of demand and substitution rates $(\boldsymbol{D}, \boldsymbol{\alpha})$ in the second group of historical data Υ_2.**

Step 4: **Determine the optimal $k^*, k^{\alpha*}$.** For each $(k, k^{\alpha}) \in \mathcal{K}$, we compute the qth percentile of $\{\widehat{\Pi}(\boldsymbol{Q}^*(k, k^{\alpha}), \boldsymbol{D}, \boldsymbol{\alpha})\}_{(\boldsymbol{D},\boldsymbol{\alpha})\in\Upsilon_2}$, and denote it as $\widehat{\Pi}^{q\%}(k, k^{\alpha})$. Given two nonnegative weights $w_1, w_2 \in \mathbb{R}_+$, we choose the optimal budgets of uncertainty $k^*, k^{\alpha*}$ which achieve the smallest weighted value $w_1 k + w_2 k^{\alpha}$ such that $v^*(k, k^{\alpha}) \leq \widehat{\Pi}^{q\%}(k, k^{\alpha})$.

3. Equivalent Reformulation and Model Properties

In this section, we study R-MNMS under cardinality-constrained uncertainty set and derive its equivalent reformulation. We also provide upper bounds of optimal order quantities and show that computing the worst-case total profit for given order quantities, in general, is NP-hard.

Throughout the rest of the paper, we will make the following assumption.

Assumption 1. *Suppose that $k^{\alpha} = n^2 - n$ in the substitution uncertainty set $\mathcal{U}_{\alpha}$.*

Assumption 1 implies that the substitution uncertainty set $\mathcal{U}_{\alpha}$ is purely a box. We make this assumption for the following reasons: (i) first of all, it is often more difficult to estimate substitution rates $\tilde{\boldsymbol{\alpha}}$ than the demand; (ii) second, under this assumption, we can derive some interesting analytical results; and (iii) third, our exact branch and cut algorithm in Section 5 can be applied to the general k^{α}, and it follows directly from the derivation in Section 5.

3.1. Equivalent Reformulation

In this subsection, we provide an alternative formulation for Model (6).

First, we make the following observation.

Lemma 1. *For any $\boldsymbol{Q}, \tilde{\boldsymbol{D}} \in \mathbb{R}^n_+$, the profit function $\widehat{\Pi}\left(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \cdot\right)$ is monotone nondecreasing in $\tilde{\boldsymbol{\alpha}}$; and for any $\boldsymbol{Q}, \tilde{\boldsymbol{\alpha}} \in \mathbb{R}^n_+$, the profit function $\widehat{\Pi}\left(\boldsymbol{Q}, \cdot, \tilde{\boldsymbol{\alpha}}\right)$ is monotone nondecreasing in $\tilde{\boldsymbol{D}}$.*

Proof. According to Model (6), the profit function $\widehat{\Pi}(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \cdot)$ is nondecreasing in $\tilde{D}_i^s(\boldsymbol{Q})$ and from (1), the effective demand $\tilde{D}_i^s(\boldsymbol{Q})$ is also nondecreasing in $\tilde{\alpha}_{ji}$ for each product $i, j \in [n]$. Therefore, the profit function $\widehat{\Pi}(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \cdot)$ is nondecreasing in $\tilde{\boldsymbol{\alpha}}$. Similarly, from (1), $\tilde{D}_i^s(\boldsymbol{Q})$ is also nondecreasing in $\tilde{D}_i$ for each product $i \in [n]$. Therefore, the profit function $\widehat{\Pi}(\boldsymbol{Q}, \cdot, \tilde{\boldsymbol{\alpha}})$ is nondecreasing in the demand $\tilde{\boldsymbol{D}}$. □

According to Lemma 1 and Assumption 1, $\min_{\tilde{\alpha} \in \mathcal{U}_\alpha} \widehat{\Pi}(\boldsymbol{Q}, \cdot, \tilde{\boldsymbol{\alpha}})$ is achieved by $\tilde{\alpha}_{ji} = \alpha_{ji} - l_{ji}^{\alpha} := \underline{\alpha}_{ji}$ for all products $i \neq j$ and $i, j \in [n]$. In this case, Model (6) becomes

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}_+^n} \min_{\tilde{\boldsymbol{D}} \in \mathcal{U}_0} \left\{ \Pi(\boldsymbol{Q}, \tilde{\boldsymbol{D}}) := \sum_{i \in [n]} \overline{P}_i Q_i - \sum_{i \in [n]} \overline{S}_i \left(Q_i - \tilde{D}_i^s(\boldsymbol{Q}) \right)_+ \right\}, \tag{7}$$

where we let $\Pi(\boldsymbol{Q}, \tilde{\boldsymbol{D}}) = \widehat{\Pi}(\boldsymbol{Q}, \tilde{\boldsymbol{D}}, \underline{\boldsymbol{\alpha}})$.

Now we are ready to show our equivalent reformulation. The main idea of the derivation is to show that in the worst-case, the uncertainty set $\mathcal{U}_0$ can be restricted to the following mixed integer set:

$$\mathcal{U} = \left\{ \tilde{\boldsymbol{D}} : \sum_{i \in [n]} z_i \leq k, \tilde{D}_i = D_i - l_i z_i, z_i \in \{0, 1\}, \forall i \in [n] \right\}. \tag{8}$$

Clearly, set $\mathcal{U} \subseteq \mathcal{U}_0$, since for any feasible point $(\tilde{\boldsymbol{D}}, \boldsymbol{z})$ satisfying constraints in (8), let us define $\Delta_i = -l_i z_i$ for each $i \in [n]$, then $(\tilde{\boldsymbol{D}}, \boldsymbol{\Delta})$ satisfies the constraints in (3). Indeed, we can show that

Proposition 1. *R-MNMS* (7) *is equivalent to*

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}_+^n} \min_{\tilde{\boldsymbol{D}} \in \mathcal{U}} \Pi \left(\boldsymbol{Q}, \tilde{\boldsymbol{D}} \right), \tag{9}$$

where $\mathcal{U}$ *is defined in* (8)*.*

Proof. Let v_1 denote the optimal value of Model (9), then we only need to show $v_1 = v^*$.

(i) $v_1 \geq v^*$. For any $\tilde{\boldsymbol{D}} \in \mathcal{U}_0$, there exists $\boldsymbol{\Delta}$ such that $\|\boldsymbol{\Delta}\|_0 \leq k, \tilde{D}_i = D_i + \Delta_i, -l_i \leq \Delta_i \leq u_i$. Let us define binary variable $z_i = \begin{cases} 0, & \text{if } \Delta_i = 0 \\ 1, & \text{if } \Delta_i \neq 0 \end{cases}$ for each $i \in [n]$. Since $\|\boldsymbol{\Delta}\|_0 \leq k$, thus we must have $\sum_{i \in [n]} z_i \leq k$. Let us define $\tilde{D}_i^* = D_i - l_i z_i$ for each $i \in [n]$. Clearly, we have $\tilde{\boldsymbol{D}}^* \in \mathcal{U}$ and $\tilde{\boldsymbol{D}}^* \leq \tilde{\boldsymbol{D}}$. For any fixed $\boldsymbol{Q} \in \mathbb{R}_+^n$, by Lemma 1, we know that the profit function $\Pi \left(\boldsymbol{Q}, \tilde{\boldsymbol{D}} \right)$ is nondecreasing in the demand $\tilde{\boldsymbol{D}}$. Thus, $\Pi \left(\boldsymbol{Q}, \tilde{\boldsymbol{D}} \right) \geq \Pi \left(\boldsymbol{Q}, \tilde{\boldsymbol{D}}^* \right)$, which implies $\min_{\tilde{\boldsymbol{D}} \in \mathcal{U}_0} \Pi \left(\boldsymbol{Q}, \tilde{\boldsymbol{D}} \right) \geq \min_{\tilde{\boldsymbol{D}} \in \mathcal{U}} \Pi \left(\boldsymbol{Q}, \tilde{\boldsymbol{D}} \right)$ for any $\boldsymbol{Q} \in \mathbb{R}_+^n$. This proves $v_1 \geq v^*$.

(ii) $v_1 \leq v^*$. Since $\mathcal{U}_0 \supseteq \mathcal{U}$, thus for any $\boldsymbol{Q} \in \mathbb{R}_+^n$, $\min_{\tilde{\boldsymbol{D}} \in \mathcal{U}_0} \Pi \left(\boldsymbol{Q}, \tilde{\boldsymbol{D}} \right) \leq \min_{\tilde{\boldsymbol{D}} \in \mathcal{U}} \Pi \left(\boldsymbol{Q}, \tilde{\boldsymbol{D}} \right)$, thus, $v_1 \leq v^*$.

□

From Proposition 1, by substituting $\tilde{D}_i = D_i - l_i z_i$ in (6) and defining the following cardinality set

$$X = \left\{ \boldsymbol{z} : \sum_{i\in[n]} z_i \le k, z_i \in \{0,1\} \right\}, \tag{10}$$

then we can have the following equivalent formulation of R-MNMS:

$$v^* = \max_{\boldsymbol{Q}\in\mathbb{R}^n_+} \left\{ f(\boldsymbol{Q}) := \sum_{i\in[n]} \overline{P}_i Q_i - R(\boldsymbol{Q}) \right\}, \tag{11a}$$

where

$$R(\boldsymbol{Q}) := \max_{\boldsymbol{z}\in X} \sum_{i\in[n]} \overline{S}_i \left(Q_i - D_i + l_i z_i - \sum_{j\in[n]} \underline{\alpha}_{ji} \left(D_j - l_j z_j - Q_j\right)_+ \right)_+ . \tag{11b}$$

This new equivalent formulation (11) allows us to compute the worst-case profit function via an integer program rather than a nonconvex program, which can be further reduced to a mixed integer linear program (MILP) in Section 5.

One direct benefit of formulation (11) is that we can easily derive upper bounds of optimal order quantities. The result can be proved by contradiction.

Proposition 2. *There exists an optimal solution $\boldsymbol{Q}^*$ to R-MNMS such that for each product $i \in [n]$, $Q_i^* \le M_i$, where $M_i = D_i + \sum_{j\in[n]} \underline{\alpha}_{ji} D_j$.*

Proof. See Appendix A.1. □

This result is very useful to derive an equivalent MILP formulation of R-MNMS in Section 5.

3.2. Complexity of the Inner Maximization Problem (11b)

It has been shown in [52] that even if $k = 0, k^\alpha = 0$, solving R-MNMS can be NP-hard. In this subsection, we will show that the inner maximization problem (11b) of R-MNMS is also NP-hard.

First, observe that

$$\left(D_j - l_j z_j - Q_j\right)_+ = \begin{cases} \left(D_j - l_j - Q_j\right)_+, & \text{if } z_j = 1 \\ \left(D_j - Q_j\right)_+, & \text{if } z_j = 0 \end{cases}$$

for each $j \in [n]$, thus this observation allows us to linearize nonlinear expressions $\{(D_j - l_j z_j - Q_j)_+\}_{j\in[n]}$ and to rewrite (11b) as

$$R(\boldsymbol{Q}) = \max_{\boldsymbol{z}\in X} \left\{ \sum_{i\in[n]} \overline{S}_i \left[Q_i - D_i + l_i z_i - \sum_{j\in[n]} \underline{\alpha}_{ji} \left(\left(D_j - l_j - Q_j\right)_+ z_j + (D_j - Q_j)_+ (1 - z_j) \right) \right]_+ \right\}. \tag{12}$$

Next, we show that the inner maximization problem (12) is NP-hard via a reduction to the well known clique problem.

Theorem 1. *The inner maximization problem* (12) *in general is NP-hard.*

Proof. See Appendix A.2. □

Theorem 1 shows that unlike many robust optimization problems, it might be difficult to derive a tractable form for the general inner maximization problem (12). Thus, instead, in Section 4, we propose three special cases such that both inner maximization (12) and R-MNMS are tractable. For general R-MNMS, we propose an equivalent MILP reformulation and develop exact and approximate algorithms to solve it, which will be presented in Section 5.

4. Three Special Cases: Closed-form Optimal Solutions

In this section, we will study three different special cases of R-MNMS (11) and derive their closed-form optimal solutions.

4.1. Special Case I*:* $n = 2, k = 1$

In this section, we study R-MNMS with only two products (i.e., $n = 2$) and the budget of uncertainty is equal to 1 (i.e., $k = 1$ in set X defined in (10)). Note that if $k = 0$ or 2, it reduces to Special Case III, which will be discussed in Section 4.3. Under this setting, R-MNMS (11) becomes:

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}^2_+} \left\{ \sum_{i \in [2]} \overline{P}_i Q_i - \max_{\boldsymbol{z} \in X} \sum_{i \in [2]} \overline{S}_i \left(Q_i - D_i + l_i z_i - \sum_{j \in [2]} \underline{\alpha}_{ji} \left(D_j - l_j z_j - Q_j \right)_+ \right)_+ \right\}, \quad (13)$$

and $X = \{\boldsymbol{z} : z_1 + z_2 \leq 1, z_i \in \{0, 1\}, \forall i \in [2]\}$. To simplify our closed-form solutions, we further make the following assumption.

Assumption 2. *Suppose that* $D_2 \underline{\alpha}_{21} \geq l_1 \geq \underline{\alpha}_{21} l_2, D_1 \underline{\alpha}_{12} \geq l_2 \geq \underline{\alpha}_{12} l_1$.

Assumption 2 postulates that the demand deviation of one product cannot be smaller than the substitution part of the other product's demand deviation and cannot be larger than the substitution part of the other product's nominal demand. Please note that our analysis is general and can be also applied to the other parametric settings without satisfying Assumption 2. However, for the brevity of this paper, we will stick to this assumption in this subsection.

The next theorem presents our main findings of the optimal order quantities for this special case under Assumption 2. The key ideas to these results are: (1) to divide the feasible regions into 9 subregions by comparing Q_i with $D_i - l_i$ and D_i for each $i \in [2]$; (2) for each subregion, R-MNMS (13) becomes a concave maximization problem with a piecewise linear objective function, thus one of its optimal solutions can be achieved by an extreme point; and (3) for each subregion, there are not too many potential optimal solutions, thus, we enumerate all the candidate solutions and find the one which achieves the highest total profit across all the 9 subregions.

Theorem 2. *Suppose $n = 2$, $k = 1$, and Assumption 2 holds, then the optimal order quantities $\boldsymbol{Q}^* = (Q_1^*, Q_2^*)$ are characterized by the following three cases:*

Case 1: If $\overline{P}_1 \leq \overline{P}_2\underline{\alpha}_{12}$ and $\overline{P}_2 \geq \overline{P}_1\underline{\alpha}_{21}$, then $(Q_1^*, Q_2^*) = (0, D_2 - l_2 + \underline{\alpha}_{12}D_1)$.

Case 2: If $\overline{P}_2 \leq \overline{P}_1\underline{\alpha}_{21}$ and $\overline{P}_1 \geq \overline{P}_2\underline{\alpha}_{12}$, then $(Q_1^*, Q_2^*) = (D_1 - l_1 + \underline{\alpha}_{21}D_2, 0)$.

Case 3: If $\overline{P}_1 \geq \overline{P}_2\underline{\alpha}_{12}$ and $\overline{P}_2 \geq \overline{P}_1\underline{\alpha}_{21}$, then we have the following two sub-cases:

Sub-case 3.1: If $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$, then $(Q_1^*, Q_2^*) = \left(D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}, D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}\right)$ or $(Q_1^*, Q_2^*) = \left(D_1, D_2 - \frac{\overline{S}_2 l_2 - \overline{S}_1 l_1}{\overline{S}_2 - \overline{S}_1\underline{\alpha}_{21}}\right)$.

Sub-case 3.2: If $\overline{S}_1 l_1 \leq \overline{S}_2 l_2$, then $(Q_1^*, Q_2^*) = \left(D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}, D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}\right)$ or $(Q_1^*, Q_2^*) = \left(D_1 - \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2\underline{\alpha}_{12}}, D_2\right)$.

Proof. See Appendix A.3. □

Theorem 2 provides a complete characterization of optimal order quantities of the two-product case, which highly depend on the comparison between the marginal profit of product i and the profit generated by using product j to substitute product i. In particular, we make the following remarks.

Remark 1. (i) In Case 1, suppose that the marginal profit of product 1 is lower than the profit generated by using product 2 to substitute product 1, but the marginal profit of product 2 is higher than the profit generated by using product 1 to substitute product 2, i.e., product 2 is much more profitable than product 1. Thus, in this case, the retailer should only order product 2 to satisfy their customers' demand as well as to satisfy part of the customers' demand for product 1 by substitution. In this case, the worst-case demand of product 2 is $D_2 - l_2$ while the worst-case demand of product 1 is equal to the nominal demand D_1.

(ii) The interpretation of Case 2 is similar and thus is omitted for brevity.

(iii) In Case 3, if the marginal profit of one product is higher than the profit generated by using the other product to substitute this product (i.e., both products are similarly profitable), then the optimal order quantities depend on the relationship between $\overline{S}_1 l_1$ and $\overline{S}_2 l_2$. One special case is that when $s_i = c_i$ for each product $i \in [2]$, i.e., the salvage value of each product is equal to its unit production cost, the optimal order quantity of product 1 is $Q_1^* = D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}$ and the optimal order quantity of product 2 is $Q_2^* = D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}$, while the worst-case demand of products 1 and 2 can be $(D_1, D_2 - l_2)$ or $(D_1 - l_1, D_2)$, respectively. If there is a tie between two solutions in Sub-case 3.1 or Sub-case 3.2, then one can randomly pick one solution as both of them are optimal.

(iv) It is impossible to have the case that $\overline{P}_1 < \overline{P}_2\underline{\alpha}_{12}$, $\overline{P}_2 < \overline{P}_1\underline{\alpha}_{21}$, which implies $1 < \underline{\alpha}_{12}\underline{\alpha}_{21}$, contradicting the assumption that all the substitution rates are between 0 and 1.

4.2. Special Case II*:* $\underline{\boldsymbol{\alpha}} = \mathbf{0}$

In this subsection, we analyze robust multi-product newsvendor problem without substitution, i.e., $\underline{\boldsymbol{\alpha}} = \mathbf{0}$. In this setting, the effective demand becomes $\tilde{D}_i^s(\boldsymbol{Q}) = \tilde{D}_i = D_i - l_i z_i$. Thus, R-MNMS (11) reduces to:

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}_+^n} \left\{ f(\boldsymbol{Q}) := \sum_{i \in [n]} \overline{P}_i Q_i - \max_{\boldsymbol{z} \in X} \sum_{i \in [n]} \overline{S}_i \left(Q_i - D_i + l_i z_i\right)_+ \right\}, \tag{14}$$

where set X is defined in (10). We first make the following observation.

Lemma 2. *There exists an optimal solution $\boldsymbol{Q}^*$ of Model* (14) *such that $D_i - l_i \le Q_i^* \le D_i$ for all $i \in [n]$.*

Proof. For notational convenience, let us define $\boldsymbol{Q}_{-i} = [Q_1, \cdots, Q_{i-1}, Q_{i+1}, \cdots, Q_n]^\top$ to be the vector of the remaining elements of $\boldsymbol{Q}$. It is sufficient to show that for any fixed $\boldsymbol{Q}_{-i} \in \mathbb{R}_+^{n-1}$, the objective function of Model (14), $f(Q_i, \boldsymbol{Q}_{-i})$, is monotone nondecreasing in Q_i when $Q_i \in [0, D_i - l_i]$ and monotone nonincreasing in Q_i when $Q_i \in [D_i, +\infty)$. Indeed, we note that

$$\begin{aligned}
& f(Q_i, \boldsymbol{Q}_{-i}) \\
&= \sum_{\tau \in [n] \setminus \{i\}} \overline{P}_\tau Q_\tau + \overline{P}_i Q_i - \max_{\boldsymbol{z} \in X} \left(\sum_{\tau \in [n] \setminus \{i\}} \overline{S}_\tau \left(Q_\tau - D_\tau + l_\tau z_\tau\right)_+ + \overline{S}_i \left(Q_i - D_i + l_i z_i\right)_+ \right) \\
&= \begin{cases} \sum\limits_{\tau \in [n] \setminus \{i\}} \overline{P}_\tau Q_\tau - \max\limits_{\boldsymbol{z} \in X} \sum\limits_{\tau \in [n] \setminus \{i\}} \overline{S}_\tau \left(Q_\tau - D_\tau + l_\tau z_\tau\right)_+ + \overline{P}_i Q_i, & \text{if } Q_i \in [0, D_i - l_i], \\ \sum\limits_{\tau \in [n] \setminus \{i\}} \overline{P}_\tau Q_\tau - \max\limits_{\boldsymbol{z} \in X} \left(\sum\limits_{\tau \in [n] \setminus \{i\}} \overline{S}_\tau \left(Q_\tau - D_\tau + l_\tau z_\tau\right)_+ - D_i + l_i z_i \right) + \left(\overline{P}_i - \overline{S}_i\right) Q_i, & \text{if } Q_i \in [D_i, +\infty). \end{cases}
\end{aligned}$$

Clearly, from the above equation, we know that if $Q_i \in [0, D_i - l_i]$, the coefficient of Q_i is $\overline{P}_i$, which is nonnegative, while if $Q_i \in [D_i, +\infty)$, the coefficient of Q_i is $\overline{P}_i - \overline{S}_i$, which is nonpositive. Thus, $f(Q_i, \boldsymbol{Q}_{-i})$ is nondecreasing on Q_i when $Q_i \in [0, D_i - l_i]$ and nonincreasing on Q_i when $Q_i \in [D_i, +\infty)$. This completes the proof. $\square$

According to Lemma 2, without loss of generality, we can assume in Model (14), $\boldsymbol{Q} \in [\boldsymbol{D} - \boldsymbol{l}, \boldsymbol{D}]$. Thus, for each $i \in [n]$, $(Q_i - D_i + l_i z_i)_+ = \begin{cases} 0, & \text{if } z_i = 0 \\ Q_i - D_i + l_i, & \text{if } z_i = 1 \end{cases} = (Q_i - D_i + l_i)\, z_i.$

Therefore, Model (14) is equivalent to

$$v^* = \max_{\boldsymbol{Q}\in[\boldsymbol{D}-\boldsymbol{l},\boldsymbol{D}]} \left(\sum_{i\in[n]} \overline{P}_i Q_i - \max_{\boldsymbol{z}\in X} \sum_{i\in[n]} \overline{S}_i \left(Q_i - D_i + l_i\right) z_i \right), \tag{15}$$

where X is defined in (10).

Suppose that $\{(1),(2),\cdots,(n)\}$ is a permutation of $[n]$ such that $\overline{S}_{(1)}l_{(1)} \geq \overline{S}_{(2)}l_{(2)} \geq \cdots \geq \overline{S}_{(n)}l_{(n)}$. We can obtain a closed-form optimal solution to Model (15) as follows.

Theorem 3. *When $\underline{\boldsymbol{\alpha}} = 0$, the optimal solutions $\boldsymbol{Q^*}$ of Model (15) are characterized as follows:*

(i) If $\sum_{i\in[n]} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, then $Q_i^* = D_i - l_i$, and $v^* = \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right)$.

(ii) If $\sum_{i\in[n]} \frac{\overline{P}_i}{\overline{S}_i} > k$,

$$Q_i^* = \begin{cases} D_i - l_i + \frac{\overline{S}_{(t+1)}l_{(t+1)}}{\overline{S}_i}, & \text{if } i \in T \\ D_i, & \text{if } i \in [n]\setminus T \end{cases},$$

and

$$v^* = \sum_{i\in[n]\setminus T} \overline{P}_i l_i - \overline{S}_{(t+1)}l_{(t+1)}k + \sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \overline{S}_{(t+1)}l_{(t+1)} + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right),$$

where set $T := \{(1),(2),\cdots,(t)\}$ satisfying $\sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, $\sum_{i\in T\cup\{(t+1)\}} \frac{\overline{P}_i}{\overline{S}_i} > k$.

Proof. See Appendix A.4.

□

Theorem 3 reveals the impact of the budget of uncertainty on the optimal order quantities. Indeed, if $\sum_{i\in[n]} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, i.e., the budget of uncertainty k is no smaller than the sum of the critical ratios of all the products, then in this case, the optimal order quantity for each product is equal to the lower bound of the demand, i.e., $Q_i^* = D_i - l_i$ for each $i \in [n]$. Hence, this implies that when the products are not very profitable or the accuracy of demand forecasting is relatively low, then the decision of the retailer should be conservative to hedge against unnecessary loss from demand forecasting. Suppose that $\sum_{i\in[n]} \frac{\overline{P}_i}{\overline{S}_i} > k$, i.e, the budget of uncertainty is smaller than the sum of critical ratios of all the products, or equivalently, relatively a small number of demand can be allowed to deviate from the nominal demand $\boldsymbol{D}$. Also, note that for each product $i \in [n]$, the value of $\overline{S}_i l_i$ can be interpreted as the risk of lost sales for product i when its order quantity is D_i with the worst-case demand $D_i - l_i$ (i.e, the sum of underage cost and overage cost multiplies the demand deviation). In this case, for each product $i \in T$ whose risk of lost sales is larger than a threshold $\overline{S}_{(t+1)}l_{(t+1)}$, its order quantities should be equal to $D_i - l_i + \frac{\overline{S}_{(t+1)}l_{(t+1)}}{\overline{S}_i}$; otherwise, it should be D_i. The threshold $\overline{S}_{(t+1)}l_{(t+1)}$ can be determined by searching for the product such that sum of the critical ratios of the products whose risk is higher than product $(t+1)$ is no larger than the budget of uncertainty k, but including the critical ratio of this product into the sum will be above k. This result tells that the products with lower risk of lost sales should be ordered up to the nominal demand, while those with higher risk should be ordered less than the nominal demand.

4.3. Special Case III*: $k = n$*

When the budget of uncertainty is equal to n, i.e., $k = n$, the uncertainty set $\mathcal{U}$ becomes

$$\mathcal{U} = \left\{ \tilde{\boldsymbol{D}} : \tilde{D}_i = D_i - l_i z_i, \sum_{i \in [n]} z_i \leq n, z_i \in \{0, 1\}, \forall i \in [n] \right\}.$$

From Lemma 1, we know that the profit function $\Pi(\boldsymbol{Q}, \tilde{\boldsymbol{D}})$ is nonincreasing in $\tilde{\boldsymbol{D}}$, thus at the optimality, we must have $z_i = 1$ for all $i \in [n]$ in the inner maximization problem (11b), i.e., the worst-case demand in this special case will always be equal to $\boldsymbol{D} - \boldsymbol{l}$. Thus, Model (11) becomes

$$v^* = \max_{\boldsymbol{Q} \in \mathbb{R}_+^n} \sum_{i \in [n]} \overline{P}_i Q_i - \sum_{i \in [n]} \overline{S}_i \left[Q_i - D_i + l_i - \sum_{j \in [n]} \underline{\alpha}_{ji} \left(D_j - l_j - Q_j\right)_+ \right]_+ . \tag{16}$$

Note that Model (16) is a multi-product newsvendor model with substitution when the demand is deterministic and is equal to $\boldsymbol{D} - \boldsymbol{l}$. According to the recent work in [52], the optimal order quantities of Model (16) can be completely characterized as follows (For more details, please refer to [52]).

Theorem 4. *(Theorem 1, [52]) When $k = n$, the optimal order quantities $\boldsymbol{Q}^*$ and the optimal total profit v^* are characterized as follows:*

(i)

$$Q_j^* = \begin{cases} D_j^s(\boldsymbol{Q}^*) = D_j - l_j + \sum\limits_{i \in \Gamma^*} \underline{\alpha}_{ij}(D_i - l_i), & \text{if } f(\Gamma^* \cup \{j\}) < f(\Gamma^*) \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

for each $j \in [n]$, where $[n] \setminus \operatorname{supp}(\boldsymbol{Q}^*) = \Gamma^*$, i.e., $\Gamma^* = \{i \in [n] : Q_i^* = 0\}$; and

(ii)

$$v^* = \max_{\Gamma \subseteq [n]} \left\{ f(\Gamma) := \sum_{j \in \Gamma} \sum_{i \in [n] \setminus \Gamma} \underline{\alpha}_{ji} \overline{P}_i (D_j - l_j) + \sum_{i \in [n] \setminus \Gamma} \overline{P}_i (D_i - l_i) \right\} := f(\Gamma^*), \tag{18}$$

In Theorem 4, if the budget of uncertainty is equal to the number of products, then for each product $j \in [n]$, its optimal order quantity $\boldsymbol{Q}_j^*$ is equal to its effective demand if its marginal profit $\overline{P}_j$ is larger than or equal to the sum of the profits generated by using other products to substitute it, and 0, otherwise. This suggests that the retailer does not need to order a product if its marginal profit is relatively low and should order up to its effective demand, otherwise. Also, in (18), the first term is the sum of the total profit for selling product $i \in [n] \setminus \Gamma$ to meet the demand of its substitutable products $j \in \Gamma$ and the second term is the profit of selling product $i \in [n] \setminus \Gamma$ to meet its own demand. Finally, please note that although we completely characterize the optimal order quantities for all the products, obtaining these value is in general NP-hard (cf., [52]).

Another interesting observation from Theorem 4 is that the optimal order quantity for each product can be equal to their worst-case demand, i.e., $Q_j^* = D_j - l_j$ for each product $j \in [n]$, under the following assumptions.

Corollary 1. *Suppose (1) $\overline{P}_i = \overline{P}_j, \forall i, j \in [n]$ and (2) for each product $j \in [n]$, $\sum_{i \in [n]} \underline{\alpha}_{ji} < 1$. Then $Q_j^* = D_j - l_j$ for all $j \in [n]$.*

Proof. Note that from Theorem 4, the optimal subset $\Gamma^* = \emptyset$. Therefore, $Q_j^* = D_j^s(\boldsymbol{Q}^*) = D_j - l_j$ for all $j \in [n]$. □

Corollary 1 shows that if all the products share the same underage cost and cannot be completely substituted by the others, then the optimal order quantities are equal to the worst-case demand, i.e., $Q_j^* = D_j - l_j$ for each product $j \in [n]$.

Finally, we remark that if $k = 0$, then the results in Theorem 4 will also hold simply by replacing $l_i = 0$ for each $i \in [n]$.

5. Solution Approaches

Note that the inner maximization Model (11b) is a nonconvex and nonsmooth optimization problem. In this section, we will introduce equivalent MILP formulations for R-MNMS (11) and its inner maximization Model (11b) by linearizing the nonconvex terms in the profit function. These equivalent formulations allow us to develop an effective branch and cut algorithm and an alternative conservative approximation to solve R-MNMS.

5.1. An Equivalent MILP Formulation of the Inner Maximization Problem

In this subsection, we will present an MILP formulation, which is equivalent to the inner maximization problem (11b)[1]. To begin with, in (11b), let us define two new variables

$$u_j = (D_j - l_j - Q_j)_+, \quad \psi_j = (D_j - Q_j)_+$$

for each $j \in [n]$. Clearly, we have $\psi_j \geq u_j$ for each $j \in [n]$. For simplicity, we still use the function $R(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi})$ to denote the optimal value of inner maximization problem (11b) for any given $\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}$,

[1] For the general k^{α}, we can derive the similar MILP formulation, which can be found in Appendix B.

i.e., the inner maximization problem becomes

$$R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) = \max_{\boldsymbol{z}} \sum_{i\in[n]} \overline{S}_i \left[Q_i - D_i + l_i z_i - \sum_{j\in[n]} \underline{\alpha}_{ji} \left(u_j z_j + \psi_j (1-z_j) \right) \right]_+ , \tag{19a}$$

$$\text{s.t.} \sum_{i\in[n]} z_i \le k, \tag{19b}$$

$$z_i \in \{0,1\}, \forall i \in [n]. \tag{19c}$$

Note that Model (19) is a convex integer maximization problem. Thus, we will further linearize the objective function into a linear form. To do so, for each $i \in [n]$, let us define a binary variable $x_i = 1$, if $Q_i - D_i + l_i z_i - \sum_j \underline{\alpha}_{ji} \left(u_j z_j + \psi_j (1-z_j) \right) \ge 0$, and 0, otherwise. Thus, Model (19) is equivalent to

$$R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) = \max_{\boldsymbol{x},\boldsymbol{z}} \sum_{i\in[n]} \overline{S}_i \left[Q_i - D_i + l_i z_i - \sum_{j\in[n]} \underline{\alpha}_{ji} \left(u_j z_j + \psi_j (1-z_j) \right) \right] x_i, \tag{20a}$$

$$\text{s.t.} \sum_{i\in[n]} z_i \le k, \tag{20b}$$

$$x_i, z_i \in \{0,1\}, \forall i \in [n]. \tag{20c}$$

The above Model (20) now becomes a binary bilinear program, which can be further linearized by introducing new variables representing the bilinear terms. The final reformulation result is shown below.

Proposition 3. *The inner maximization problem* (19) *is equivalent to*

$$R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) = \max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \sum_{i\in[n]} \overline{S}_i \left[(Q_i - D_i) x_i + l_i y_{ii} - \sum_{j\in[n]} \underline{\alpha}_{ji} \left(u_j y_{ji} + \psi_j (x_i - y_{ji}) \right) \right] \tag{21a}$$

$$s.t. \sum_{i\in[n]} z_i \le k. \tag{21b}$$

$$y_{ji} \le x_i, \forall i,j \in [n], \tag{21c}$$

$$y_{ji} \le z_j, \forall i,j \in [n], \tag{21d}$$

$$z_i, x_i \in \{0,1\}, y_{ji} \ge 0, \forall i,j \in [n]. \tag{21e}$$

Proof. See Appendix A.5. □

5.2. Reformulation of R-MNMS and branch and cut algorithm

Next we are going to investigate an MILP reformulation for R-MNMS (11), which is amenable for a branch and cut algorithm. First, from Proposition 2, without loss of generality, we can assume that the order quantities $\boldsymbol{Q}$ can be upper bounded by $\boldsymbol{M}$. Thus, for each product $i \in [n]$, its order quantity Q_i must belong to one of the following three intervals: $[0, D_i - l_i], [D_i - l_i, D_i], [D_i, M_i]$ (we break the boundary points arbitrarily). For notational convenience, let us denote $D_i^{(0)} = 0$, $D_i^{(1)} = D_i - l_i$, $D_i^{(2)} = D_i$, and $D_i^{(3)} = M_i$. Next, we introduce one binary variable for each interval to indicate whether Q_i is in this interval or not, i.e., we let $\chi_i^{(e)} = 1$ if $Q_i \in [D_i^{(e-1)}, D_i^{(e)}]$ for each $e \in [3]$; and 0, otherwise. And we let

$$\sum_{e \in [3]} \chi_i^{(e)} = 1, \tag{22a}$$

to enforce that Q_i indeed belongs to only one interval. Correspondingly, for each product $i \in [n]$ and $e \in [3]$, we further introduce another variable $w_i^{(e)}$ to be equal to Q_i if $Q_i \in [D_i^{(e-1)}, D_i^{(e)}]$, and 0, otherwise. That is,

$$D_i^{(e-1)} \chi_i^{(e)} \leq w_i^{(e)} \leq D_i^{(e)} \chi_i^{(e)}, \forall i \in [n], e \in [3], \tag{22b}$$

$$\sum_{e \in [3]} w_i^{(e)} = Q_i, \forall i \in [n]. \tag{22c}$$

Next, we can express u_i and ψ_i (recall that $u_i = (D_i - l_i - Q_i)_+$ and $\psi_i = (D_i - Q_i)_+$) as linear functions of variables $\{\chi_i^{(e)}\}_{e \in [2]}$ and $\{w_i^{(e)}\}_{e \in [2]}$ for each product $i \in [n]$, i.e.,

$$u_i = (D_i - l_i)\chi_i^{(1)} - w_i^{(1)}, \forall i \in [n], \tag{22d}$$

$$\psi_i = D_i \sum_{e \in [2]} \chi_i^{(e)} - \sum_{e \in [2]} w_i^{(e)}, \forall i \in [n], \tag{22e}$$

Clearly, in (22d), if $Q_i > D_i - l_i$, then u_i is equal to 0 since both $\chi_i^{(1)} = 0, w_i^{(1)} = 0$ and otherwise, it is equal to $D_i - l_i - Q_i$. And in (22e), if $Q_i > D_i$, then ψ_i is equal to 0 since $\chi_i^{(1)} = \chi_i^{(2)} = 0, w_i^{(1)} = w_i^{(2)} = 0$, and otherwise, it is equal to $D_i - Q_i$. For the inner maximization problem (21), let us also define function $g(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ to be its objective function, i.e.,

$$g(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \sum_{i \in [n]} \overline{S}_i \left[(Q_i - D_i) x_i + l_i y_{ii} - \sum_{j \in [n]} \underline{\alpha}_{ji} \left(u_j y_{ji} + \psi_j (x_i - y_{ji}) \right) \right],$$

and set Ξ to be its feasible region, i.e.,

$$\Xi = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) : (21\text{b}) - (21\text{e})\}.$$

In view of the above development, we have the following equivalent MILP formulation of R-

MNMS (11):

$$v^* = \max_{\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{\chi},\boldsymbol{w},\eta} \quad \sum_{i\in[n]} \overline{P}_i Q_i - \eta, \tag{23a}$$

$$\text{s.t.} \quad \eta \geq g\left(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\right), \forall(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in \Xi, \tag{23b}$$

$$w_i^{(e)}, u_i, \psi_i \geq 0, \chi_i^{(e)} \in \{0,1\}, \forall i \in [n], e \in [3]. \tag{23c}$$

$$(22\text{a}) - (22\text{e}).$$

Note that in (23b), there can be exponentially many constraints. Therefore, we propose a branch and cut algorithm to solve Model (23). To begin with, suppose we are given a subset $\widehat{\Xi} \subseteq \Xi$, which can be empty, then the master problem is formulated as below:

$$\max_{\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{\chi},\boldsymbol{w},\eta} \left\{ \sum_{i\in[n]} \overline{P}_i Q_i - \eta : \eta \geq g\left(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\right), \forall(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in \widehat{\Xi}, (22\text{a}) - (22\text{e}), (23\text{c}) \right\}. \tag{24}$$

Clearly, Model (24) is a relaxation of Model (23), since $\widehat{\Xi} \subseteq \Xi$. Given an optimal solution $\left(\widehat{\boldsymbol{Q}},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{\psi}},\widehat{\boldsymbol{\chi}},\widehat{\boldsymbol{w}},\widehat{\eta}\right)$ to the master problem (24) to check whether this solution is optimal to original Model (23) or not, it is sufficient to check whether it satisfies constraints (23b), i.e., solve the inner maximization problem (21) by letting $(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) = (\widehat{\boldsymbol{Q}},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{\psi}})$ as below:

$$R(\widehat{\boldsymbol{Q}},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{\psi}}) = \max_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in\Xi} \left\{ g\left(\widehat{\boldsymbol{Q}},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{\psi}},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\right)\right\}, \tag{25}$$

and check if $\widehat{\eta} \geq R(\widehat{\boldsymbol{Q}},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{\psi}})$ or not. If $\widehat{\eta} \geq R(\widehat{\boldsymbol{Q}},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{\psi}})$, then $\left(\widehat{\boldsymbol{Q}},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{\psi}},\widehat{\boldsymbol{\chi}},\widehat{\boldsymbol{w}},\widehat{\eta}\right)$ is optimal to Model (23). Otherwise, let $(\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}},\widehat{\boldsymbol{z}})$ be an optimal solution to Model (25). Then add a new constraint

$$\eta \geq g(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}},\widehat{\boldsymbol{z}})$$

into the master problem (24) and continue. Note that this solution procedure can be integrated with branch and bound, which is known as "branch and cut" (cf. [35, 43, 7]).

Below, we summarize the proposed branch and cut algorithm to solve Model (23), i.e., at each branch and bound node, we proceed the following cut generating procedure until achieving optimality.

Step 0: Initialize set $\widehat{\Xi} = \emptyset$.

Step 1: Solve the proposed master problem (24) with an optimal solution $\left(\widehat{\boldsymbol{Q}},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{\psi}},\widehat{\boldsymbol{\chi}},\widehat{\boldsymbol{w}},\widehat{\eta}\right)$.

Step 2: Solve Model (25), denote its optimal solution by $(\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}},\widehat{\boldsymbol{z}})$ and optimal value $R(\widehat{\boldsymbol{Q}},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{\psi}})$.

Step 3: There are two cases:

Csse 1: If $\widehat{\eta} \geq R(\widehat{\boldsymbol{Q}},\widehat{\boldsymbol{u}},\widehat{\boldsymbol{\psi}})$, set $\boldsymbol{Q}^* \leftarrow \widehat{\boldsymbol{Q}}, \boldsymbol{u}^* \leftarrow \widehat{\boldsymbol{u}}, \boldsymbol{\psi}^* \leftarrow \widehat{\boldsymbol{\psi}}, \boldsymbol{\chi}^* \leftarrow \widehat{\boldsymbol{\chi}}, \boldsymbol{w}^* \leftarrow \widehat{\boldsymbol{w}}$, $\eta^* \leftarrow \widehat{\eta}$, stop and output the optimal solution $(\boldsymbol{Q}^*,\boldsymbol{u}^*,\boldsymbol{\psi}^*,\boldsymbol{\chi}^*,\boldsymbol{w}^*,\eta^*)$.

Csse 2: If $\widehat{\eta} < R(\widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{u}}, \widehat{\boldsymbol{\psi}})$, then augment set $\widehat{\Xi} = \widehat{\Xi} \cup (\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}})$, and go to Step 1.

Note that although this branch and cut algorithm will terminate in a finite number of steps since there are only a finite number of points in set Ξ, as well as finite number of binary variables in the master problem. However, to generate a new constraint at Step 2 might be very time-consuming since it involves solving an MILP (25), i.e., the inner maximization problem (21). In the remaining part of this section, we will replace this MILP (25) by its continuous relaxation and derive a conservative approximation for R-MNMS.

5.3. Conservative Approximation

In practice, branch and cut algorithm might not be efficient to solve very large-scale problem instances. In this section, we propose a simple but very effective conservative approximation to solve R-MNMS (23), i.e., the optimal solution from conservative approximation is a feasible solution to R-MNMS (23). We also provide some sufficient conditions under which this conservative approximation yields an exact optimal solution to R-MNMS (23).

To derive the conservative approximation, we simply relax variables $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ in set Ξ to be continuous in R-MNMS (23), then we can obtain the following lower bound, i.e., a conservative approximation to Model (23):

$$v^{CA} = \max_{\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{\chi},\boldsymbol{w},\eta} \left\{ \sum_{i\in[n]} \overline{P}_i Q_i - \eta : \eta \geq g\left(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\right), \forall (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \Xi_C, \text{(22a)} - \text{(22e)}, \text{(23c)} \right\}, \quad (26)$$

where Ξ_C denotes the continuous relaxation of set X.

Note that the constraints $\eta \geq g(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}), \forall (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \Xi_C$ is equivalent to

$$\eta \geq \max_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in\Xi_C} \left\{ g\left(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\right) \right\},$$

where the right-hand side is a linear program with nonempty and bounded feasible region for any given $(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi})$. Therefore, according to the strong duality of linear program, we can replace the max operator by its dual, i.e., an equivalent min operator, and further change the min operator with the existence one. Let $\varpi, \boldsymbol{\sigma}, \boldsymbol{\rho}, \boldsymbol{\zeta}, \boldsymbol{\xi}$ be the dual variables associated with constraints (21b),(21c), (21d), $\boldsymbol{z} \leq \boldsymbol{e}$ and $\boldsymbol{x} \leq \boldsymbol{e}$, respectively. Then the conservative approximation (26) is equivalent to the following MILP:

$$v^{CA} = \max_{\boldsymbol{Q},\boldsymbol{u},\boldsymbol{v},\eta,\varpi,\boldsymbol{\sigma},\boldsymbol{\rho},\boldsymbol{\zeta},\boldsymbol{\xi}} \sum_{i\in[n]} \overline{P}_i Q_i - \eta, \quad (27a)$$

$$\text{s.t. } \eta \geq k\varpi + \sum_{i\in[n]} (\zeta_i + \xi_i), \quad (27b)$$

$$\varpi + \zeta_j - \sum_{i\in[n]} \rho_{ji} \geq 0, \forall j \in [n], \quad (27c)$$

$$\sigma_{ji} + \rho_{ji} \geq -\overline{S}_i \underline{\alpha}_{ji}(u_j - \psi_j), \forall i, j \in [n], i \neq j, \tag{27d}$$

$$\sigma_{jj} + \rho_{jj} \geq \overline{S}_j l_j, \forall j \in [n], \tag{27e}$$

$$\xi_i - \sum_{j \in [n]} \sigma_{ji} \geq \overline{S}_i (Q_i - D_i) - \overline{S}_i \sum_{j \in [n]} \underline{\alpha}_{ji} \psi_j, \forall i \in [n], \tag{27f}$$

$$\zeta_i, \xi_i, \sigma_{ij}, \rho_{ij} \geq 0, \forall i, j \in [n], \tag{27g}$$

$$(22a) - (22e), (23c).$$

The following result summarizes the above development of the conservative approximation and also shows that under some sufficient conditions, this approximation can be exact, i.e., $v^{CA} = v^*$.

Theorem 5. *Let v^{CA} denote the optimal value of Model* (27)*. Then*

(i) $v^{CA} \leq v^*$; and

(ii) $v^{CA} = v^*$, if one of the following conditions holds: (1) $\underline{\boldsymbol{\alpha}} = \mathbf{0}$, or (2) $n = k$.

Proof. See Appendix A.6. □

From Theorem 5, we see that the conservative approximation (27) provides a feasible solution to R-MNMS (23). In addition, Theorem 5 tells that the conservative approximation can find a very good-quality solution, which can even be optimal to R-MNMS (23). We will illustrate these facts in Section 6.

6. Computational Study

In this section, we test the performances of branch and cut algorithm and conservative approximation to solve R-MNMS (23). Also, we test the reliability of R-MNMS (23) compared with its risk neutral counterpart.

6.1. Effectiveness of Algorithms

We considered instances with $n = 10$ and $n = 20$ products. For each $n \in \{10, 20\}$, we generated 10 random instances, where for each product $i \in [n]$, the nominal demand D_i is between 50 and 100, the unit price p_i ranged from 85 to 95, unit cost c_i varied from 40 to 50, and the salvage value s_i was between 22 and 30. All the products were assumed to be similar, we assume $k^\alpha = n^2 - n$, and thus the substitution rates were generated uniformly between 0 and 1, satisfying $\sum_{i \in [n]} \underline{\alpha}_{ji} = 0.8$ and $\underline{\alpha}_{jj} = 0$ for each $j \in [n]$. The lower bound of the demand was set to be proportional to the nominal demand, i.e., $\boldsymbol{l} = \theta \boldsymbol{D}$, where $\theta \in (0, 1)$ is called "deviation ratio". We tested these instances with the deviation ratio $\theta \in \{0.2, 0.4\}$ and the budget of uncertainty $k \in \{5, 10\}$. Both approaches were coded in Python 2.7 with calls to Gurobi 7.5 on a personal computer with 2.3

GHz Intel Core i5 processor and 8G of memory. The CPU time limit of Gurobi was set to be 3600 seconds.

Table 1 and Table 2 display the computational results of branch and cut algorithm and conservative approximation method with $n = 10$ and $n = 20$, respectively. For the branch and cut algorithm, Opt.val denotes the optimal value if available, LB and UB denote best lower and upper bounds, Gap denotes the optimality gap, computed as (UB-LB)/UB; for conservative approximation, C.val denotes its output objective value, and A-Gap represents its optimality gap, computed as (UB-C.val)/UB (note that UB is equal to the Opt.val if available).

From Table 1, we see that when $n = 10$, both approaches finds good-quality feasible solutions. As explained in Section 5.3, the solutions obtained from conservative approximation method are equal to the solutions calculated from branch and cut algorithm when $k = n = 10$, i.e., A-Gap=0. In Table 2, we note that when the number of products increases to 20, the conservative approximation method can find good-quality feasible solutions within the time limit. Oftentimes, the conservative approximation solution can be even better than that obtained by the branch and cut algorithm. Also from Table 2, we see that if the budget of uncertainty k increases and $k \leq \frac{n}{2}$, i.e., the number of possible realizations of products' demand grows, then the computational time tends to be longer. Also, since a larger k implies that a larger number of products whose worst-case demand can be equal to their lower bounds, therefore we can anticipate that the total profit becomes smaller. Since θ denotes how much the worst-case demand can deviate from the nominal demand, the increase of θ implies that the variance of random demand grows, which means a chance of being understock or overstock becomes larger and leads to a smaller total profit.

6.2. Robustness of Model (23)

In this subsection, we illustrate how to find the optimal budget of uncertainty and also test robustness of Model (23). Suppose that there are 10 products. The values of $\boldsymbol{p}, \boldsymbol{c}$, $\boldsymbol{s}$, $\underline{\boldsymbol{\alpha}}$, and k^α are the same as those in Section 6.1. We also assumed that there are 200 historical data and we split them into two groups, Υ_1, Υ_2, with equal size. These historical demand were generated by sampling from independent uniformly random variables between 20 and 80. We choose the candidate set $\mathcal{K}$ of budget of uncertainty k as $\{0, 1, \cdots, 10\}$. According to Section 2.2 with percentile $q = 10$, we found the optimal budget of uncertainty $k^* = 6$, which is the smallest $k \in \mathcal{K}$ such that $v^*(k, k^\alpha) \leq \widehat{\Pi}^{10\%}(k, k^\alpha)$ as shown in Figure 1.

Table 1: Computational results of branch and cut algorithm and conservative approximation method with $n = 10$

k	θ	Instances	Branch and Cut		Conservative Approximation		
			Time	Opt.val	Time	C.val	A-Gap (%)
5	0.2	1	7.41	29366.4	0.54	29246.3	0.41
		2	1.45	30903.2	0.48	30738.8	0.53
		3	7.39	27824.7	0.58	27742.2	0.30
		4	1.97	27509.3	0.45	27431.1	0.28
		5	8.62	33482.8	0.52	33362.9	0.36
		6	1.10	30826.7	0.36	30730.1	0.31
		7	7.69	30652.2	0.42	30614.1	0.12
		8	1.23	29820.5	0.35	29651.1	0.57
		9	7.81	31935.9	0.46	31876.4	0.19
		10	1.20	33228.9	0.33	33163.8	0.20
Average			4.59	30555.1	0.45	30455.7	0.33
5	0.4	1	7.57	24225.8	0.45	23985.6	0.99
		2	1.24	25434.4	0.44	25105.6	1.29
		3	10.16	22961.9	0.45	22793.5	0.74
		4	1.16	22841.6	0.39	22685.1	0.69
		5	5.85	27626.2	0.35	27386.8	0.87
		6	1.32	25459.5	0.43	25266.2	0.76
		7	7.96	25480.5	0.33	25404.2	0.30
		8	1.15	24566.1	0.31	24227.1	1.38
		9	13.32	26447.3	0.33	26329.8	0.44
		10	2.06	27463.7	0.36	27333.6	0.47
Average			5.18	25250.7	0.38	25051.8	0.79
10	0.2	1	10.29	27605.6	0.45	27605.6	0
		2	1.44	29097.6	0.37	29097.6	0
		3	6.67	26152.8	0.32	26152.8	0
		4	1.39	25741.6	0.34	25741.6	0
		5	9.76	31471.2	0.46	31471.2	0
		6	1.28	28955.2	0.30	28955.2	0
		7	11.58	28659.2	0.34	28659.2	0
		8	1.14	28060.0	0.35	28060.0	0
		9	9.13	29938.4	0.40	29938.4	0
		10	1.13	31195.2	0.33	31195.2	0
Average			5.38	28687.7	0.37	28687.7	0
10	0.4	1	6.62	20704.2	0.33	20704.2	0
		2	1.23	21823.2	0.41	21823.2	0
		3	7.75	19614.6	0.34	19614.6	0
		4	1.31	19306.2	0.39	19306.2	0
		5	7.84	23603.4	0.49	23603.4	0
		6	1.26	21716.4	0.34	21716.4	0
		7	10.12	21494.4	0.46	21494.4	0
		8	1.66	21045.0	0.38	21045.0	0
		9	7.91	22453.8	0.29	22453.8	0
		10	1.49	23396.4	0.35	23396.4	0
Average			4.72	21515.8	0.38	21515.8	0

Table 2: Computational results of branch and cut algorithm and conservative approximation method with $n = 20$

k	θ	Instances	Branch and Cut				Conservative Approximation		
			Time	LB	UB	Gap (%)	Time	C.val	A-Gap (%)
5	0.2	1	3600	62696.6	64708.6	3.21	426	63598.0	1.72
		2	3600	57808.9	59814.1	3.47	376	58676.9	1.90
		3	3600	67405.0	69220.8	2.69	386	68271.3	1.37
		4	3600	63595.0	65665.0	3.26	474	64289.4	2.10
		5	3600	68212.6	70193.6	2.90	483	68970.1	1.74
		6	3600	63162.1	65226.0	3.27	389	64158.1	1.64
		7	3600	57161.3	58977.5	3.18	443	57873.8	1.87
		8	3600	59865.9	61533.3	2.79	399	60482.0	1.71
		9	3600	63039.5	64734.2	2.69	297	64015.2	1.11
		10	3600	63736.1	65823.4	3.28	445	64440.0	2.10
Average			3600	62668.3	64589.6	2.98	412	63477.5	1.73
5	0.4	1	3600	56885.9	59198.5	4.07	209	57605.3	2.69
		2	3600	52545.4	55277.5	5.20	164	53302.7	3.57
		3	3600	61266.0	63481.5	3.62	210	62016.9	2.31
		4	3600	57894.0	60005.2	3.65	171	58491.9	2.52
		5	3600	62189.9	64349.5	3.47	214	62787.7	2.43
		6	3600	57600.8	59577.6	3.43	162	58411.6	1.96
		7	3600	51958.4	53876.0	3.69	188	52501.2	2.55
		8	3600	54290.0	56740.3	4.51	166	54681.1	3.63
		9	3600	57439.3	60157.6	4.73	162	58052.8	3.50
		10	3600	57968.5	60021.7	3.54	193	58443.9	2.63
Average			3600	57003.8	59268.5	4.54	184	57629.5	3.96
10	0.2	1	3600	58550.7	61391.8	4.85	677	58760.5	4.29
		2	3600	53961.4	55228.0	2.35	676	54308.9	1.66
		3	3600	62883.8	66951.2	6.47	674	62988.8	5.92
		4	3600	59129.6	62307.3	5.37	744	59423.0	4.63
		5	3600	63215.9	65995.8	5.37	775	63774.9	3.37
		6	3600	58661.4	61816.1	5.38	588	59310.5	4.05
		7	3600	53268.3	56034.5	4.40	679	53631.6	4.29
		8	3600	55993.3	58219.7	3.98	584	56143.3	3.57
		9	3600	58682.7	61631.3	5.03	519	59298.9	3.79
		10	3600	59474.8	62217.1	4.61	667	59677.8	4.08
Average			3600	58382.2	61179.3	3.84	658	58731.8	2.78
10	0.4	1	3600	48742.5	50596.0	3.80	192	48090.8	4.95
		2	3600	44928.1	46423.0	3.33	132	44626.2	3.87
		3	3600	52462.0	54713.1	4.29	162	51612.7	5.67
		4	3600	49276.9	51014.4	3.53	149	48758.0	4.42
		5	3600	52576.1	55947.0	6.41	137	52511.7	6.14
		6	3600	48995.8	50832.1	3.75	113	48792.0	4.01
		7	3600	44435.2	45413.2	2.20	133	44066.6	2.97
		8	3600	46552.2	47417.1	1.86	141	45977.6	3.04
		9	3600	49148.4	50407.4	2.56	117	48861.3	3.07
		10	3600	49663.0	50784.3	2.26	183	48956.5	3.60
Average			3600	48678.0	50354.8	3.27	146	48225.3	4.20

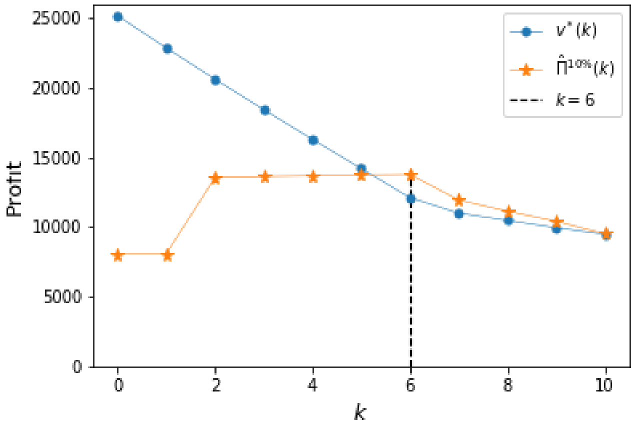

Figure 1: 10th percentile of profits for Model (2) by plugging in the optimal order quantities of robust model (23) for different k.

We also tested the reliability of the solution from robust Model (23) by comparing with the risk neutral one studied in [52], which has the following form:

$$v^{rn} = \max_{\boldsymbol{Q} \in \mathbb{R}^n_+} \left\{ \sum_{i \in [n]} \overline{P}_i Q_i - \mathbb{E}_{\mathbb{P}} \left[\sum_{i \in [n]} \overline{S}_i \left(Q_i - \tilde{D}_i^s(\boldsymbol{Q}) \right)_+ \right] \right\}, \tag{28}$$

where $\mathbb{P}$ denotes a particular probability distribution.

We first used the demand data in Υ_2 to obtain the optimal order quantities of robust Model (23) with $k^* = 6$. Also, we obtained the optimal order quantities from the risk neutral Model (28) by solving the sample average approximation (SAA) with the demand realizations from set Υ_2. To compare the quality of both solutions, we assumed that the underlying true probability distribution of each product's demand is independent Gaussian $\mathcal{N}(\mu, \sigma^2)$ truncated at the interval $[20, 80]$. We selected different parametric pairs (μ, σ^2) of Gaussian random vectors, and for each pair (μ, σ^2), we generated 10^5 i.i.d. samples to evaluate the solutions from robust Model (23) and risk neutral Model (28) and also to compute their statistical confidence intervals. The computational results are presented in Table 3.

From Table 3, we see that if the data are very limited and unable to predict the underlying true probability distribution or if the underlying true probability distribution is not the same as the one we stick to, then the solution from robust Model (23) is more reliable than that from risk neutral Model (28). On the other hand, if the underlying true distribution is close to the one we predict using the historical data (e.g., in the cases of Gaussian$(40, 40^2)$ and Gaussian$(40, 50^2)$), then risk neutral Model (28) can be more accurate. In practice, if there are limited data or the demand is changing rapidly, then we recommend the robust Model (23), and if there are plenty of historical data and products' demand is quite stable, then risk neutral Model (28) is more desirable.

Table 3: Mean and 95% Confidence Interval (CI) objective value of robust Model (23) and risk neutral Model (28) under different Gaussian distributions.

Distribution	Mean and 95% CI of Robust Model	Mean and 95% CI of Risk Neutral Model
Gaussian$(20, 10^2)$	6416.22 ± 6.29	5066.66 ± 6.33
Gaussian$(20, 40^2)$	11559.16 ± 15.64	10951.76 ± 17.74
Gaussian$(20, 50^2)$	12316.00 ± 16.38	11929.89 ± 18.96
Gaussian$(30, 10^2)$	10875.82 ± 9.24	9554.59 ± 9.30
Gaussian$(30, 50^2)$	13643.80 ± 16.16	13387.78 ± 18.70
Gaussian$(30, 80^2)$	14013.59 ± 16.63	13959.36 ± 19.53
Gaussian$(40, 10^2)$	16618.50 ± 9.61	15553.12 ± 10.27
Gaussian$(40, 40^2)$	15780.94 ± 15.98	15964.15 ± 18.90
Gaussian$(40, 50^2)$	15734.04 ± 16.32	16037.57 ± 19.47

7. Conclusion

This paper studies the robust multi-product newsvendor problem with substitution (R-MNMS), where the demand and substitution are under cardinality-constrained uncertainty sets. We first prove that evaluating the worst-case total profit for given order quantities, in general, is NP-hard. Next, we identify three solvable special cases of R-MNMS and derive their closed-form optimal solutions. For a general R-MNMS, we propose a mixed integer linear program formulation which can be solved by a branch and cut algorithm. We also develop a conservative approximation method to solve R-MNMS, and show that under certain conditions, its optimal solution can also be optimal to R-MNMS. Finally, we conduct numerical studies to illustrate the effectiveness and solution quality of the proposed algorithms. Please note that the cardinality uncertainty set is essential for the results in this paper, which might not hold if one changes to other uncertainty sets. One possible future direction is to extend the robust model to other interesting data-driven uncertainty sets, for example, moment-based uncertainty sets [40, 50, 23]. Another direction is to incorporate pricing decision into R-MNMS, i.e., to study joint inventory and pricing optimization in R-MNMS.

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Appendix A. Proofs

A.1 Proof of Proposition 2

Proposition 2. *There exists an optimal solution* $\boldsymbol{Q}^*$ *to R-MNMS such that for each product* $i \in [n]$, $Q_i^* \leq M_i$, *where* $M_i = D_i + \sum_{j\in[n]} \underline{\alpha}_{ji} D_j$.

Proof. We prove the result by contradiction. Suppose for any optimal solution $\boldsymbol{Q}^*$, there exists a product $i \in [n]$ such that $Q_i^* > M_i$. Let set $\mathbb{B} := \{i \in [n] : Q_i^* > M_i\}$. Hence, $\mathbb{B} \neq \emptyset$. Let us define a new solution $\widehat{\boldsymbol{Q}}$ such that $\widehat{Q}_i = \begin{cases} M_i, & \text{if } i \in \mathbb{B} \\ Q_i^*, & \text{otherwise} \end{cases}$ for each product $i \in [n]$. Clearly, $\widehat{Q}_i \leq M_i$ for each $i \in [n]$. Then the objective value $f(\widehat{\boldsymbol{Q}})$ is equal to

$$
\begin{aligned}
f(\widehat{\boldsymbol{Q}}) =& \sum_{i\in\mathbb{B}} \overline{P}_i M_i + \sum_{i\in[n]\setminus\mathbb{B}} \overline{P}_i Q_i^* \\
& - \max_{\boldsymbol{z}\in X} \Bigg\{ \sum_{i\in\mathbb{B}} \overline{S}_i \left(M_i - D_i + l_i z_i - \sum_{j\in\mathbb{B}} \underline{\alpha}_{ji} \left(D_j - l_j z_j - M_j\right)_+ - \sum_{j\in[n]\setminus\mathbb{B}} \underline{\alpha}_{ji} \left(D_j - l_j z_j - Q_j^*\right)_+ \right)_+ \\
& + \sum_{i\in[n]\setminus\mathbb{B}} \overline{S}_i \left(Q_i^* - D_i + l_i z_i - \sum_{j\in\mathbb{B}} \underline{\alpha}_{ji} \left(D_j - l_j z_j - M_j\right)_+ - \sum_{j\in[n]\setminus\mathbb{B}} \underline{\alpha}_{ji} \left(D_j - l_j z_j - Q_j^*\right)_+ \right)_+ \Bigg\} \\
=& \sum_{i\in\mathbb{B}} \overline{P}_i M_i + \sum_{i\in[n]\setminus\mathbb{B}} \overline{P}_i Q_i^* + \sum_{i\in\mathbb{B}} \overline{S}_i \left(Q_i^* - M_i\right) \\
& - \max_{\boldsymbol{z}\in X} \Bigg\{ \sum_{i\in\mathbb{B}} \overline{S}_i \left(Q_i^* - D_i + l_i z_i - \sum_{j\in[n]\setminus\mathbb{B}} \underline{\alpha}_{ji} \left(D_j - l_j z_j - Q_j^*\right)_+ \right) \\
& + \sum_{i\in[n]\setminus\mathbb{B}} \overline{S}_i \left(Q_i^* - D_i + l_i z_i - \sum_{j\in[n]\setminus\mathbb{B}} \underline{\alpha}_{ji} \left(D_j - l_j z_j - Q_j^*\right)_+ \right)_+ \Bigg\} \\
=& \sum_{i\in[n]} \overline{P}_i Q_i^* - \max_{\boldsymbol{z}\in X} \left\{ \sum_{i\in[n]} \overline{S}_i \left(Q_i^* - D_i + l_i z_i - \sum_{j\in[n]} \underline{\alpha}_{ji} \left(D_j - l_j z_j - Q_j^*\right)_+ \right)_+ \right\} \\
& + \sum_{i\in\mathbb{B}} \left(\overline{S}_i - \overline{P}_i\right) \left(Q_i^* - M_i\right) \\
=& v^* + \sum_{i\in\mathbb{B}} \left(\overline{S}_i - \overline{P}_i\right) \left(Q_i^* - M_i\right) \\
\geq& v^*,
\end{aligned}
$$

where the second equality is because of $M_j = D_j + \sum_{i\in[n]} \underline{\alpha}_{ji} D_i$ for all $j \in \mathbb{B}$, the third equality is because for each $j \in \mathbb{B}$, we have $Q_j^* > M_j = D_j + \sum_{i\in[n]} \underline{\alpha}_{ij} D_i$, the forth equality is due to the optimality of $\boldsymbol{Q}^*$, and the last inequality holds because $\overline{S}_i \geq \overline{P}_i$ and $Q_i^* > M_i$ for each $i \in [n]$. This implies $\widehat{\boldsymbol{Q}}$ is also an optimal solution, a contradiction. □

A.2 Proof of Theorem 1

Theorem 1. *The inner maximization problem* (12) *in general is NP-hard.*

Proof. We prove this result from a reduction of clique problem to be a special case of Model (12).

(Clique Problem) Given an undirected graph $G(V,E)$, does it have a size-τ clique?

Let us consider a special instance of the inner maximization problem (12): suppose that there are $n = |V| + |E|$ products, and for each product $i \in E$, we let $\overline{S}_i = 1, Q_i = D_i, l_i = 1$, while for each product $j \in V$, we let $\overline{S}_j = 1, Q_j = D_j - l_j, l_j = 1$. Additionally, the substitution rate matrix $\underline{\boldsymbol{\alpha}}$ is defined as

$$\underline{\alpha}_{ji} = \begin{cases} 1, & \text{if edge } i \in E \text{ contains node } j \in V \\ 0, & \text{otherwise} \end{cases}$$

for all $i, j \in V \cup E$. Let the budget of uncertainty $k = \frac{\tau(\tau+1)}{2}$. Under this setting, the inner maximization problem (12) reduces to

$$R\left(\boldsymbol{Q}\right) = \max_{\boldsymbol{z}} \sum_{i \in E} \left(z_i^{(E)} - \sum_{j \in V} \underline{\alpha}_{ji} \left(1 - z_j^{(V)}\right)\right)_+ + \sum_{j \in V} z_j^{(V)}, \tag{29a}$$

$$\text{s.t.} \sum_{j \in V} z_j^{(V)} + \sum_{i \in E} z_i^{(E)} \leq \frac{\tau(\tau+1)}{2}, \tag{29b}$$

$$z_j^{(V)}, z_i^{(E)} \in \{0,1\}. \tag{29c}$$

It is sufficient to show that the Clique Problem is equivalent to Model (29), i.e., we only need to show the following claim.

Claim 1. *There is a clique with τ nodes in the undirected graph $G(V,E)$ if and only if $R\left(\boldsymbol{Q}\right) = \frac{\tau(\tau+1)}{2}$.*

Proof. Before we prove the result, let us denote $\boldsymbol{z}^*$ as an optimal solution of Model (29), and also define the following two sets: $V^* = \{j \in V : (z_j^{(V)})^* = 1\}$, $E^* = \{i \in E : (z_i^{(E)})^* = 1\}$, i.e., $\widehat{G}(V^*, E^*)$ is a substructure of $G(V,E)$. Note that $\widehat{G}(V^*, E^*)$ might not be a graph since we might not choose enough nodes to cover all the edges, i.e., there might exist an edge in set E^* but not both of its two nodes are selected in set V^*. Thus, $R\left(\boldsymbol{Q}\right)$ is equal to

$$R\left(\boldsymbol{Q}\right) = \sum_{i \in E^*} \left(1 - \sum_{j \in V} \underline{\alpha}_{ji} \left(1 - (z_j^{(V)})^*\right)\right)_+ + \sum_{j \in V^*} (z_j^{(V)})^* = \sum_{i \in E^*} \left(1 - \sum_{j \in V \setminus V^*} \underline{\alpha}_{ji}\right)_+ + |V^*|. \tag{30}$$

From (30), we have the following inequality:

$$R(\boldsymbol{Q}) = \sum_{i\in E^*}\left(1-\sum_{j\in V\setminus V^*}\underline{\alpha}_{ji}\right)_+ + |V^*| \le \sum_{i\in E}\left(1-\sum_{j\in V\setminus V^*}\underline{\alpha}_{ji}\right)_+ + |V^*| \le \binom{|V^*|}{2} + |V^*|, \quad (31)$$

where the first inequality is due to $E^* \subseteq E$, and the second inequality is because of $\left(1-\sum_{j\in V\setminus V^*}\underline{\alpha}_{ji}\right)_+ = 0$ if at least one of the two nodes from edge i is not covered by set V^*.

Now we are ready to prove the main results.

"only if". Suppose that there exists a size-τ clique (V_τ, E_τ) in the graph $G(V,E)$. Let us denote a binary vector $\widehat{\boldsymbol{z}}$ as

$$\widehat{z}_j^{(V)} = \begin{cases} 1, & j \in V_\tau \\ 0, & \text{otherwise} \end{cases}, \forall i \in V, \quad \widehat{z}_i^{(E)} = \begin{cases} 1, & i \in E_\tau \\ 0, & \text{otherwise} \end{cases}, \forall j \in E.$$

Clearly, $\widehat{\boldsymbol{z}}$ is a feasible solution to Model (29), with an objective value equal to $\frac{\tau(\tau+1)}{2}$. Thus, $R(\boldsymbol{Q}) \ge \frac{\tau(\tau+1)}{2}$.

Now suppose that $R(\boldsymbol{Q}) > \frac{\tau(\tau+1)}{2}$. According to the objective function (29a), we must have

$$|E^*| + |V^*| = \sum_{i\in E}(z_i^{(E)})^* + \sum_{j\in V}(z_j^{(V)})^* \ge R(\boldsymbol{Q}) > \frac{\tau(\tau+1)}{2}.$$

Also, the constraint (29b) implies that

$$|E^*| + |V^*| \le \frac{\tau(\tau+1)}{2},$$

a contradiction.

"if". Suppose that $R(\boldsymbol{Q}) = \frac{\tau(\tau+1)}{2}$. According to (30), we must have

$$\frac{\tau(\tau+1)}{2} = R(\boldsymbol{Q}) = \sum_{i\in E^*}\left(1-\sum_{j\in V\setminus V^*}\underline{\alpha}_{ji}\right)_+ + |V^*| \le |E^*| + |V^*|$$

and by (31), we also have

$$\frac{\tau(\tau+1)}{2} = R(\boldsymbol{Q}) \le \binom{|V^*|}{2} + |V^*|,$$

On the other hand, the constraint (29b) implies that $|V^*| + |E^*| \le \frac{\tau(\tau+1)}{2}$. Thus, we must have $|V^*| + |E^*| = \frac{\tau(\tau+1)}{2}$. Suppose that the substructure $\widehat{G}(V^*, E^*)$ is not a clique, then there exists $i_0 = (u_0, v_0) \in E^*$ such that at least one of its nodes is not chosen, i.e., $\left(1 - \sum_{j \in V \setminus V^*} \underline{\alpha}_{ji_0}\right)_+ = 0$. Thus, by (30), we have

$$\frac{\tau(\tau+1)}{2} = R(\boldsymbol{Q}) = \sum_{i \in E^* \setminus \{i_0\}} \left(1 - \sum_{j \in V \setminus V^*} \underline{\alpha}_{ji}\right)_+ + |V^*| \leq |E^*| - 1 + |V^*| < \frac{\tau(\tau+1)}{2},$$

a contradiction.

□

□

A.3 Proof of Theorem 2

Theorem 2. *Suppose $n = 2$, $k = 1$, and Assumption 2 holds, then the optimal order quantities $\boldsymbol{Q}^* = (Q_1^*, Q_2^*)$ are characterized by the following three cases:*

Case 1: If $\overline{P}_1 \leq \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \geq \overline{P}_1 \underline{\alpha}_{21}$, then $(Q_1^*, Q_2^*) = (0, D_2 - l_2 + \underline{\alpha}_{12} D_1)$.

Case 2: If $\overline{P}_2 \leq \overline{P}_1 \underline{\alpha}_{21}$ and $\overline{P}_1 \geq \overline{P}_2 \underline{\alpha}_{12}$, then $(Q_1^*, Q_2^*) = (D_1 - l_1 + \underline{\alpha}_{21} D_2, 0)$.

Case 3: If $\overline{P}_1 \geq \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \geq \overline{P}_1 \underline{\alpha}_{21}$, then we have the following two sub-cases:

Sub-case 3.1: If $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$, then $(Q_1^*, Q_2^*) = \left(D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12} \underline{\alpha}_{21}}, D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12} \underline{\alpha}_{21}}\right)$ or $(Q_1^*, Q_2^*) = \left(D_1, D_2 - \frac{\overline{S}_2 l_2 - \overline{S}_1 l_1}{\overline{S}_2 - \overline{S}_1 \underline{\alpha}_{21}}\right)$.

Sub-case 3.2: If $\overline{S}_1 l_1 \leq \overline{S}_2 l_2$, then $(Q_1^*, Q_2^*) = \left(D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12} \underline{\alpha}_{21}}, D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12} \underline{\alpha}_{21}}\right)$ or $(Q_1^*, Q_2^*) = \left(D_1 - \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}}, D_2\right)$.

Proof. According to Model (13), we have

$$v^* = \max_{Q_1, Q_2 \geq 0} \left\{ f(\boldsymbol{Q}) = \overline{P}_1 Q_1 + \overline{P}_2 Q_2 - R(\boldsymbol{Q}) \right\}, \tag{32}$$

and

$$\begin{aligned}
R(\boldsymbol{Q}) =& \max_{\boldsymbol{z} \in X} \sum_{i \in [2]} \overline{S}_i \left[Q_i - D_i + l_i z_i - \sum_{j \in [2]} \underline{\alpha}_{ji} \left((D_j - l_j - Q_j)_+ z_j + (D_j - Q_j)_+ (1 - z_j) \right) \right]_+ \\
=& \max \Big\{ \overline{S}_1 \left(Q_1 - D_1 + l_1 - \underline{\alpha}_{21} (D_2 - Q_2)_+ \right)_+ + \overline{S}_2 \left(Q_2 - D_2 - \underline{\alpha}_{12} (D_1 - l_1 - Q_1)_+ \right)_+, \\
& \overline{S}_1 \left(Q_1 - D_1 - \underline{\alpha}_{21} (D_2 - l_2 - Q_2)_+ \right)_+ + \overline{S}_2 \left(Q_2 - D_2 + l_2 - \underline{\alpha}_{12} (D_1 - Q_1)_+ \right)_+, \\
& \overline{S}_1 \left(Q_1 - D_1 - \underline{\alpha}_{21} (D_2 - l_2 - Q_2)_+ \right)_+ + \overline{S}_2 \left(Q_2 - D_2 - \underline{\alpha}_{12} (D_1 - Q_1)_+ \right)_+ \Big\} \\
=& \max \Big\{ \overline{S}_1 \left(Q_1 - D_1 + l_1 - \underline{\alpha}_{21} (D_2 - Q_2)_+ \right)_+ + \overline{S}_2 \left(Q_2 - D_2 - \underline{\alpha}_{12} (D_1 - l_1 - Q_1)_+ \right)_+,
\end{aligned}$$

$$\overline{S}_1\left(Q_1-D_1-\underline{\alpha}_{21}\left(D_2-l_2-Q_2\right)_+\right)_+ + \overline{S}_2\left(Q_2-D_2+l_2-\underline{\alpha}_{12}\left(D_1-Q_1\right)_+\right)_+\Big\},$$

where the second equality is due to $X=\left\{\boldsymbol{z}:\sum_{i\in[2]} z_i\leq 1, z_i\in\{0,1\},\forall i\in[2]\right\}=\{(0,1),(1,0),(0,0)\}$ and the third equality is because $\overline{S}_1\left(Q_1-D_1-\underline{\alpha}_{21}\left(D_2-l_2-Q_2\right)_+\right)_+ + \overline{S}_2\left(Q_2-D_2-\underline{\alpha}_{12}\left(D_1-Q_1\right)_+\right)_+$ is dominated by the other two.

Note that for each $i\in[2]$, the optimal order quantity Q_i^* must belong to one of the three intervals $[0,D_i-l_i]$, $[D_i-l_i,D_i]$, or $[D_i,+\infty)$. Thus, we can divide the feasible region into 9 subregions (see Figure 2 for an illustration), where under each subregion, function $R(\boldsymbol{Q})$ becomes piecewise convex, thus Model (32) is solvable. Therefore, we can optimize Model (32) over each subregion, and the solution with the largest objective value corresponds to an optimal solution to the original problem (32). Therefore, we need to discuss the 9 cases, corresponding to 9 subregions.

Before we derive the main results, we observe a characterization of an optimal solution of maximizing a piecewise concave function over a box.

Observation 1. *Given an integer number τ, consider the following piecewise concave optimization program:*

$$\max_{\boldsymbol{x}}\left\{\min_{i\in[\tau]}(\boldsymbol{c}^i)^\top\boldsymbol{x}:\boldsymbol{0}\leq\boldsymbol{x}\leq\boldsymbol{U}\right\}.$$

Then an optimal solution of the above optimization problem can be one of the following points:

(i) the extreme points of the box $[\boldsymbol{0},\boldsymbol{U}]$; and

(ii) the intersection point of any affine system $(\boldsymbol{c}^i)^\top\boldsymbol{x}=(\boldsymbol{c}^j)^\top\boldsymbol{x}$ for all $i,j\in\mathbb{B}\subseteq[\tau]$ with $2\leq|\mathbb{B}|\leq n$ and the boundary of the box $[\boldsymbol{0},\boldsymbol{U}]$.

(iii) the unique solution of the affine system $(\boldsymbol{c}^i)^\top\boldsymbol{x}=(\boldsymbol{c}^j)^\top\boldsymbol{x}$ for all $i,j\in\mathbb{B}$ with $|\mathbb{B}|=n+1$, which is in the box $[\boldsymbol{0},\boldsymbol{U}]$.

Proof. Note that the piecewise concave optimization program can be written as the following linear program:

$$\max_{\boldsymbol{x}}\left\{w:w\leq(\boldsymbol{c}^i)^\top\boldsymbol{x},\forall i\in[\tau],\boldsymbol{0}\leq\boldsymbol{x}\leq\boldsymbol{U}\right\}.$$

The conclusion follows by the fact that one optimal solution of the above linear program must be an extreme point, and condition (i), (ii), and (iii) exactly characterize all the extreme points. □

Now we are ready to discuss the following 9 cases.

Case 1. Suppose $(Q_1,Q_2)\in\Omega_1$, i.e., $0\leq Q_1\leq D_1-l_1$, $0\leq Q_2\leq D_2-l_2$.

In this case, $R(\boldsymbol{Q})=0$ and Model (32) becomes:

$$\max_{\boldsymbol{Q}}\left\{f(\boldsymbol{Q})=\overline{P}_1Q_1+\overline{P}_2Q_2:0\leq Q_1\leq D_1-l_1,0\leq Q_2\leq D_2-l_2\right\}.$$

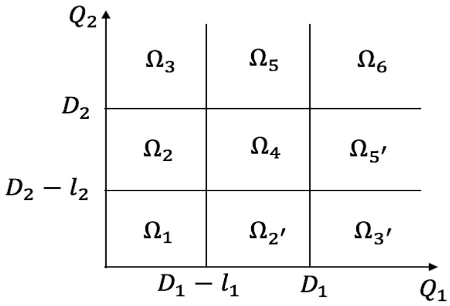

Figure 2: Decomposition of feasible solution regions into 9 sub-regions, where $\Omega_1 = \{(Q_1, Q_2) : 0 \le Q_1 \le D_1 - l_1, 0 \le Q_2 \le D_2 - l_2\}$, $\Omega_2 = \{(Q_1, Q_2) : 0 \le Q_1 \le D_1 - l_1, D_2 - l_2 \le Q_2 \le D_2\}$, $\Omega_3 = \{(Q_1, Q_2) : 0 \le Q_1 \le D_1 - l_1, D_2 \le Q_2\}$, $\Omega_4 = \{(Q_1, Q_2) : D_1 - l_1 \le Q_1 \le D_1, D_2 - l_2 \le Q_2 \le D_2\}$, $\Omega_5 = \{(Q_1, Q_2) : D_1 - l_1 \le Q_1 \le D_1, D_2 \le Q_2\}$, $\Omega_6 = \{(Q_1, Q_2) : D_1 \le Q_1, D_2 \le Q_2\}$, $\Omega_{2'} = \{(Q_1, Q_2) : D_1 - l_1 \le Q_1 \le D_1, 0 \le Q_2 \le D_2 - l_2\}$, $\Omega_{3'} = \{(Q_1, Q_2) : D_1 \le Q_1, 0 \le Q_2 \le D_2 - l_2\}$, $\Omega_{5'} = \{(Q_1, Q_2) : D_1 \le Q_1, D_2 - l_2 \le Q_2 \le D_2\}$.

Clearly, the optimal solution of the above linear program is $\boldsymbol{t}_1^1 = (D_1 - l_1, D_2 - l_2)$, and its optimal total profit is

$$f(\boldsymbol{t}_1^1) = \overline{P}_1(D_1 - l_1) + \overline{P}_2(D_2 - l_2).$$

Case 2. Suppose $(Q_1, Q_2) \in \Omega_2$, i.e., $0 \le Q_1 \le D_1 - l_1, D_2 - l_2 \le Q_2 \le D_2$.

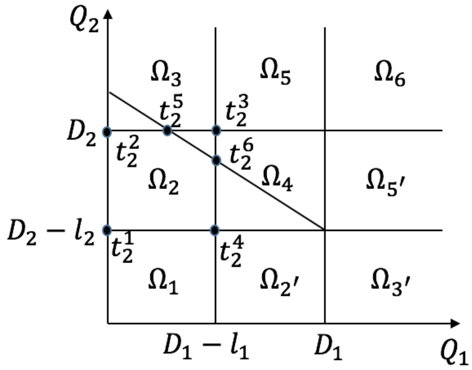

Figure 3: Illustration of possible solutions of Case 2.

In this case, we have $R(\boldsymbol{Q}) = \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))_+$ and Model (32) becomes:

$$\max_{\boldsymbol{Q}} \left\{ f(\boldsymbol{Q}) = \overline{P}_1 Q_1 + \overline{P}_2 Q_2 - \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))_+ : 0 \le Q_1 \le D_1 - l_1, D_2 - l_2 \le Q_2 \le D_2 \right\}.$$

According to Observation 1, the optimal solution of above optimization problem can be one of the following points: (1) extreme points of Ω_2 i.e., $\boldsymbol{t}_2^1, \boldsymbol{t}_2^2, \boldsymbol{t}_2^3, \boldsymbol{t}_2^4$; and (2) the intersection points of linear equation $Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1) = 0$ with feasible regions, i.e., $\boldsymbol{t}_2^5, \boldsymbol{t}_2^6$ (see Figure 3 for an illustration). These potential optimal solutions and their corresponding total profits are listed in Table 4.

Table 4: The possible solutions and their total profits in Case 2

Possible solutions	Total profit
$\boldsymbol{t}_2^1 = (0, D_2 - l_2)$	$f(\boldsymbol{t}_2^1) = \overline{P}_2(D_2 - l_2)$
$\boldsymbol{t}_2^2 = (0, D_2)$	$f(\boldsymbol{t}_2^2) = \overline{P}_2 D_2$
$\boldsymbol{t}_2^3 = (D_1 - l_1, D_2)$	$f(\boldsymbol{t}_2^3) = \overline{P}_1(D_1 - l_1) + \overline{P}_2 D_2 - \overline{S}_2(l_2 - \underline{\alpha}_{12} l_1)$
$\boldsymbol{t}_1^1 = \boldsymbol{t}_2^4 = (D_1 - l_1, D_2 - l_2)$	$f(\boldsymbol{t}_2^4) = f(\boldsymbol{t}_1^1) = \overline{P}_1(D_1 - l_1) + \overline{P}_2(D_2 - l_2)$
$\boldsymbol{t}_2^5 = (D_1 - l_2/\underline{\alpha}_{12}, D_2)$	$f(\boldsymbol{t}_2^5) = \overline{P}_1(D_1 - l_2/\underline{\alpha}_{12}) + \overline{P}_2 D_2$
$\boldsymbol{t}_2^6 = (D_1 - l_1, D_2 - l_2 + \underline{\alpha}_{12} l_1)$	$f(\boldsymbol{t}_2^6) = \overline{P}_1(D_1 - l_1) + \overline{P}_2(D_2 - l_2 + \underline{\alpha}_{12} l_1)$

It remains to compare these solutions. Clearly, we have

- $f(\boldsymbol{t}_2^2) \geq f(\boldsymbol{t}_2^1)$, since $\overline{P}_2, l_2 \geq 0$,
- $f(\boldsymbol{t}_2^5) - f(\boldsymbol{t}_2^2) = \overline{P}_1(D_1 - l_2/\underline{\alpha}_{12}) \geq 0$ due to Assumption 2 that $\underline{\alpha}_{12} D_1 - l_2 \geq 0$,
- $f(\boldsymbol{t}_2^6) - f(\boldsymbol{t}_2^3) = \overline{P}_2(-l_2 + \underline{\alpha}_{12} l_1) + \overline{S}_2(l_2 - \alpha_{12} l_1) = (-\overline{P}_2 + \overline{S}_2)(l_2 - \underline{\alpha}_{12} l_1) \geq 0$ due to $\overline{S}_2 \geq \overline{P}_2$ and Assumption 2 that $l_2 - \underline{\alpha}_{12} l_1 \geq 0$,
- $f(\boldsymbol{t}_2^6) - f(\boldsymbol{t}_2^4) = \overline{P}_2 \underline{\alpha}_{12} l_1 \geq 0$, and
- $f(\boldsymbol{t}_2^6) - f(\boldsymbol{t}_2^5) = -\overline{P}_1 l_1 + \overline{P}_2(-l_2 + \underline{\alpha}_{12} l_1) + \overline{P}_1 \frac{l_2}{\underline{\alpha}_{12}} = \frac{1}{\underline{\alpha}_{12}}(\overline{P}_1 - \overline{P}_2 \underline{\alpha}_{12})(l_2 - \underline{\alpha}_{12} l_1) \begin{cases} \geq 0, & \text{if } \overline{P}_1 \geq \overline{P}_2 \underline{\alpha}_{12} \\ \leq 0, & \text{if } \overline{P}_1 \leq \overline{P}_2 \underline{\alpha}_{12} \end{cases}$, due to Assumption 2 that $l_2 - \underline{\alpha}_{12} l_1 \geq 0$.

From the above comparison, we can draw the following conclusion on the best solution in the subregions Ω_1 and Ω_2:

(i) If $\overline{P}_1 - \overline{P}_2 \underline{\alpha}_{12} \leq 0$, then the point $\boldsymbol{t}_2^5$ dominates the other points in the subregions Ω_1 and Ω_2, since $f(\boldsymbol{t}_2^5) \geq f(\boldsymbol{t}_2^2) \geq f(\boldsymbol{t}_2^1), f(\boldsymbol{t}_2^5) \geq f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_2^4)$, and $f(\boldsymbol{t}_2^5) \geq f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_2^3)$.

(ii) If $\overline{P}_1 - \overline{P}_2 \underline{\alpha}_{12} \geq 0$, then the point $\boldsymbol{t}_2^6$ dominates the other points in the subregions Ω_1 and Ω_2, since $f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_2^5) \geq f(\boldsymbol{t}_2^2) \geq f(\boldsymbol{t}_2^1), f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_2^4)$, and $f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_2^3)$.

Case 3. Suppose $(Q_1, Q_2) \in \Omega_3$**, i.e.,** $0 \leq Q_1 \leq D_1 - l_1, D_2 \leq Q_2$**.**

In this case, $R(\boldsymbol{Q}) = \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))_+$ and Model (32) becomes:

$$\max_{\boldsymbol{Q}} \left\{ f(\boldsymbol{Q}) = \overline{P}_1 Q_1 + \overline{P}_2 Q_2 - \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))_+ : 0 \leq Q_1 \leq D_1 - l_1, D_2 \leq Q_2 \right\}.$$

According to Observation 1, the optimal solution can be one of the following points: (1) the extreme points in Ω_3, i.e., $\boldsymbol{t}_3^1, \boldsymbol{t}_3^2$; and (2) the intersection points of linear $Q_2-D_2+l_2-\underline{\alpha}_{12}(D_1-Q_1)=0$ and the boundary of Ω_3, i.e., $\boldsymbol{t}_3^3, \boldsymbol{t}_3^4$ (see Figure 4 for an illustration). These solutions and their corresponding total profits are listed in Table 5.

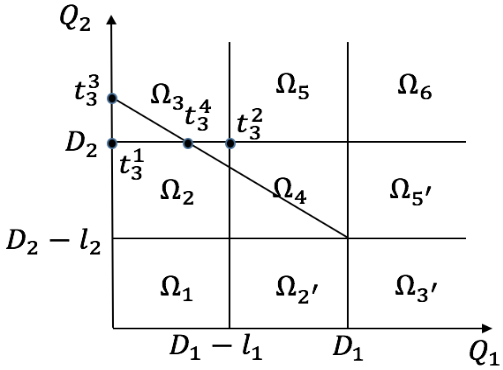

Figure 4: Possible solutions in Case 3

Table 5: Possible solutions and their total profits in Case 3

Possible solutions	Total profit
$\boldsymbol{t}_3^1=\boldsymbol{t}_2^2=(0,D_2)$	$f(\boldsymbol{t}_3^1)=f(\boldsymbol{t}_2^2)=\overline{P}_2D_2$
$\boldsymbol{t}_3^2=\boldsymbol{t}_2^3=(D_1-l_1,D_2)$	$f(\boldsymbol{t}_3^2)=f(\boldsymbol{t}_2^3)=\overline{P}_1(D_1-l_1)+\overline{P}_2D_2-\overline{S}_2(l_2-\underline{\alpha}_{12}l_1)$
$\boldsymbol{t}_3^3=(0,D_2-l_2+\underline{\alpha}_{12}D_1)$	$f(\boldsymbol{t}_3^3)=\overline{P}_2(D_2-l_2+\underline{\alpha}_{12}D_1)$
$\boldsymbol{t}_3^4=\boldsymbol{t}_2^5=(D_1-l_2/\underline{\alpha}_{12},D_2)$	$f(\boldsymbol{t}_3^4)=f(\boldsymbol{t}_2^5)=\overline{P}_1(D_1-l_2/\underline{\alpha}_{12})+\overline{P}_2D_2$

In view of the results in Case 1 and Case 2, it remains to compare $f(\boldsymbol{t}_2^5), f(\boldsymbol{t}_3^3)$ and also $f(\boldsymbol{t}_2^6), f(\boldsymbol{t}_3^3)$. Clearly, we have

- $f(\boldsymbol{t}_2^5)-f(\boldsymbol{t}_3^3)=\overline{P}_1(D_1-l_2/\underline{\alpha}_{12})-\overline{P}_2(-l_2+\underline{\alpha}_{12}D_1)=\frac{1}{\underline{\alpha}_{12}}(\overline{P}_1-\overline{P}_2\underline{\alpha}_{12})(\underline{\alpha}_{12}D_1-l_2)\begin{cases}\geq 0, & \text{if } \overline{P}_1\geq\overline{P}_2\underline{\alpha}_{12}\\ \leq 0, & \text{if } \overline{P}_1\leq\overline{P}_2\underline{\alpha}_{12}\end{cases},$ due to Assumption 2 that $\underline{\alpha}_{12}D_1\geq l_2$,
- $f(\boldsymbol{t}_2^6)-f(\boldsymbol{t}_3^3)=\overline{P}_1(D_1-l_1)+\overline{P}_2\underline{\alpha}_{12}(l_1-D_1)=(\overline{P}_1-\overline{P}_2\underline{\alpha}_{12})(D_1-l_1)\begin{cases}\geq 0, & \text{if } \overline{P}_1\geq\overline{P}_2\underline{\alpha}_{12}\\ \leq 0, & \text{if } \overline{P}_1\leq\overline{P}_2\underline{\alpha}_{12}\end{cases}$ since $D_l\geq l_1$,

From the above comparison as well as the results of Case 1 and Case 2, we can draw the following conclusion on the best solution in subregions Ω_1, Ω_2 and Ω_3:

(i) If $\overline{P}_1 \geq \overline{P}_2\underline{\alpha}_{12}$, then $\boldsymbol{t}_2^6$ dominates all the other points in Ω_1, Ω_2, and Ω_3 since $f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_2^5) \geq f(\boldsymbol{t}_3^3)$.

(ii) If $\overline{P}_1 \leq \overline{P}_2\underline{\alpha}_{12}$, then $\boldsymbol{t}_3^3$ dominates all the other points in Ω_1, Ω_2, and Ω_3 since $f(\boldsymbol{t}_2^6) \leq f(\boldsymbol{t}_2^5) \leq f(\boldsymbol{t}_3^3)$.

Case 4. Suppose $(Q_1, Q_2) \in \Omega_4$**, i.e.,** $D_1 - l_1 \leq Q_1 \leq D_1, D_2 - l_2 \leq Q_2 \leq D_2$**.**

In this case, $R(\boldsymbol{Q}) = \max\{\overline{S}_1(Q_1 - D_1 + l_1 - \underline{\alpha}_{21}(D_2 - Q_2))_+, \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))_+\}$ and Model (32) becomes:

$$\max_{\boldsymbol{Q}} \left\{ f(\boldsymbol{Q}) = \overline{P}_1 Q_1 + \overline{P}_2 Q_2 - R(\boldsymbol{Q}) : D_1 - l_1 \leq Q_1 \leq D_1, D_2 - l_2 \leq Q_2 \leq D_2 \right\}.$$

According to Observation 1, the optimal solution can be one of the following points: (1) the extreme points of Ω_4, i.e., $\boldsymbol{t}_4^1, \boldsymbol{t}_4^2, \boldsymbol{t}_4^3, \boldsymbol{t}_4^4$; (2) the intersection points of the line $Q_1 - D_1 + l_1 - \underline{\alpha}_{11}(D_2 - Q_2) = 0$ with the boundary of Ω_4, i.e., $\boldsymbol{t}_4^2, \boldsymbol{t}_4^5$; (3) the intersection points of the line $Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1) = 0$ with the boundary of Ω_4, i.e., $\boldsymbol{t}_4^4, \boldsymbol{t}_4^6$; and (4) the intersection point of two lines $\overline{S}_1(Q_1 - D_1 + l_1 - \underline{\alpha}_{21}(D_2 - Q_2))_+ = 0$ and $\overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))_+ = 0$, i.e., $\boldsymbol{t}_4^7$ (see Figure 5 for an illustration). These solutions and their total profits are listed in Table 6.

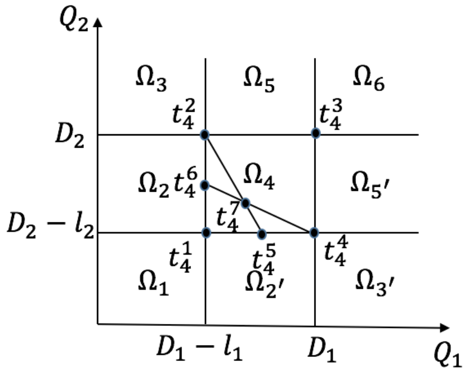

Figure 5: Possible solutions in Case 4

Table 6: Possible solutions and their total profits in Case 4

Possible solutions	Total profit
$\boldsymbol{t}_4^1 = \boldsymbol{t}_2^4 = \boldsymbol{t}_1^1 = (D_1 - l_1, D_2 - l_2)$	$f(\boldsymbol{t}_4^1) = f(\boldsymbol{t}_2^4) = f(\boldsymbol{t}_1^1) = \overline{P}_1(D_1 - l_1) + \overline{P}_2(D_2 - l_2)$
$\boldsymbol{t}_4^2 = \boldsymbol{t}_3^2 = \boldsymbol{t}_2^3 = (D_1 - l_1, D_2)$	$f(\boldsymbol{t}_4^2) = f(\boldsymbol{t}_3^2) = f(\boldsymbol{t}_2^3) = \overline{P}_1(D_1 - l_1) + \overline{P}_2 D_2 - \overline{S}_2(l_2 - \underline{\alpha}_{12} l_1)$
$\boldsymbol{t}_4^3 = (D_1, D_2)$	$f(\boldsymbol{t}_4^3) = \overline{P}_1 D_1 + \overline{P}_2 D_2 - \max\{\overline{S}_1 l_1, \overline{S}_2 l_2\}$
$\boldsymbol{t}_4^4 = (D_1, D_2 - l_2)$	$f(\boldsymbol{t}_4^4) = \overline{P}_1 D_1 + \overline{P}_2(D_2 - l_2) - \overline{S}_1(l_1 - \underline{\alpha}_{21} l_2)$
$\boldsymbol{t}_4^5 = (D_1 - l_1 + \underline{\alpha}_{21} l_2, D_2 - l_2)$	$f(\boldsymbol{t}_4^5) = \overline{P}_1(D_1 - l_1 + \underline{\alpha}_{21} l_2) + \overline{P}_2(D_2 - l_2)$
$\boldsymbol{t}_4^6 = \boldsymbol{t}_2^6 = (D_1 - l_1, D_2 - l_2 + \underline{\alpha}_{12} l_1)$	$f(\boldsymbol{t}_4^6) = f(\boldsymbol{t}_2^6) = \overline{P}_1(D_1 - l_1) + \overline{P}_2 (D_2 - l_2 + \underline{\alpha}_{12} l_1)$
$\boldsymbol{t}_4^7 = \left(D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}, D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}\right)$	$f(\boldsymbol{t}_4^7) = \overline{P}_1 \left(D_1 - \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}\right) + \overline{P}_2 \left(D_2 - \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}}\right)$

In view of the results in Case 1- Case 3, we know that point $\boldsymbol{t}_2^6$ or $\boldsymbol{t}_3^3$ dominates all the other points of Ω_1, Ω_2, and Ω_3, so we only need to compare $f(\boldsymbol{t}_2^6)$, $f(\boldsymbol{t}_3^3)$, $f(\boldsymbol{t}_4^3)$, $f(\boldsymbol{t}_4^4)$, $f(\boldsymbol{t}_4^5)$, $f(\boldsymbol{t}_4^7)$.

- $f(\boldsymbol{t}_4^4) - f(\boldsymbol{t}_4^5) = (\overline{P}_1 - \overline{S}_1)(l_1 - \underline{\alpha}_{21} l_2) \le 0$ due to Assumption 2 that $l_1 - \underline{\alpha}_{21} l_2 \ge 0$ and the fact that $\overline{P}_1 \le \overline{S}_1$,
- If $\overline{P}_1 \le \overline{P}_2 \underline{\alpha}_{12}$, then

$$\begin{aligned} f(\boldsymbol{t}_4^3) - f(\boldsymbol{t}_3^3) &= \overline{P}_1 D_1 - \max\{\overline{S}_1 l_1, \overline{S}_2 l_2\} + \overline{P}_2 l_2 - \overline{P}_2 \underline{\alpha}_{12} D_1 \\ &= (\overline{P}_1 - \overline{P}_2 \underline{\alpha}_{12}) D_1 + (\overline{P}_2 l_2 - \max\{\overline{S}_1 l_1, \overline{S}_2 l_2\}) \le 0, \end{aligned}$$

  where the inequality is because of $D_1 \ge 0$ and $\overline{P}_2 l_2 \le \overline{S}_2 l_2 \le \max\{\overline{S}_1 l_1, \overline{S}_2 l_2\}$. Otherwise, $f(\boldsymbol{t}_4^3)$ and $f(\boldsymbol{t}_3^3)$ are incomparable.

- Compare $f(\boldsymbol{t}_2^6)$ with $f(\boldsymbol{t}_4^7)$

$$\begin{aligned} f(\boldsymbol{t}_2^6) - f(\boldsymbol{t}_4^7) &= \overline{P}_2(-l_2 + \underline{\alpha}_{12} l_1) + \overline{P}_1(-l_1) + \overline{P}_1 \frac{l_1 - \underline{\alpha}_{21} l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} + \overline{P}_2 \frac{l_2 - \underline{\alpha}_{12} l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} \\ &= -\underline{\alpha}_{21} \frac{(\overline{P}_1 - \overline{P}_2 \underline{\alpha}_{12})(l_2 - l_1 \underline{\alpha}_{12})}{1 - \underline{\alpha}_{21}\underline{\alpha}_{21}} \begin{cases} \le 0, & \text{if } \overline{P}_1 \ge \overline{P}_2 \underline{\alpha}_{12} \\ \ge 0, & \text{if } \overline{P}_1 \le \overline{P}_2 \underline{\alpha}_{12} \end{cases}, \end{aligned}$$

  where the inequalities are due to Assumption 2 that $l_2 \ge \underline{\alpha}_{12} l_1$ and the fact that $0 \le \underline{\alpha}_{21} \le 1, 0 \le \underline{\alpha}_{12} \le 1$.

- If $\overline{P}_1 \le \overline{P}_2 \underline{\alpha}_{12}$, then

$$f(\boldsymbol{t}_4^5) - f(\boldsymbol{t}_3^3) = \overline{P}_1(D_1 - l_1 + \underline{\alpha}_{21} l_2) - \overline{P}_2 \underline{\alpha}_{12} D_1 = (\overline{P}_1 - \overline{P}_2 \underline{\alpha}_{12}) D_1 - \overline{P}_1(l_1 - \underline{\alpha}_{21} l_2) \le 0,$$

  where the inequality is due to Assumption 2 that $l_1 \ge \underline{\alpha}_{21} l_2$ and the fact that $\overline{P}_1 \ge 0, D_1 \ge 0$.

- Compare $f(\boldsymbol{t}_2^6)$ with $f(\boldsymbol{t}_3^3)$

$$f(\boldsymbol{t}_2^6) - f(\boldsymbol{t}_3^3) = \overline{P}_1(D_1 - l_1) + \overline{P}_2\underline{\alpha}_{12}l_1 - \overline{P}_2\underline{\alpha}_{12}D_1 = (\overline{P}_1 - \overline{P}_2\underline{\alpha}_{12})(D_1 - l_1)$$
$$\begin{cases} \geq 0, & \text{if } \overline{P}_1 \geq \overline{P}_2\underline{\alpha}_{12} \\ \leq 0, & \text{if } \overline{P}_1 \leq \overline{P}_2\underline{\alpha}_{12} \end{cases},$$

  where the inequalities are due to the fact that $D_1 \geq l_1$.

- Compare $f(\boldsymbol{t}_4^5)$ with $f(\boldsymbol{t}_4^7)$

$$\begin{aligned} f(\boldsymbol{t}_4^5) - f(\boldsymbol{t}_4^7) &= \overline{P}_1(-l_1 + \underline{\alpha}_{21}l_2) + \overline{P}_2(-l_2) + \overline{P}_1\frac{l_1 - \underline{\alpha}_{21}l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} + \overline{P}_2\frac{l_2 - \underline{\alpha}_{12}l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} \\ &= -\underline{\alpha}_{12}\frac{(\overline{P}_2 - \overline{P}_1\underline{\alpha}_{21})(l_1 - l_2\underline{\alpha}_{21})}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} \begin{cases} \leq 0, & \text{if } \overline{P}_2 \geq \overline{P}_1\underline{\alpha}_{21} \\ \geq 0, & \text{if } \overline{P}_2 \leq \overline{P}_1\underline{\alpha}_{21} \end{cases}, \end{aligned}$$

  where the inequalities are due to Assumption 2 that $l_1 \geq \underline{\alpha}_{21}l_2$ and the fact that $0 \leq \underline{\alpha}_{21} \leq 1, 0 \leq \underline{\alpha}_{12} \leq 1$.

- Compare $f(\boldsymbol{t}_2^5)$ with $f(\boldsymbol{t}_4^7)$

$$f(\boldsymbol{t}_2^5) - f(\boldsymbol{t}_4^7) = -\overline{P}_1\frac{l_2}{\underline{\alpha}_{12}} + \overline{P}_1\frac{l_1 - \underline{\alpha}_{21}l_2}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} + \overline{P}_2\frac{l_2 - \underline{\alpha}_{12}l_1}{1 - \underline{\alpha}_{12}\underline{\alpha}_{21}} = -\frac{(\overline{P}_1 - \overline{P}_2\underline{\alpha}_{12})(l_2 - l_1\underline{\alpha}_{12})}{\underline{\alpha}_{12}(1 - \underline{\alpha}_{21}\underline{\alpha}_{21})}$$
$$\begin{cases} \leq 0, & \text{if } \overline{P}_1 \geq \overline{P}_2\underline{\alpha}_{12} \\ \geq 0, & \text{if } \overline{P}_1 \leq \overline{P}_2\underline{\alpha}_{12} \end{cases},$$

  where the inequalities are due to Assumption 2 that $l_2 \geq \underline{\alpha}_{12}l_1$ and the fact that $0 \leq \underline{\alpha}_{21} \leq 1, 0 \leq \underline{\alpha}_{12} \leq 1$.

From the above comparison as well as the results of Case 1-Case 3, we can draw the following conclusion on the best solution in subregions Ω_1, Ω_2, Ω_3, and Ω_4:

(i) If $\overline{P}_1 \leq \overline{P}_2\underline{\alpha}_{12}$, then we must have $\overline{P}_2 \geq \overline{P}_1\underline{\alpha}_{21}$ since $\underline{\alpha}_{12}, \underline{\alpha}_{21} \in [0, 1]$, and $\boldsymbol{t}_3^3$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, and Ω_4, since $f(\boldsymbol{t}_3^3) \geq f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_4^7)$, $f(\boldsymbol{t}_4^5) \leq f(\boldsymbol{t}_3^3)$, and $f(\boldsymbol{t}_4^3) \leq f(\boldsymbol{t}_3^3)$.

(ii) If $\overline{P}_1 \geq \overline{P}_2\underline{\alpha}_{12}$ and $\overline{P}_2 \leq \overline{P}_1\underline{\alpha}_{21}$, then $\boldsymbol{t}_4^3$ or $\boldsymbol{t}_4^5$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, and Ω_4, since $f(\boldsymbol{t}_4^5) \geq f(\boldsymbol{t}_4^7) \geq f(\boldsymbol{t}_2^5)$, and $f(\boldsymbol{t}_4^7) \geq f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_3^3)$.

(iii) If $\overline{P}_1 \geq \overline{P}_2\underline{\alpha}_{12}$ and $\overline{P}_2 \geq \overline{P}_1\underline{\alpha}_{21}$, then $\boldsymbol{t}_4^3$ or $\boldsymbol{t}_4^7$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, and Ω_4, since $f(\boldsymbol{t}_4^7) \geq f(\boldsymbol{t}_3^3)$, $f(\boldsymbol{t}_4^7) \geq f(\boldsymbol{t}_2^6) \geq f(\boldsymbol{t}_3^3)$, and $f(\boldsymbol{t}_4^7) \geq f(\boldsymbol{t}_4^5)$.

Case 5. Suppose $(Q_1, Q_2) \in \Omega_5$**, i.e.,** $D_1 - l_1 \leq Q_1 \leq D_1, D_2 \leq Q_2$**.**

In this case, $R(\boldsymbol{Q}) = \max\{\overline{S}_1(Q_1 - D_1 + l_1) + \overline{S}_2(Q_2 - D_2), \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}\,(D_1 - Q_1))\}$ and Model (32) becomes:

$$\max_{\boldsymbol{Q}} \left\{ f(\boldsymbol{Q}) = \overline{P}_1 Q_1 + \overline{P}_2 Q_2 - R(\boldsymbol{Q}) : D_1 - l_1 \leq Q_1 \leq D_1, D_2 \leq Q_2 \right\}.$$

According to Observation 1, the optimal solution can be one of the following points: (1) the extreme points of Ω_5, i.e, $\boldsymbol{t}_5^1, \boldsymbol{t}_5^2$; (2) the intersection point of line $\overline{S}_1(Q_1 - D_1 + l_1) + \overline{S}_2(Q_2 - D_2) = \overline{S}_2(Q_2 - D_2 + l_2 - \underline{\alpha}_{12}(D_1 - Q_1))$ and the boundary of Ω_5, i.e., $\boldsymbol{t}_5^3$ (See Figure 6 for an illustration). Note that $\boldsymbol{t}_5^3 \in \Omega_5$ if $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$, otherwise, $\boldsymbol{t}_5^3 \notin \Omega_5$. These possible solutions are listed in Table 7.

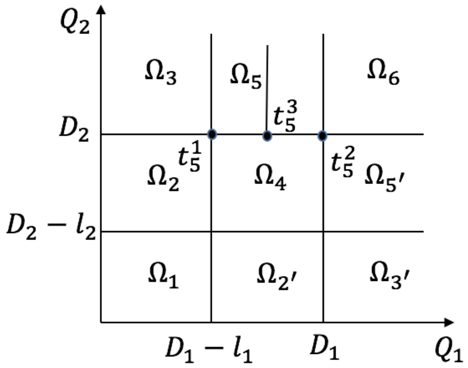

Figure 6: Possible solutions in Case 5

Table 7: Possible solutions and their total profits in Case 5

Possible solutions	Total profit
$\boldsymbol{t}_5^1 = \boldsymbol{t}_4^2 = \boldsymbol{t}_3^2 = \boldsymbol{t}_2^3 = (D_1 - l_1, D_2)$	$f(\boldsymbol{t}_5^1) = f(\boldsymbol{t}_4^2) = f(\boldsymbol{t}_3^2) = f(\boldsymbol{t}_2^3) = \overline{P}_1(D_1 - l_1) + \overline{P}_2 D_2 - \overline{S}_2(l_2 - \underline{\alpha}_{12} l_1)$
$\boldsymbol{t}_5^2 = \boldsymbol{t}_4^3 = (D_1, D_2)$	$f(\boldsymbol{t}_5^2) = f(\boldsymbol{t}_4^3) = \overline{P}_1 D_1 + \overline{P}_2 D_2 - \max\left\{\overline{S}_1 l_1, \overline{S}_2 l_2\right\}$
$\boldsymbol{t}_5^3 = (D_1 - \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}}, D_2)$	$f(\boldsymbol{t}_5^3) = \overline{P}_1 D_1 + \overline{P}_2 D_2 - \overline{P}_1 \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} - \overline{S}_1 \overline{S}_2 \frac{l_2 - l_1 \underline{\alpha}_{12}}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}}$

In view of the results in Case 1- Case 4, the only new point is $\boldsymbol{t}_5^3$, which is in subregion 5 if $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$. Thus, suppose that $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$, we will compare $f(\boldsymbol{t}_5^3)$ with $f(\boldsymbol{t}_3^3)$, $f(\boldsymbol{t}_4^3)$.

- Compare $f(\boldsymbol{t}_4^3)$ with $f(\boldsymbol{t}_5^3)$:

$$\begin{aligned} f(\boldsymbol{t}_4^3) - f(\boldsymbol{t}_5^3) &= \overline{P}_1 D_1 + \overline{P}_2 D_2 - \overline{S}_1 l_1 - \overline{P}_1 D_1 - \overline{P}_2 D_2 + \overline{P}_1 \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} + \overline{S}_1 \overline{S}_2 \frac{l_2 - l_1 \underline{\alpha}_{12}}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} \\ &= -\frac{(\overline{S}_1 - \overline{P}_1)(\overline{S}_1 l_1 - \overline{S}_2 l_2)}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} \leq 0, \end{aligned}$$

where the inequality is due to $\overline{S}_1 \geq \overline{P}_1$ and $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$.

- Compare $f(\boldsymbol{t}_3^3)$ with $f(\boldsymbol{t}_5^3)$

$$\begin{aligned}
f(\boldsymbol{t}_3^3) - f(\boldsymbol{t}_5^3) &= \overline{P}_2\left(D_2 - l_2 + \underline{\alpha}_{12} D_1\right) - \overline{P}_1 D_1 - \overline{P}_2 D_2 + \overline{P}_1 \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} + \overline{S}_1 \overline{S}_2 \frac{l_2 - l_1 \underline{\alpha}_{12}}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} \\
&\geq \overline{P}_2\left(-l_2 + \underline{\alpha}_{12} D_1\right) - \overline{P}_1 D_1 + \overline{P}_1 \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} + \overline{S}_1 \overline{P}_2 \frac{l_2 - l_1 \underline{\alpha}_{12}}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} \\
&= \left(\overline{P}_2 \underline{\alpha}_{12} - \overline{P}_1\right)\left(D_1 - \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}}\right).
\end{aligned}$$

Since $D_1 - \frac{\overline{S}_1 l_1 - \overline{S}_2 l_2}{\overline{S}_1 - \overline{S}_2 \underline{\alpha}_{12}} \geq D_1 - l_1 \geq 0$ and $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$, thus, $f(\boldsymbol{t}_3^3) \geq f(\boldsymbol{t}_5^3)$, if $\overline{P}_1 \leq \overline{P}_2 \underline{\alpha}_{12}$.

From the above comparison results as well as the results of Case 1-Case 4, we can draw the following conclusion on the best solution in subregions Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5:

(i) If $\overline{P}_1 \leq \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \geq \overline{P}_1 \underline{\alpha}_{21}$, then $\boldsymbol{t}_3^3$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5, since $f(\boldsymbol{t}_3^3) \geq f(\boldsymbol{t}_5^3)$.

(ii) If $\overline{P}_1 \geq \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \leq \overline{P}_1 \underline{\alpha}_{21}$, then

  (a) if $\overline{S}_1 l_1 \leq \overline{S}_2 l_2$, then $\boldsymbol{t}_4^3$ or $\boldsymbol{t}_4^5$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5, and

  (b) if $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$, then $\boldsymbol{t}_5^3$ or $\boldsymbol{t}_4^5$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5.

(iii) If $\overline{P}_1 \geq \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \geq \overline{P}_1 \underline{\alpha}_{21}$, then

  (a) if $\overline{S}_1 l_1 \leq \overline{S}_2 l_2$, then $\boldsymbol{t}_4^3$ or $\boldsymbol{t}_4^7$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5, and

  (b) if $\overline{S}_1 l_1 \geq \overline{S}_2 l_2$, then $\boldsymbol{t}_5^3$ or $\boldsymbol{t}_4^7$ dominates all the other points in subregions Ω_1, Ω_2, Ω_3, Ω_4, and Ω_5.

Case 6. Suppose $(Q_1, Q_2) \in \Omega_6$, i.e., $D_1 \leq Q_1, D_2 \leq Q_2$.

In this case, $R(\boldsymbol{Q}) = \max\{\overline{S}_1(Q_1 - D_1 + l_1) + \overline{S}_2(Q_2 - D_2), \overline{S}_1(Q_1 - D_1) + \overline{S}_2(Q_2 - D_2 + l_2)\}$ and Model (32) becomes:

$$\max_{\boldsymbol{Q}} \left\{ f(\boldsymbol{Q}) = \overline{P}_1 Q_1 + \overline{P}_2 Q_2 - R(\boldsymbol{Q}) : D_1 \leq Q_1, D_2 \leq Q_2 \right\}.$$

According to Observation 1, the optimal solution can only be $\boldsymbol{t}_6^1$, which is listed in Table 8.

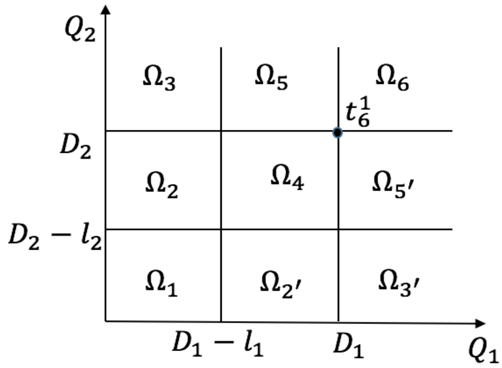

Figure 7: Possible solutions in Case 6

Table 8: Possible solutions and their total profits in Case 6

Possible solutions	Total profit
$t_6^1 = t_5^2 = t_4^3 = (D_1, D_2)$	$f(\boldsymbol{t}_6^1) = f(\boldsymbol{t}_5^2) = f(\boldsymbol{t}_4^3) = \overline{P}_1 D_1 + \overline{P}_2 D_2 - \max\left\{\overline{S}_1 l_1, \overline{S}_2 l_2\right\}$

Note that there is no new optimal solution generated in the case; thus, the conclusion in Case 5 still follows.

Next, for the Cases $2'$, $3'$, $5'$, since they are symmetric to Cases 2, 3, 5, thus we will directly write down the possible solutions.

Case $2'$. Suppose $(Q_1, Q_2) \in \Omega_{2'}$, i.e., $D_1 \le Q_1, D_2 \le Q_2$.

Case $2'$ is symmetric to Case 2, and its possible solutions are listed in Table 9.

Table 9: Possible solutions and their total profits in Case 2'

Possible solutions	Total profit
$t_{2'}^1 = (D_1 - l_1, 0)$	$f(\boldsymbol{t}_{2'}^1) = \overline{P}_1(D_1 - l_1)$
$t_{2'}^2 = (D_1, 0)$	$f(\boldsymbol{t}_{2'}^2) = \overline{P}_1 D_1$
$t_{2'}^3 = (D_1, D_2 - l_2)$	$f(\boldsymbol{t}_{2'}^3) = \overline{P}_2(D_2 - l_2) + \overline{P}_1 D_1 - \overline{S}_1(l_1 - \underline{\alpha}_{21} l_2)$
$t_{2'}^4 = t_4^1 = t_2^4 = t_1^1 = (D_1 - l_1, D_2 - l_2)$	$f(\boldsymbol{t}_{2'}^4) = f(\boldsymbol{t}_4^1) = f(\boldsymbol{t}_2^4) = f(\boldsymbol{t}_1^1) = \overline{P}_2(D_2 - l_2) + \overline{P}_1(D_1 - l_1)$
$t_{2'}^5 = (D_1, D_2 - l_1/\underline{\alpha}_{21})$	$f(\boldsymbol{t}_{2'}^5) = \overline{P}_2(D_2 - l_1/\underline{\alpha}_{21}) + \overline{P}_1 D_1$
$t_{2'}^6 = (D_1 - l_1 + \underline{\alpha}_{21} l_2, D_2 - l_2)$	$f(\boldsymbol{t}_{2'}^6) = \overline{P}_2(D_2 - l_2) + \overline{P}_1(D_1 - l_1 + \underline{\alpha}_{21} l_2)$

Case $3'$. Suppose $(Q_1, Q_2) \in \Omega_{3'}$, i.e., $D_1 \le Q_1, 0 \le Q_2 \le D_2 - l_2$.

Case $3'$ is symmetric to Case 3 and its possible solutions are listed in Table 10.

Table 10: Possible solutions and their total profits in Case 3'

Possible solutions	Total profit
$t^1_{3'} = t^2_{2'} = (D_1, 0)$	$f(\boldsymbol{t}^1_{3'}) = f(\boldsymbol{t}^2_{2'}) = \overline{P}_1 D_1$
$t^2_{3'} = t^3_{2'} = (D_1, D_2 - l_2)$	$f(\boldsymbol{t}^2_{3'}) = f(\boldsymbol{t}^3_{2'}) = \overline{P}_2(D_2 - l_2) + \overline{P}_1 D_1 - \overline{S}_1(l_1 - \underline{\alpha}_{21} l_2)$
$t^3_{3'} = (D_1 - l_1 + \underline{\alpha}_{21} D_2, 0)$	$f(\boldsymbol{t}^3_{3'}) = \overline{P}_1(D_1 - l_1 + \underline{\alpha}_{21} D_2)$
$t^4_{3'} = t^5_{2'} = (D_1, D_2 - l_1/\underline{\alpha}_{21})$	$f(\boldsymbol{t}^4_{3'}) = f(\boldsymbol{t}^5_{2'}) = \overline{P}_2(D_2 - l_1/\underline{\alpha}_{21}) + \overline{P}_1 D_1$

Case $5'$. Suppose $(Q_1, Q_2) \in \Omega_{5'}$, i.e., $D_1 \le Q_1, D_2 - l_2 \le Q_2 \le D_2$.

Case $5'$ is symmetric to Case 5 and its possible solutions are listed in Table 11.

Table 11: Possible solutions and their total profits for Case 5'

Possible solutions	Total profit
$t^1_{5'} = t^2_4 = t^2_{3'} = t^3_{2'} = (D_1, D_2 - l_2)$	$f(\boldsymbol{t}^1_{5'}) = f(\boldsymbol{t}^2_4) = f(\boldsymbol{t}^2_{3'}) = f(\boldsymbol{t}^3_{2'}) = \overline{P}_2(D_2 - l_2) + \overline{P}_1 D_1 - \overline{S}_1(l_1 - \underline{\alpha}_{21} l_2)$
$t^2_{5'} = t^2_5 = t^1_4 = (D_1, D_2)$	$f(\boldsymbol{t}^2_{5'}) = f(\boldsymbol{t}^2_5) = f(\boldsymbol{t}^1_4) = \overline{P}_2 D_2 + \overline{P}_1 D_1 - \max\left\{\overline{S}_2 l_2, \overline{S}_1 l_1\right\}$
$t^3_{5'} = (D_1, D_2 - \frac{\overline{S}_2 l_2 - \overline{S}_1 l_1}{\overline{S}_2 - \overline{S}_1 \underline{\alpha}_{21}})$	$f(\boldsymbol{t}^3_{5'}) = \overline{P}_1 D_1 + \overline{P}_2 D_2 - \overline{P}_2 \frac{\overline{S}_2 l_2 - \overline{S}_1 l_1}{\overline{S}_2 - \overline{S}_1 \underline{\alpha}_{21}} - \overline{S}_1 \overline{S}_2 \frac{l_1 - l_2 \underline{\alpha}_{21}}{\overline{S}_2 - \overline{S}_1 \underline{\alpha}_{21}}$

Based on the results in Case 1-Case 6, thus symmetricly, we can also draw the following conclusions in the subregions Ω_1, $\Omega_{2'}$, $\Omega_{3'}$, Ω_4, $\Omega_{5'}$ and Ω_6:

(i) If $\overline{P}_1 \le \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \ge \overline{P}_1 \underline{\alpha}_{21}$, then

  (a) if $\overline{S}_1 l_1 \ge \overline{S}_2 l_2$, then $\boldsymbol{t}^3_4$ or $\boldsymbol{t}^5_4$ dominates all the other points in subregions Ω_1, $\Omega_{2'}$, $\Omega_{3'}$, Ω_4, and $\Omega_{5'}$.

  (b) if $\overline{S}_1 l_1 \le \overline{S}_2 l_2$, then $\boldsymbol{t}^3_{5'}$ or $\boldsymbol{t}^5_4$ dominates all the other points in subregions Ω_1, $\Omega_{2'}$, $\Omega_{3'}$, Ω_4, and $\Omega_{5'}$, since $f(\boldsymbol{t}^3_4) \le f(\boldsymbol{t}^3_{5'})$.

(ii) If $\overline{P}_1 \ge \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \le \overline{P}_1 \underline{\alpha}_{21}$, then $\boldsymbol{t}^3_{3'}$ dominates all the other points in subregions Ω_1, $\Omega_{2'}$, $\Omega_{3'}$, Ω_4, and $\Omega_{5'}$.

(iii) If $\overline{P}_1 \ge \overline{P}_2 \underline{\alpha}_{12}$ and $\overline{P}_2 \ge \overline{P}_1 \underline{\alpha}_{21}$, then

  (a) if $\overline{S}_1 l_1 \ge \overline{S}_2 l_2$, then $\boldsymbol{t}^3_4$ or $\boldsymbol{t}^7_4$ dominates all the other points in subregions Ω_1, $\Omega_{2'}$, $\Omega_{3'}$, Ω_4, and $\Omega_{5'}$, and

  (b) if $\overline{S}_1 l_1 \le \overline{S}_2 l_2$, then $\boldsymbol{t}^3_{5'}$ or $\boldsymbol{t}^7_4$ dominates all the other points in subregions Ω_1, $\Omega_{2'}$, $\Omega_{3'}$, Ω_4, and $\Omega_{5'}$, since $f(\boldsymbol{t}^3_4) \le f(\boldsymbol{t}^3_{5'})$.

Thus, combining all the comparison results, we can conclude that

Case 1: If $\overline{P}_1 \le \overline{P}_2\underline{\alpha}_{12}$ and $\overline{P}_2 \ge \overline{P}_1\underline{\alpha}_{21}$, then $\boldsymbol{t}_3^3$ dominates all the other points of the 9 subregions, since $f(\boldsymbol{t}_4^5) \le f(\boldsymbol{t}_3^3)$, $f(\boldsymbol{t}_4^3) \le f(\boldsymbol{t}_3^3)$ from the results of Case 4, and when $\overline{S}_1 l_1 \le \overline{S}_2 l_2$, we have

$$\begin{aligned} f(\boldsymbol{t}_{5'}^3) - f(\boldsymbol{t}_3^3) &= \overline{P}_1 D_1 + \overline{P}_2 D_2 - \overline{P}_2 \frac{\overline{S}_2 l_2 - \overline{S}_1 l_1}{\overline{S}_2 - \overline{S}_1\underline{\alpha}_{21}} - \overline{S}_1\overline{S}_2 \frac{l_1 - l_2\underline{\alpha}_{21}}{\overline{S}_2 - \overline{S}_1\underline{\alpha}_{21}} - \left(\overline{P}_2(D_2 - l_2 + \underline{\alpha}_{12} D_1)\right) \\ &= \left(\overline{P}_1 - \overline{P}_2\underline{\alpha}_{12}\right) D_1 + \overline{S}_1\left(\overline{S}_2 - \overline{P}_2\right) \frac{l_2\underline{\alpha}_{21} - l_1}{\overline{S}_2 - \overline{S}_1\underline{\alpha}_{21}} \le 0, \end{aligned}$$

where the inequality is due to $\overline{P}_1 \le \overline{P}_2\underline{\alpha}_{12}$, $\overline{S}_1 \ge 0, \overline{S}_2 - \overline{P}_2 \ge 0$, $l_2\underline{\alpha}_{21} \le l_1$ and $\overline{S}_2 \ge \overline{S}_1\underline{\alpha}_{21}$ (due to $\overline{S}_1 l_1 \le \overline{S}_2 l_2$ and $l_2\underline{\alpha}_{21} \le l_1$).

Case 2: If $\overline{P}_2 \le \overline{P}_1\underline{\alpha}_{21}$ and $\overline{P}_1 \ge \overline{P}_2\underline{\alpha}_{12}$, then $\boldsymbol{t}_{3'}^3$ dominates all the other points of the 9 subregions, since $f(\boldsymbol{t}_4^5) \le f(\boldsymbol{t}_{3'}^3)$, $f(\boldsymbol{t}_4^3) \le f(\boldsymbol{t}_{3'}^3)$ which is due to the symmetric results from Case 4 that $f(\boldsymbol{t}_4^5) \le f(\boldsymbol{t}_3^3)$, $f(\boldsymbol{t}_4^3) \le f(\boldsymbol{t}_3^3)$, and when $\overline{S}_1 l_1 \ge \overline{S}_2 l_2$, by the results that $f(\boldsymbol{t}_{5'}^3) \le f(\boldsymbol{t}_3^3)$, we must have $f(\boldsymbol{t}_5^3) \le f(\boldsymbol{t}_{3'}^3)$.

Case 3: If $\overline{P}_1 \ge \overline{P}_2\underline{\alpha}_{12}$, $\overline{P}_2 \ge \overline{P}_1\underline{\alpha}_{21}$, there we can separate the results into two sub-cases:

Sub-case 3.1: If $\overline{S}_1 l_1 \ge \overline{S}_2 l_2$, $\boldsymbol{t}_5^3$ or $\boldsymbol{t}_4^7$ dominates all the other points, since $f(\boldsymbol{t}_5^3) \ge f(\boldsymbol{t}_4^3)$ from Case 5;

Sub-case 3.2: If $\overline{S}_1 l_1 \le \overline{S}_2 l_2$, $\boldsymbol{t}_{5'}^3$ or $\boldsymbol{t}_4^7$ dominates all the other points, since $f(\boldsymbol{t}_{5'}^3) \ge f(\boldsymbol{t}_4^3)$ by symmetry.

This completes the proof.

□

A.4 Proof of Theorem 3

Theorem 3. *When $\underline{\boldsymbol{\alpha}} = 0$, the optimal solutions $\boldsymbol{Q}^*$ of Model* (15) *are characterized as follows:*

(i) If $\sum_{i\in[n]} \frac{\overline{P}_i}{\overline{S}_i} \le k$, then $Q_i^* = D_i - l_i$, and $v^* = \sum_{i\in[n]} \overline{P}_i (D_i - l_i)$.

(ii) If $\sum_{i\in[n]} \frac{\overline{P}_i}{\overline{S}_i} > k$,

$$Q_i^* = \begin{cases} D_i - l_i + \frac{\overline{S}_{(t+1)} l_{(t+1)}}{\overline{S}_i}, & \text{if } i \in T \\ D_i, & \text{if } i \in [n] \setminus T \end{cases},$$

and

$$v^* = \sum_{i\in[n]\setminus T} \overline{P}_i l_i - \overline{S}_{(t+1)} l_{(t+1)} k + \sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \overline{S}_{(t+1)} l_{(t+1)} + \sum_{i\in[n]} \overline{P}_i (D_i - l_i),$$

where set $T := \{(1), (2), \cdots, (t)\}$ satisfying $\sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \le k$, $\sum_{i\in T\cup\{(t+1)\}} \frac{\overline{P}_i}{\overline{S}_i} > k$.

Proof. Let $\widehat{X} = \{\boldsymbol{z} : \sum_{i\in[n]} z_i \leq k, z_i \in [0,1], \forall i \in [n]\}$, which is a well-known integral polytope. Thus, $\operatorname{conv}(X) = \widehat{X}$ and the inner maximization problem of (14) is equivalent to maximize a linear function of set $\widehat{X}$. Thus, we have

$$v^* = \max_{\boldsymbol{Q}\in[\boldsymbol{D}-\boldsymbol{l},\boldsymbol{D}]} \left(\sum_{i\in[n]} \overline{P}_i Q_i - \max_{\boldsymbol{z}\in\widehat{X}} \sum_{i\in[n]} \overline{S}_i \left(Q_i - D_i + l_i\right) z_i \right). \tag{33}$$

Let $q_i = Q_i - D_i + l_i$ for each $i \in [n]$, then Model (33) is equivalent to

$$v^* = \max_{\boldsymbol{q}\in[\boldsymbol{0},\boldsymbol{l}]} \min_{\boldsymbol{z}\in\widehat{X}} \left(\sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right) q_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \right). \tag{34}$$

Let λ be the dual variable associated with constraint $\sum_{i\in[n]} z_i \leq k$ and β_i be the dual variable associated with constraint $z_i \leq 1$ for each $i \in [n]$. Then by reformulating the inner maximization into its dual form, Model (34) is equivalent to

$$v^* = \max_{\boldsymbol{q},\lambda,\boldsymbol{\beta}} k\lambda + \sum_{i\in[n]} \beta_i + \sum_{i\in[n]} \overline{P}_i q_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \tag{35a}$$

$$\text{s.t. } \lambda + \beta_i \leq -\overline{S}_i q_i, i \in [n], \tag{35b}$$

$$0 \leq q_i \leq l_i, i \in [n], \tag{35c}$$

$$\lambda, \beta_i \leq 0, i \in [n]. \tag{35d}$$

In Model (34), since the objective function is concave in $\boldsymbol{q}$ and convex in $\boldsymbol{z}$, and set $\widehat{X}$ is convex compact set, thus according to Sion's minimax theorem (cf., [45]), we can equivalently reformulate Model (34) by switching the min with max operators as follows:

$$v^* = \min_{\boldsymbol{z}\in\widehat{X}} \max_{\boldsymbol{q}\in[\boldsymbol{0},\boldsymbol{l}]} \left\{ \sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right) q_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \right\}.$$

Note that $\max_{q_i\in[0,l_i]} \left(\sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right) q_i\right) = \sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right)_+ l_i$ for each $i \in [n]$. Thus, Model (34) is equivalent to

$$v^* = \min_{\boldsymbol{z}\in\widehat{X}} \left\{ \sum_{i\in[n]} \left(\overline{P}_i - \overline{S}_i z_i\right)_+ l_i + \sum_{i\in[n]} \overline{P}_i \left(D_i - l_i\right) \right\}. \tag{36}$$

we also observe that

Claim 2. *Model* (36) *is equivalent to*

$$v^* = \min_{\boldsymbol{z} \in \widehat{X}_1} \left\{ \sum_{i \in [n]} \left(\overline{P}_i - \overline{S}_i z_i \right) l_i + \sum_{i \in [n]} \overline{P}_i \left(D_i - l_i \right) \right\}, \tag{37}$$

where $\widehat{X}_1 = \left\{ \boldsymbol{z} : \sum_{i \in [n]} z_i \leq k, 0 \leq z_i \leq \frac{\bar{P}_i}{\bar{S}_i} \right\}$.

Proof. Let v_1^* denote the optimal value of Model (37). To prove Model (36) is equivalent to Model (37), we only need to show $v^* = v_1^*$.

$v^* \geq v_1^*$. Given an optimal solution $\boldsymbol{z}^*$ of Model (36), let us define set $\mathcal{J}_1 = \{ i \in [n] : 0 \leq z_i^* \leq \frac{\bar{P}_i}{\bar{S}_i} \}$ and $\mathcal{J}_2 = \{ i \in [n] : \frac{\bar{P}_i}{\bar{S}_i} < z_i^* \leq 1 \}$. Clearly, $\mathcal{J}_1 \cup \mathcal{J}_2 = [n]$ and $\mathcal{J}_1 \cap \mathcal{J}_2 = \emptyset$. Next, we define a new solution $\widehat{\boldsymbol{z}}$ such that

$$\widehat{z}_i = \begin{cases} z_i^*, & \text{if } i \in \mathcal{J}_1 \\ \frac{\bar{P}_i}{\bar{S}_i}, & \text{otherwise} \end{cases},$$

for each $i \in [n]$. Clearly, $\widehat{\boldsymbol{z}} \in \widehat{X}_1$. We also have

$$\begin{aligned} v^* &= \sum_{i \in [n]} \left(\overline{P}_i - \overline{S}_i z_i^* \right)_+ l_i + \sum_{i \in [n]} \overline{P}_i \left(D_i - l_i \right) \\ &= \sum_{i \in \mathcal{J}_1} \left(\overline{P}_i - \overline{S}_i z_i^* \right) l_i + \sum_{i \in [n]} \overline{P}_i \left(D_i - l_i \right) \\ &= \sum_{i \in [n]} \left(\overline{P}_i - \overline{S}_i \widehat{z}_i \right) l_i + \sum_{i \in [n]} \overline{P}_i \left(D_i - l_i \right), \end{aligned}$$

where the third equality is due to the definition of $\widehat{\boldsymbol{z}}$. Therefore, $\widehat{\boldsymbol{z}}$ is feasible to Model (37) with the same objective value v^*. Thus, we have $v^* \geq v_1^*$.

$v^* \leq v_1^*$. Since $\widehat{X}_1 \subseteq \widehat{X}$, thus, $v^* \leq v_1^*$.

□

Note that Model (37) is a continuous knapsack minimization problem and can be solved by greedy procedure (c.f., [14, 22]). Let $\boldsymbol{z}^*$ denote an optimal solution to Model (37). To obtain $\boldsymbol{z}^*$, we first sort $\{\overline{S}_i l_i\}_{i \in [n]}$ in the descending order $\overline{S}_{(1)} l_{(1)} \geq \overline{S}_{(2)} l_{(2)} \geq \cdots \geq \overline{S}_{(n)} l_{(n)}$. Next, we discuss two cases:

Case 1. If $\sum_{i \in [n]} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, then we have

$$z_i^* = \frac{\overline{P}_i}{\overline{S}_i}, \forall i \in [n],$$

and $v^* = \sum_{i \in [n]} \overline{P}_i \left(D_i - l_i \right)$. On the other hand, in (35), let us consider the following feasible solution $\lambda^* = 0, \beta_i^* = 0, q_i^* = 0$ for all $i \in [n]$ with objective value equal to

$\sum_{i\in[n]} \overline{P}_i (D_i - l_i)$. Therefore, $(\boldsymbol{Q}^*, \lambda^*, \boldsymbol{\beta}^*)$ is optimal to (35). Hence, the optimal order quantity for each product $i \in [n]$ is

$$Q_i^* = q_i^* + D_i - l_i = D_i - l_i.$$

Case 2. If $\sum_i \frac{\overline{P}_i}{\overline{S}_i} > k$, then let us define set $T := \{(1), (2), \ldots, (t)\}$ such that $\sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \leq k$, $\sum_{i\in T\cup(t+1)} \frac{\overline{P}_i}{\overline{S}_i} > k$. Then we have

$$z_i^* = \begin{cases} \frac{\overline{P}_i}{\overline{S}_i} & \text{if } i \in T \\ k - \sum_{\tau\in T} \frac{\overline{P}_\tau}{\overline{S}_\tau} & \text{if } i = (t+1) \\ 0 & \text{otherwise} \end{cases},$$

for each $i \in [n]$, and

$$\begin{aligned} v^* &= \sum_{i\in[n]\setminus T} \left(\overline{P}_i - \overline{S}_i z_i^*\right) l_i + \sum_{i\in[n]} \overline{P}_i (D_i - l_i) \\ &= \overline{P}_{(t+1)} l_{(t+1)} - \overline{S}_{(t+1)} \left(k - \sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i}\right) l_{(t+1)} + \sum_{i\in\{[n]\setminus T\cup(t+1)\}} \overline{P}_i l_i + \sum_{i\in[n]} \overline{P}_i (D_i - l_i) \\ &= \sum_{i\in[n]\setminus T} \overline{P}_i l_i - \overline{S}_{(t+1)} l_{(t+1)} k + \sum_{i\in T} \frac{\overline{P}_i}{\overline{S}_i} \overline{S}_{(t+1)} l_{(t+1)} + \sum_{i\in[n]} \overline{P}_i (D_i - l_i) \end{aligned}$$

Next $\lambda^* = -\overline{S}_{t+1} l_{t+1}, \beta_i^* = 0$ and

$$q_i^* = \begin{cases} \frac{\overline{S}_{t+1} l_{t+1}}{\overline{S}_i}, & \text{if } i \in T \\ l_i, & \text{if } i \in [n] \setminus T \end{cases}$$

for each product $i \in [n]$. Clearly, $(\boldsymbol{Q}^*, \lambda^*, \boldsymbol{\beta}^*)$ is feasible to (35) with objective value equal to v^*. Therefore, $(\boldsymbol{Q}^*, \lambda^*, \boldsymbol{\beta}^*)$ is optimal to (35). Hence, the optimal order quantity for each product $i \in [n]$ is

$$Q_i^* = q_i^* + D_i - l_i = \begin{cases} D_i - l_i + \frac{\overline{S}_{t+1} l_{t+1}}{\overline{S}_i}, & \text{if } i \in T \\ D_i, & \text{if } i \in [n] \setminus T \end{cases}.$$

□

A.5 Proof of Proposition 3

Proposition 3. *The inner maximization problem* (19) *is equivalent to*

$$R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) = \max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \sum_{i\in[n]} \overline{S}_i \left[(Q_i - D_i)x_i + l_i y_{ii} - \sum_{j\in[n]} \underline{\alpha}_{ji} \left(u_j y_{ji} + \psi_j(x_i - y_{ji})\right)\right] \tag{21a}$$

$$s.t. \sum_{i\in[n]} z_i \le k. \tag{21b}$$

$$y_{ji} \le x_i, \forall i,j \in [n], \tag{21c}$$

$$y_{ji} \le z_j, \forall i,j \in [n], \tag{21d}$$

$$z_i, x_i \in \{0,1\}, y_{ji} \ge 0, \forall i,j \in [n]. \tag{21e}$$

Proof. Let $\widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$ denote the optimal value of Model (21). It is sufficient to show that $R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) = \widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$ for any feasible $(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) \in \mathbb{R}_+^{3n}$.

$R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) \le \widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$. Suppose $(\boldsymbol{x}^*,\boldsymbol{y}^*) \in \{0,1\}^{2n}$ is an optimal solution of Model (20). Define $y_{ji}^* = z_j^* x_i^*$ for each $i,j \in [n]$. Clearly, $(\boldsymbol{x}^*,\boldsymbol{y}^*,\boldsymbol{z}^*)$ is feasible to Model (21) and

$$\begin{aligned} R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) &= \sum_{i\in[n]} \overline{S}_i \left[Q_i - D_i + l_i z_i^* - \sum_j \underline{\alpha}_{ji} \left(u_j z_j^* + v_j(1 - z_j^*)\right)\right] x_i^* \\ &= \sum_{i\in[n]} \overline{S}_i \left(Q_i - D_i)x_i^* + l_i y_{ii}^* - \sum_{j\in[n]} \underline{\alpha}_{ji} \left(u_j y_{ji}^* + \psi_j(x_i^* - y_{ji}^*)\right)\right], \end{aligned}$$

i.e., it yields the same objective value as $R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$. Thus, $R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) \le \widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$.

$R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) \ge \widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$. Suppose $(\boldsymbol{x}^*,\boldsymbol{y}^*,\boldsymbol{z}^*)$ is an optimal solution to Model (21). Since both $\boldsymbol{x}^*$ and $\boldsymbol{z}^*$ are binary, thus according to constraints (21c) and (21d), we must have $y_{ji}^* \le z_j^* x_i^* = \min\{z_j^*, x_i^*\}$ for each $i,j \in [n]$. Therefore,

$$\begin{aligned} \widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) &= \sum_{i\in[n]} \overline{S}_i \left[(Q_i - D_i)x_i^* + l_i y_{ii}^* - \sum_{j\in[n]} \underline{\alpha}_{ji} \left(u_j y_{ji}^* + \psi_j(x_i^* - y_{ji}^*)\right)\right] \\ &\le \sum_{i\in[n]} \overline{S}_i \left[Q_i - D_i + l_i z_i^* - \sum_j \underline{\alpha}_{ji} \left(u_j z_j^* + v_j(1 - z_j^*)\right)\right] x_i^* \end{aligned}$$

where the first inequality is due to the coefficients of $\{y_{ji}^*\}_{j,i\in[n]}$ are all nonnegative, i.e., $\overline{S}_i l_i \ge 0$ and $\psi_j \ge u_j$ for all $i,j \in [n]$. Hence, $(\boldsymbol{x}^*,\boldsymbol{z}^*)$ is feasible to Model (20) and yields an objective value at least as large as $\widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$, which implies that $R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) \ge \widehat{R}(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi})$.

□

A.6 Proof of Theorem 5

Theorem 5. *Let v^{CA} denote the optimal value of Model* (27)*. Then*

(i) $v^{CA} \leq v^*$*; and*

(ii) $v^{CA} = v^*$*, if one of the following conditions holds: (1)* $\underline{\boldsymbol{\alpha}} = \mathbf{0}$*, or (2)* $n = k$*.*

Proof. $v^* \geq v^{CA}$ holds since in (27), we replace set Ξ to be its continuous relaxation Ξ_C. It remains to show that $v^* = v^{CA}$ if $\underline{\boldsymbol{\alpha}} = \mathbf{0}$ or $n = k$. To prove this result, it is sufficient to show that for any given $(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi})$ satisfying constraints (22a) – (22e), (23b) – (23c), the following linear program

$$\max_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in\Xi_C} g\left(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\right)$$

has an integral optimal solution, i.e., the continuous relaxation of Model (21) has an integral optimal solution.

$\underline{\boldsymbol{\alpha}} = \mathbf{0}$. In this case, Model (21) is equivalent to

$$R(\boldsymbol{Q}, \boldsymbol{u}, \boldsymbol{\psi}) = \max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \sum_{i\in[n]} \overline{S}_i \left[(Q_i - D_i)x_i + l_i y_{ii}\right] \tag{40a}$$

$$\text{s.t.} \sum_{i\in[n]} z_i \leq k. \tag{40b}$$

$$y_{ii} \leq x_i, \forall i \in [n], \tag{40c}$$

$$y_{ii} \leq z_i, \forall i \in [n], \tag{40d}$$

$$z_i, x_i \in \{0, 1\}, y_{ii} \geq 0, \forall i \in [n]. \tag{40e}$$

We let $\widehat{\Xi}_C$ denote the continuous relaxation of the feasible region of Model (40), where we relax $\boldsymbol{x}, \boldsymbol{z}$ to be continuous. Then, it is sufficient to show that $\widehat{\Xi}_C$ is an integral polytope.

First of all, let us write the constraints (40b) – (40e) in the matrix form as below:

$$\begin{bmatrix} \boldsymbol{e}^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{y} \\ \boldsymbol{x} \end{bmatrix} \leq \begin{bmatrix} k \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{41}$$

where $\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$, $\boldsymbol{y} = \begin{bmatrix} y_{11} \\ y_{22} \\ \vdots \\ y_{nn} \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. By Theorem 19.3 on Page 269 of [41], to prove

$\mathcal{P}$ is a integral polytope, it is sufficient to prove $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{e}^\top & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}$ is totally unimodular (TU). Indeed, if $\boldsymbol{A}$ is TU, and since $\begin{bmatrix} k \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix}$ is integral, thus, $\mathcal{P}$ is an integral polytope. Hence, the continuous relaxation of Model (40), which is a linear program, has an integral optimal solution. According to Theorem 19.3 on Page 269 of [41], to prove $\boldsymbol{A}$ is a totally unimodular matrix, it is sufficient to prove that for any $S \subseteq [3n]$, there exist S_1 and S_2 such that $S_1 \cap S_2 = \emptyset, S_1 \cup S_2 = S, \sum_{i \in S_1} \boldsymbol{A}_{\cdot i} - \sum_{i \in S_2} \boldsymbol{A}_{\cdot i} \in \{-1, 0, 1\}^{2n+1}$.

We let $\widehat{S}_1 = S \cap \{1, 2, \cdots, n\} = \{i_1, \cdots, i_{|\overline{S}_1|}\}$, $T_z^1 = \{i_\tau\}_{\tau \leq |\overline{S}_1|, \tau \text{ is odd}}$ and $T_z^2 = \{i_\tau\}_{\tau \leq |\overline{S}_1|, \tau \text{ is even}}$. Also, we let $\widehat{S}_2 = S \cap \{n+1, \cdots, 2n\}$, $\widehat{S}_3 = S \cap \{2n+1, \cdots, 3n\}$, $T_y^1 = \left\{ j \in \widehat{S}_2 : j - n \in T_z^1 \right\}$, $T_y^2 = \widehat{S}_2 \setminus T_y^1$, $T_x^1 = \left\{ j \in \widehat{S}_3 : j - n \in T_y^1 \right\}$, $T_x^2 = \overline{S}_3 \setminus T_x^1$. Clearly, we have $S_1 = T_z^1 \cup T_y^1 \cup T_x^1$, $S_2 = S \setminus S_1$. For such S_1 and S_2, we have $\sum_{i \in S_1} \boldsymbol{A}_{\cdot i} - \sum_{i \in S_2} \boldsymbol{A}_{\cdot i} \in \{-1, 0, 1\}^{2n+1}$.

$k = n$. From the discussion in Section 4.3, we already know at the optimality, we must have $z_i^* = 1$ for all $i \in [n]$ when $k = n$. In (21a), the coefficient of y_{ji} is $\sum_{j \in [n]} \underline{\alpha}_{ji}(\psi_j - u_j) \geq 0$, since $\psi_j \geq u_j$ for each $j, i \in [n]$ and $j \neq i$. Also, the coefficient of y_{ii} is l_i, which is nonnegative, for each $i \in [n]$. Thus, at the optimality of the continuous relaxation of Model (21), we must have $y_{ji} = \min(x_i, z_j) = \min(x_i, 1) = x_i$ for all $i, j \in [n]$. Then, the continuous relaxation of Model (21) is equivalent to

$$
\begin{aligned}
&\max_{\boldsymbol{x} \in [0,1]^n} \sum_{i \in [n]} \overline{S}_i \left((Q_i - D_i) x_i + l_i y_{ii} - \sum_{j \in [n]} \underline{\alpha}_{ji} \left(u_j y_{ji} + \psi_j (x_i - y_{ji}) \right) \right) \\
= &\max_{\boldsymbol{x} \in [0,1]^n} \sum_{i \in [n]} \overline{S}_i \left((Q_i - D_i + l_i) + \sum_{j \in [n]} \underline{\alpha}_{ji} (Q_j - D_j + l_j)_+ \right) x_i,
\end{aligned}
$$

which is a linear program over a unit box. Thus, there exists an optimal solution $\boldsymbol{x}^*$ of the above linear program, which corresponds to an extreme point of the box $[0, 1]^n$, i.e., $x_i^* \in \{0, 1\}$ for all $i \in [n]$. Thus, $y_{ji}^* = x_i^* \in \{0, 1\}$ for all $i, j \in [n]$. Therefore, the continuous relaxation of Model (21) has an integral optimal solution $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*)$.

□

Appendix B. MILP Reformulation of Model (19) for General k^α

Suppose k^α is general, i.e., Assumption 1 does not hold. Then the formulation (19) becomes

$$R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) = \max_{\boldsymbol{z}} \sum_{i\in[n]} \overline{S}_i \left[Q_i - D_i + l_i z_i - \sum_{j\in[n]} (\alpha_{ji} - l_{ji}^\alpha z_{ji}^\alpha)\,(u_j z_j + \psi_j(1-z_j)) \right]_+ , \tag{42a}$$

$$\text{s.t.} \sum_{i\in[n]} z_i \le k, \tag{42b}$$

$$\sum_{j\in[n]}\sum_{i\in[n]} z_{ji}^\alpha \le k^\alpha, \tag{42c}$$

$$z_i \in \{0,1\}, \forall i \in [n], \tag{42d}$$

$$z_{ji}^\alpha \in \{0,1\}, \forall j,i \in [n]. \tag{42e}$$

Similar to Proposition 3, We can reformulate (42) as an MILP.

Proposition 4. *The inner maximization problem* (42) *is equivalent to*

$$\begin{aligned} R(\boldsymbol{Q},\boldsymbol{u},\boldsymbol{\psi}) = \max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \sum_{i\in[n]} \overline{S}_i \Bigg\{ & (Q_i - D_i)\,x_i + l_i y_{ii} - \sum_{j\in[n]} \alpha_{ji}\left[u_j y_{ji} + \psi_j\,(x_i - y_{ji})\right] \\ & + \sum_{j\in[n]} l_{ji}^\alpha \left[u_j T_{ji}^\alpha + \psi_j\left(B_{ji}^\alpha - T_{ji}^\alpha\right)\right] \Bigg\} \\ s.t. \quad & \sum_{i\in[n]} z_i \le k. \\ & \sum_{j\in[n]}\sum_{i\in[n]} z_{ji}^\alpha \le k^\alpha, \\ & y_{ji} \le x_i, \quad y_{ji} \le z_j, \forall i,j \in [n], \\ & T_{ji}^\alpha \le z_{ji}^\alpha, T_{ji}^\alpha \le y_{ji}, \forall i,j\in[n], \\ & B_{ji}^\alpha \le z_{ji}^\alpha, B_{ji}^\alpha \le x_i, B_{ji}^\alpha \ge z_{ji}^\alpha + x_i - 1, \forall i,j \in [n] \\ & z_{ji}^\alpha, z_i, x_i \in \{0,1\}, y_{ji}, T_{ji}^\alpha \ge 0, \forall i,j \in [n]. \end{aligned}$$

Proof. The proof is similar to that of Proposition 3, i.e., we eliminate the bilinear terms by introducing variables $y_{ji} = x_i z_j, T_{ji}^\alpha = z_{ji}^\alpha y_{ji}, B_{ji}^\alpha = z_{ji}^\alpha x_i$ for each $i,j \in [n]$, and then applying McCormick inequalities [28]. □