n-Step cutset inequalities: facets for multi-module capacitated network design problem

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Abstract. Many real-world decision-making problems can be modeled as network design problems, especially on networks with capacity requirements on links. In network design problems, decisions are made on installation of flow transfer capacities on the links and routing of flow from a set of source nodes to a set of sink nodes through the links. Many network design problems of this type that have been studied involve different types of capacity sizes (modules), which we refer to as the multi-module capacitated network design (MMND) problem. In this paper, we present a new family of inequalities for MMND through polyhedral analysis of a mixed integer set closely related to MMND. We show that various classes of cutset inequalities in the literature are special cases of these inequalities. We also study the strength of this family of inequalities and identify conditions under which they are facet-defining. These inequalities are then tested on MMND problem instances, and our computational results show that this family of inequalities are very effective for solving MMND problems. Generalizations of these inequalities for some variants of MMND are also discussed.

Keywords. Network Design; Cutting planes; n-Step cutset inequalities; Multi-module capacitated network design; Cutset polyhedron; n-Step MIR

1 Introduction

More and more decision-making problems in today's world, ranging from distribution networks of online retailers to data networks of cloud providers, have been modeled as capacitated network design problems. In the capacitated network design problem, decisions are made on installation of flow transfer capacities on the links and routing of flow from a set of source nodes to a set of sink nodes through the links of the network. The objective is to minimize the cost of installing capacities and transferring the flow from source nodes to sink nodes.

Telecommunication networks are prime examples of capacitated networks and a major motivation to study capacitated network design in the literature. In these networks, the nodes are clients, servers or any device that broadcasts, receives, or distributes signals or data. The end-users of the network have certain demand for speed which is measured by bit-rates. In order to satisfy such demand to ensure the quality of service, telecommunication service providers install transmission facilities such as fiber-optic cables between the nodes of the network to transmit signals or data. Such transmission facilities each have a certain capacity on the maximum bit rate they are able to transmit, known as its bandwidth. Similar analogies for flow and capacity can be found in other applications, such as transportation networks and power grid networks.

In many applications, the flow transfer capacities are available in the form of modules of different sizes. Each module size has a predetermined cost. The cost structure usually follows economy of scale, i.e., a unit of capacity costs less in larger modules compared to smaller ones. Examples of such capacities in telecommunication networks are fiber-optic cables of different bandwidths. Typically, a set of cable types with different bandwidths are available to procure [13], and the cost of procuring cables constitutes a major part of the total cost for network design. In such situations, the decision-making about installing capacities becomes more complicated than the case where we have only a single-sized capacity module, as we need to determine the composition of capacity modules of different sizes to be installed on each link. In this paper, we are interested in this type of problems, referred to as the *multi-module capacitated network design* (MMND) problem.

MMND can be defined on directed or undirected networks and formulated as mixed integer programs (MIPs). On a directed network, the MIP is formulated as follows:

let $G := (V, A)$ be a directed graph, where V and A are the set of nodes and arcs of G, respectively. For any non-empty $U, W \subset V$, let $\delta(U, W)$ denote the set of arcs from the nodes in U to the nodes in W. For any $v \in V$, let $\delta^+(v) := \delta(V \setminus \{v\}, v)$ and $\delta^-(v) := \delta(v, V \setminus \{v\})$ be the set of arcs that have v as their head and tail, respectively. Assume we have M differently sized capacity modules, indexed by $1, \ldots, M$. Let $C_1, \ldots, C_M > 0$ be the sizes of these capacity modules, respectively. Without loss of generality, we assume that $C_1 > C_2 > \ldots > C_M$. The cost of installing one capacity module t, $t = 1, \ldots, M$ on arc a is f_t^a. We assume there is a single type of commodity with multiple sources and multiple sinks over the network. A demand d_v is associated with each node such that $\sum_{v \in V} d_v = 0$. For sink nodes we have $d_v > 0$ and for all other nodes we have $d_v \leq 0$. Given the unit cost of flow along arc $a \in A$, denoted by h_a, and the pre-installed capacity on arc $a \in A$, denoted by g^a, MMND on a directed network can be formulated as the following mixed integer program:

$$\min \sum_{a \in A} (h_a x_a + \sum_{t=1}^{M} f_t^a y_t^a) \tag{1}$$

$$\sum_{a \in \delta^+(v)} x_a - \sum_{a \in \delta^-(v)} x_a = d_v, v \in V \tag{2}$$

$$x_a \leq \sum_{t=1}^{M} C_t y_t^a + g^a, a \in A \tag{3}$$

$$(x, y) \in \mathbb{R}_+^{|A|} \times \mathbb{Z}_+^{M|A|}, \tag{4}$$

where the flow variable x_a is the number of flow units to be transferred along arc $a \in A$, and the capacity variable y_t^a is the number of capacity module t to be installed on arc $a \in A$, $t = 1, \ldots, M$. We refer to the problem defined by (1)-(4) as the directed MMND.

For undirected networks, two types of MMND problems have been studied, which we refer to as the undirected MMND and the bidirected MMND. Let $H := (V, E)$ be an undirected graph, where E is the set of (undirected) edges. For each edge $e \in E$, we introduce a pair of anti-parallel (directed) arcs e^+ and e^-. Let A be the set of all such arcs, i.e., $A := \{e^+ = ij, e^- = ji : e = (i, j) \in E\}$. For any $v \in V$, $\delta^+(v)$ and $\delta^-(v)$ are defined similarly to those in the directed MMND. In both models, let the capacity variable y_t^e be the number of capacity module t to be installed on edge $e \in E, t = 1, \ldots, M$, and the flow variables x_{e^+}, x_{e^-} be the number of flow units to be transferred along the arcs e^+ and e^- corresponding to e. Let h_a denote unit flow cost on arc $a \in A$, f_t^e denote the cost of installing one capacity module t on edge $e \in E$, and g^e denote the pre-installed capacity on edge $e \in E$. In the undirected MMND, the summation of flows in both directed arcs corresponding to an edge is bounded by the edge capacity. Therefore, this problem is formulated as follows:

$$\min \sum_{a \in A} h_a x_a + \sum_{e \in E} \sum_{t=1}^{M} f_t^e y_t^e \tag{5}$$

$$\sum_{a \in \delta^+(v)} x_a - \sum_{a \in \delta^-(v)} x_a = d_v, v \in V \tag{6}$$

$$x_{e^+} + x_{e^-} \leq \sum_{t=1}^{M} C_t y_t^e + g^e, e \in E \tag{7}$$

$$(x, y) \in \mathbb{R}_+^{2|E|} \times \mathbb{Z}_+^{M|E|}. \tag{8}$$

In the bidirected MMND, the flow in each of the two arcs corresponding to an edge is bounded by the edge capacity. Therefore, this problem is formulated by (5), (6), (8) and the following constraint

$$\max\{x_{e^+}, x_{e^-}\} \leq \sum_{t=1}^{M} C_t y_t^e + g^e, e \in E \tag{9}$$

instead of (7). The bidirected MMND can be transformed to the directed MMND where the capacities on the forward and backward arcs between a pair of nodes are the same. See [22, 23] for more details on these problems.

Telecommunication networks have had a significant role in motivating the study of MMND and its variants in the MIP literature from different perspectives; see for example [8, 9, 10, 12, 14, 19, 20, 22, 23, 24]. A considerable number of studies have addressed the MIP formulations of MMND from the cutting plane perspective [8, 9, 12, 19, 20, 22, 23]. In these studies, the so-called cutset inequalities are among the most effective classes of inequalities for network design problems. This class of inequalities are derived for the convex hull of a certain relaxation of directed/undirected/bidrected MMND, called the *cutset polyhedron* for the respective problem (see Section 2 for more details).

Magnanti and Mirchandani [19] studied the cutset inequalities that include only capacity variables for the undirected MMND with $M \leq 3$, where the module sizes are *divisible*, i.e., $C_2|C_1$ and $C_3|C_2$. Bienstock and Günlük [8] proposed the flow-cut-set inequalities to include flow variables for the bidirected MMND with $M = 2$ divisible capacity modules. Chopra et al. [9] presented the cut-set inequalities for directed MMND with $M \leq 2$ divisible capacity modules. Atamtürk [2] generalized the cut-set inequalities to the multifacility cut-set inequalities for the directed MMND with any M (not necessarily divisible) capacity modules. Raack et al. [22, 23] generalized the multifacility cut-set inequalities for the undirected and the bidrected MMND. Table 1 summarizes some pertinent features of the aforementioned studies.

Table 1: Summary of major relevant studies on network design problems

Reference	Problem type*	M*	n*	Inequalities	
				Name in reference	Name in this paper
[19]	u	≤ 3, div	≤ 2	cutset	1&2-step capacity cutset
[8]	b	2, div	1	flow-cut-set	1-step flow cutset
[9]	d	≤ 2, div	1	cut-set	1-step cutset
[2]	d	any	1	multifacility cut-set	1-step cutset
[23]	d,u,b	any	1	flow cutset	1-step cutset
This paper	d,u,b	any	any	-	n-step cutset

*: "d", "u", and "b" denote directed, undirected, and bidirected, respectively. "div" means only divisible module sizes were studied. M denotes the number of capacity modules in the problem and n is the number of capacity modules used to derive the inequality.

Almost all aforementioned cutset inequalities were derived, or can be shown to be derivable, by applying the traditional mixed integer rounding (MIR), referred to as the 1-step MIR, on the base inequalities formed by certain aggregation and relaxation of the defining constraints of the corresponding cutset polyhedron (see [21] and [26] for more details on 1-step MIR inequalities). More precisely, the cutset inequalities in [8, 23] were derived using 1-step MIR, and the cutset inequalities in [2, 9, 19], although not derived directly using 1-step MIR, can be easily shown that can be derived using 1-step MIR as well.

We note that in derivation of most of the aforementioned inequalities, the information of only one of the capacity modules is used (as noted in Table 1), even though information of all of the capacity modules is important for MMND. The only exception is the cutset facet for the 3-module problem in [19] involving only capacity variables, which, as we will show later, can be derived using 2-step MIR inequalities.

Motivated by this observation, in this paper, we develop cutset inequalities for MMND using the information of all the capacity modules for any number of capacity modules, show their theoretical strength, and demonstrate they are computationally very effective in solving MMND, especially compared to cutset inequalities derived based on the information of a single capacity module. In developing these inequalities, we employ the n-step MIR theory. Kianfar and Fathi [16] presented the n-step MIR inequalities for the mixed integer knapsack set (see Section 2 for more details). The (1-step) MIR inequalities [21, 26] and the 2-step

MIR inequalities [11] are special cases of the n-step MIR inequalities for $n = 1$ and 2, respectively. Generalizations of the n-step MIR inequalities have been shown to be facet-defining for several generalizations of the mixed integer knapsack set [3, 6, 7, 17, 25]. Inequalities derived based on the n-step MIR theory and its generalizations have been previously proven to be effective cuts for the single node capacitated network design problem [15] and the multi-module lot-sizing (MMLS) problem [5, 6, 25].

Our contributions in this paper are as follows:

- We propose a new family of valid inequalities for MMND, referred to as the *n-step cutset inequalities*, for any integer $n \leq M$. The n-step cutset inequality uses the information of n capacity modules. As a result, our inequalities can use the information of any desired number of capacity modules, and particularly, all the capacity modules when $n = M$. For the directed MMND, we show that the cut-set inequality in [9] and the multifacility cut-set inequality in [2, 23] are special cases of the n-step cutset inequality.

- We introduce the *n-step flow cutset inequality* and the *n-step capacity cutset inequality* as special cases of the n-step cutset inequality. We show that the n-step cutset inequality, the n-step flow cutset inequality, and the n-step capacity cutset inequality are facet-defining under certain conditions for the directed cutset polyhedron. Based on a result in [23], these inequalities are also facet-defining for the convex hull of the directed MMND.

- We design a cutting plane algorithm to add n-step cutset inequalities to MMND problem instances. Each iteration of the cutting plane algorithm calls a polynomial time separation heuristic to progressively generate a violated cut if any, followed by a cut selection procedure.

- We show that the n-step cutset, flow cutset, and capacity cutset inequalities, applied using our cutting plane algorithm and separation heuristic, are computationally very effective in solving directed MMND instances. For our 2-module problem instances, the average total solution time (including cut generation) with 2-step cutset cuts added was 0.35 times that of CPLEX 12.7 in its default setting. This time was also 0.59 times the solution time when only 1-step cutset cuts (which only use the information of a single capacity module) were added. With the 2-step cutset cuts, the number of branch-and-bound nodes was 0.23 times the number of nodes with default CPLEX and 0.38 times that with only 1-step cutset cuts. For the 3-module problem instances, the average total solution time with 3-step cutset cuts added was 0.45 times that of CPLEX in its default setting, 0.45 times the solution time when only 1-step cutset cuts were added, and 0.56 times the solution time when only 2-step cutset cuts were added. With the 3-step cutset cuts, the number of branch-and-bound nodes was 0.32 times the number of nodes with default CPLEX, 0.42 times that with only 1-step cutset cuts, and 0.55 times that with only 2-step cutset cuts.

- We generalize the n-step cutset inequalities for the undirected MMND and the bidirected MMND problems. The generalized n-step cutset inequalities can be shown to be facet-defining for the convex hulls of the undirected and the bidirected MMND, as well as their respective cutset polyhedra. We show that the cutset inequality in [19] and the flow-cut-set inequality in [8] are special cases of the n-step cutset inequalities for the undirected MMND and the bidirected MMND, respectively.

- We discuss the generalization the n-step cutset inequalities for all types of MMND with more than one commodities.

The rest of this paper is organized as follows: in Section 2 we provide a brief background on the cutset inequalities as well as the n-step MIR inequalities as needed for this paper. In Section 3 we introduce the n-step cutset inequality and prove its validity for MMND problems. In Section 4 we prove these inequalities are facet-defining under certain conditions, and in Section 5, we present our computational experiments showing their effectiveness in solving MMND problems. In Section 6 we present generalizations of the n-step cutset inequalities to the undirected and bidirected MMND. In Section 7 we discuss generalizations of the n-step cutset inequalities to MMND problems in multi-commodity scenarios. We conclude in Section 8.

2 Necessary background

In this section, we first briefly review the cutset inequalities previously discussed in the literature, and then follow by a review of the n-step MIR theory to the extent that is required in this paper.

2.1 Cutset inequalities

Let X^d, X^u, and X^b be the convex hulls of the set of feasible solutions to the directed, undirected, and bidirected MMND, respectively, i.e.,

$$\begin{aligned} X^d &:= conv\{(x,y) : (x,y) \text{ satisfies } (2), (3), \text{and } (4)\}, \\ X^u &:= conv\{(x,y) : (x,y) \text{ satisfies } (6), (7), \text{and } (8)\}, \\ X^b &:= conv\{(x,y) : (x,y) \text{ satisfies } (6), (8), \text{and } (9)\}. \end{aligned}$$

As mentioned in Section 1, cutset inequalities for the directed, undirected, and bidirected MMND are in fact valid inequalities for certain relaxations of X^d, X^u, and X^b, respectively, which are referred to as the cutset polyhedron for the respective problem [2]. For the directed MMND, the cutset polyhedron is defined as follows. Let $U \subset V$ be a non-empty strict subset of V and $\overline{U} := V \setminus U$. Also, let $A_U^+ := \delta(U, \overline{U})$, $A_U^- := \delta(\overline{U}, U)$, and $A_U := A_U^+ \cup A_U^-$.

Let $d_U = \sum_{v \in U} d_v$. Without loss of generality, we assume $d_U \geq 0$, because if $d_U < 0$, then from $\sum_{v \in V} d_v = 0$, we have $\sum_{v \in \overline{U}} d_v > 0$, in which case we can switch U and $\overline{U}$. The cutset polyhedron corresponding to the partition $(U, \overline{U})$ for the directed MMND is defined as

$$P_U^d := conv\Big\{(x,y) \in \mathbb{R}_+^{|A_U|} \times \mathbb{Z}_+^{M|A_U|} : \tag{10}$$

$$\sum_{a \in A_U^+} x_a - \sum_{a \in A_U^-} x_a = d_U, \tag{11}$$

$$x_a \leq \sum_{t=1}^{M} C_t y_t^a + g^a, a \in A_U \Big\}, \tag{12}$$

where (11) is obtained by aggregating (2) over $v \in U$. Notice that P_U^d is a relaxation of X^d. Therefore any valid inequality for P_U^d is also valid for X^d.

For the undirected MMND, the cutset polyhedron can be similarly defined for a partition $(U, \overline{U})$ of V. Let E_U be the set of edges crossing the partition. Also, recall that for the undirected case, a pair of anti-parallel (directed) arcs e^+ and e^- are introduced for each edge $e \in E$, and $A := \{e^+ = ij, e^- = ji : e = (i,j) \in E\}$. A_U^+, A_U^-, A_U, and d_U are defined similarly to the directed MMND case. The cutset polyhedron for the undirected MMND is defined as

$$P_U^u := conv\Big\{(x,y) \in \mathbb{R}_+^{|A_U|} \times \mathbb{Z}_+^{M|E_U|} : \tag{13}$$

$$\sum_{a \in A_U^+} x_a - \sum_{a \in A_U^-} x_a = d_U, \tag{14}$$

$$x_{e^+} + x_{e^-} \leq \sum_{t=1}^{M} C_t y_t^e + g^e, e \in E_U \Big\}. \tag{15}$$

Naturally, for the bidirected MMND, the cutset polyhedron P_U^b is the same as P_U^u except that (15) is replaced with

$$\max\{x_{e^+}, x_{e^-}\} \leq \sum_{t=1}^{M} C_t y_t^e + g^e, e \in E_U. \tag{16}$$

Magnanti and Mirchandani [19] considered the undirected MMND problem having a single source $v_s \in V$ with supply $d_{v_s} < 0$, a single sink $v_t \in V$ with demand $d_{v_t} = -d_{v_s}$, and $d_v = 0$ for $v \in V \setminus \{v_s, v_t\}$, for $M \leq 3$ divisible capacity modules (this problem was referred to as the network loading problem in [19]). No pre-installed capacities were assumed. For $M = 3$, the capacity modules were assumed to be $(C_1, C_2, C_3) = (\lambda C, C, 1)$, where C and λ are positive integers.

Defining $p_0 := d_U - C\lfloor d_U/C \rfloor$ and $p_1 := (d_U - \lambda C \lfloor d_U/\lambda C \rfloor - p_0)/C$, the following inequalities were shown to be valid and facet-defining for the convex hull of this particular form of 3-module undirected MMND:

$$\sum_{e \in E_U} \left(y_3^e + p_0 y_2^e + \lambda p_0 y_1^e \right) \geq p_0 \left\lceil \frac{d_U}{C} \right\rceil, \tag{17}$$

$$\sum_{e \in E_U} \left(y_3^e + \min(p_1 C + p_0, C) y_2^e + (p_1 C + p_0) y_1^e \right) \geq (p_1 C + p_0) \left\lceil \frac{d_U}{\lambda C} \right\rceil, \tag{18}$$

$$\sum_{e \in E_U} \left(y_3^e + p_0 y_2^e + p_0 (p_1 + 1) y_1^e \right) \geq p_0 (p_1 + 1) \left\lceil \frac{d_U}{\lambda C} \right\rceil. \tag{19}$$

These inequalities were referred to as the cutset inequalities in [19]. We refer to (17) and (18) as 1-step capacity cutset inequalities and (19) as the 2-step capacity cutset inequality in this paper. It was shown in [19] that for certain cost vectors, adding these inequalities to the linear programming relaxation of the network loading problem yields integer optimal solutions.

Bienstock and Günlük [8] studied the bidirected MMND problem assuming $M = 2$ and $(C_1, C_2) = (C, 1)$. Given a partition $(U, \overline{U})$ of V, let $E_\subset$ be a subset of E_U. Each edge $e \in E_\subset$ can be represented by its two antiparallel arcs e^+ and e^-. Let $A_\subset$ be the set of all such arcs, i.e., $A_\subset := \{e^+ = ij, e^- = ji : e = (i,j) \in E_\subset\}$. Let $A_\subset^+ \subseteq A_\subset$ be the set of arcs that have tails in U and heads in $\overline{U}$ and $A_\subset^- := A_\subset \setminus A_\subset^+$. Defining $d_U := \sum_{v \in U} d_v$ and $\overline{p_0} := d_U - \sum_{e \in E_\subset} g^e - C \left\lfloor (d_U - \sum_{e \in E_\subset} g^e)/C \right\rfloor$, they introduced the flow-cut-set inequality of the form

$$\sum_{a \in A_U^+ \setminus A_\subset^+} x_a + \sum_{e \in E_\subset} (y_2^e + \overline{p_0} y_1^e) \geq \overline{p_0} \left\lceil \frac{d_U}{C} \right\rceil. \tag{20}$$

They showed that the flow-cut-set inequalities define facets of the convex hull of the 2-module bidirected MMND under certain conditions. We refer to these inequalities as the 1-step flow cutset inequalities in this paper.

Chopra et al. [9] studied the directed MMND problem with the same single-source and single-sink assumption as that in [19]. This problem assumed $M \leq 2$ capacity modules and no pre-installed capacities. For $M = 2$, the capacity modules were assumed to be $(C_1, C_2) = (C, 1)$. They showed that the 1-module directed MMND problem is NP-hard, and the 2-module directed MMND problem is NP-hard even when the flow costs $h_a = 0$ for all $a \in A$. For a given partition $(U, \overline{U})$ of V such that the source $v_s \in \overline{U}$ and the sink $v_t \in U$, let d_U and p_0 be defined the same as those in [19]. For $A_\subset^+ \subseteq A_U^+$ and $A_\subset^- \subseteq A_U^-$, they showed the following inequality is valid for the 2-module directed MMND:

$$\sum_{a \in A_U^+ \setminus A_\subset^+} x_a - \sum_{a \in A_\subset^-} x_a + \sum_{a \in A_\subset^+} \left(y_2^a + p_0 y_1^a \right) + \sum_{a \in A_\subset^-} \left((C - p_0) y_1^a + y_2^a \right) \geq p_0 \left\lceil \frac{d_U}{C} \right\rceil. \tag{21}$$

(21) was referred to as the cut-set inequality in [9]. We refer to (21) as the 1-step cutset inequality in this paper.

Atamtürk [2] studied P_U^d directly (assuming no pre-installed capacities). The cut-set inequality (21) was generalized to the multifacility cut-set inequality for P_U^d with any fixed number of modules. For $A_\subset^+ \subseteq A_U^+$ and $A_\subset^- \subseteq A_U^-$, the multifacility cut-set inequality has the form

$$\sum_{t=1}^{M} \phi_s^+(C_t) \sum_{a \in A_\subset^+} y_t^a + \sum_{t=1}^{M} \phi_s^-(C_t) \sum_{a \in A_\subset^-} y_t^a + \sum_{a \in A_U^+ \setminus A_\subset^+} x_a - \sum_{a \in A_\subset^-} x_a \geq p_s \left\lceil \frac{d_U}{C_s} \right\rceil, \tag{22}$$

where for some $s \in \{1, \ldots, M\}$, $p_s := d_U - C_s \lfloor d_U/C_s \rfloor$, $\phi_s^+(C_t) := \min\{C_t - \lfloor C_t/C_s \rfloor (C_s - (d_U - C_s \lfloor d_U/C_s \rfloor)), \lceil C_t/C_s \rceil (d_U - C_s \lfloor d_U/C_s \rfloor)\}$, $\phi_s^-(C_t) := \min\{C_t - \lfloor C_t/C_s \rfloor (d_U - C_s \lfloor d_U/C_s \rfloor), \lceil C_t/C_s \rceil (C_s - (d_U - C_s \lfloor d_U/C_s \rfloor))\}$.

Atamtürk [2] showed that the multifacility cut-set inequalities define facets of P_U^d under certain conditions. Raack et al. [22, 23] generalized these results for the undirected and the bidirected MMND. They also provided conditions under which facet-defining inequalities of P_U^d, P_U^u, and P_U^b are also facet-defining for X^d, X^u, and X^b, respectively.

2.2 n-step MIR inequalities

In this section, we briefly review the n-step MIR inequalities. Kianfar and Fathi [16] developed the n-step MIR inequalities for the mixed integer knapsack set

$$P^{MIK} = \Big\{(z, \epsilon) \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+ : \sum_{t \in I} a_t z_t + \epsilon \geq b_0\Big\},$$

where $a_t, b_0 \in \mathbb{R}$. Given $n \in \{1, \ldots, |I|\}$, let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a sequence of real numbers such that $\alpha_1 > \alpha_2 > \ldots > \alpha_n > 0$ (the same assumption holds wherever α is defined throughout the rest of this paper). For any $r \in \mathbb{R}$, define the recursive reminders

$$r^{(k)} := r^{(k-1)} - \alpha_k \left\lfloor \frac{r^{(k-1)}}{\alpha_k} \right\rfloor, k = 1, \ldots, n, \tag{23}$$

where $r^{(0)} := r$. Define $\sum_k^l (.) := 0$ and $\prod_k^l (.) := 1$ if $k > l$ for any $k, l \in \mathbb{Z}$. The n-step MIR inequality for P^{MIK} is

$$\sum_{t \in I} \mu^n_{\alpha, b_0}(a_t) z_t + \epsilon \geq \mu^n_{\alpha, b_0}(b_0), \tag{24}$$

where for any $r \in \mathbb{R}$, the n-step MIR function is defined as

$$\mu^n_{\alpha, b_0}(r) = \begin{cases} {b_0}^{(n)} \sum\limits_{k=1}^{m} \prod\limits_{l=k+1}^{n} \left\lceil \frac{{b_0}^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{r^{(k-1)}}{\alpha_k} \right\rfloor + {b_0}^{(n)} \prod\limits_{l=m+2}^{n} \left\lceil \frac{{b_0}^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{r^{(m)}}{\alpha_{m+1}} \right\rceil \\ \qquad\qquad\qquad \text{if } r \in \mathcal{L}^n_m; m = 0, 1, \ldots, n-1, \\ {b_0}^{(n)} \sum\limits_{k=1}^{n} \prod\limits_{l=k+1}^{n} \left\lceil \frac{{b_0}^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{r^{(k-1)}}{\alpha_k} \right\rfloor + r^{(n)} \text{ if } r \in \mathcal{L}^n_n, \end{cases} \tag{25}$$

where $\mathbb{R}$ is partitioned by

$$\begin{aligned} &\mathcal{L}^n_m = \{r \in \mathbb{R} : r^{(k)} < {b_0}^{(k)}, k = 1, \ldots, m, r^{(m+1)} \geq {b_0}^{(m+1)}\} \text{ for } m = 0, \ldots, n-1; \\ &\mathcal{L}^n_n = \{r \in \mathbb{R} : r^{(k)} < {b_0}^{(k)}, k = 1, \ldots, n\}. \end{aligned} \tag{26}$$

Kianfar and Fathi [16] showed that (24) is valid for P^{MIK} if the *n-step MIR conditions*

$$\alpha_k \left\lceil \frac{{b_0}^{(k-1)}}{\alpha_k} \right\rceil \leq \alpha_{k-1}, k = 2, \ldots, n \tag{27}$$

hold, and is facet-defining for the convex hull of P^{MIK} under several additional conditions (see Corollary 1 of [3] and Theorem 10 of [15]).

3 n-step cutset inequalities

In this section, we introduce a new family of valid inequalities for X^d and P_U^d. We refer to them as the *n-step cutset inequalities*.

Theorem 1 *Let $(U, \overline{U})$ be a partition of V and P_U^d be the corresponding cutset polyhedron. For $A_\subset^+ \subseteq A_U^+$ and $A_\subset^- \subseteq A_U^-$, define $D := d_U - \sum_{a \in A_\subset^+} g^a + \sum_{a \in A_\subset^-} g^a$. Given $n \in \{1, \ldots, M\}$ and $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$,* *if the n-step MIR conditions* (27) *are satisfied, i.e.,* $\alpha_k \left\lceil D^{(k-1)}/\alpha_k \right\rceil \leq \alpha_{k-1}, k = 2, \ldots, n$*, the n-step cutset inequality*

$$\sum_{t=1}^{M} \mu^n_{\alpha,D}(C_t) \sum_{a \in A^+_{\subset}} y^a_t + \sum_{t=1}^{M} \left(C_t + \mu^n_{\alpha,D}(-C_t) \right) \sum_{a \in A^-_{\subset}} y^a_t + \sum_{a \in A^+_U \setminus A^+_{\subset}} x_a - \sum_{a \in A^-_{\subset}} x_a \geq \mu^n_{\alpha,D}(D) - \sum_{a \in A^-_{\subset}} g^a \quad (28)$$

is valid for P^d_U.

Proof. Rewrite the flow conservation constraint of P^d_U (11) as

$$\sum_{a \in A^+_{\subset}} x_a + \sum_{a \in A^+_U \setminus A^+_{\subset}} x_a - \sum_{a \in A^-_{\subset}} x_a - \sum_{a \in A^-_U \setminus A^-_{\subset}} x_a = d_U.$$

Relaxing x_a, $a \in A^+_{\subset}$ using the capacity constraints (12) and x_a, $a \in A^-_U \setminus A^-_{\subset}$ using the nonnegativity constraints $x_a \geq 0$, we have

$$\sum_{t=1}^{M} C_t \sum_{a \in A^+_{\subset}} y^a_t + \sum_{a \in A^+_U \setminus A^+_{\subset}} x_a - \sum_{a \in A^-_{\subset}} x_a \geq d_U - \sum_{a \in A^+_{\subset}} g^a. \quad (29)$$

Adding and subtracting the capacity constraints (12) for $a \in A^-_{\subset}$, (29) can be rewritten as

$$\sum_{t=1}^{M} C_t \sum_{a \in A^+_{\subset}} y^a_t + \sum_{t=1}^{M} (-C_t) \sum_{a \in A^-_{\subset}} y^a_t + \sum_{a \in A^+_U \setminus A^+_{\subset}} x_a + \sum_{a \in A^-_{\subset}} (\sum_{t=1}^{M} C_t y^a_t + g^a - x_a) \geq D. \quad (30)$$

Now the expression $\sum_{a \in A^+_U \setminus A^+_{\subset}} x_a + \sum_{a \in A^-_{\subset}} (\sum_{t=1}^{M} C_t y^a_t + g^a - x_a)$ is non-negative and can be treated as ϵ in P^{MIK}. Also, D can be treated as b_0 in P^{MIK}. Let $I^+ := \{1, \ldots, M\}$. For $t \in I^+$, the expression $\sum_{a \in A^+_{\subset}} y^a_t$ is a non-negative integer and can be treated as z_t in P^{MIK}, and C_t can be treated as a_t in P^{MIK}. Similarly, let $I^- := \{1, \ldots, M\}$, then for $t \in I^-$, the expression $\sum_{a \in A^-_{\subset}} y^a_t$ can be treated as z_t in P^{MIK}, and $(-C_t)$ can be treated as a_t in P^{MIK}. Let $I = I^+ \cup I^-$, then (30) can be rewritten as the defining inequality of P^{MIK}. Since by assumption the n-step MIR conditions hold, by applying the n-step MIR inequality and replace z_t, a_t, ϵ, and b_0 with their aforementioned corresponding expressions, the resulting inequality is valid for P^d_U. By reorganizing terms the resulting inequality is (28). □

Remark 1 *If* $\alpha_1, \ldots, \alpha_n$ *are divisible, the n-step MIR conditions* (27) *are automatically satisfied regardless of the value of* D.

Remark 2 *The n-step cutset inequality is also valid for a variant of* P^d_U *with variable capacities, where* (12) *is replaced by the constraints* $x_a \leq C_a y^a + g^a, a = 1, \ldots, M$. *This can be shown by rewriting these constraints as* $x_a \leq \sum_{t \in A_U} C_t y^a_t + g^a$, $y^a_t = 0$ *for* $t \neq a$*, and thus constructing an instance of* P^d_U *with additional constraints. A special case of such variant with* $A^-_U = \emptyset$ *is discussed in [3, 18].*

Special cases We illustrate several special cases of the n-step cutset inequalities (28) by setting different values of n, α, $A^+_{\subset}$, and $A^-_{\subset}$.

- **Cut-set inequality.** The cut-set inequality (21) in [9] is obtained by setting $n = 1, \alpha = C_1$ in (28).
- **Multifacility cut-set inequality.** For the multifacility cut-set inequality (22) in [2], we note that although the functions $\phi^+_s(\cdot)$ and $\phi^+_s(\cdot)$ depend on the values of all the capacity modules $C_1, \ldots, C_M$, they can be derived using 1-step MIR, a single-parameter theory. Given $s \in \{1, \ldots, M\}$, this inequality can be obtained by setting $n = 1$ and $\alpha = C_s$ in (28). To see this, we substitute $n = 1$ and $\alpha = C_s$ into the n-step MIR function (25), and we have that for any $r \in \mathbb{R}_+$,

$$\mu^1_{C_s,D}(r) = \begin{cases} D^{(1)} \left\lceil \frac{r}{C_s} \right\rceil = \left\lceil \frac{r}{C_s} \right\rceil \left(D - C_s \left\lfloor \frac{D}{C_s} \right\rfloor \right) & \text{if } r^{(1)} \geq D^{(1)}, \\ D^{(1)} \left\lfloor \frac{r}{C_s} \right\rfloor + r^{(1)} = r - \left\lfloor \frac{r}{C_s} \right\rfloor \left(C_s - (D - C_s \left\lfloor \frac{D}{C_s} \right\rfloor) \right) & \text{if } r^{(1)} < D^{(1)}, \end{cases}$$

and

$$\mu^1_{C_s,D}(-r) = \begin{cases} D^{(1)}\left\lceil \frac{-r}{C_s}\right\rceil = -\left(D - C_s\left\lfloor \frac{D}{C_s}\right\rfloor\right)\left\lfloor \frac{r}{C_s}\right\rfloor & \text{if } (-r)^{(1)} \geq D^{(1)}, \\ D^{(1)}\left\lfloor \frac{-r}{C_s}\right\rfloor + (-r)^{(1)} = C_s\left\lceil \frac{r}{C_s}\right\rceil - r - \left(D - C_s\left\lfloor \frac{D}{C_s}\right\rfloor\right)\left\lceil \frac{r}{C_s}\right\rceil & \text{if } (-r)^{(1)} < D^{(1)}. \end{cases}$$

Substituting the above 1-step MIR function values into (28) results in the multifacility cut-set inequality (22).

- ***n*-Step flow cutset inequality.** By setting $A^-_{\subset} = \emptyset$ in (28), we get

$$\sum_{t=1}^{M} \mu^n_{\alpha,D}(C_t) \sum_{a\in A^+_{\subset}} y^a_t + \sum_{a\in A^+_U\setminus A^+_{\subset}} x_a \geq \mu^n_{\alpha,D}(D). \tag{31}$$

  We refer to (31) as the *n-step flow cutset* inequality. A similar inequality can be obtained for the bidirected MMND, of which the flow-cut-set inequality (20) in [8] is a special case (see Section 6).

- ***n*-Step capacity cutset inequality.** By setting $A^+_{\subset} = A^+_U$ and $A^-_{\subset} = \emptyset$ in (28), we get

$$\sum_{t=1}^{M} \mu^n_{\alpha,D}(C_t) \sum_{a\in A^+_U} y^a_t \geq \mu^n_{\alpha,D}(D). \tag{32}$$

  We refer to (32) as the *n-step capacity cutset* inequality. A similar inequality can be obtained for the undirected MMND, of which the cutset inequalities (17), (18) and (19) in [19] are special cases (see Section 6).

Example 1 *Consider a directed cutset polyhedron with 3 capacity modules, where*

$$P^d_U = conv\Big\{(x,y)\in\mathbb{R}^4_+\times\mathbb{Z}^{12}_+ : \\ x_1 + x_2 + x_3 - x_4 = 50, \\ x_a \leq 27y^a_1 + 13y^a_2 + 6y^a_3, a = 1,\ldots,4.\Big\}$$

In this example, we have $(C_1,C_2,C_3) = (27,13,6)$, $A^+_U = \{1,2\}$, $A^-_U = \{3,4\}$, $d_U = 50$, and $g^a = 0$ for all $a \in A_U$. Let $A^+_{\subset} = \{2\}, A^+_U \setminus A^+_{\subset} = \{1\}, A^-_{\subset} = \{3\}$, and $A^-_U \setminus A^-_{\subset} = \{4\}$. We illustrate the n-step cutset inequalities and their special cases for 3 combinations of n and α: the 1-step cutset inequalities where $n = 1, \alpha = C_1$, the 2-step cutset inequalities where $n = 2, \alpha = \{C_1, C_2\}$, and the 3-step cutset inequalities where $n = 3, \alpha = \{C_1, C_2, C_3\}$. We present these inequalities by listing a sequence of points in P^d_U with fractional values for y. Each point satisfies all preceding inequalities and is followed by an inequality that cuts off this fractional point. Only non-zero elements are mentioned for these points.
$p^0 : x_2 = 50, y^2_1 = 50/27$. *The **1-step capacity cutset inequality***

$$23(y^1_1 + y^2_1) + 13(y^1_2 + y^2_2) + 6(y^1_3 + y^2_3) \geq 46$$

cuts off p^0 *by* 92/27.
$p^1 : x_2 = 50, y^2_1 = 1, y^2_2 = 23/13$. *The **2-step capacity cutset inequality***

$$20(y^1_1 + y^2_1) + 10(y^1_2 + y^2_2) + 6(y^1_3 + y^2_3) \geq 40$$

cuts off p^1 *by* 30/13.
$p^2 : x_2 = 50, y^2_1 = 1, y^2_2 = 1, y^2_3 = 10/6$. *The **3-step capacity cutset inequality***

$$16(y^1_1 + y^2_1) + 8(y^1_2 + y^2_2) + 4(y^1_3 + y^2_3) \geq 32$$

cuts off p^2 *by* 8/6.
$p^3 : x_2 = 30, y_1^2 = 30/27, x_1 = 20, y_1^1 = 24/27$. *The* ***1-step flow cutset inequality***

$$23y_1^2 + 13y_2^2 + 6y_3^2 + x_1 \geq 46$$

cuts off p^3 *by* 12/27.
$p^4 : x_2 = 41, y_1^2 = 30/27, y_2^2 = 14/13, x_1 = 9, y_1^1 = 6/13$. *The* ***2-step flow cutset inequality***

$$20y_1^2 + 10y_2^2 + 6y_3^2 + x_1 \geq 40$$

cuts off p^4 *by* 3/13.
$p^5 : x_2 = 47, y_1^2 = 1, y_2^2 = 1, y_3^2 = 7/6, x_1 = 3, y_1^1 = 5/24$. *The* ***3-step flow cutset inequality***

$$16y_1^2 + 8y_2^2 + 4y_3^2 + x_1 \geq 32$$

cuts off p^5 *by* 1/3.
$p^6 : x_2 = 54, y_1^2 = 2, x_3 = 4, y_1^3 = 4/27$. *The* ***1-step cutset inequality***

$$23y_1^2 + 13y_2^2 + 6y_3^2 + 4y_1^3 + 4y_2^3 + 4y_3^3 + x_1 - x_3 \geq 46$$

cuts off p^6 *by* 92/27.
$p^7 : x_2 = 67, y_1^2 = 2, y_2^2 = 1, x_3 = 17, y_2^3 = 17/13$. *The* ***2-step cutset inequality***

$$20y_1^2 + 10y_2^2 + 6y_3^2 + 7y_1^3 + 4y_2^3 + 4y_3^3 + x_1 - x_3 \geq 40$$

cuts off p^7 *by* 23/13.
$p^8 : x_2 = 56, y_1^2 = 2, y_3^2 = 2/6, x_3 = 6, y_3^3 = 1$. *The* ***3-step cutset inequality***

$$16y_1^2 + 8y_2^2 + 4y_3^2 + 11y_1^3 + 6y_2^3 + 4y_3^3 + x_1 - x_3 \geq 32$$

cuts off p^8 *by* 2/3.

4 Facet-defining n-step cutset inequalities

In this section, we study the facet-defining properties of the n-step cutset inequalities. Specifically, we give sufficient conditions under which the n-step cutset inequality (28), the n-step flow cutset inequality (31), and the n-step capacity cutset inequality (32) are facet-defining for P_U^d as well as X^d. Let $(U, \overline{U})$ be a partition of V and P_U^d be the corresponding cutset polyhedron. Given $A_{\subset}^+ \subseteq A_U^+$, $A_{\subset}^- \subseteq A_U^-$, let $D := d_U - \sum_{a \in A_{\subset}^+} g^a + \sum_{a \in A_{\subset}^-} g^a$. In order to prove the results, we define the following points and directions. Notice that for all directions and points we illustrate below, only nonzero values are mentioned.

Definition 1 *Let* i, j, ξ, ω *be indices of arcs. Define the following points :*

(a) For any $i \in A_{\subset}^+, j \in A_U^+ \setminus A_{\subset}^+$, *the points* $\mathcal{A}_l^{i,j}, l = 1, \dots, n$:

$$y_t^i = \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor, t = 1, \dots, l, x_i = \sum_{t=1}^{l} \alpha_t \left\lfloor \frac{d^{(t-1)}}{\alpha_t} \right\rfloor + g^i,$$
$$x_{i'} = g^{i'}, i' \in A_{\subset}^+ \setminus \{i\} \cup A_{\subset}^-, y_l^j = 1, x_j = D^{(l)},$$

and the points $\mathcal{A}_l^{i,j}, l = n+1, \dots, M$:

$$y_t^i = \begin{cases} \left\lfloor \frac{d^{(t-1)}}{\alpha_t} \right\rfloor, & t = 1, \dots, n, \\ 1, & t = l, \end{cases} \quad x_i = \sum_{t=1}^{n} \alpha_t \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + \min\{C_l, D^{(n)}\} + g^i,$$
$$x_{i'} = g^{i'}, i' \in A_{\subset}^+ \setminus \{i\} \cup A_{\subset}^-, y_n^j = 1, x_j = \max\{0, D^{(n)} - C_l\}.$$

(b) For any $i \in A_{\subset}^{+}$, the points $\mathcal{B}_l^i, l = 1, \dots, n$:

$$y_t^i = \begin{cases} \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor, \ t = 1, \dots, l-1, \\ \left\lceil \frac{D^{(t-1)}}{\alpha_t} \right\rceil, \ t = l, \end{cases} \quad x_i = D + g^i, x_{i'} = g^{i'}, i' \in A_{\subset}^{+} \setminus \{i\} \cup A_{\subset}^{-},$$

and the points $\mathcal{B}_l^i, l = n+1, \dots, M$:

$$y_t^i = \begin{cases} \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor, & t = 1, \dots, n, \\ 1, & t = l, \end{cases} \quad x_i = D + g^i, x_{i'} = g^{i'}, i' \in A_{\subset}^{+} \setminus \{i\} \cup A_{\subset}^{-}.$$

(c) For any $i \in A_{\subset}^{+}, \xi \in A_{\subset}^{-}, \omega \in A_U^{-} \setminus A_{\subset}^{-}$, the points $\mathcal{C}_l^{i,\xi,\omega}, l = 2, \dots, n$:

$$y_t^i = \begin{cases} \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + 1, \ t = 1, \\ \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor, \ t = 2, \dots, n+1-l, \\ \left\lceil \frac{D^{(t-1)}}{\alpha_t} \right\rceil - 1, \ t = n+2-l, \end{cases} \quad x_i = \sum_{t=1}^{n+2-l} \alpha_t \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + \alpha_1 + g^i,$$

$$x_{i'} = g^{i'}, i' \in A_{\subset}^{+} \setminus \{i\}, y_l^{\xi} = 1, x_{\xi} = \alpha_l + g^{\xi}, x_{\xi'} = g^{\xi'}, \xi' \in A_{\subset}^{-} \setminus \{\xi\},$$
$$y_{n+2-l}^{\omega} = 1, x_{\omega} = \alpha_1 - \alpha_l - D^{(n+2-l)}.$$

(d) For any $i \in A_{\subset}^{+}, \omega \in A_U^{-} \setminus A_{\subset}^{-}$, the point $\mathcal{F}^{i,\omega}$:

$$y_1^i = \left\lceil \frac{D}{\alpha_1} \right\rceil, x_i = \alpha_1 \left\lceil \frac{D}{\alpha_1} \right\rceil + g^i, x_{i'} = g^{i'}, i' \in A_{\subset}^{+} \setminus \{i\},$$
$$y_1^{\omega} = 1, x_{\omega} = \alpha_1 \left\lceil \frac{D}{\alpha_1} \right\rceil - D, x_{\xi} = g^{\xi}, \xi \in A_{\subset}^{-}.$$

(e) For any $i \in A_{\subset}^{+}, j \in A_U^{+} \setminus A_{\subset}^{+}, \xi \in A_{\subset}^{-}$, the point $\mathcal{G}^{i,j,\xi}$:

$$y_t^i = \begin{cases} \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + 1, \ t = 1, \\ \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor, \ t = 2, \dots, n, \end{cases} \quad x_i = \sum_{t=1}^{n} \alpha_t \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + \alpha_1 + g^i,$$
$$x_{i'} = g^{i'}, i' \in A_{\subset}^{+} \setminus \{i\}, y_n^j = 1, x_j = D^{(n)}, y_1^{\xi} = 1, x_{\xi} = \alpha_1 + g^{\xi},$$
$$x_{\xi'} = g^{\xi'}, \xi' \in A_{\subset}^{-} \setminus \{\xi\}.$$

(f) For any $i \in A_{\subset}^{+}, \xi \in A_{\subset}^{-}$, the point $\mathcal{H}^{i,\xi}$:

$$y_t^i = \begin{cases} \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + 1, & t = 1, \\ \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor, & t = 2, \dots, n, \end{cases} \quad x_i = \sum_{t=1}^{n} \alpha_t \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + \alpha_1 + g^i,$$
$$x_{i'} = g^{i'}, i' \in A_{\subset}^{+} \setminus \{i\}, y_1^{\xi} = 1, x_{\xi} = \alpha_1 - D^{(n)} + g^{\xi},$$
$$x_{\xi'} = g^{\xi'}, \xi' \in A_{\subset}^{-} \setminus \{\xi\}.$$

(g) For $i \in A_U, t = 1, \dots, n$, the direction $\mathcal{E}_t^i$ where $y_t^i = 1$.

Next, we provide several intermediate results that will be used to prove the main result of this section.

Proposition 1 *Given $n \in \{1, \dots, M\}$ and $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, for any $r \in \mathbb{R}$,*

(a) $r = \sum_{k=1}^{t} \alpha_k \left\lfloor \frac{r^{(k-1)}}{\alpha_k} \right\rfloor + r^{(t)}$, $t = 1, \dots, n$,

(b) $r \leq \sum_{k=1}^{t} \alpha_k \left\lfloor \frac{r^{(k-1)}}{\alpha_k} \right\rfloor + \alpha_{t+1} \left\lceil \frac{r^{(t)}}{\alpha_{t+1}} \right\rceil, t = 1, \ldots, n-1.$

Lemma 1 *Let $\sum_{t=1}^{0}(\cdot) := 0$ and $\prod_{t=1}^{0}(\cdot) := 1$. Given $n \in \{1, \ldots, M\}$ and $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, for any $r \in \mathbb{R}$,*

(a) $\sum_{t=1}^{l-1} \prod_{k=t+1}^{n} \left\lceil \frac{r^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{r^{(t-1)}}{\alpha_t} \right\rfloor + \prod_{k=l}^{n} \left\lceil \frac{r^{(k-1)}}{\alpha_k} \right\rceil = \prod_{k=1}^{n} \left\lceil \frac{r^{(k-1)}}{\alpha_k} \right\rceil, l = 1, \ldots, n,$

(b) $\sum_{t=1}^{n} \prod_{k=t+1}^{n} \left\lceil \frac{r^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{r^{(t-1)}}{\alpha_t} \right\rfloor = \prod_{k=1}^{n} \left\lceil \frac{r^{(k-1)}}{\alpha_k} \right\rceil - 1.$

Proof. See Lemma 6 of [25]. □

Lemma 2 *Given $n \in \{1, \ldots, M\}$, $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, and $D \in \mathbb{R}$ such that $D^{(n)} > 0$, for any $r \in \mathbb{R}$ such that $0 < r \leq \alpha_1$,*

(a) If $D^{(s)} \leq r \leq \alpha_t, t \in \{1, \ldots, n\}$, then $\mu_{\alpha,D}^{n}(r) = D^{(n)} \prod_{k=t+1}^{n} \lceil D^{(k-1)}/\alpha_k \rceil$.

(b) If $0 < r \leq D^{(n)}$, then $\mu_{\alpha,D}^{n}(r) = r$.

(c) If $r = \alpha_1$, then $\mu_{\alpha,D}^{n}(-r) = -D^{(n)} \prod_{k=2}^{n} \lceil D^{(k-1)}/\alpha_k \rceil$.

(d) If $D^{(s)} \leq \alpha_1 - r \leq \alpha_t$ for some $t \in \{2, \ldots, n\}$, then $\mu_{\alpha,D}^{n}(-r) =$
$D^{(n)} \prod_{k=t+1}^{n} \lceil D^{(k-1)}/\alpha_k \rceil - D^{(n)} \prod_{k=2}^{n} \lceil D^{(k-1)}/\alpha_k \rceil$.

Proof. See Appendix. □

Now we are ready to present the main results of this section.

Theorem 2 *Let $(U, \overline{U})$ be a partition of V and P_U^d be the corresponding cutset polyhedron. For $A_{\subset}^{+} \subseteq A_U^{+}$ and $A_{\subset}^{-} \subseteq A_U^{-}$, let $D := d_U - \sum_{a \in A_{\subset}^{+}} g^a + \sum_{a \in A_{\subset}^{-}} g^a$. Given $n \in \{1, \ldots, M\}$ and $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, the n-step cutset inequality (28) is facet-defining for P_U^d if*

(a) $n = M, \alpha = \{C_1, \ldots, C_M\},$

(b) $D^{(n+2-t)} < \alpha_1 - \alpha_t \leq \alpha_{n+2-t}$ *for* $t = 2, \ldots, n, D^{(n)} > 0,$

(c) $\frac{D^{(t-1)}}{\alpha_t} < \left\lceil \frac{D^{(t-1)}}{\alpha_t} \right\rceil \leq \frac{\alpha_{t-1}}{\alpha_t}, t = 2, \ldots, n,$

(d) $A_{\subset}^{+} \neq \emptyset, A_U^{+} \setminus A_{\subset}^{+} \neq \emptyset, A_{\subset}^{-} \neq \emptyset, A_U^{-} \setminus A_{\subset}^{-} \neq \emptyset.$

Proof. Define $\prod_{k=n+2}^{n}(\cdot) := 0$ and $\prod_{k=n+1}^{n}(\cdot) := 1$. Consider the hyperplane corresponding to (28). By substituting values of the n-step MIR function (25) corresponding to the ones of Lemma 2 under conditions (a) and (b), the hyperplane can be rewritten as

$$\sum_{t=1}^{n} D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A_{\subset}^{+}} y_t^a + \sum_{t=1}^{n} \left(\alpha_t + D^{(n)} \prod_{k=n+3-t}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil - D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \right) \sum_{a \in A_{\subset}^{-}} y_t^a$$
$$+ \sum_{a \in A_U^{+} \setminus A_{\subset}^{+}} x_a - \sum_{a \in A_{\subset}^{-}} x_a = D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil - \sum_{a \in A_{\subset}^{-}} g^a. \tag{33}$$

Let

$$\sum_{t=1}^{n}\sum_{a\in A_U}\beta_t^a y_t^a+\sum_{a\in A_U}\pi_a x_a=\theta \tag{34}$$

be a hyperplane passing through the face of P_U^d defined by (33). We prove (34) must be a scalar multiple of (33) plus the flow balance equality (11).

For any $i\in(A_U^+\setminus A_\subset^+)\cup(A_U^-\setminus A_\subset^-)$, $t=1,\dots,n$, consider the direction $\mathcal{E}_t^i$. $\mathcal{E}_t^i$ is an unbounded direction for both P_U^d and (33), and hence a direction for the face defined by (34). This implies that $\beta_t^i=0$ for all $i\in(A_U^+\setminus A_\subset^+)\cup(A_U^-\setminus A_\subset^-)$, $t=1,\dots,n$.

Now, for any $i\in A_\subset^+$, $l=1,\dots,n$, and $\omega\in A_U^-\setminus A_\subset^-$, consider the points $\mathcal{B}_l^i$ and $\mathcal{F}^{i,\omega}$. It is easy to check that $\mathcal{B}_l^i$ and $\mathcal{F}^{i,\omega}$ are in P_U^d by Proposition 1, and by (a) of Lemma 1, $\mathcal{B}_l^i$ and $\mathcal{F}^{i,\omega}$ satisfy (33). Then $\mathcal{B}_l^i$ and $\mathcal{F}^{i,\omega}$ must satisfy (34). Now, for any $i\in A_\subset^+$ and $\omega\in A_U^-\setminus A_\subset^-$, by substituting $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$ into (34) and subtracting one equality from the other, we have $\pi_i(\alpha_1\lceil D/\alpha_1\rceil-D)+\pi_\omega(\alpha_1\lceil D/\alpha_1\rceil-D)=0$, which implies that $\pi_i=-\pi_\omega$ for $i\in A_\subset^+$ and $\omega\in A_U^-\setminus A_\subset^-$. Now, since all points of P_U^d satisfy flow balance equality (11), we may add multiples of the flow balance equality to facet-defining inequalities without changing them. Therefore, without loss of generality, we assume that $\pi_\gamma=0$ for some $\gamma\in A_\subset^+$. This implies that

$$\pi_i=0, i\in A_\subset^+\cup(A_U^-\setminus A_\subset^-). \tag{35}$$

Next, for any $i\in A_\subset^+$, $j\in A_U^+\setminus A_\subset^+$, and $\xi\in A_\subset^-$, consider the points $\mathcal{G}^{i,j,\xi}$ and $\mathcal{H}^{i,\xi}$. It is easy to check that $\mathcal{G}^{i,j,\xi}$ and $\mathcal{H}^{i,\xi}$ are in P_U^d by Proposition 1, and by (b) of Lemma 1, $\mathcal{G}^{i,j,\xi}$ and $\mathcal{H}^{i,\xi}$ satisfy (33). Then $\mathcal{G}^{i,j,\xi}$ and $\mathcal{H}^{i,\xi}$ must satisfy (34). Now, for any $i\in A_\subset^+$, $j\in A_U^+\setminus A_\subset^+$, and $\xi\in A_\subset^-$, by substituting $\mathcal{G}^{i,j,\xi}$ and $\mathcal{H}^{i,\xi}$ into (34) and subtracting one equality from the other, we have $(\pi_\xi+\pi_j)D^{(n)}=0$, which implies that $\pi_\xi=-\pi_j$ for $\xi\in A_\subset^-$ and $j\in A_U^+\setminus A_\subset^+$. Thus, $\exists\tau\in A_U^+\setminus A_\subset^+$ such that

$$\pi_j=\pi_\tau, j\in A_U^+\setminus A_\subset^+, \pi_\xi=-\pi_\tau, \xi\in A_\subset^-. \tag{36}$$

Now, for any $i\in A_\subset^+$ and $j\in A_U^+\setminus A_\subset^+$, consider the point $\mathcal{A}_n^{i,j}$. It is easy to check that $\mathcal{A}_n^{i,j}$ is in P_U^d by Proposition 1, and by (b) of Lemma 1, $\mathcal{A}_n^{i,j}$ satisfies (33). Then $\mathcal{A}_n^{i,j}$ must satisfy (34). For any $i\in A_\subset^+$ and $j\in A_U^+\setminus A_\subset^+$, by substituting $\mathcal{A}_n^{i,j}$ and $\mathcal{B}_n^i$ into (34) and subtracting one equality from the other, we obtain

$$\beta_n^i=\pi_\tau D^{(n)}, i\in A_\subset^+. \tag{37}$$

Now, if we substitute $\mathcal{B}_n^i$, $\mathcal{B}_{n-1}^i,\dots$, $\mathcal{B}_1^i$ one after another into (34) and subtract one equality from another, we obtain $\beta_{l-1}^i=\beta_l^i\lceil D^{(l-1)}/\alpha_l\rceil$ for $l=n,n-1,\dots,2$, which implies

$$\beta_t^i=\pi_\tau D^{(n)}\prod_{k=t+1}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil, i\in A_\subset^+, t=1,\dots,n. \tag{38}$$

Next, for any $i\in A_\subset^+$, $j\in A_U^+\setminus A_\subset^+$, and $\xi\in A_\subset^-$, consider the points $\mathcal{A}_n^{i,j}$ and $\mathcal{H}^{i,\xi}$. Substituting $\mathcal{A}_n^{i,j}$ and $\mathcal{H}^{i,\xi}$ into (34), and subtracting one equality from the other, we have

$$\beta_1^\xi=\pi_\tau\Big(\alpha_1-D^{(n)}\prod_{k=2}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil\Big), \xi\in A_\subset^-. \tag{39}$$

Next, for any $i\in A_\subset^+$, $\xi\in A_\subset^-$, and $\omega\in A_U^-\setminus A_\subset^-$, consider the points $\mathcal{C}_l^{i,\xi,\omega}$, $l=2,\dots,n$. It is easy to check that $\mathcal{C}_l^{i,\xi,\omega}$ is in P_U^d by Proposition 1, and by (a) of Lemma 1, $\mathcal{C}_l^{i,\xi,\omega}$ satisfy (33). Then $\mathcal{C}_l^{i,\xi,\omega}$ must satisfy (34). If we substitute $\mathcal{H}^{i,\xi}$ and $\mathcal{C}_2^{i,\xi,\omega}$ into (34) and subtract one equality from the other, we have $\beta_2^\xi=\beta_1^\xi+\pi_\tau\big(\alpha_2-(\alpha_1-D^{(n)})\big)$, which implies that

$$\beta_2^\xi=\pi_\tau\Big(\alpha_2+D^{(n)}-D^{(n)}\prod_{k=2}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil\Big), \xi\in A_\subset^-. \tag{40}$$

If we substitute $\mathcal{H}^{i,\xi}$ and $\mathcal{C}_3^{i,\xi,\omega}$ into (34) and subtract one equality from the other, we have $\beta_3^{\xi} = \beta_1^{\xi} + \pi_\tau\big(D^{(n)}\left\lfloor D^{(n-1)}/\alpha_n\right\rfloor - (\alpha_1 - D^{(n)} - \alpha_3)\big)$, which implies

$$\beta_3^{\xi} = \pi_\tau\bigg(\alpha_3 + D^{(n)}\left\lceil\frac{D^{(n-1)}}{\alpha_n}\right\rceil - D^{(n)}\prod_{k=2}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil\bigg), \xi \in A_{\subset}^{-}. \tag{41}$$

By substituting $\mathcal{C}_3^{i,\xi,\omega}$, $\mathcal{C}_4^{i,\xi,\omega}$, …, $\mathcal{C}_n^{i,\xi,\omega}$ one after another into (34) and subtracting one equality from another, we have

$$\beta_{l+1}^{\xi}{=}\beta_l^{\xi} + \pi_\tau\bigg(D^{(n)}\prod_{k=n+3-l}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil\left\lfloor\frac{D^{(n+1-l)}}{\alpha_{n+2-l}}\right\rfloor - \alpha_l + \alpha_{l+1}\bigg), l = 3,\dots,n-1,$$

which implies

$$\beta_l^{\xi}{=}\pi_\tau\bigg(\alpha_l + D^{(n)}\prod_{k=n+3-l}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil - D^{(n)}\prod_{k=2}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil\bigg), i \in A_{\subset}^{-}, l = 3,\dots,n. \tag{42}$$

By (35)(36)(37)(38)(39)(40) and (42), (34) is reduced to

$$\begin{aligned}&\sum_{t=1}^{n}\pi_\tau D^{(n)}\prod_{k=t+1}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil\sum_{a\in A_{\subset}^{+}} y_t^a + \sum_{t=1}^{n}\pi_\tau\bigg(\alpha_t + D^{(n)}\prod_{k=n+3-t}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil \\ &- D^{(n)}\prod_{k=2}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil\bigg)\sum_{a\in A_{\subset}^{-}} y_t^a + \sum_{a\in A_U^{+}\setminus A_{\subset}^{+}}\pi_\tau x_a - \sum_{a\in A_{\subset}^{-}}\pi_\tau x_a = \theta.\end{aligned} \tag{43}$$

Finally, by substituting $\mathcal{B}_n^i$ for some $i \in A_{\subset}^{+}$ into (43), we have

$$\theta = \pi_\tau\bigg(D^{(n)}\prod_{k=1}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil - \sum_{a\in A_{\subset}^{-}} g^a\bigg), \tag{44}$$

which reduces (43) to

$$\begin{aligned}&\sum_{t=1}^{n}\pi_\tau D^{(n)}\prod_{k=t+1}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil\sum_{a\in A_{\subset}^{+}} y_t^a + \sum_{t=1}^{n}\pi_\tau\bigg(\alpha_t + D^{(n)}\prod_{k=n+3-t}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil \\ &-D^{(n)}\prod_{k=2}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil\bigg)\sum_{a\in A_{\subset}^{-}} y_t^a + \sum_{a\in A_U^{+}\setminus A_{\subset}^{+}}\pi_\tau x_a - \sum_{a\in A_{\subset}^{-}}\pi_\tau x_a = \pi_\tau\bigg(D^{(n)}\prod_{k=1}^{n}\left\lceil\frac{D^{(k-1)}}{\alpha_k}\right\rceil - \sum_{a\in A_{\subset}^{-}} g^a\bigg).\end{aligned} \tag{45}$$

(45) is a scalar multiple of (33) (the scalar is π_τ). This completes the proof. □

Next, we give sufficient conditions for the n-step flow cutset inequality (31) to be facet-defining for P_U^d. Note that although the n-step flow cutset inequality (31) is a special case of the n-step cutset inequality (28), their facet-defining conditions are exclusive.

Theorem 3 *Let $(U, \overline{U})$ be a partition of V and P_U^d be the corresponding cutset polyhedron. For $A_{\subset}^{+} \subseteq A_U^{+}$, let $D := d_U - \sum_{a\in A_{\subset}^{+}} g^a$. Given $n \in \{1,\dots,M\}$ and $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, the n-step flow cutset inequality (31) is facet-defining for P_U^d if*

(a) $\alpha = \{C_1,\dots,C_n\}$,

(b) $D^{(n)} > 0$,

(c) $\frac{D^{(t-1)}}{\alpha_t} < \left\lceil\frac{D^{(t-1)}}{\alpha_t}\right\rceil \le \frac{\alpha_{t-1}}{\alpha_t}, t = 2,\dots,n,$

(d) $A_{\subset}^{+} \neq \emptyset, A_U^{+} \setminus A_{\subset}^{+} \neq \emptyset$.

Proof. See Appendix. □

Finally, we give sufficient conditions for the n-step capacity cutset inequality (32) to be facet-defining for P_U^d.

Theorem 4 *Let $(U, \overline{U})$ be a partition of V and P_U^d be the corresponding cutset polyhedron. Let $D := d_U - \sum_{a \in A_U^+} g^a$. Given $n \in \{1, \dots, M\}$ and $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, the n-step capacity cutset inequality* (32) *is facet-defining for P_U^d if*

(a) $\alpha = \{C_1, \dots, C_n\}$,

(b) $0 < D^{(n)} \leq \alpha_t, t = 1, \dots, M$,

(c) $\frac{D^{(t-1)}}{\alpha_t} < \left\lceil \frac{D^{(t-1)}}{\alpha_t} \right\rceil \leq \frac{\alpha_{t-1}}{\alpha_t}, t = 2, \dots, n.$

Proof. See Appendix. □

The n-step cutset inequalities are facet-defining not only for P_U^d, but also for X^d under additional conditions. This is straightforward by the result of Raack et al. in [22]. Note that this result was presented for MMND without pre-installed capacities on arcs, i.e., $g^a = 0$ for all $a \in A$, but the same result and proof hold for MMND with pre-installed capacities.

Lemma 3 ([22]) *Let $(U, \overline{U})$ be a partition of V. For any $V' \subset V$, define $G[V'] := (V', A_{V'})$ where $A_{V'} := \{a = ij \in A : i, j \in V'\}$. Let*

$$\sum_{t=1}^{M} \sum_{a \in A} \beta_t^a y_t^a + \sum_{a \in A} \pi_a x_a = \theta$$

be a facet-defining inequality of P_U^d. Then it is also facet-defining for X^d if both $G[U]$ and $G[\overline{U}]$ are strongly connected.

By this lemma, together with Theorem 2, Theorem 3, and Theorem 4, we have the following:

Corollary 1 *The n-step cutset (resp. flow cutset, capacity cutset) inequality is facet-defining for X^d if in addition to the conditions in Theorem 2 (resp. Theorem 3, Theorem 4), $G[U]$ and $G[\overline{U}]$ are strongly connected.*

5 Computational results

In this section, we examine the effectiveness of the n-step cutset inequalities on our randomly generated directed MMND test instances. We illustrate the random graph generation procedure in Section 5.1, the separation heuristic in Section 5.2, the cutting plane algorithm to add the cuts in Section 5.3, and the setup and results of our experiments in Section 5.4. In our computations, we assume no pre-installed capacities on arcs, i.e., $g^a = 0, a \in A$.

5.1 Graph generation

Our idea of generating random directed graphs is similar to the ones in [20, 24]. Each graph has 50 nodes, whose coordinates are uniformly distributed on a 100×100 region in the Euclidean plane. We randomly choose 5 of the nodes to be the sources, and 30 of the nodes to be the sinks. The other 15 nodes are assigned to be transshipment nodes. Real-life graphs have low arc degree, and arcs with large length should be avoided [24]. Therefore, we randomly choose an out degree for each sink or transshipment node to be equal to 2, 3, 4, or 5 with a probability of 0.2, 0.3, 0.3, and 0.2. These probabilities are adjusted for source nodes to have higher out degree on average, so that sink nodes have higher chances of being reachable from the source nodes. We set the upper bound of the arc length to be 50. For each node v, we iteratively add a directed arc from v to the node closest to v if no directed arc from v is present, until the degree requirement of v is satisfied, or there are no more nodes within the range of length 50. Then, for each source node, we check if every other node of the graph can be reached from this source node. If not, we reject this graph and generate a new one. If a valid graph is generated, we then create an instance of directed MMND by assigning demand to nodes and creating capacity modules, as discussed in Section 5.4.

5.2 Separation

Given an LP relaxation optimal solution $(\hat{x}, \hat{y})$ of a directed MMND instance, the number of n-step cutset inequalities (28) is exponential with respect to choices of $A_U, A_\subset^+, A_\subset^-$, n, and α. Finding the most violated inequality with respect to $A_U, A_\subset^+, A_\subset^-$, n, and α simultaneously is an NP-hard problem even for the special case where $M = 1$ with a single source and a single sink [2] in the network. In our experiments, we set specific values for α and n in each run, and we use a simple heuristic to determine A_U (see Section 5.3). Given A_U, n, and α, finding the most violated n-step cutset inequality can be done in linear time by setting $A_\subset^+$ and $A_\subset^-$ as follows:

$$A_\subset^+ = \Big\{a \in A_U^+ : \sum_{t=1}^{M} \mu_{\alpha,D}^n(C_t)\hat{y_t^a} \le \hat{x_a}\Big\},$$

$$A_\subset^- = \Big\{a \in A_U^- : \sum_{t=1}^{M}(C_t + \mu_{\alpha,D}^n(-C_t))\hat{y_t^a} < \hat{x_a}\Big\}.$$

However, previous computational efforts on cutset inequalities [4, 8, 23] and our tests on the n-step cutset inequalities suggested the following observations:

- The n-step capacity cutset inequalities contribute the most on reducing time and integrality gap for network design problems.
- The most violated n-step cutset inequalities do not necessarily perform the best in reducing the solution time.

Following these observations in our experiments with the n-step cutset inequalities, in this paper, we design a new separation heuristic that

- prioritizes generating violated n-step capacity cutset inequalities over n-step flow cutset and cutset inequalities.
- prioritizes generating violated n-step cutset inequalities with the least number of flow variables over the most violated n-step cutset inequality.

In the separation heuristic, for each given A_U, n, and α, we consider generating the n-step capacity cutset inequality and the n-step cutset inequalities in a hierarchical manner. We first check if the corresponding n-step capacity cutset inequality is violated by $(\hat{x}, \hat{y})$. If so, the separation procedure returns with the n-step capacity cutset inequality. Notice that the n-step capacity cutset inequality is a special case of the n-step cutset inequality where $A_\subset^+ = A_U^+$, $A_U^+ \backslash A_\subset^+ = \emptyset$, $A_\subset^- = \emptyset$, and $A_U^- \backslash A_\subset^- = A_U^-$. Therefore, if the n-step capacity cutset inequality is not violated, we may construct a violated n-step cutset inequality by progressively moving the arcs from $A_\subset^+$ to $A_U^+ \backslash A_\subset^+$ and from $A_U^- \backslash A_\subset^-$ to $A_\subset^-$. We choose such arcs based on the following criteria. Let $\sigma := \sum_{t=1}^M \mu_{\alpha,D}^n(C_t) \sum_{a\in A_\subset^+} \hat{y_t^a} + \sum_{t=1}^M \big(C_t + \mu_{\alpha,D}^n(-C_t)\big) \sum_{a\in A_\subset^-} \hat{y_t^a} + \sum_{a\in A_U^+ \backslash A_\subset^+} \hat{x_a} - \sum_{a\in A_\subset^-} \hat{x_a} - \mu_{\alpha,D}^n(D)$ be the slack of the n-step cutset inequality. At first, we have $\sigma = \sum_{t=1}^M \mu_{\alpha,D}^n(C_t) \sum_{a\in A_U^+} \hat{y_t^a} - \mu_{\alpha,D}^n(D) > 0$. Let w_a be the slack for arc $a \in A_\subset^+ \cup (A_U^- \setminus A_\subset^-)$, which is defined as

$$w_a := \begin{cases} \hat{x_a} - \sum_{t=1}^M \mu_{\alpha,D}^n(C_t)\hat{y_t^a}, a \in A_\subset^+ \\ \sum_{t=1}^M (C_t + \mu_{\alpha,D}^n(-C_t))\hat{y_t^a} - \hat{x_a}, a \in A_U^- \setminus A_\subset^-. \end{cases} \tag{46}$$

Let $\bar{a} = argmin_{a\in A_\subset^+ \cup (A_U^- \backslash A_\subset^-)} w_a$. If $w_{\bar{a}} \ge 0$ and $\sigma \ge 0$, then we conclude that no violated inequality can be generated for the current combination of A_U, n, and α. If $w_{\bar{a}} < 0$, we move the corresponding arc $\bar{a}$ from $A_\subset^+$ to $A_U^+ \setminus A_\subset^+$ if $\bar{a} \in A_\subset^+$, or from $A_U^- \setminus A_\subset^-$ to $A_\subset^-$ if $a \in A_U^- \setminus A_\subset^-$. By doing so, the slack σ of the resulting n-step cutset inequality is decreased by $-w_a$. We repeat the above process until $\sigma < 0$, at which point the separation procedure returns with a violated n-step cutset inequality, or conclude that no violated inequality can be generated when $w_{\bar{a}} \geq 0$ and $\sigma \geq 0$. Notice that if $A_{\subset}^{-} = \emptyset$ in the resulting inequality, then the resulting violated inequality is an n-step flow cutset inequality.

The above cut generating procedure is summarized in Algorithm 1.

Algorithm 1 Separation Heuristic

Input: Current LP relaxation optimal solution $(\hat{x}, \hat{y})$, A_U, n, and α
Output: Coefficients $\eta_\zeta = (\pi, \beta) \in \mathbb{R}^{|A|} \times \mathbb{R}^{M|A|}$ of a cut

1. Let $(\pi, \beta) = 0$
2. Let $\beta_t^a = \mu_{\alpha,D}^n(C_t), a \in A_U^+, t = 1, \ldots, M$
3. If $\sum_{t=1}^{M} \beta_t^a \sum_{a \in A_U^+} \hat{y}_t^a < \mu_{\alpha,D}^n(D)$

   Stop and output (π, β)

   Else

   Let $\sigma = \sum_{t=1}^{M} \mu_{\alpha,D}^n(C_t) \sum_{a \in A_U^+} \hat{y}_t^a - \mu_{\alpha,D}^n(D)$ and go to 4
4. Let w_a be calculated as in (46) for $a \in A_{\subset}^{+} \cup (A_U^- \setminus A_{\subset}^-)$
5. While $\sigma \geq 0$

   Let $\bar{a} = argmin_{a \in A_{\subset}^{+} \cup (A_U^- \setminus A_{\subset}^-)} \{w_a\}$

   If $w_{\bar{a}} \geq 0$

       Stop; no cutset cut can be generated

   Else

       If $\bar{a} \in A_{\subset}^{+}$

           Let $\beta_t^{\bar{a}} = 0, t = 1, \ldots, M$, $\pi_{\bar{a}} = 1$

           Move $\bar{a}$ to $A_U^+ \setminus A_{\subset}^{+}$

       Else

           Let $\beta_t^{\bar{a}} = C_t + \mu_{\alpha,D}^n(-C_t), t = 1, \ldots, M$, $\pi_{\bar{a}} = -1$;

           Move $\bar{a}$ to $A_{\subset}^{-}$

       Let $\sigma = \sigma + w_{\bar{a}}$
6. Stop and output (π, β)

5.3 Cutting Plane Algorithm

In our experiments, we add the n-step cutset inequalities using a cutting plane algorithm to tighten the formulation of the MMND instances and then solve the instances by CPLEX. Each iteration of the cutting plane algorithm starts with a choice of partition $(U, \overline{U})$ of V. For our experiments, we simply considered all partitions $(U, \overline{U})$ of V such that $1 \leq |U| \leq 4$. This was based on the observation in [2] and our computations that most of the violated inequalities are generated from uneven partitions. At each iteration, given the LP relaxation optimal solution $(\hat{x}, \hat{y})$, A_U corresponding to $(U, \overline{U})$, and fixed n and α, the separation procedure in Section 5.2 was then called and at most one violated n-step cutset inequality was generated.

In our experiments, we noticed that violated n-step cutset inequalities can be found for most combinations of A_U, n and α. Based on previous computational efforts [4, 8, 23] and our experiments, adding too many n-step cutset inequalities to the formulation may significantly increase the solution time. We limit the

Algorithm 2 Cutting Plane Algorithm

Input: n and α

1. Let $R = \emptyset$
2. Solve LP relaxation and get optimal solution $(\hat{x}, \hat{y})$
3. For each U such that $1 \leq |U| \leq 4$

    Call Algorithm 1 w.r.t. $(\hat{x}, \hat{y})$, A_U, n, α, and get new cut ζ

    For each $p \in R$

        Let $o_p = |\eta_p^T \eta_\zeta| / (||\eta_p|| \cdot ||\eta_\zeta||)$

        If $o_p >$ **threshold**

            Continue to next iteration of 3

    Add cut ζ to formulation

    Let $R = R \cup \{\zeta\}$

    Solve LP relaxation and get optimal solution $(\hat{x}, \hat{y})$
4. Remove inactive cuts w.r.t. $(\hat{x}, \hat{y})$

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number of n-step cutset inequalities added to the formulation by using a technique similar to that in [1] to select a small number of inequalities to add to the formulation. It calculates the orthogonality of the newly generated cuts with respect to previously added cuts, and aims at selecting a nearly orthogonal subset of cutting planes, which cut as deep as possible into the current LP relaxation polyhedron.

The cut selection process is as follows. Let ζ be the newly generated n-step cutset inequality, and R be the set of all previously added cuts to the formulation, indexed by p. Let η_ζ and $\eta_p, p \in R$ be the coefficient vectors of their corresponding inequalities. The orthogonality of ζ with respect to p is calculated by $o_p = |\eta_p^T \eta_\zeta| / (||\eta_p|| \cdot ||\eta_\zeta||)$, and the orthogonality of ζ with respect to the set of all previously added cuts R is defined as $o_\zeta = \max_{p \in R} o_p$. We only add the newly generated cut to the formulation if o_ζ is less than or equal to a fixed threshold. In our experiments, this threshold was tuned to be 0.3. Practically, given the cut generated by the separation, we iterate $p \in R$ to calculate o_p, and if $o_p > 0.3$, we skip the current iteration and go to the next iteration with another choice of $(U, \overline{U})$.

The LP relaxation problem was reoptimized when a new cut was added to the formulation. Then the next iteration of the cutting plane algorithm starts with another choice of $(U, \overline{U})$ with the updated LP relaxation optimal solution. This process was repeated for all of our choices of $(U, \overline{U})$. Finally, cuts that were inactive at the final LP relaxation optimal solution were removed.

The cutting plane algorithm is summarized in Algorithm 2.

5.4 Experimental setup and results

We first generated random directed graphs following the steps of Section 5.1. For each valid graph, we created an instance of MMND by assigning demand to nodes and creating the capacity modules. The demand of each sink node was chosen from *uniform*[10, 190]. The negative of the aggregated demand over all sinks was then randomly split among the sources. The unit flow cost h_a for each arc $a \in A$ was equal to its length, rounding down to the nearest integer. For each 2-module MMND instance, we assigned to it one of the 3 sets of capacity modules: (130, 50), (170, 70), and (200, 80). We also assigned to it one of the 2 sets of costs associated with these capacity modules: (10000, 5000) and (18000, 9000) (we assumed the module cost to be the same for every arc, i.e., $f_t^a = f_t$, $a \in A$). For each 3-module MMND instance, we assigned to it one of the 3 sets of capacity modules: (130, 50, 20), (170, 70, 30), and (200, 80, 30). We also assigned to it one of the 3 sets of costs associated with these capacity modules: (10000, 5000, 2500), (18000, 9000, 5000), and (25000, 13000, 9000). The summary of the instances is listed in Table 2.

For each 2-module and 3-module MMND instance, we performed several runs. In the first run, we solved

Table 2: Summary of MMND problem instances

Instance	Capacity modules	Capacity module costs
2_1_1	(130, 50)	(10000, 5000)
2_1_2	(130, 50)	(18000, 9000)
2_2_1	(170, 70)	(10000, 5000)
2_2_2	(170, 70)	(18000, 9000)
2_3_1	(200, 80)	(10000, 5000)
2_3_2	(200, 80)	(18000, 9000)
3_1_1	(130, 50, 20)	(10000, 5000, 2500)
3_1_2	(130, 50, 20)	(18000, 9000, 5000)
3_1_3	(130, 50, 20)	(25000, 13000, 9000)
3_2_1	(170, 70, 30)	(10000, 5000, 2500)
3_2_2	(170, 70, 30)	(18000, 9000, 5000)
3_2_3	(170, 70, 30)	(25000, 13000, 9000)
3_3_1	(200, 80, 30)	(10000, 5000, 2500)
3_3_2	(200, 80, 30)	(18000, 9000, 5000)
3_3_3	(200, 80, 30)	(25000, 13000, 9000)

it using CPLEX in its default settings. The corresponding results are under label DEF in Table 3 and 4. For subsequent runs, we added the n-step cutset inequalities to the formulation using the cutting plane algorithm described in Section 5.3 and solved the instance with the added cuts using CPLEX in its default settings. We added n-step cutset inequalities with different values for n and α in different runs and compared the performance of these cuts. For each 2-module MMND instance, two subsequent runs were performed, where the 1-step cutset inequalities with $n = 1, \alpha = C_1$, and the 2-step cutset inequalities with $n = 2, \alpha = \{C_1, C_2\}$, were added to the formulation, respectively. Their respective results are under labels 1CUT and 2CUT in Table 3. For each 3-module MMND instance, three subsequent runs were performed, where the 1-step cutset inequalities with $n = 1, \alpha = C_1$, the 2-step cutset inequalities with $n = 2, \alpha = \{C_1, C_2\}$, and the 3-step cutset inequalities with $n = 3, \alpha = \{C_1, C_2, C_3\}$, were added to the formulation, respectively. Their respective results are under labels 1CUT, 2CUT, and 3CUT in Table 4.

We implemented the instance generation and the cutting plane algorithm in C++, and the instances were solved by CPLEX 12.7. All the experiments were run on a PC with Intel Core i7 2.50GHz processor with 4 cores and 16 GB of RAM. The time limit for CPLEX was set to be 2 hours. The results are listed in Table 3 and 4.

Table 3 summarizes the computational results on the 2-module MMND instances. Each row reports the average results for 10 instances of the corresponding instance category.

We report the following statistics if applicable: under DEF, the time (in seconds) to solve the instance (T); the number of branch-and-bound nodes reported by CPLEX (*Nodes*); the initial integrality gap, calculated by $G_0 = 100 \times (zmip - zlp)/zmip$, where zlp and $zmip$ are the optimal objective values of the LP relaxation and MIP, respectively. For each type of cuts, we report the number of active cuts added to the formulation (*Cuts*); the number of branch-and-bound nodes reported by CPLEX after adding the cuts (*Nodes*); the percentage of the integrality gap closed by our cuts, i.e., $G\% = 100 \times (zcut - zlp)/(zmip - zlp)$, where zlp, $zcut$, and $zmip$ are the optimal objective values of the LP relaxation without the cuts, LP relaxation with the cuts, and MIP, respectively; the time (in seconds) to generate the customized cuts (T_{Cut}); the time (in seconds) to solve the instance excluding the cut generation time (T_{Opt}); and the total solution time including the cut generation time (T). In DEF, $T = T_{Opt}$.

For 2-module MMND instances, we noticed significant improvement in the time and nodes required to solve the instances by adding the 2-step cutset inequalities. On average, the gap closed by the 2-step cutset inequalities was 79.2%. The average total solution time (including cut generation) with our 2-step cutset cuts was 0.35 times that of CPLEX 12.7 in default settings, and the number of branch-and-bound nodes was 0.23 times that of the default CPLEX. The best performance was on the category with capacity modules (200, 80) and costs (18000, 9000), where the average total solution time with the 2-step cuts was 0.11 times

Table 3: Results of computational experiments on 2-module MMND instances

Instance	DEF			1CUT						2CUT					
	T_{Opt}	*Nodes*	G_0	*Cuts*	T_{Cut}	T_{Opt}	T	*Nodes*	$G\%$	*Cuts*	T_{Cut}	T_{Opt}	T	*Nodes*	$G\%$
2_1_1	458	1713521	17	45	14	106	120	502724	46	83	35	122	154	486370	77
2_1_2	672	2490527	18	46	14	313	327	1520851	46	83	38	390	417	1280358	77
2_2_1	569	2200308	22	44	15	126	142	575316	47	76	40	110	150	472718	79
2_2_2	554	3363122	24	45	15	374	390	1920436	47	77	42	194	236	778173	79
2_3_1	279	1405855	22	50	18	391	408	1375045	51	92	44	56	101	214235	81
2_3_2	762	3425010	24	48	17	536	553	2937331	51	90	44	36	81	131356	82

Table 4: Results of computational experiments on 3-module MMND instances

Instance	DEF			1CUT						2CUT						3CUT					
	T_{Opt}	*Nodes*	G_0	*Cuts*	T_{Cut}	T_{Opt}	T	*Nodes*	$G\%$	*Cuts*	T_{Cut}	T_{Opt}	T	*Nodes*	$G\%$	*Cuts*	T_{Cut}	T_{Opt}	T	*Nodes*	$G\%$
3_1_1	953	3184686	15	62	26	1493	1519	3859727	54	82	45	771	816	2539234	57	85	43	223	266	643022	72
3_1_2	804	1121292	17	63	28	1471	1499	3440327	54	91	55	955	1010	1858099	62	89	49	478	527	1209506	73
3_1_3	827	1751619	20	62	26	598	624	1292236	56	94	55	414	469	926358	72	94	47	165	212	356813	75
3_2_1	1105	6626769	14	61	20	476	496	1968267	49	74	36	551	587	2860707	52	84	42	208	250	904037	73
3_2_2	1056	4360067	16	62	22	428	450	1457589	50	82	42	457	499	1655077	59	90	45	344	389	1170541	74
3_2_3	826	1438662	19	64	24	866	889	2164659	53	92	61	783	844	1074849	70	89	44	624	667	1636625	74
3_3_1	31	122339	19	58	20	36	56	128792	52	96	56	41	97	161283	60	88	38	18	57	61685	77
3_3_2	66	306930	21	65	27	58	85	201923	53	99	63	38	102	88630	65	93	40	29	69	70178	77
3_3_3	75	257924	25	69	29	82	112	248610	57	101	67	65	133	149331	74	97	46	79	125	147462	77

that of CPLEX 12.7 in its default settings, and the number of branch-and-bound nodes was 0.04 times that of the default CPLEX.

Furthermore, in 4 of 6 categories, the 2-step cutset inequalities outperformed the 1-step cutset inequalities in terms of the solution time excluding cut-generation time T_{Opt}, and in 3 of them, the 2-step cutset inequalities also had advantages in terms of the total solution time T. For all categories, the instances with the 2-step cutset inequalities required less number of nodes to solve than the instances with the 1-step cutset inequalities. On average, the total solution time (including cut generation) with our 2-step cutset inequalities was 0.59 times that with the 1-step cutset inequalities, and the number of branch-and-bound nodes was 0.38 times that with the 1-step cutset inequalities. The integrality gap closed by our 2-step cutset inequalities was 1.6 times that closed by the 1-step cutset inequalities.

Table 4 summarizes the results on the 3-module MMND instances. For instances with the capacity modules (200, 80, 30) which were easier to solve, the average total solution time by adding the 3-step cutset inequalities was slightly worse than that of CPLEX in its default settings because of relatively long cut generation time. For harder instances, however, the improvement by adding the 3-step cutset inequalities was significant over CPLEX in its default settings. On average, the total solution time (including cut generation) with our 3-step cutset inequalities was 0.45 times that of CPLEX 12.7 in default settings, 0.45 times that with only 1-step cutset inequalities added, and 0.56 times that with only 2-step cutset inequalities added. The number of branch-and-bound nodes with our 3-step cutset inequalities was 0.32 times that of default CPLEX, 0.42 times that with only 1-step cutset inequalities, and 0.55 times that with only 2-step cutset inequalities. The gap closed by the 3-step cutset cuts was 74.8%, which was 1.4 times that by the 1-step cutset cuts, and 1.2 times that by the 2-step cutset cuts.

Therefore, we conclude that the 2-step cutset inequalities are effective in solving 2-module MMND instances, and the 3-step cutset inequalities are effective in solving 3-module MMND instances. Moreover, they are more effective than the n-step cutset inequalities that use information of less number of capacity modules.

6 n-step cutset inequalities for undirected and bidirected MMND

Our results for the directed MMND can be easily generalized for the undirected and the bidirected MMND.

Given a partition $(U, \overline{U})$ of V, let $S_1, S_2 \subset E_U$. Each edge $e \in E_U$ is represented by its two antiparallel arcs e^+ and e^-. Let A_1 be the set of such arcs corresponding to edges in S_1, $A_1^+ \subseteq A_1$ be the set of arcs in A_1 who have tails in $\overline{U}$ and heads in U, and $A_1^- := A_1 \setminus A_1^+$ (and define A_2, A_2^+, and A_2^- similarly for S_2).

We have the following.

Theorem 5 *Given a partition $(U,\overline{U})$ of V, let $S_1, S_2 \subset E_U$. Define $D := d_U - \sum_{e\in S_1} g^e + \sum_{e\in S_2} g^e$. Given $n \in \{1,\dots,M\}$, $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, if the n-step MIR conditions (27) are satisfied, i.e., $\alpha_k \left\lceil D^{(k-1)}/\alpha_k \right\rceil \le \alpha_{k-1}, k = 2,\dots,n$, the n-step cutset inequality*

$$\sum_{t=1}^{M} \mu^n_{\alpha,D}(C_t) \sum_{e\in S_1} y^e_t + \sum_{t=1}^{M} \Big(C_t + \mu^n_{\alpha,D}(-C_t) \Big) \sum_{e\in S_2} y^e_t + \sum_{a\in A^+_U \setminus A^+_1} x_a - \sum_{a\in A^-_2} x_{ij} \ge \mu^n_{\alpha,D}(D) - \sum_{e\in S_2} g^e \quad (47)$$

is valid for X^u and X^b.

Proof. We show it for X^u and X^b respectively. For X^u, adding the nonnegativity constraints $x_a \ge 0, a \in A^+_U \setminus A^+_2$ to (14), we have

$$\sum_{a\in A^+_1} x_a + \sum_{a\in A^+_U\setminus A^+_1} x_a - \sum_{a\in A^-_2} x_a \ge d_U. \quad (48)$$

The rest of the proof is similar to that of Theorem 1.

For X^b, as mentioned in Section 1, X^b is a special case of X^d where the arcs sharing the same edge have the same capacity. Therefore the n-step cutset inequality (28) is valid for X^b. Set $A^+_1 = A^+_{\subset}$, $A^-_2 = A^-_{\subset}$, $S_1 = \{e : e^+ \in A^+_1 \text{ or } e^- \in A^+_1\}$, and $S_2 = \{e : e^+ \in A^+_2 \text{ or } e^- \in A^+_2\}$. Then (28) becomes (47). □

Remark 3 *The n-step flow cutset inequality, obtained by setting $S_2 = \emptyset$ in (47), is valid for X^u and X^b. The n-step capacity cutset inequality, obtained by setting $S_1 = E_U$ and $S_2 = \emptyset$ in (47), is valid for X^u and X^b.*

Special Cases We illustrate several special cases of the n-step cutset inequalities for X^u and X^b in the literature.

- **Cutset inequality.** The cutset inequality (17) is obtained by setting $n = 1, \alpha = C_2, S_1 = E_U$, and $S_2 = \emptyset$ in (47). The cutset inequality (18) is obtained by setting $n = 1, \alpha = C_1, S_1 = E_U$, and $S_2 = \emptyset$ in (47). The cutset inequality (19) is in fact a 2-step MIR inequality [11]. This inequality can be rewritten as

$$D^{(2)} \left\lceil \frac{D^{(1)}}{C_2} \right\rceil \sum_{e\in E_U} y^e_1 + D^{(2)} \sum_{e\in E_U} y^e_2 + \sum_{e\in E_U} y^e_3 \ge D^{(2)} \left\lceil \frac{D^{(1)}}{C_2} \right\rceil \left\lceil \frac{D}{C_1} \right\rceil. \quad (49)$$

  (49) is obtained by setting $n = 2, \alpha = \{C_1, C_2\}, S_1 = E_U$, and $S_2 = \emptyset$ in (47). It was also mentioned in [19] that for the network loading problem with any number of divisible capacity modules and $C_M = 1$, (49) can be generalized to

$$\sum_{t=1}^{M} D^{(M-1)} \prod_{k=t+1}^{M-1} \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil \sum_{e\in E_U} y^e_t + \sum_{e\in E_U} y^e_M \ge D^{(M-1)} \prod_{k=t+1}^{M-1} \left\lceil \frac{D^{(k-1)}}{C_t} \right\rceil. \quad (50)$$

  This inequality is obtained by setting $n = M-1, \alpha = \{C_1,\dots,C_M\}$, $S_1 = E_U$, and $S_2 = \emptyset$ in (47). To our knowledge, (49) and (50) are the only inequalities in the family of n-step cutset inequalities where $n > 1$ in the literature.

- **Flow-cut-set inequality.** The flow-cut-set inequality (20) is obtained by setting $n = 1, \alpha = C_1$, and $S_2 = \emptyset$ in (47).

We now present the conditions under which the n-step cutset inequalities are facet-defining for P^u_U and P^b_U. Their proofs are similar to that for P^d_U and are omitted.

Theorem 6 *The n-step cutset inequality is facet-defining for P^u_U and P^b_U if conditions (a), (b), and (c) of Theorem 2 hold, and $S_1 \ne \emptyset, S_2 \ne \emptyset, A^+_U \setminus A^+_1 \ne \emptyset, A^-_2 \ne \emptyset$.*

Theorem 7 *The n-step flow cutset inequality is facet-defining for P_U^u and P_U^b if conditions (a), (b), and (c) of Theorem 3 hold, and $S_1 \neq \emptyset, A_U^+ \setminus A_1^+ \neq \emptyset$.*

Theorem 8 *The n-step capacity cutset inequality is facet-defining for P_U^u and P_U^b if conditions (a), (b), and (c) of Theorem 4 hold.*

Based on a result similar to Lemma 3 for X^u and X^b (see Theorem 3.4 in [23]), we have the following.

Corollary 2 *The n-step cutset inequality is facet-defining for X^u (resp. X^b) if it is facet-defining for P_U^u (resp. P_U^b), and the graphs induced by U and $\overline{U}$ are connected.*

7 Multi-commodity directed MMND problem

Understanding the polyhedral structure of MMND motivates us to study the multi-commodity MMND problem (MCMMND). Multi-commodity networks often arise in the backbone of telecommunication networks [4]. In this section, we discuss how to generalize the n-step cutset inequalities for directed, undirected, and bidirected MCMMND.

For MCMMND, the network structures and the capacity modules can be defined similarly to those of MMND. Instead of a single commodity of demand, MCMMND have a set of commodities Q. Each commodity is identified by a single-source-single-sink pair of nodes, i.e., for each $q \in Q$, there is a single sink node $v_t^q \in V$ with demand $d^q > 0$ and a single source node $v_s^q \in V$ with supply $-d^q$.

For the directed MCMMND, let h_a^q be the unit cost of flow along arc $a \in A$ for commodity $q \in Q$. The mixed integer programming formulation for the directed MCMMND is

$$\min \sum_{a\in A}(\sum_{q\in Q} h_a^q x_a^q + \sum_{t=1}^{M} f_t^a y_t^a) \tag{51}$$

$$\sum_{a\in\delta^+(v)} x_a^q - \sum_{a\in\delta^-(v)} x_a^q = d_v^q, v \in V, q \in Q \tag{52}$$

$$\sum_{q\in Q} x_a^q \leq \sum_{t=1}^{M} C_t y_t^a + g^a, a \in A \tag{53}$$

$$(x,y) \in \mathbb{R}_+^{|Q||A|} \times \mathbb{Z}_+^{M|Q||A|}, \tag{54}$$

where x_a^q is now the number of flow units transferred along arc a for commodity q. Let Y^d be the convex hull of the set defined by (52)-(54). Given a partition $(U, \overline{U})$ of V, the corresponding cutset polyhedron is

$$\overline{P}_U^d := conv\Big\{(x,y) \in \mathbb{R}_+^{|Q||E_U|} \times \mathbb{Z}_+^{M|E_U|} : \tag{55}$$

$$\sum_{a\in A_U^+} x_a^q - \sum_{a\in A_U^-} x_a^q = d_U^q, q \in Q \tag{56}$$

$$\sum_{q\in Q} x_a^q \leq \sum_{t=1}^{M} C_t y_t^a + g^a, a \in A_U \quad \Big\}, \tag{57}$$

where $d_U^q := \sum_{v\in U} d_v^q$.

We present generalization of the n-step cutset inequality for the directed MCMMND in our next theorem.

Theorem 9 *Let $(U,\overline{U})$ be a partition of V. For $K \subseteq Q$, $A_\subset^+ \subseteq A_U^+$, and $A_\subset^- \subseteq A_U^-$, define $\overline{D} := \sum_{q\in K} d_U^q - \sum_{a\in A_\subset^+} g^a + \sum_{a\in A_\subset^-} g^a$. Given $n \in \{1,\dots,M\}$ and $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, if the n-step MIR conditions (27)* *are satisfied, i.e.,* $\alpha_t \left\lceil \overline{D}^{(t-1)}/\alpha_t \right\rceil \le \alpha_{t-1}, t = 2, \dots, n,$ *the n-step cutset inequality*

$$\sum_{t=1}^{M} \mu^n_{\alpha,\overline{D}}(C_t) \sum_{a \in A^+_C} y^a_t + \sum_{t=1}^{M} \left(C_t + \mu^n_{\alpha,\overline{D}}(-C_t)\right) \sum_{a \in A^-_C} y^a_t + \sum_{a \in A^+_U \setminus A^+_C} \sum_{q \in K} x^q_a - \sum_{a \in A^-_C} \sum_{q \in K} x^q_a \ge \mu^n_{\alpha,\overline{D}}(\overline{D}) - \sum_{a \in A^-_C} g^a \tag{58}$$

is valid for Y^d *and* $\overline{P}^d_U$.

Proof. By aggregating the flow conservation constraints (56) over $q \in K$, relaxing the capacity constraints (57) to $\sum_{q \in K} x^q_a \le \sum_{t=1}^{M} C_t y^a_t + g^a$ for $a \in A_U$, making change of variables $x_a = \sum_{q \in K} x^q_a, a \in A_U$, and letting $d_U = \sum_{q \in K} d^q_U$, we can construct a directed cutset polyhedron P^d_U from $\overline{P}^d_U$. The n-step cutset inequality (28) is valid for P^d_U, and if we rewrite the n-step cutset inequality with $x_a = \sum_{q \in K} x^q_a, a \in A_U$ and $D = \overline{D}$, the resulting inequality is (58). □

The multi-commodity undirected and bidirected MMND can be defined similarly to the multi-commodity directed MMND. Let Y^u and Y^b be the convex hulls of the multi-commodity undirected MMND and the multi-commodity bidrected MMND, respectively. We present in our next theorem the n-step cutset inequalities for Y^u and Y^b. The proof is similar to that of Theorem 9 and is thus omitted.

Theorem 10 *Let* $(U, \overline{U})$ *be a partition of* V. *For* $K \subseteq Q$, $S_1, S_2 \subseteq E_U$, *and* $A^+_C \subseteq A^+_U$, *let* $\overline{D} := \sum_{q \in K} d^q_U - \sum_{e \in S_1} g^e + \sum_{e \in S_2} g^e$. *Given* $n \in \{1, \dots, M\}$ *and* $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, *if the n-step MIR conditions* (27) *are satisfied, i.e.,* $\alpha_t \left\lceil \overline{D}^{(t-1)}/\alpha_t \right\rceil \le \alpha_{t-1}, t = 2, \dots, n,$ *the n-step cutset inequality*

$$\sum_{t=1}^{M} \mu^n_{\alpha,\overline{D}}(C_t) \sum_{e \in S_1} y^e_t + \sum_{t=1}^{M} \left(C_t + \mu^n_{\alpha,\overline{D}}(-C_t)\right) \sum_{e \in S_2} y^e_t + \sum_{a \in A^+_U \setminus A^+_1} \sum_{q \in K} x^q_a - \sum_{a \in A^-_2} \sum_{q \in K} x^q_a \ge \mu^n_{\alpha,\overline{D}}(\overline{D}) - \sum_{e \in S_2} g^e \tag{59}$$

is valid for Y^u *and* Y^b.

Whether the generalized n-step cutset inequalities are facet-defining for Y^d, Y^u, and Y^b under certain conditions is an open question and requires further polyhedral study in future research.

8 Concluding remarks

We studied the multi-module capacitated network design problem (MMND) by developing a new family of inequalities, the n-step cutset inequalities, from cutset polyhedron, the convex hull of a mixed integer set closely related to MMND. We showed that cutset inequalities previously presented in the literature are special cases of the n-step cutset inequalities, and we also proved that the n-step cutset inequalities are facet-defining for the cutset polyhedron as well as the convex hull of MMND under certain conditions. Our computational results showed that these inequalities are very effective in solving MMND problems.

We are interested in two future directions following this research: the polyhedral structure of the multi-commodity multi-module capacitated network design problem (MCMMND), and the multi-module survivable network design problem (MM-SND) where the network is capable of recovering from link failures.

Acknowledgements. This work is supported by the National Science Foundation Grant CMMI-1435526, which is gratefully acknowledged.

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Appendix

Proof of Lemma 2.

(a) See Lemma 1 of [15].

(b) See Lemma 1 of [15].

(c) In this case $(-r)^{(1)} = \ldots = (-r)^{(n)} = 0$ and $(-r) \in \mathcal{L}_n^n$, so $\mu^n_{\alpha,D}(-r) = D^{(n)} \prod_{k=2}^{n} \lceil D^{(k-1)}/\alpha_k \rceil \lfloor -r/\alpha_1 \rfloor = -D^{(n)} \prod_{k=2}^{n} \lceil D^{(k-1)}/\alpha_k \rceil \lceil r/\alpha_1 \rceil = -D^{(n)} \prod_{k=2}^{n} \lceil D^{(k-1)}/\alpha_k \rceil$.

(d) This can be proved similarly to Lemma 1 of [15]. Since $D^{(t)} \le \alpha_1 - r \le \alpha_t$, $(-r)^{(1)} = \ldots = (-r)^{(t-1)} = \alpha_1 - r$. Let ψ be the smallest integer such that $(-u)^{(\psi+1)} \ge D^{(\psi+1)}$ holds, and let $\psi = n$ otherwise. Thus $(-r) \in \mathcal{L}^n_\psi$. By definition $D^{(1)} \ge \ldots \ge D^{(n)}$, if $\alpha_1 - r \ge D^{(t)}$, then $\psi + 1 \le t$. Thus we have $(-r)^{(1)} = \ldots = (-r)^{(\psi)} = \ldots = (-r)^{(t-1)} = \alpha_1 - r \le \alpha_t < \ldots < \alpha_\psi < \ldots < \alpha_1$. Then

$$\begin{aligned}
\mu^n_{\alpha,D}(-r) &= D^{(n)} \sum_{k=1}^{\psi} \prod_{l=k+1}^{n} \left\lceil \frac{D^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{(-r)^{(k-1)}}{\alpha_k} \right\rfloor + D^{(n)} \prod_{l=\psi+2}^{n} \left\lceil \frac{D^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{(-r)^{(\psi)}}{\alpha_{\psi+1}} \right\rceil \\
&= D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{-r}{\alpha_1} \right\rfloor + D^{(n)} \prod_{l=\psi+2}^{n} \left\lceil \frac{D^{(l-1)}}{\alpha_l} \right\rceil \\
&= -D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \left\lceil \frac{r}{\alpha_1} \right\rceil + D^{(n)} \prod_{l=\psi+2}^{n} \left\lceil \frac{D^{(l-1)}}{\alpha_l} \right\rceil \\
&= -D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil + D^{(n)} \prod_{l=t+1}^{n} \left\lceil \frac{D^{(l-1)}}{\alpha_l} \right\rceil .
\end{aligned}$$

The last equality holds because if $\psi \le t-2$, then $D^{(t-1)} \le \ldots \le D^{(\psi+1)} \le \alpha_1 - r < \alpha_t < \ldots < \alpha_{\psi+2}$, then $\lceil D^{(l-1)}/\alpha_l \rceil = 1, l = \psi+2, \ldots, t$.

Proof of Theorem 3.

Proof. Let ρ be the index of the last capacity module whose size is larger than $D^{(n)}$, i.e., $\rho = max\{t \in \{1,\dots,n\} : \alpha_t > D^{(n)}\}$. Then we have $\alpha_t > D^{(n)}$, $t = n+1,\dots,\rho$ and $\alpha_t \le D^{(n)}$, $t = \rho+1,\dots,M$. Consider the hyperplane corresponding to (31). By substituting values of the n-step MIR function (25) corresponding to the ones of Lemma 2 under conditions (a) and (b), the hyperplane can be rewritten as

$$\sum_{t=1}^{n} D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a\in A_{\subset}^{+}} y_t^a + \sum_{t=n+1}^{\rho} D^{(n)} \sum_{a\in A_{\subset}^{+}} y_t^a + \sum_{t=\rho+1}^{M} C_t \sum_{a\in A_{\subset}^{+}} y_t^a + \sum_{a\in A_U^+\setminus A_{\subset}^{+}} x_a = D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil . \tag{60}$$

Let

$$\sum_{t=1}^{M} \sum_{a\in A_U} \beta_t^a y_t^a + \sum_{a\in A_U} \pi_a x_a = \theta \tag{61}$$

be a hyperplane passing through the face defined by (60). We prove that (61) is a scalar multiple of (60) plus the flow balance equality (11).

For any $i \in (A_U^+ \setminus A_{\subset}^+) \cup A_U^-$, $t = 1,\dots,M$, consider the direction $\mathcal{E}_t^i$. $\mathcal{E}_t^i$ is an unbounded direction for both P_U^d and (60), and hence a direction for the face defined by (61). This implies that $\beta_t^i = 0$, $i \in (A_U^+ \setminus A_{\subset}^+) \cup A_U^-$, $t = 1,\dots,M$.

Next, for any $i \in A_{\subset}^+$ and $\omega \in A_U^-$, consider the points $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$. It is easy to check that $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$ satisfy (60) by (a) of Lemma 1, and by Proposition 1, $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$ are in P_U^d. Then $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$ must satisfy (61). By substituting $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$ into (61) and subtracting one equality from the other, we have $(\alpha_1 \lceil D/\alpha_1 \rceil - D)(\pi_i + \pi_j) = 0$, which implies that $\pi_i = -\pi_j$ for any $i \in A_{\subset}^+$, $\omega \in A_U^-$.

Now, since all points of P_U^d satisfy the flow balance equality (11), we may add multiples of the flow balance equality to facet-defining inequalities without changing them. Therefore, without loss of generality, we assume that $\pi_\gamma = 0$ for some $\gamma \in A_{\subset}^+$. This implies that

$$\pi_i = 0, i \in A_{\subset}^+ \cup A_U^-. \tag{62}$$

Next, for any $i \in A_{\subset}^+, j \in A_U^+ \setminus A_{\subset}^+$, consider the points $\mathcal{B}_n^i$ and $\mathcal{A}_n^{i,j}$. It is easy to check that they are in P_U^d by Proposition 1, and by $(a)(b)$ of Lemma 1, they satisfy (60). Then they must satisfy (61). By substituting them into (61), and subtracting one equality from the other, we have $\beta_n^i = D^{(n)}\pi_j$. Since this is true for any $j \in A_U^+ \setminus A_{\subset}^+$, this implies that $\exists \tau \in A_U^+ \setminus A_{\subset}^+$ such that

$$\pi_j = \pi_\tau, j \in A_U^+ \setminus A_{\subset}^+, \beta_n^i = D^{(n)}\pi_\tau, i \in A_{\subset}^+. \tag{63}$$

Now, for any $i \in A_{\subset}^+$, $j \in A_U^+ \setminus A_{\subset}^+$, consider the points $\mathcal{A}_n^{i,j}$ and $\mathcal{A}_l^{i,j}$, $l \in \{n+1,\dots,\rho\}$. It is easy to check that they are in P_U^d by Proposition 1, and by $(a)(b)$ of Lemma 1, they all satisfy (60). Then they must satisfy (61). By substituting them into (61), and subtracting one equality from the other, we have

$$\beta_l^i = D^{(n)}\pi_\tau, i \in A_{\subset}^+, l = n+1,\dots,\rho. \tag{64}$$

Next, for any $i \in A_{\subset}^+$, $j \in A_U^+ \setminus A_{\subset}^+$, consider the points $\mathcal{A}_n^{i,j}$ and $\mathcal{A}_l^{i,j}, l \in \{\rho+1,\dots,M\}$. It is easy to check they are in P_U^d by Proposition 1, and by (b) of Lemma 1, they all satisfy (60). Then they must satisfy (61). By substituting them into (61), and subtracting one equality from the other, we have

$$\beta_l^i = C_l\pi_\tau, i \in A_{\subset}^+, l = \rho+1,\dots,M. \tag{65}$$

Next, for any $i \in A_{\subset}^+$, consider the points $\mathcal{B}_l^i$, $\mathcal{B}_{l-1}^i$, $l = 2,\dots,n$. It is easy to check that they are in P_U^d by Proposition 1, and by (a) of Lemma 1, they all satisfy (60). Then they must satisfy (61). By substituting them one after another into (61), and subtracting one equality from the other, we have $\mathcal{B}_{l-1}^i = \mathcal{B}_l^i \lceil D^{(l-1)}/\alpha_l \rceil$, $l = 2,\dots,n$. Since $\beta_n^i = D^{(n)}\pi_\tau$, this implies that

$$\beta_l^i = D^{(n)} \prod_{t=l+1}^{n} \left\lceil \frac{D^{(t-1)}}{\alpha_t} \right\rceil \pi_\tau, i \in A_{\subset}^+, l = 1,\dots,n. \tag{66}$$

So far, (61) has been reduced to

$$\sum_{t=1}^{n} D^{(n)}\pi_\tau \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a\in A_\subset^+} y_t^a + \sum_{t=n+1}^{\rho} D^{(n)}\pi_\tau \sum_{a\in A_\subset^+} y_t^a + \sum_{t=\rho+1}^{M} C_t\pi_\tau \sum_{a\in A_\subset^+} y_t^a + \sum_{a\in A_U^+\setminus A_\subset^+} \pi_\tau x_a = \theta. \tag{67}$$

Finally, by substituting $\mathcal{B}_1^i$ for some $i \in A_\subset^+$ into (67), we have $\theta = D^{(n)}\pi_\tau \prod_{k=1}^{n} \lceil D^{(k-1)}/\alpha_k \rceil$, which reduces (67) to

$$\sum_{t=1}^{n} D^{(n)}\pi_\tau \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a\in A_\subset^+} y_t^a + \sum_{t=n+1}^{\rho} D^{(n)}\pi_\tau \sum_{a\in A_\subset^+} y_t^a + \sum_{t=\rho+1}^{M} C_t\pi_\tau \sum_{a\in A_\subset^+} y_t^a + \sum_{a\in A_U^+\setminus A_\subset^+} \pi_\tau x_a = D^{(n)}\pi_\tau \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil. \tag{68}$$

(68) is a scalar multiple of (60) (the scalar is π_τ). This completes the proof. □

Proof of Theorem 4.

Proof. Consider the hyperplane corresponding to (32). By substituting values of the n-step MIR function (25) corresponding to the ones of Lemma 2 under conditions (a) and (b), the hyperplane can be rewritten as

$$\sum_{t=1}^{n} D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a\in A_U^+} y_t^a + \sum_{t=n+1}^{M} D^{(n)} \sum_{a\in A_U^+} y_t^a = D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil. \tag{69}$$

Let

$$\sum_{t=1}^{M} \sum_{a\in A_U} \beta_t^a y_t^a + \sum_{a\in A_U} \pi_a x_a = \theta \tag{70}$$

be a hyperplane passing through the face defined by (69). We prove that (70) is a scalar multiple of (69) plus the flow balance equality (11).

For any $i \in A_U^-$, $t = 1,\dots,M$, consider the direction $\mathcal{E}_t^i$. $\mathcal{E}_t^i$ is an unbounded direction for both P_U^d and (69), and hence a direction for the face defined by (70). This imples that $\beta_t^i = 0$, $i \in A_U^-$, $t = 1,\dots,M$.

Next, for any $i \in A_U^+$ and $\omega \in A_U^-$, consider the points $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$. By similar argument to the proof of Theorem 3, we have $\pi_i = -\pi_j$ for any $i \in A_U^+$ and $\omega \in A_U^-$. Now since we may add multiples of the flow balance equality to facet-defining inequalities without changing them, by similar argument to the proof of Theorem 3, we have $\pi_i = 0$, $i \in A_U$.

Next, for any $i \in A_U^+$, consider the points $\mathcal{B}_l^i$, $l = 1,\dots,n$. By similar argument to the proof of Theorem 3, if we substitute $\mathcal{B}_n^i$ and $\mathcal{B}_{n-1}^i$ into (70) and subtract one equality from the other, we have $\beta_{n-1}^i = \lceil d^{(n-1)}/\alpha_n \rceil \beta_n^i$. If we substitute $\mathcal{B}_n^i$, $\mathcal{B}_{n-1}^i$, …, $\mathcal{B}_1^i$ one after another into (70) and subtract one equality from another, we have $\beta_{l-1}^i = \lceil D^{(l-1)}/\alpha_l \rceil \beta_l^i$, $l = 2,\dots,n$, which implies $\beta_l^i = \prod_{k=l+1}^{n} \lceil D^{(k-1)}/\alpha_k \rceil \beta_n^i$, $i \in A_U^+$, $l = 1,\dots,n$.

Next, for any $i \in A_U^+$, consider the points $\mathcal{B}_l^i$, $l = n+1,\dots,M$. By similar arguments to the proof of Theorem 3, $\mathcal{B}_l^i$ satisfy (70). By substituting $\mathcal{B}_n^i$ and $\mathcal{B}_l^i$ into (70) and subtracting one equality from the other, we have $\beta_l^i = \beta_n^i, i \in A_U^+$, $l = n+1,\dots,M$.

Now, if $|A_U^+| = 1$, (70) is reduced to

$$\sum_{t=1}^{n} \beta_n^i \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil y_t^i + \sum_{t=n+1}^{M} \beta_n^i y_t^i = \theta \tag{71}$$

where $\{i\} = A_U^+$. Otherwise if $|A_U^+| > 1$, then for any $i, j \in A_U^+$, if we substitute the points $\mathcal{B}_1^i$ and $\mathcal{B}_1^j$ into (70) and subtract one equality from the other, we have $\beta_1^i = \beta_1^j$. Therefore $\beta_l^i = \beta_l^j$, $l = 1, \ldots, M$. Since our choices of i and j are arbitrary, $\exists \tau \in A_U^+$ such that $\beta_l^i = \beta_l^\tau$, $l = 1, \ldots, M$ for any $i \in A_U^+$. Then (70) is reduced to

$$\sum_{t=1}^{n} \beta_n^\tau \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A_U^+} y_t^a + \sum_{t=n+1}^{M} \beta_n^\tau \sum_{a \in A_U^+} y_t^a = \theta, \tag{72}$$

of which (71) is a special case.

Finally, if we substitute $\mathcal{B}_1^\tau$ into (72), we have $\theta = \prod_{k=1}^{n} \left\lceil D^{(k-1)}/\alpha_k \right\rceil \beta_n^\tau$, which reduces (72) to

$$\sum_{t=1}^{n} \beta_n^\tau \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A_U^+} y_t^a + \sum_{t=n+1}^{M} \beta_n^\tau \sum_{a \in A_U^+} y_t^a = \beta_n^\tau \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil. \tag{73}$$

(73) is a scalar multiple of (69) (the scalar is $\beta_n^\tau / D^{(n)}$). This completes the proof. $\square$