

EQUIVARIANT PERTURBATION IN
GOMORY AND JOHNSON’S INFINITE GROUP PROBLEM
VII. INVERSE SEMIGROUP THEORY, CLOSURES,
DECOMPOSITION OF PERTURBATIONS

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ABSTRACT. In this self-contained paper, we present a theory of the piecewise linear minimal valid functions for the 1-row Gomory–Johnson infinite group problem. The non-extreme minimal valid functions are those that admit effective perturbations. We give a precise description of the space of these perturbations as a direct sum of certain finite- and infinite-dimensional subspaces. The infinite-dimensional subspaces have partial symmetries; to describe them, we develop a theory of inverse semigroups of partial bijections, interacting with the functional equations satisfied by the perturbations. Our paper provides the foundation for grid-free algorithms for the Gomory–Johnson model, in particular for testing extremality of piecewise linear functions whose breakpoints are rational numbers with huge denominators.

1. INTRODUCTION

1.1. **Finite group relaxations $R_f(P, Z)$ of integer programs and hierarchies of valid inequalities.** A powerful method to derive cutting planes for unstructured integer linear optimization problems is to study relaxations with more structure and convenient properties. The pioneering relaxation in this line of research on general-purpose cutting planes is Gomory’s *finite group relaxation* [10], whose convex hull is known as the *corner polyhedron*.

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The relaxations are structured around the simplex method, applied to the continuous relaxation, and are therefore suitable for generating cuts in a linear-programming-based cutting-plane procedure. The group relaxation is obtained by forgetting about the nonnegativity of all basic variables, retaining only their integrality. Viewed in the space of nonbasic variables, the equations of the simplex tableau are replaced by congruences modulo the abelian group (\mathbb{Z} -module) generated by the columns of the basis matrix. Quotienting out by this group, one obtains a “group equation,” which gives the relaxation its name. Further relaxations are obtained by picking just one or a few rows of the system, or more generally by condensing the system by means of group homomorphisms; see [10] for its remarks on the use of (additive) group characters.

In the present paper, we restrict ourselves to 1-row (“cyclic”) group relaxations, which after aggregation of non-basic variables with identical coefficients can be brought to the form

$$(1.1) \quad \begin{aligned} \sum_{p \in P} y(p) p &\in f + \mathbb{Z} \\ y(p) &\in \mathbb{Z}_+ \quad \text{for all } p \in P \end{aligned}$$

where P is a finite subset of an additive group $G = \frac{1}{q}\mathbb{Z} \supset \mathbb{Z}$ and $f \in G \setminus \mathbb{Z}$, so $f + \mathbb{Z}$ is a coset of the subgroup \mathbb{Z} in G . This is called Gomory’s *finite (cyclic) group problem*. We denote the convex hull of y by $R_f(P, \mathbb{Z})$; it is a polyhedron of blocking type. Therefore every nontrivial valid linear inequality can be written in the form $\langle \pi, y \rangle := \sum_{p \in P} \pi(p)y(p) \geq 1$; then we call π a *valid function*. If $\pi' \leq \pi$ are two valid functions, then the valid inequality $\langle \pi, y \rangle \geq 1$ is a conic combination of $\langle \pi', y \rangle \geq 1$ and nonnegativity inequalities $y(p) \geq 0$. Thus it suffices to consider the *minimal (valid) functions* π , defined by the property

$$(M) \quad \begin{aligned} &\text{if } \pi' \text{ is valid and } \pi' \leq \pi \\ &\text{then } \pi' = \pi. \end{aligned}$$

A stronger notion is that of *extreme functions* π , defined by the property

$$(E) \quad \begin{aligned} &\text{if } \pi^+ \text{ and } \pi^- \text{ are minimal and } \pi = \frac{1}{2}(\pi^+ + \pi^-) \\ &\text{then } \pi = \pi^+ = \pi^-. \end{aligned}$$

Extreme functions correspond to facet-defining inequalities for $R_f(P, \mathbb{Z})$. Following the traditions of polyhedral combinatorics, we are interested in describing families of extreme functions and making them available for cutting-plane algorithms.

1.2. Master problems $R_f(G, \mathbb{Z})$ and the subadditive characterization of minimal functions. Gomory’s approach was to consider *master problems* for this purpose. The sets of solutions y to 1-row group relaxations $R_f(P)$ for subsets P of the same group G inject into the *master group*

relaxation

$$(1.2) \quad \sum_{p \in G} y(p) p \in f + \mathbb{Z}$$

$y: G \rightarrow \mathbb{Z}_+$ has finite support

by setting $y(p) = 0$ for $p \notin P$. We denote its convex hull by $R_f(G, \mathbb{Z})$. This is an infinite-dimensional set. By Gomory's master theorem [10, Theorem 13], every extreme function π' for $R_f(P, \mathbb{Z})$ is obtained from some extreme function π for a master problem $R_f(G, \mathbb{Z})$ with $P \subseteq G$ by restricting the function, $\pi' = \pi|_P$. Moreover, Gomory [10] gave a characterization of the minimal functions for the master problem $R_f(G, \mathbb{Z})$ by the following functional inequalities and equations:

$$(1.3a) \quad \pi(x) \geq 0 \quad \text{for } x \in G,$$

$$(1.3b) \quad \pi(x+z) = \pi(x) \quad \text{for } x \in G, z \in \mathbb{Z} \quad (\text{periodicity})$$

$$(1.3c) \quad \pi(0) = 0, \pi(f) = 1,$$

$$(1.3d) \quad \Delta\pi(x, y) \geq 0 \quad \text{for } x, y \in G \quad (\text{subadditivity}),$$

$$(1.3e) \quad \Delta\pi(x, f-x) = 0 \quad \text{for } x \in G \quad (\text{symmetry condition}),$$

where $\Delta\pi(x, y) = \pi(x) + \pi(y) - \pi(x+y)$ is the *subadditivity slack* function. By quotienting out by \mathbb{Z} , this system describes a polyhedron in $\mathbb{R}^{G/\mathbb{Z}}$. Extreme functions are then simply the vertices of this polyhedron; thus some of the subadditivity inequalities $\Delta\pi(x, y) \geq 0$ are tight, i.e., additivity holds.

1.3. Continuous interpolations of extreme functions and the infinite group problem $R_f(\mathbb{R}, \mathbb{Z})$. Gomory and Johnson, in their seminal papers [11, 12], noted that many extreme functions for finite master group problems follow simple patterns that become apparent in the piecewise linear interpolations of these functions. The simplest pattern is that of the well-known two-slope function giving the Gomory mixed integer cut (`gmic` )¹, which can be found in all finite group problems; see Figure 1 (left).¹ Gomory and Johnson initiated a program to study such functions of a real variable systematically. The technical framework is that of the *infinite group problem*, in which the group G in (1.2) is enlarged from $\frac{1}{q}\mathbb{Z}$ to \mathbb{R} . Gomory and Johnson proved that the characterization (1.3) of minimal functions still holds in this setting.

For an extreme function $\pi|_G$ for a finite master problem $R_f(G, \mathbb{Z})$, the piecewise linear interpolation $\pi = \text{interpolate_to_infinite_group}(\pi|_G)$ is a minimal function, but not necessarily extreme. (A partial converse is true; the restriction of a continuous piecewise linear extreme function π for $R_f(\mathbb{R}, \mathbb{Z})$ to a group G that includes all breakpoints of π is extreme for $R_f(G, \mathbb{Z})$.) There

¹A function name shown in sans serif font refers to the software [21], which includes the Electronic Compendium of Extreme Functions [17].

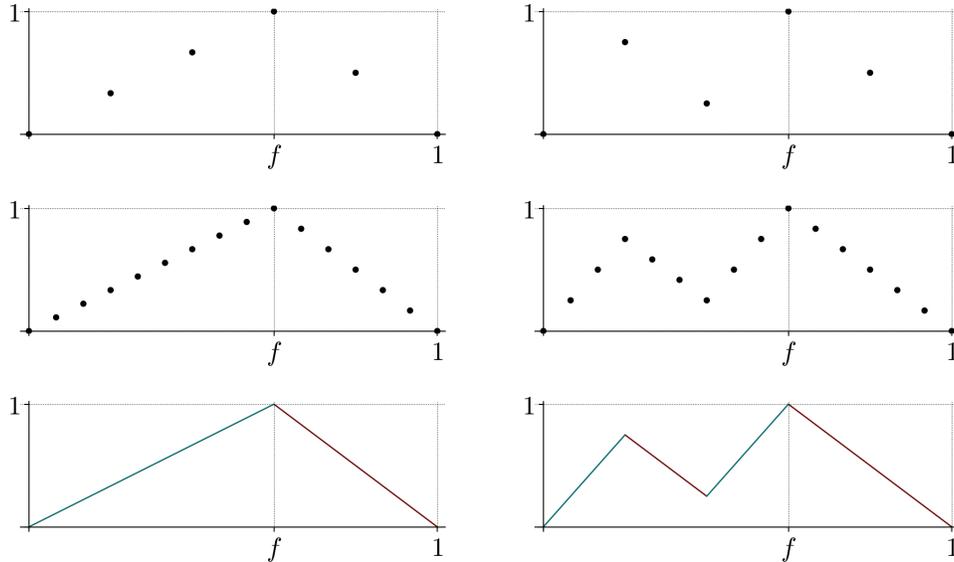


FIGURE 1. Extreme functions for finite master group problems following simple patterns that become apparent in the piecewise linear interpolations. *Left*, the Gomory mixed-integer cut (gmic). *Right*, another two-slope extreme function.

is a possible viewpoint on the extreme functions for the infinite group problems as “robust” cut-generating functions that ignore the specific number-theoretic properties of a particular group problem $R_f(\frac{1}{q}\mathbb{Z}, \mathbb{Z})$. As a matter of fact, in a numerical implementation, the value q and exact rational value of f would not be readily available.

A natural algorithmic focus lies on piecewise linear valid functions, though a part of the literature [1, 2] also studies more complicated functions. (When we discuss piecewise linear functions in this paper, we include the discontinuous case with possible jumps at breakpoints, which includes important examples such as the Gomory fractional cut, `gomory_fractional` \leftarrow ..)

For \mathbb{Z} -periodic piecewise linear functions, the characterization (1.3) of minimal functions gives a simple algorithm for testing minimality, based on enumerating the vertices of a certain polyhedral complex; see [3, section 2.2] and [16, section 5]. For testing the extremality of a piecewise linear minimal function, however, in contrast to the finite group case, we cannot directly use polyhedral methods any more. Since the quotient \mathbb{R}/\mathbb{Z} is not finite, we have to use infinite-dimensional methods of functional equations and inequalities. The most important lemma from this theory is the Gomory–Johnson interval lemma, variants of which has been used in virtually all proofs of extremality in the literature.

1.4. The space $\tilde{\Pi}^\pi$ of effective perturbations $\tilde{\pi}$ of a minimal valid function. Recall that by definition (E), a minimal valid function π is extreme if it cannot be written as a convex combination of two other minimal valid functions π^+ , π^- . A fruitful approach to extremality testing, introduced by Basu et al. in Part I of the present series of papers [3], has been to consider the difference function (*perturbation*) $\bar{\pi} = \pi^+ - \pi$, which allows us to write $\pi^1 = \pi + \bar{\pi}$ and $\pi^2 = \pi - \bar{\pi}$. (Recently, Di Summa [9] obtained a breakthrough result on the question of piecewise linearity of extreme functions using this approach.) It is convenient to build a space from the notion of perturbation functions. Following Part V [16, section 6], we define the space

$$(1.4) \quad \tilde{\Pi}^\pi = \{ \tilde{\pi}: \mathbb{R} \rightarrow \mathbb{R} \mid \exists \epsilon > 0 \text{ s.t. } \pi^\pm = \pi \pm \epsilon \tilde{\pi} \text{ are minimal valid} \}$$

of *effective perturbation functions* for the minimal valid function π . This is a vector space (Lemma 9.11), a subspace of the space of bounded functions. The function π is extreme if and only if the space $\tilde{\Pi}^\pi$ is trivial.

If additivity ($\Delta\pi(x, y) = 0$) holds for some (x, y) , then by convexity also $\Delta\tilde{\pi}(x, y) = 0$ holds for every effective perturbation $\tilde{\pi} \in \tilde{\Pi}^\pi$. This is also true for additivity in the limit [3, Lemma 2.7]; see also [16, Lemma 6.1]. Because π is assumed to be piecewise linear, the infinite system of functional equations describing additivity and limit-additivity of $\tilde{\pi}$ can be structured (“combinatorialized”) according to a certain polyhedral complex [3, 16].

1.5. Finite-dimensional and equivariant perturbations. In Part I of the present series, Basu et al. [3] gave the first algorithm to decide extremality of a piecewise linear function with rational breakpoints in some “grid” (group) $G = \frac{1}{q}\mathbb{Z}$.

In a first step, one tests whether there exists a nontrivial perturbation for π in the finite-dimensional subspace of $\tilde{\Pi}^\pi$ that consists of the functions `interpolate_to_infinite_group`($\tilde{\pi}|_G$), where $\tilde{\pi}|_G$ is an effective perturbation function for the restriction $\pi|_G$ to the finite group problem $R_f(G, \mathbb{Z})$.

Otherwise, one may assume that $\tilde{\pi}|_G = 0$. Under this assumption, the interval lemma forces $\tilde{\pi}|_C = 0$ for certain *directly covered* intervals C . Basu et al.’s crucial observation was that if there are any remaining *uncovered* intervals, then one-dimensional families of additivity equations impose a type of symmetry of the perturbation function. By analyzing the required symmetry, one can construct a perturbation function and prove nonextremality of π .

Consider the additivity equations

$$(1.5) \quad \Delta\tilde{\pi}(x, t) = \tilde{\pi}(x) + \tilde{\pi}(t) - \tilde{\pi}(x + t) = 0, \quad \text{for } x \in D,$$

where D is an interval and $t \in \frac{1}{q}\mathbb{Z}$ is a grid point. Because $\tilde{\pi}(t) = 0$, this simplifies to

$$(1.6a) \quad \tilde{\pi}(x) = \tilde{\pi}(x + t) \quad \text{for } x \in D.$$

We then say that $\tilde{\pi}$ is *invariant* under the action of the *translation* $\tau_t: x \mapsto x+t$ (restricted to the interval D). Likewise, a second type of one-dimensional families of additivity equations simplifies to

$$(1.6b) \quad \tilde{\pi}(x) = -\tilde{\pi}(r-x) \quad \text{for } x \in D.$$

Here a negative sign comes in. We call $\rho_r: x \mapsto r-x$ a *reflection*. By assigning a *character* $\chi(\tau_t) = +1$ and $\chi(\rho_r) = -1$ to the translations and reflections, we can unify equations (1.6) as

$$(1.7) \quad \tilde{\pi}(x) = \chi(\gamma) \tilde{\pi}(\gamma(x)) \quad \text{for } x \in D,$$

where γ is either a translation or a reflection. We then say that $\tilde{\pi}$ is *equivariant* under the action of γ .

By analyzing the group $\Gamma \subset \text{Aff}(\mathbb{R})$ generated by all relevant translations and reflections, Basu et al. constructed a universal template function $\psi: \mathbb{R} \rightarrow \mathbb{R}$, a continuous piecewise linear function with breakpoints in $\frac{1}{4q}\mathbb{Z}$, which is equivariant under the action of the group Γ . Taking

$$(1.8) \quad \tilde{\pi}(x) = \begin{cases} \psi(x) & \text{for } x \text{ in uncovered intervals,} \\ 0 & \text{for } x \text{ in covered intervals} \end{cases}$$

then gives an effective perturbation function. (A revised construction in Basu et al.'s survey [6, section 8.2] gives a continuous piecewise linear function $\tilde{\pi}$ with breakpoints in $\frac{1}{3q}\mathbb{Z}$.)

1.6. Contributions of the present paper. It has been a long-term research project to develop a complete, *grid-free* algorithmic theory and software implementation for piecewise linear minimal valid functions, extending the reach of the grid-based extremality test introduced in Part I of the series [3], which we described in subsection 1.5 above. While Parts II–IV develop a grid-based theory for 2-row relaxations, Part V of our series [16] returned to the one-row case. It introduced our software [21] and prepared the grid-free theory with several results. Part VI of the series [20] discussed the case of piecewise linear functions that are discontinuous on both sides of the origin and have irrational breakpoints. The present paper, part VII of the series, and a computational companion paper, part VIII of the series, are the culmination of the project for the case of piecewise linear functions of one variable.

1.6.1. *Method: Inverse semigroups as the language of partial symmetries.*

Group actions are the standard language to describe symmetries of mathematical objects. The use of group actions was fruitful in Part I of our series to obtain the first algorithm for testing extremality. However, group actions do not provide a complete theory of the effective perturbations. This becomes most apparent in [3, section 5], where Basu et al. introduce a family of extreme functions with irrational breakpoints, `bhk_irrational` . Here the group Γ generated by the translations and reflections only gives the

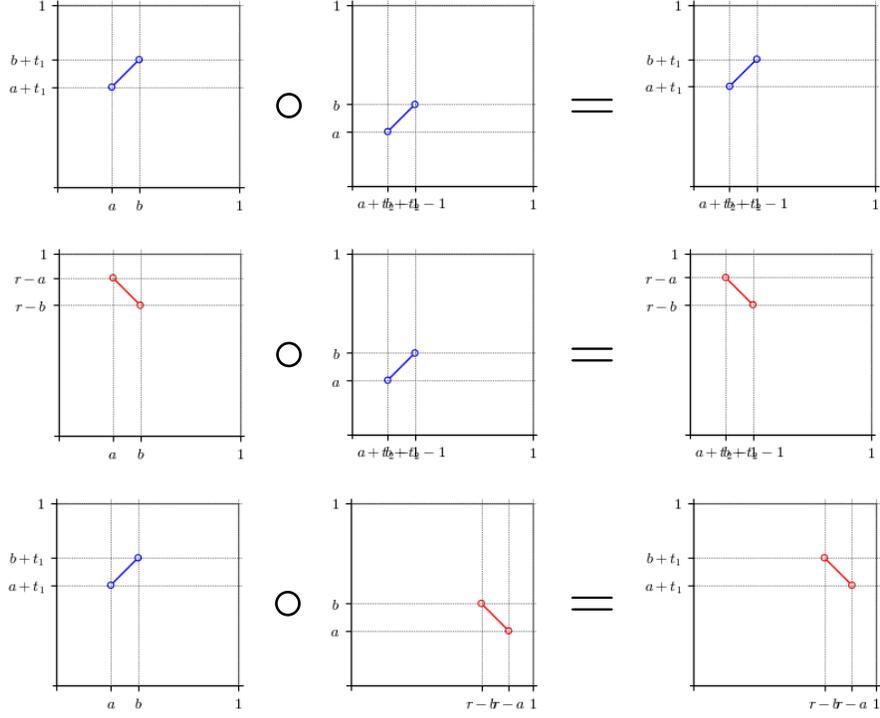


FIGURE 2. Operations of the inverse semigroup I: Composition

correct result when a certain non-group-theoretic reachability condition [3, Assumption 5.1, Lemma 5.2] is satisfied.

The underlying reason is that the restriction of the translations and reflections to the interval domains D in (1.6) is not considered in the reflection group. Indeed, what the translations and reflections describe is not a full symmetry of the perturbation function, but only a *partial symmetry* within the uncovered intervals.

The correct language to describe partial symmetries is the less well-known theory of inverse-semigroup actions. An *inverse semigroup* $(\Gamma, \circ, \cdot^{-1})$, following [22, page 7], is a semigroup, i.e., a set Γ closed under an associative operation \circ , satisfying the additional axiom that

$$\begin{aligned}
 (\exists! \text{ inverse}) \quad & \text{for every } \omega \in \Gamma, \text{ there exists a unique element } \omega^{-1} \in \Gamma \\
 & \text{such that } \omega = \omega \circ \omega^{-1} \circ \omega \text{ and } \omega^{-1} = \omega^{-1} \circ \omega \circ \omega^{-1}.
 \end{aligned}$$

The equations in the axiom describe the familiar properties of a pseudoinverse, but due to the required uniqueness, we will simply refer to ω^{-1} as the *inverse* of ω . In his monograph [22], Lawson points out that

the relationship between inverse semigroups and partial symmetries is a generalization of the relation between groups and symmetries.

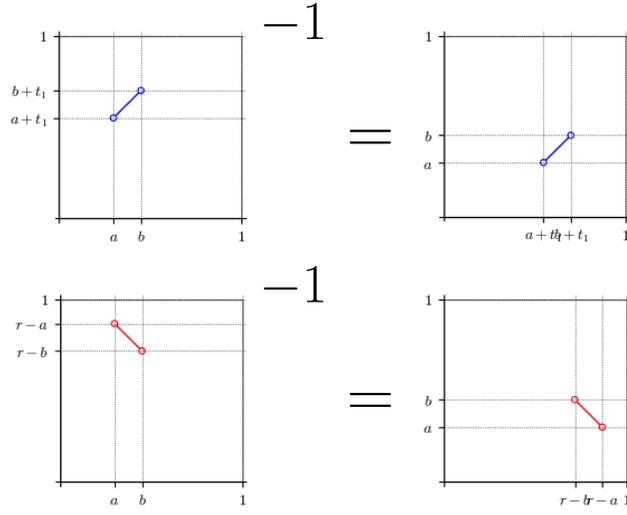
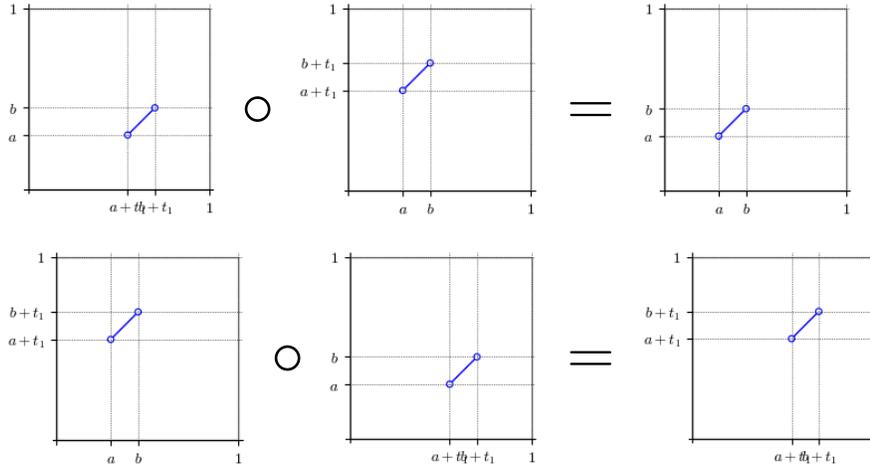


FIGURE 3. Operations of the inverse semigroup II: Inverse

FIGURE 4. Operations of the inverse semigroup III: Composition with inverse. The partial identities $\tau_0|_D$ are the idempotents of the inverse semigroup.

Concretely, inverse semigroups arise as semigroups of partial bijections of a set, where the operation \circ is the composition and \cdot^{-1} is the inverse of a partial bijection. We define the restrictions of the previously defined translations and reflections to open intervals D . We denote them by $\tau_t|_D$ and $\rho_r|_D$ and consider them as partial bijections of \mathbb{R} to itself, with domains $\text{dom}(\tau_t|_D) = D = \text{dom}(\rho_r|_D)$ and images $\text{im}(\tau_t|_D) = \tau_t(D) = D + t$ and $\text{im}(\rho_r|_D) = \rho_r(D) = r - D$. We refer to these partial bijections as *moves*.

The composition of two moves $\gamma_1|_{D_1}$ and $\gamma_2|_{D_2}$ is defined as

$$(1.9) \quad \gamma_2|_{D_2} \circ \gamma_1|_{D_1} = \gamma_2 \circ \gamma_1|_{D_1 \cap \gamma_1^{-1}(D_2)};$$

see Figure 2. The domain of the composition is either an open interval or the empty set. (By definition, there are exactly two empty moves: the empty translation $\tau|_{\emptyset}$ and the empty reflection $\rho|_{\emptyset}$.) Note that the inverse of a move $\gamma|_D$, given by $(\gamma|_D)^{-1} = \gamma^{-1}|_{\gamma(D)}$, is not an inverse in a group-theoretic sense: The compositions

$$(1.10) \quad \gamma|_D \circ (\gamma|_D)^{-1} = \tau_0|_{\gamma(D)} \quad \text{and} \quad (\gamma|_D)^{-1} \circ \gamma|_D = \tau_0|_D$$

are only *partial identities* (restrictions of the identity τ_0 to intervals) and therefore not neutral elements but merely idempotents (Figure 4).

We develop methods that center around inverse semigroups of moves and their generating sets. We study the set of moves that are respected by the effective perturbations of a given minimal function π . We analyze the closure properties (axioms) that it satisfies: algebraically, it is an inverse semigroup; but additional order-theoretic and analytic closure properties come in. Starting from an initial set (*move ensemble*) Ω^0 , we can then form the closure with respect to the axioms. We call it the *moves closure* of Ω^0 (or *closed move semigroup* generated by Ω^0) and denote it by $\text{clsemi}_A(\Omega^0)$.

In the first part of the paper, we develop these methods in full generality, without using any specific properties of the Gomory–Johnson model. Then we turn to the study of piecewise linear functions; here we make the assumption of continuity from at least one side of the origin.

For all piecewise linear functions with rational breakpoints, we will show that $\text{clsemi}_A(\Omega^0)$ has a simple structure: Its graph consists of a finite union of line segments and rectangles. (We say that it is *finitely presented*.) It will become clear that we can compute $\text{clsemi}_A(\Omega^0)$ in finitely many steps using a completion-type algorithm, using only the algebraic and order-theoretic axioms, by manipulating finite presentations of generating systems. However, this algorithm is not the focus of the present paper: We defer all computational questions to the forthcoming companion paper [14].

Instead, an important point of our paper is that finitely presented closures $\text{clsemi}_A(\Omega^0)$ arise in a more general context, through the interplay of the order-theoretic, algebraic, and analytic closure properties. Move ensembles whose graphs are connected open sets extend to open rectangles already in the joined semigroup (Corollary 4.9). Our key theorem using the analytic properties is Theorem 7.9: Rectangles appear in the closure whenever there is a convergent sequence of moves. (In part I of our series, we have observed a glimpse of this phenomenon already, in a specific arithmetic context.) Empirically, for all families of piecewise linear minimal valid functions in the literature (see [17] for an electronic compendium), even if the breakpoints are irrational, the closure has a finite presentation. This includes the function `bhk_irrational` , which we mentioned above. Again, we defer

questions regarding the computation of this closure, which then needs to use the additional axioms, to our forthcoming paper [14].

1.6.2. *Result: Precise description of the space of equivariant perturbations.* Under the above assumptions, the finite presentation of $\text{clsemi}_A(\Omega^0)$ allows us to read off a precise description of the space of equivariant perturbations as a direct sum decomposition of vector subspaces (Theorem 10.30).

One component in the decomposition is a finite-dimensional space, consisting of (possibly discontinuous) piecewise linear functions. In contrast to the grid-based algorithm, the set of breakpoints of these functions is not fixed, but it is computed by our algorithm. The finite-dimensional space is then described by a system of finitely many linear equations (Lemma 10.25).

Then, for each of the finitely many *uncovered components* (defined in section 10), there is a component that is an infinite-dimensional space isomorphic to the space of Lipschitz functions on a compact interval that vanish on the boundary. More specifically, our algorithm computes an open interval D , the *fundamental domain*, on which we take the space of Lipschitz functions $\tilde{\pi}$ that vanish on the boundary ∂D . Additionally there are finitely many moves $\gamma_j|_D$ with pairwise disjoint images $\gamma_j(D)$ that together extend the functions equivariantly to the whole uncovered component. Outside of the component, the functions in this space are zero. This is Theorem 10.28.

This description of the space strengthens previous results. The method of Part I [3], described in subsection 1.5, guarantees to construct a piecewise linear effective perturbation if the space is nontrivial; but it does not provide a complete description of the space. A theorem regarding direct sum decomposition appeared in [5, Theorem 3.14], but it is limited to the grid case.

We remark that the precise description of the perturbation space of a minimal function π enables us to strengthen (*lift*) it by constructing a direction in the space of effective perturbations. By our theorem, the problem of finding such a direction decomposes into subproblems; one finite-dimensional, the others independent variational problems over Lipschitz functions.

1.6.3. *Computational implications: Grid-free algorithms, natural proofs.* We only sketch the computational implications of the present paper because we will elaborate on them in our companion paper. The inverse semigroup theory lays the foundation for grid-free algorithms for minimal valid functions, including automated extremality tests, which are detached from the finite group problem. A grid-free test is faster for functions whose breakpoints are rational numbers with huge denominators; and it enables computations for functions with irrational breakpoints. More importantly, the grid-free algorithms can give natural extremality proofs, similar to the general proof pattern of extremality proofs in the published literature. In this way, the grid-free algorithms enable automated extremality proofs for smoothly parameterized families of extreme functions, as described in [18].

1.7. Structure of the paper. In sections 2–4, we introduce moves as partial bijections of \mathbb{R} . We study *ensembles* (sets) of such moves, which can be equipped with both an order-theoretic structure (restriction and continuation) and an algebraic structure (inverse semigroups). In section 5 we describe how move ensembles and semigroups describe partial symmetries of a function by a system of functional equations. Move ensembles for bounded functions have additional properties, which we explore in section 6. Then, in section 7, we study closure properties that capture the additional properties of move ensembles for continuous functions. This development culminates in the notion of *closed move semigroups* in subsection 7.3.

We then apply this theory to compute the effective perturbation space of a piecewise linear minimal valid function π . In section 8, we introduce the *initial additive move ensemble* Ω^0 , which describes functional equations satisfied by every effective perturbation of π . For piecewise linear functions π , it is related to the *additive faces* of a polyhedral complex (section 9). Finally, in section 10, working with a finite presentation of the closed move semigroup $\text{clsemi}_A(\Omega^0)$ generated by Ω^0 , we prove the main theorem of the paper, the decomposition theorem for the space of effective perturbations of π .

We end the paper in section 11 with a discussion of the limitations of our approach and an outlook on the computational companion paper [14]. See the next pages for a detailed table of contents.

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TABLE 1. Notation for move ensembles and semigroups

$\Gamma(\mathbb{R})$	Group of unrestricted translations and reflections of \mathbb{R}	17
τ_t, ρ_r	translation, reflection	
γ	some element	
$\Gamma^{\subseteq}(\mathbb{R})$	Inverse semigroup of translations, reflections with domains	17
$\tau_t _D$	translation restricted to open interval D	
$\rho_r _D$	reflection restricted to open interval D	
$\gamma _D$	unrestricted move restricted to open interval D	
Ω	A move ensemble: a subset of $\Gamma^{\subseteq}(\mathbb{R})$	19
Ω^{inv}	... satisfying (inv)	21
Γ	A move semigroup: an inverse subsemigroup of $\Gamma^{\subseteq}(\mathbb{R})$	21
$\Omega^{\subseteq}, \Gamma^{\subseteq}$	A move ensemble, or semigroup, satisfying (restrict)	19
$\Omega^{\vee}, \Gamma^{\vee}$... satisfying (restrict), (continuation)	19
$\Omega^{\boxtimes}, \Gamma^{\boxtimes}$... satisfying (restrict), (continuation), (kaleido)	28
$\bar{\Omega}, \bar{\Gamma}$... satisfying limit axiom (lim) or (arblim)	32
$\bar{\Omega}^{\vee}, \bar{\Gamma}^{\vee}$... satisfying (extend $_A$)	
Ω^{fin}	A finite move ensemble	} finite presentation
\mathcal{C}	Connected covered components	
Ω^{red}	A reduced finite move ensemble	
\mathbb{L}, \mathbb{M}	Families of move ensembles	38

TABLE 2. List of axioms for move ensembles

(composition)	} <i>move semigroup</i>	} <i>joined</i>	} <i>closed move</i>	21
(inv)				$\Gamma = \text{isemi}(\Omega)$
(restrict)	} <i>joined ensemble</i>	} $\Gamma^{\vee} = \text{jsemi}(\Omega)$	} <i>semigroup</i>	19
(continuation)				$\Omega^{\vee} = \text{join}(\Omega)$
(kaleido)	<i>kaleidoscopic ens.</i>	Ω^{\boxtimes}	$\text{clsemi}_A(\Omega)$	28
(lim), (arblim)	<i>limits-closed ens.</i>	$\bar{\Omega} = \text{arblim}(\Omega)$		33
(extend $_A$)	<i>extended ensemble</i>	$\bar{\Omega}^{\vee} = \text{extend}_A(\Omega)$		35

2. TRANSLATION AND REFLECTION MOVES. THEIR ALGEBRAIC AND ORDER-THEORETIC STRUCTURE

2.1. Group $\Gamma(\mathbb{R})$ of unrestricted translations τ_t and reflections ρ_r , character χ .

Definition 2.1. For a point $r \in \mathbb{R}$, define the (unrestricted) *reflection* $\rho_r: \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto r - x$. For a vector $t \in \mathbb{R}$, define the (unrestricted) *translation* $\tau_t: \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto x + t$.

The set $\Gamma(\mathbb{R}) = \{\rho_r, \tau_t \mid r \in \mathbb{R}, t \in \mathbb{R}\}$ of all translations and reflections, with the operations of function composition \circ and inverse \cdot^{-1} , has the structure of a group. It is a subgroup of the group $\text{Aff}(\mathbb{R})$ of regular affine transformations of \mathbb{R} .

To denote an element that can be either a translation or a reflection, we will usually use the letter γ . To recover whether an element γ is a translation or a reflection, we assign a *character* $\chi(\rho_r) = -1$ to every reflection and $\chi(\tau_t) = +1$ to every translation. The map $\gamma \mapsto \chi(\gamma)$ is a group character, i.e., a homomorphism, so compositions of elements follow the rule

$$(2.1) \quad \chi(\gamma_1 \circ \gamma_2) = \chi(\gamma_1) \cdot \chi(\gamma_2).$$

2.2. **Restricted moves $\gamma|_D \in \Gamma^{\subseteq}(\mathbb{R})$ as partial bijections of \mathbb{R} .** As we mentioned in the introduction, compared to [3], where finitely generated subgroups of $\Gamma(\mathbb{R})$ were used for the grid-based extremality test algorithm, in this paper we develop a more detailed theory using restricted moves with domains. Our terminology is based on the monograph [22] on inverse semigroups.

We begin by restricting translations and reflections $\gamma \in \Gamma(\mathbb{R})$ to open interval domains $D \subseteq \mathbb{R}$.

Definition 2.2. Let $\gamma \in \Gamma(\mathbb{R})$ be a translation or reflection, and let $D \subseteq \mathbb{R}$ be an open interval or the empty set.

- (a) The *move* $\gamma|_D$ is the partial function with domain D and image $\gamma(D)$, defined by $\gamma|_D(x) = \gamma(x)$ for $x \in D$.
- (b) The *character* of $\gamma|_D$ is the character of γ .
- (c) Two moves $\gamma_1|_{D_1}, \gamma_2|_{D_2}$ with open interval domains D_1, D_2 are *equal* if $\gamma_1 = \gamma_2$ and $D_1 = D_2$. A move with an open interval domain is not *equal* to a move with an empty domain. We identify all translations with empty domain and denote this object by $\tau|_{\emptyset}$. Likewise, we identify all reflections with empty domain and denote this object by $\rho|_{\emptyset}$. The empty translation and the empty reflection are not *equal*; they are distinct objects with $\chi(\tau|_{\emptyset}) = +1$ and $\chi(\rho|_{\emptyset}) = -1$.
- (d) The set of all moves is denoted by $\Gamma^{\subseteq}(\mathbb{R})$.

2.2.1. *Remark on the relation to pseudogroups.* Inverse semigroups of partial homeomorphisms between open subsets of a topological space are known as *pseudogroups* [22, section 1.2]. However, our theory differs in the following

ways: (1) We only allow open intervals (and the empty set) as domains of the partial functions, rather than arbitrary open subsets. The reason for our choice will become clear in section 5, where we will use moves to describe systems of functional equations. (2) Less importantly, we have two empty moves, one for each possible character, rather than a unique empty move.

2.3. Graphs of moves. We find it convenient to describe the graphs of moves. The *graph* of $\gamma|_D$ is the set

$$\text{Gr}(\gamma|_D) = \{ (x, y) \in \mathbb{R} \times \mathbb{R} \mid x \in D, \gamma(x) = y \}.$$

Figures showing the graphs have already appeared in Figure 2 and Figure 3. To emphasize that the domains of all nonempty moves are open intervals, we decorate the endpoints of the moves by hollow circles, indicating that the endpoints are not part of the graphs.

2.4. Restriction partial order \subseteq on moves. The set of all moves comes with a natural partial order.

Definition 2.3. $\gamma_1|_{D_1}$ is a *restriction* of $\gamma_2|_{D_2}$, denoted $\gamma_1|_{D_1} \subseteq \gamma_2|_{D_2}$, if $D_1 \subseteq D_2$, $\chi(\gamma_1) = \chi(\gamma_2)$, and $\gamma_1(x) = \gamma_2(x)$ for $x \in D_1$.

Thus, in this partial order, translations and reflections are incomparable. We have $\tau|_\emptyset \subseteq \tau_t|_D$ for all translations and likewise $\rho|_\emptyset \subseteq \rho_r|_D$.

Definition 2.4. Given $\gamma|_D$ and an open interval (or empty set) $D' \subseteq D$, the *restriction* of $\gamma|_D$ to D' is the move $(\gamma|_D)|_{D'} = \gamma|_{D'}$. Given an open interval (or empty set) $I' \subseteq \gamma(D)$, the *corestriction* of $\gamma|_D$ to I' is the move $I'|(\gamma|_D) = \gamma|_{D \cap \gamma^{-1}(I')}$.

2.5. Inverse semigroup structure $(\Gamma^{\subseteq}(\mathbb{R}), \circ, \cdot^{-1})$. Let $\gamma_1|_{D_1}$ and $\gamma_2|_{D_2}$ be two moves. As noted in the introduction, their composition $\gamma_2|_{D_2} \circ \gamma_1|_{D_1}$ is defined as $\gamma_2 \circ \gamma_1|_{D_1 \cap \gamma_1^{-1}(D_2)}$ (Figure 2). The domain of this partial bijection is either an open interval or the empty set; so it is again a move. It is clear that the composition operation \circ is associative. Hence the moves form a semigroup $(\Gamma^{\subseteq}(\mathbb{R}), \circ)$.

As we have noted already, a move $\gamma|_D$ also has a (unique) inverse given by $(\gamma|_D)^{-1} = \gamma^{-1}|_{\gamma(D)}$ (Figure 3) satisfying the laws (1.10) (Figure 4). Hence the moves form an inverse semigroup $(\Gamma^{\subseteq}(\mathbb{R}), \circ, \cdot^{-1})$. Its idempotent elements are exactly the *partial identities*, which are restrictions of the identity translation τ_0 to open intervals D together with the empty translation $\tau|_\emptyset$. (The empty reflection is not idempotent; we have $\rho|_\emptyset \circ \rho|_\emptyset = \tau|_\emptyset$.)

The inverse semigroup structure interacts with the restriction partial order (subsection 2.4) as follows [22, Proposition 1.1.4]. If $\gamma|_{D'} \subseteq \gamma|_D$, then $\gamma|_{D'}^{-1} \subseteq \gamma|_D^{-1}$; moreover, this restriction can be expressed as a composition with an idempotent: $\gamma|_{D'} = (\gamma|_D)|_{D'} = \gamma|_D \circ \tau_0|_{D'}$. Finally, if $\gamma_i|_{D'_i} \subseteq \gamma_i|_{D_i}$ for $i = 1, 2$, then $\gamma_2|_{D'_2} \circ \gamma_1|_{D'_1} \subseteq \gamma_2|_{D_2} \circ \gamma_1|_{D_1}$.

3. ENSEMBLES Ω OF MOVES. THEIR ORDER-THEORETIC STRUCTURE

Now we consider *move ensembles* Ω , i.e., arbitrary subsets of the inverse semigroup $\Gamma^{\subseteq}(\mathbb{R})$. We denote elements by $\gamma|_D$, where $\gamma \in \Gamma(\mathbb{R})$ is an unrestricted move and D is the domain .

3.1. Order-theoretic structure.

3.1.1. Restriction-closed move ensembles Ω^{\subseteq} .

Definition 3.1. A move ensemble Ω^{\subseteq} is *restriction-closed* if it satisfies the following axiom.

(restrict) If $\gamma|_D \in \Omega^{\subseteq}$ and $D' \subseteq D$ is an open interval or the empty set,
then $\gamma|_{D'} \in \Omega^{\subseteq}$.

For a move ensemble Ω , the restriction closure $\text{restrict}(\Omega)$ is the smallest restriction-closed move ensemble containing Ω . (It consists of all restrictions of moves of Ω .)

Remark 3.2. Throughout the paper, a superscript like \subseteq in Ω^{\subseteq} indicates an axiom that the set Ω^{\subseteq} satisfies. See Table 1 for an overview of notation used in the paper.

Example 3.3. The inverse semigroup $\Gamma^{\subseteq}(\mathbb{R})$ of all restricted translations and reflections is a restriction-closed move ensemble.

3.1.2. Join-closed move ensembles Ω^{\vee} .

Definition 3.4. A move ensemble Ω^{\vee} is (*completely*) *join-closed* if it satisfies (restrict) and the following condition.

(continuation) If there is a family $\Omega_{\mathfrak{J}} = \{\gamma|_I \mid I \in \mathfrak{J}\} \subseteq \Omega^{\vee}$
such that $D = \bigcup_{I \in \mathfrak{J}} I$ is an interval,
then $\gamma|_D \in \Omega^{\vee}$.

Definition 3.5. We define the *joined ensemble* $\text{join}(\Omega)$ of Ω as the smallest set of moves containing Ω that satisfies (continuation) and (restrict).

Lemma 3.6. For a move ensemble Ω , the joined ensemble $\text{join}(\Omega)$ consists of the following moves.

$$(3.1) \quad \left\{ \gamma|_D \mid D \subseteq \bigcup_{I \in \mathfrak{J}} I, \text{ where } \gamma|_I \in \Omega \text{ for } I \in \mathfrak{J}, D \text{ empty or open interval} \right\}.$$

Proof. This set clearly satisfies (continuation) and (restrict). On the other hand, $\text{join}(\Omega)$ needs to contain this set. \square

3.1.3. *Presentation by the set $\text{Max}(\Omega^\vee)$ of maximal elements.*

Definition 3.7. For a move ensemble Ω , let $\text{Max}(\Omega)$ denote the set of maximal elements of Ω in the restriction partial order.

Lemma 3.8. *A join-closed move ensemble Ω^\vee is equal to the restriction closure and to the joined ensemble of its maximal elements in the restriction partial order:*

$$(3.2) \quad \Omega^\vee = \text{restrict}(\text{Max}(\Omega^\vee)) = \text{join}(\text{Max}(\Omega^\vee))$$

Proof. Let $\gamma|_D \in \Omega^\vee$. Let $\mathfrak{D} = \{D' \supseteq D \mid \gamma|_{D'} \in \Omega^\vee\}$. Let $\bar{D} = \bigcup \mathfrak{D}$, an interval. Then $\bar{D} \in \Omega^\vee$ because Ω^\vee satisfies (continuation). Moreover, $\gamma|_D \subseteq \gamma|_{\bar{D}} \in \text{Max}(\Omega^\vee)$ and thus $\gamma|_D \in \text{restrict}(\text{Max}(\Omega^\vee))$. The other inclusions are trivial. \square

3.2. Move ensembles as set-valued maps $\mathbf{R} \Rightarrow \mathbf{R}$. Domains, images, restrictions.

Definition 3.9. Let Ω be a move ensemble and R be a disjoint union of open intervals, $R = \bigcup_{R' \in \mathfrak{J}} R'$. The *restriction* $\Omega|_R$ is the move ensemble consisting of the restrictions $\gamma|_{D \cap R'}$ whenever $\gamma|_D \in \Omega$, $R' \in \mathfrak{J}$, and either $D = \emptyset$ or $D \cap R' \neq \emptyset$. Similarly, we define the *corestriction* ${}_R|\Omega$ and the *double restriction* ${}_R|\Omega|_R$.

In the restrictions, domains of moves are restricted to subintervals of R . Note that by our definition, the restrictions contain empty moves if and only if Ω contains empty moves. Therefore we have the following two convenient properties:

Lemma 3.10. *For a move ensemble Ω^\subseteq that satisfies (restrict), the restrictions satisfy (restrict), and we have*

$$\begin{aligned} \Omega^\subseteq|_R &= \{ \gamma|_D \in \Omega^\subseteq \mid D \subseteq R \}, \\ {}_R|\Omega^\subseteq &= \{ \gamma|_D \in \Omega^\subseteq \mid \gamma(D) \subseteq R \}, \\ {}_R|\Omega^\subseteq|_R &= \{ \gamma|_D \in \Omega^\subseteq \mid D, \gamma(D) \subseteq R \}. \end{aligned}$$

Likewise, restrictions also preserve (continuation).

Lemma 3.11. *Let $\Omega^{\max} = \text{Max}(\Omega^\vee)$, where Ω^\vee is a joined ensemble. Then each of the restrictions $\Omega^{\max}|_R$, ${}_R|\Omega^{\max}$, and ${}_R|\Omega^{\max}|_R$ consists of the maximal elements of $\Omega^\vee|_R$, ${}_R|\Omega^\vee$, and ${}_R|\Omega^\vee|_R$, respectively.*

We define the domain and image of a move ensemble Ω , as well as the image of a set under the ensemble.

Definition 3.12. Let Ω be a move ensemble. The *domain* of Ω is

$$\text{dom}(\Omega) = \bigcup \{ D \mid \gamma|_D \in \Omega \text{ for some } \gamma \};$$

the *image* of Ω is

$$\text{im}(\Omega) = \bigcup \{ \gamma(D) \mid \gamma|_D \in \Omega \text{ for some } \gamma \}.$$

Definition 3.13. Let Ω be a move ensemble and $X \subseteq \mathbb{R}$ be a set. Define

$$\Omega(X) = \{ \gamma(x) \mid \gamma|_D \in \Omega, x \in X \cap D \}.$$

Remark 3.14. In these notions, a move ensemble behaves like a set-valued map $\Omega: \mathbb{R} \Rightarrow \mathbb{R}$.

3.3. Graphs $\text{Gr}(\Omega)$, $\text{Gr}_+(\Omega)$, $\text{Gr}_-(\Omega)$ of move ensembles Ω . We introduced graphs of moves in subsection 2.3. For a move ensemble Ω we define the *translation moves graph*

$$\text{Gr}_+(\Omega) = \bigcup \{ \text{Gr}(\gamma|_D) \mid \gamma|_D \in \Omega \text{ and } \chi(\gamma) = 1 \},$$

consisting of line segments with slopes $+1$, and the *reflection moves graph*

$$\text{Gr}_-(\Omega) = \bigcup \{ \text{Gr}(\gamma|_D) \mid \gamma|_D \in \Omega \text{ and } \chi(\gamma) = -1 \},$$

consisting of line segments with slopes -1 . The *graph* of Ω is

$$\text{Gr}(\Omega) = \text{Gr}_+(\Omega) \cup \text{Gr}_-(\Omega).$$

We also define the *character conflict graph*,

$$\text{Gr}_\pm(\Omega) = \text{Gr}_+(\Omega) \cap \text{Gr}_-(\Omega).$$

The map $\Omega \mapsto (\text{Gr}_+(\Omega), \text{Gr}_-(\Omega))$ becomes an injection if restricted to the join-closed move ensembles Ω^\vee . Hence these pairs of graphs faithfully represent all join-closed move ensembles. (In our figures showing these graphs, we superimpose the translation graph (blue) and reflection graph (red).)

We can go back from graphs to ensembles using the following notation.

Definition 3.15. Let $O \subseteq \mathbb{R}^2$. We define the (join-closed) move ensembles

$$\text{moves}_+(O) = \{ \tau_t|_D \mid \text{Gr}(\tau_t|_D) \subset O, D \text{ an open interval or empty} \},$$

$$\text{moves}_-(O) = \{ \rho_r|_D \mid \text{Gr}(\rho_r|_D) \subset O, D \text{ an open interval or empty} \},$$

$$\text{moves}(O) = \{ \gamma|_D \mid \text{Gr}(\gamma|_D) \subset O, D \text{ an open interval or empty} \}.$$

Thus, $\text{moves}(O) = \text{moves}_+(O) \cup \text{moves}_-(O)$.

4. INVERSE SEMIGROUPS GENERATED BY MOVE ENSEMBLES

Now we turn to the study of inverse semigroups generated by move ensembles.

4.1. Move semigroups Γ ; move semigroups $\text{isemi}(\Omega)$ generated by ensembles Ω .

Definition 4.1. A move ensemble Γ is a *move semigroup* (or, an *inverse subsemigroup* of $\Gamma^\subseteq(\mathbb{R})$) if it satisfies the following axioms:

$$\text{(composition)} \quad \gamma'|_{D'} \circ \gamma|_D \in \Gamma \text{ for all } \gamma|_D, \gamma'|_{D'} \in \Gamma,$$

$$\text{(inv)} \quad (\gamma|_D)^{-1} \in \Gamma \text{ for all } \gamma|_D \in \Gamma.$$

Definition 4.2. For a move ensemble Ω , the *move semigroup* $\text{isemi}(\Omega)$ generated by Ω is the smallest move semigroup containing Ω .

Definition 4.3. A move semigroup Γ is *finitely generated* if there exists a finite set Ω such that $\Gamma = \text{isemi}(\Omega)$.

Lemma 4.4. *Let Ω^{inv} be a move ensemble satisfying (inv). Then $\text{isemi}(\Omega^{\text{inv}})$ is the set of all finite compositions $\gamma^k|_{D_k} \circ \cdots \circ \gamma^1|_{D_1}$ of moves $\gamma^i|_{D_i} \in \Omega^{\text{inv}}$.*

Remark 4.5. Since the domains of moves in Ω are empty or open intervals, any move $\gamma|_D \in \text{isemi}(\Omega)$ also has a domain D that is empty or an open interval. If $\gamma|_D \in \Omega$, then the idempotent $(\gamma|_D)^{-1} \circ \gamma|_D = \tau_0|_D$ is an element of $\text{isemi}(\Omega)$. The inverse semigroup generated by the empty set is the empty set.

4.2. Move semigroups and joins; joined move semigroups $\text{jsemi}(\Omega)$ generated by ensembles Ω . Move semigroups generated by joined ensembles are not automatically join-closed. On the other hand, joining does preserve the semigroup properties.

Lemma 4.6. *Let Γ be a move semigroup. Then the joined ensemble $\text{join}(\Gamma)$ is a move semigroup. In particular, for a move ensemble Ω , we have*

$$\text{join}(\text{isemi}(\Omega)) = \text{isemi}(\text{join}(\text{isemi}(\Omega))).$$

Proof. Let $\gamma|_D, \gamma'|_{D'} \in \text{join}(\Gamma)$. We first show that $\text{join}(\Gamma)$ satisfies the axiom (composition). By equation (3.1), there exist collections \mathfrak{J} and \mathfrak{J}' of open intervals, such that $D \subseteq \bigcup_{I \in \mathfrak{J}} I$, $D' \subseteq \bigcup_{I' \in \mathfrak{J}'} I'$, and $\gamma|_I, \gamma'|_{I'} \in \Gamma$ for all $I \in \mathfrak{J}, I' \in \mathfrak{J}'$. We know that

$$\gamma'|_{I'} \circ \gamma|_I = (\gamma' \circ \gamma)|_{\gamma^{-1}(I') \cap I} \in \Gamma, \text{ for all } I \in \mathfrak{J} \text{ and } I' \in \mathfrak{J}',$$

since Γ satisfies (composition), and that

$$\gamma^{-1}(D') \cap D \subseteq \gamma^{-1}\left(\bigcup_{I' \in \mathfrak{J}'} I'\right) \cap \left(\bigcup_{I \in \mathfrak{J}} I\right) = \bigcup_{I \in \mathfrak{J}, I' \in \mathfrak{J}'} (\gamma^{-1}(I') \cap I).$$

Therefore, by equation (3.1), $\gamma'|_{D'} \circ \gamma|_D = (\gamma' \circ \gamma)|_{\gamma^{-1}(D') \cap D} \in \text{join}(\Gamma)$.

We will now show that $\text{join}(\Gamma)$ satisfies axiom (inv). We know that $(\gamma|_I)^{-1} = \gamma^{-1}|_{\gamma(I)} \in \Gamma$ for all $I \in \mathfrak{J}$, since Γ satisfies (inv), and that $\gamma(D) \subseteq \gamma(\bigcup_{I \in \mathfrak{J}} I) = \bigcup_{I \in \mathfrak{J}} \gamma(I)$. Therefore, $(\gamma|_D)^{-1} = \gamma^{-1}|_{\gamma(D)} \in \text{join}(\Gamma)$. We conclude that $\text{join}(\Gamma)$ is a move semigroup, so $\text{join}(\Gamma) = \text{isemi}(\text{join}(\Gamma))$. \square

Definition 4.7. Let Ω be a move ensemble. Then the *joined move semigroup* of Ω is defined as

$$\text{jsemi}(\Omega) = \text{join}(\text{isemi}(\Omega)).$$

4.3. Move semigroups $\text{moves}(\mathcal{O})$, $\text{moves}_+(\mathcal{O})$, $\text{moves}_-(\mathcal{O})$ generated by connected open ensembles. Finitely generated inverse semigroups, as defined in subsection 4.1, are not general enough for our purposes. As we will see later, we need to consider move ensembles Ω whose graphs are open connected sets. They generate inverse semigroups $\text{isemi}(\Omega)$ that are not finitely generated. However, they have the following simple structure (see Figure 5).

Theorem 4.8. *Let Ω be an ensemble of moves. Let $O \subseteq \mathbb{R}^2$ be a connected open set. Let $D = \text{dom}(O) := \text{dom}(\text{moves}(O))$ and $I = \text{im}(O) := \text{im}(\text{moves}(O))$.*

- (1) *If $\text{Gr}_+(\Omega)$ contains O , then $\text{Gr}_+(\text{isemi}(\Omega))$ contains $(D \cup I) \times (D \cup I)$.*
- (2) *If $\text{Gr}_-(\Omega)$ contains O , then $\text{Gr}_-(\text{isemi}(\Omega))$ contains $(D \times I) \cup (I \times D)$ and $\text{Gr}_+(\Omega)$ contains $(D \times D) \cup (I \times I)$.*
- (3) *If $\text{Gr}_\pm(\Omega)$ contains O , then $\text{Gr}_\pm(\text{isemi}(\Omega))$ contains $(D \cup I) \times (D \cup I)$.*

Proof. Part 2. We show that (2a) $\text{Gr}_-(\text{isemi}(\Omega))$ contains $D \times I$ and (2b) $\text{Gr}_+(\text{isemi}(\Omega))$ contains $D \times D$; the other two containments of $I \times D$ and $I \times I$ follow from the fact that $\text{isemi}(\Omega)$ is closed under inverse.

Let $(x, y), (x', y') \in O$ be two arbitrary points in the connected open set O . Since there is a path between (x, y) and (x', y') contained in O , and the path is compact, it is covered by finitely many open ℓ_∞ -balls $O_1, \dots, O_n \subseteq O$ with $(x_1, y_1) := (x, y) \in O_1, (x_2, y_2) \in O_1 \cap O_2, \dots, (x_n, y_n) \in O_{n-1} \cap O_n$ and $(x_{n+1}, y_{n+1}) := (x', y') \in O_n$. Since $(x_1, y_1), (x_2, y_1), (x_2, y_2), \dots, (x_n, y_n), (x_{n+1}, y_n), (x_{n+1}, y_{n+1}) \in O$, there exist $\rho_{r_1}|_{D_1}, \rho_{r'_1}|_{D'_1}, \rho_{r_2}|_{D_2}, \dots, \rho_{r_n}|_{D_n}, \rho_{r'_n}|_{D'_n}$ and $\rho_{r_{n+1}}|_{D_{n+1}} \in \Omega$ such that $\rho_{r_i}|_{D_i}(x_i) = y_i$ for $i = 1, \dots, n+1$ and $\rho_{r'_i}|_{D'_i}(x_{i+1}) = y_i$ for $i = 1, \dots, n$. Notice that the inverse restricted reflections $(\rho_{r'_i}|_{D'_i})^{-1} \in \text{isemi}(\Omega)$ with $(\rho_{r'_i}|_{D'_i})^{-1}(y_i) = x_{i+1}$ for $i = 1, \dots, n$. We have

$$x_1 \xrightarrow{\rho_{r_1}|_{D_1}} y_1 \xrightarrow{(\rho_{r'_1}|_{D'_1})^{-1}} x_2 \mapsto \dots \mapsto y_n \xrightarrow{(\rho_{r'_n}|_{D'_n})^{-1}} x_{n+1} \xrightarrow{\rho_{r_{n+1}}|_{D_{n+1}}} y_{n+1}.$$

The composition of the $2n+1$ reflections

$$\rho_r|_{D_r} := \rho_{r_{n+1}}|_{D_{n+1}} \circ (\rho_{r'_n}|_{D'_n})^{-1} \circ \rho_{r_n}|_{D_n} \circ \dots \circ (\rho_{r'_1}|_{D'_1})^{-1} \circ \rho_{r_1}|_{D_1}$$

is a restricted reflection, satisfying that $\rho_r|_{D_r} \in \text{isemi}(\Omega)$ and $\rho_r|_{D_r}(x) = y'$. Therefore, (2a) holds. The composition of the $2n$ reflections

$$\tau_t|_{D_t} := (\rho_{r'_n}|_{D'_n})^{-1} \circ \rho_{r_n}|_{D_n} \circ \dots \circ (\rho_{r'_1}|_{D'_1})^{-1} \circ \rho_{r_1}|_{D_1}$$

is a restricted translation, satisfying that $\tau_t|_{D_t} \in \text{isemi}(\Omega)$ and $\tau_t|_{D_t}(x) = x'$. Therefore, (2b) holds.

Part 1 follows exactly the same proof as part 2 using instead restricted translations $\tau_{t_1}|_{D_1}, \tau_{t'_1}|_{D'_1}, \tau_{t_2}|_{D_2}, \dots, \tau_{t_n}|_{D_n}, \tau_{t'_n}|_{D'_n}, \tau_{t_{n+1}}|_{D_{n+1}} \in \Omega$.

Part 3. Let $(x, y), (x', y') \in O$. By part 1 and 2, there exist restricted translation and reflection $\tau_t|_{D_t}, \rho_r|_{D_r} \in \text{isemi}(\Omega)$ such that $x \xrightarrow{\tau_t|_{D_t}} y \xrightarrow{\rho_r|_{D_r}} x'$. The composition $\rho_r|_{D_r} \circ \tau_t|_{D_t}$ is a restricted reflection in $\text{isemi}(\Omega)$. Therefore, $\text{Gr}_-(\text{isemi}(\Omega))$ contains $D \times D$. By part 1, part 2 and the fact that $\text{isemi}(\Omega)$ is closed under inverse, we obtain that part 3 holds. \square

The following corollary sharpens the result.

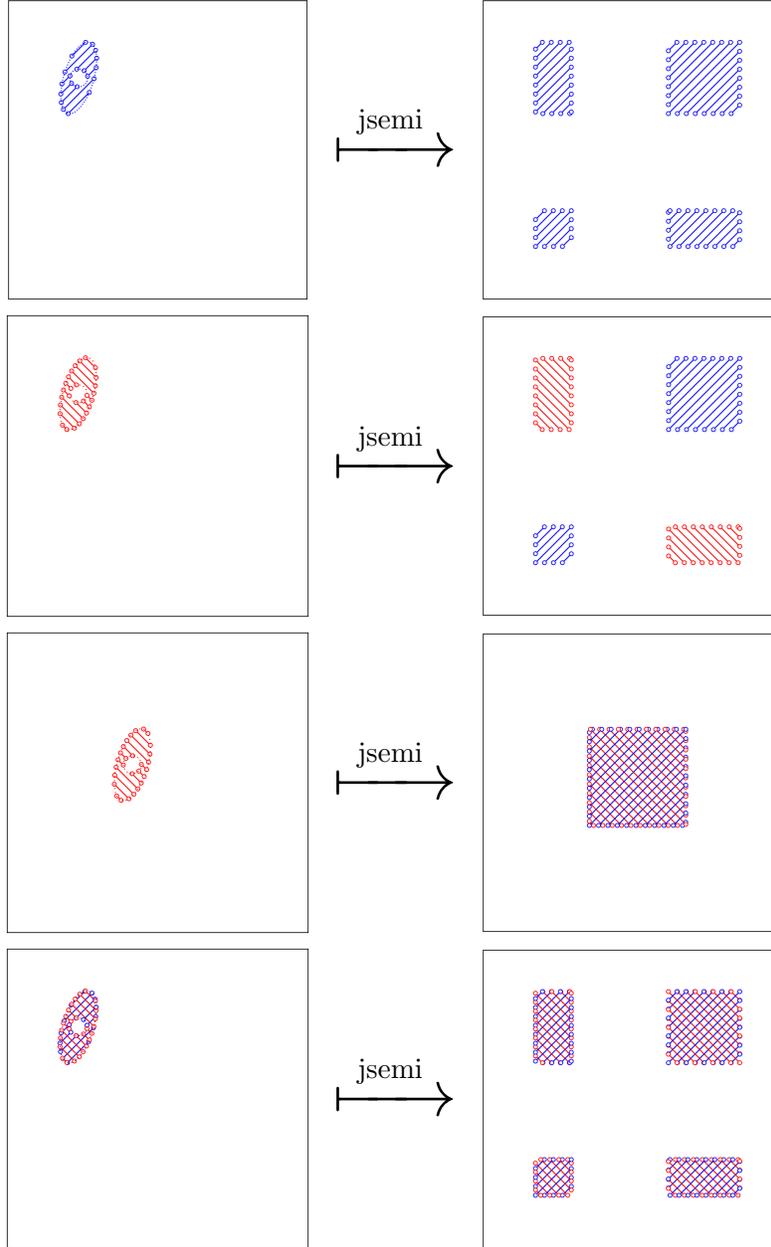


FIGURE 5. Illustrations for Theorem 4.8 and Corollary 4.9. Here, only a finite set of moves is considered. If, however, an infinite set is used by considering all moves in the O -shaped set in the left plots, then the entire rectangles would be filled in on the right plots.

Corollary 4.9. *Let $O \subseteq \mathbb{R}^2$ be a connected open set, with $D = \text{dom}(O) = \text{dom}(\text{moves}(O))$ and $I = \text{im}(O) = \text{im}(\text{moves}(O))$.*

- (1) $\text{jsemi}(\text{moves}_+(O)) = \text{moves}_+((D \cup I) \times (D \cup I)).$
- (2a) $\text{jsemi}(\text{moves}_-(O)) = \text{moves}_-((D \times I) \cup (I \times D)) \cup \text{moves}_+((D \times D) \cup (I \times I)),$ *if $D \cap I = \emptyset$.*
- (2b) $\text{jsemi}(\text{moves}_-(O)) = \text{moves}_-((D \cup I) \times (D \cup I)),$ *if $D \cap I \neq \emptyset$.*
- (3) $\text{jsemi}(\text{moves}(O)) = \text{moves}_+((D \cup I) \times (D \cup I)).$

Proof. By applying Theorem 4.8-(1), (2) and (3) to $\Omega = \text{moves}_+(O)$, $\Omega = \text{moves}_-(O)$ and $\Omega = \text{moves}(O)$, we obtain that $\text{jsemi}(\Omega)$ on the left-hand side of the equation in (1), (2a) and (3) contains the move ensemble on the right-hand side, respectively. In case (2b) where $D \cap I \neq \emptyset$, by applying Theorem 4.8-(2) to $\Omega = \text{moves}_-(O)$, we have that $\text{jsemi}(\Omega)$ contains $\text{moves}_-((D \cup I) \times (D \cap I))$. It then follows from Theorem 4.8-(3) that $\text{jsemi}(\Omega)$ contains the right-hand side of (2b). Conversely, the right-hand side of the equation in each case is a joined move semigroup that contains Ω . Hence, the equality holds. \square

Remark 4.10. Theorem 4.8 suggests to consider the following class of generating ensembles for inverse semigroups. Take a finite ensemble $\Omega^{\text{fin}} = \{\gamma^1|_{D_1}, \dots, \gamma^n|_{D_n}\}$ together with a finite list of infinite ensembles of the form $\text{moves}_+(D_i \times I_i)$, $i = n+1, \dots, n+m$ and $\text{moves}_-(D_i \times I_i)$, $i = n+m+1, \dots, n+m+\ell$, where D_i and I_i are open intervals. However, we suppress the details of this. In section 6, an additional assumption will allow us to use a more convenient class of generating ensembles.

5. Ω -EQUIVARIANT FUNCTIONS

5.1. Spaces of Ω -equivariant functions. Move ensembles encode a system of functional equations as follows.

Definition 5.1. Let Ω be a move ensemble and let $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be a function.

- (a) We say that θ is *affinely Ω -equivariant* (in short, θ *respects* Ω) provided that for every $\gamma|_D \in \Omega$ there exists a constant $c_{\gamma|_D}^\theta$ such that

$$(5.1) \quad \theta(\gamma|_D(x)) = \chi(\gamma)\theta(x) + c_{\gamma|_D}^\theta \quad \text{for } x \in D,$$

where $\chi(\gamma) = \pm 1$ is the character of γ .

- (b) If all constants $c_{\gamma|_D}^\theta$ can be chosen to be zero, then we say that θ is *Ω -equivariant* (or, *equivariant under the action of Ω*).

Throughout the paper, we will be working with affinely Ω -equivariant functions. At the very end, in section 10, an important space of Ω -equivariant functions will appear.

Remark 5.2. It now becomes clear why singletons $\{x\}$ are not allowed as the domain D of a move. The functional equation (5.1) would degenerate to a single equation with an independent constant $c_{\gamma|\{x\}}^\theta$. The equation and the constant can be eliminated from the system.

Some trivial relations between the constants $c_{\gamma|_D}^\theta$ are induced by the restriction partial order on moves (subsection 2.4). If $\emptyset \neq D \subset D'$, thus $\gamma|_D \subseteq \gamma|_{D'}$ and $D \neq \emptyset$, then necessarily $c_{\gamma|_D}^\theta = c_{\gamma|_{D'}}^\theta$.

Thus it is natural to work with restriction-closed ensembles, as defined in subsection 3.1.1.

Lemma 5.3. *For a space Θ of functions, we denote by Θ^Ω the set of affinely Ω -equivariant functions in Θ . If Θ is a vector space, then so is Θ^Ω .*

Proof. Let $\theta_1, \theta_2 \in \Theta$ and $a_1, a_2 \in \mathbb{R}$. Let $\theta = a_1\theta_1 + a_2\theta_2$. Then $\theta \in \Theta$. Moreover, let $c_{\gamma|_D}^{\theta_1}$ for $\gamma|_D \in \Omega$ and $c_{\gamma|_D}^{\theta_2}$ for $\gamma|_D \in \Omega$ be the families of constants that satisfy (5.1) for θ_1 and θ_2 , respectively. Then $c_{\gamma|_D}^\theta = a_1c_{\gamma|_D}^{\theta_1} + a_2c_{\gamma|_D}^{\theta_2}$ for $\gamma|_D \in \Omega$ is a family of constants that satisfy (5.1) for θ . \square

5.2. Join-closed semigroup Γ^{resp} of moves respected by given functions.

Definition 5.4. For a function $\theta: \text{dom}(\theta) \rightarrow \mathbb{R}$, we denote the ensemble of moves respected by θ as

$$\Gamma^{\text{resp}}(\theta) = \{ \gamma|_D \in \Gamma^\subseteq(\mathbb{R}) \mid D, \gamma(D) \subseteq \text{dom}(\theta), \exists c_{\gamma|_D}^\theta \in \mathbb{R} \text{ s.t. (5.1) holds} \}.$$

(Clearly $\Gamma^{\text{resp}}(\theta)$ is the largest move ensemble that θ respects.) For a space Θ' of functions, we denote $\Gamma^{\text{resp}}(\Theta') = \bigcap_{\theta \in \Theta'} \Gamma^{\text{resp}}(\theta)$.

Theorem 5.5. *Let Ω be a move ensemble. If a function θ respects Ω , then θ respects the joined semigroup $\text{jsemi}(\Omega)$.*

To prove this, we use the following lemma.

Lemma 5.6. *Let \mathfrak{J} be a collection of open intervals that cover the open interval (l, u) . If a function g is constant over each interval I from the collection \mathfrak{J} , then g is constant over (l, u) .*

Proof. Let $m = \frac{l+u}{2}$ and $a = g(m)$. Consider the interval $J = \{ y \in (l, m) \mid g(x) = a \text{ for all } x \in [y, m] \}$. Since m is contained in some open interval $I \in \mathfrak{J}$ and $g(x) = a$ for $x \in I$, we know that J is non-empty. Let $l' = \inf J$. We now show that $l = l'$. Suppose that $l \neq l'$. Then there exists an open interval $I \in \mathfrak{J}$ such that $l' \in I$, and g is constant over I . Since $I \cap J \neq \emptyset$ and $g(x) = a$ for $x \in J$, we have that $g(x) = a$ for $x \in I$, a contradiction to $l' = \inf J$. Hence $g(x) = a$ for all $l < x \leq m$. Similarly, one shows that $g(x) = a$ for all $m \leq x < u$. Therefore, g is constant over (l, u) . \square

Proof of Theorem 5.5. Let $\gamma|_D \in \text{jsemi}(\Omega)$. Thus, there exists a collection \mathfrak{J} of open intervals, such that $D = \bigcup_{I \in \mathfrak{J}} I$ and $\gamma|_I \in \text{isemi}(\Omega)$ for each $I \in \mathfrak{J}$.

Define $g(x) = \theta(\gamma(x)) - \chi(\gamma)\theta(x)$ for $x \in D$. We first show that g is constant over each interval $I \in \mathfrak{J}$. Let $I \in \mathfrak{J}$. Since $\gamma|_I \in \text{isemi}(\Omega)$, we can write it in the form $\gamma|_I = \gamma_k|_{D_k} \circ \gamma_{k-1}|_{D_{k-1}} \circ \cdots \circ \gamma_1|_{D_1}$, where $\gamma_1|_{D_1}, \gamma_2|_{D_2}, \dots, \gamma_k|_{D_k} \in \Omega$. Let $x_0 \in I$ and denote $x_i = \gamma_i(x_{i-1})$ for $i =$

$1, 2, \dots, k$. Then, $x_i \in D_{i+1}$ for $i = 0, 1, \dots, k-1$, and $x_k = \gamma|_I(x_0) = \gamma(x_0)$. Since θ respects Ω , for $i = 1, 2, \dots, k$, we have that

$$\theta(x_i) = \chi(\gamma_i)\theta(x_{i-1}) + c_i^\theta,$$

where the constants c_i^θ are independent of the choice of $x_0 \in I$. We also know that $\chi(\gamma) = \chi(\gamma_1)\chi(\gamma_2)\dots\chi(\gamma_k)$. Therefore,

$$\begin{aligned} g(x_0) &= \theta(\gamma(x_0)) - \chi(\gamma)\theta(x_0) \\ &= \theta(x_k) - \chi(\gamma_k)\chi(\gamma_{k-1})\dots\chi(\gamma_1)\theta(x_0) = \sum_{j=1}^k \left(\prod_{i=j+1}^k \chi(\gamma_i) \right) c_j^\theta \end{aligned}$$

is constant for $x_0 \in I$.

Then, it follows from Lemma 5.6 that g is constant over D . \square

Corollary 5.7. *For a function θ , the ensemble $\Gamma^{\text{resp}}(\theta)$ defined in Definition 5.4 is a join-closed move semigroup. The same holds for the ensemble $\Gamma^{\text{resp}}(\Theta')$, where Θ' is a space of functions.*

6. KALEIDOSCOPIC JOINED ENSEMBLES AND BOUNDED FUNCTIONS. FINITE PRESENTATIONS BY MOVES AND COMPONENTS

6.1. Cauchy–Pexider functional equation $f(x) + g(y) = h(x + y)$.

Recall from subsection 5.1 that move ensembles encode systems of functional equations. We now bring a first result on functional equations to use. The following result on the Cauchy–Pexider functional equation on bounded domains appeared in [5, Theorem 4.3]. Here we state it for functions of a single real variable. It is a variant of the Gomory–Johnson interval lemma, which has been used throughout the extreme functions literature. Note that it requires a weak assumption regarding the function space. Boundedness is sufficient; see [5] for a more detailed discussion.

Lemma 6.1 (Convex additivity domain lemma). *Let $f, g, h: \mathbb{R} \rightarrow \mathbb{R}$ be bounded functions and let $E \subseteq \mathbb{R}^2$ be open, convex, and bounded. Suppose that*

$$f(x) + g(y) = h(x + y) \quad \text{for all } (x, y) \in E.$$

Define the projections

$$p_1(x, y) = x, \quad p_2(x, y) = y, \quad p_3(x, y) = x + y$$

as functions from \mathbb{R}^2 to \mathbb{R} . Then f, g, h are affine with the same slopes on the domains $p_1(E), p_2(E), p_3(E)$, respectively.

6.2. Kaleidoscopic move ensembles. When we are only interested in bounded functions that respect a move ensemble Ω , then it follows from Lemma 6.1 that we can replace Ω by a move ensemble Ω^\boxtimes with more convenient properties.

Lemma 6.2. *Let $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be a bounded function. Let $D, I \subseteq \mathbb{R}$ be open intervals. The following are equivalent:*

- (1) θ respects $\text{moves}_+(D \times I)$,
- (2) θ respects $\text{moves}_-(D \times I)$,
- (3) θ respects $\text{moves}(D \times I)$,
- (4) θ is affine on D and I with the same slope.

Proof. We first show that (1) implies (4). By assumption, the function θ satisfies equation (5.1) for all $\tau_t|_{D_t}$, where $t \in \{y - x \mid x \in D, y \in I\}$ and $D_t = \{x \in D \mid x + t \in I\}$. Thus, there exists a function $c: I + (-D) \rightarrow \mathbb{R}$ such that

$$\theta(x + t) = \theta(x) + c(t) \quad \text{for all } (x, x + t) \in D \times I.$$

The function c is bounded because θ is. Then, by Lemma 6.1 with $f = h = \theta$ and $g = c$, we have that θ affine on D and I with the same slope. The proofs that each of (2) and (3) implies (4) are similar; we omit them.

Now we show that (4) implies (1). Fix $t = y - x$ for some $x \in D, y \in I$. Since θ is affine on D and I with the same slope, there exist scalars a, b, b' such that $\theta(x) = a \cdot x + b$ for all $x \in D$ and $\theta(x) = a \cdot x + b'$ for all $x \in I$. But then for all $x \in D$ such that $x + t \in I$, we have that $\theta(x + t) - \theta(x) = a \cdot t$, which is constant. Therefore, θ respects $\tau_t|_{D_t}$. Again the proofs that (4) also implies (2) and (3) is similar and we omit them. \square

Motivated by these results, we make the following definitions.

Definition 6.3. A move ensemble Ω^\boxtimes is a *kaleidoscopic joined ensemble* if it satisfies (restrict), (continuation), and the following axiom:

$$\begin{aligned} \text{(kaleido)} \quad & \text{for open intervals } D, I \subseteq \mathbb{R} \\ & \text{moves}_+(D \times I) \subseteq \Omega^\boxtimes \text{ if and only if } \text{moves}_-(D \times I) \subseteq \Omega^\boxtimes. \end{aligned}$$

6.3. Covered intervals, connected covered components.

Definition 6.4. For a kaleidoscopic joined ensemble Ω^\boxtimes and an open interval D such that $\text{moves}(D \times D) \subseteq \Omega^\boxtimes$, we say that D is a *covered interval* in Ω^\boxtimes .

Let Γ^\boxtimes be a kaleidoscopic joined move semigroup. For two open intervals D_1, D_2 , if

$$\text{moves}((D_1 \cup D_2) \times (D_1 \cup D_2)) \subseteq \Gamma^\boxtimes,$$

then we say that both D_1 and D_2 are covered intervals in the same *connected covered component* of Γ^\boxtimes . (Here the word ‘‘connected’’ does not refer to the topology of \mathbb{R} , in contrast to subsection 4.3.) It follows from Corollary 4.9 that this is an equivalence relation. However, we want to define the notion of a *connected covered component* also for kaleidoscopic joined ensembles Ω^\boxtimes that are not semigroups. In this case there is no equivalence relation (transitivity fails), but we still use the word ‘‘components’’ in the following definition.

Definition 6.5. Let Ω^\boxtimes be a kaleidoscopic joined ensemble. Let C be an open set such that $\text{moves}(C \times C) \subseteq \Omega^\boxtimes$. Then C is called a *connected covered component* of Ω^\boxtimes . Any two covered intervals $D_1, D_2 \subseteq C$ are said to be *connected* by the component C .

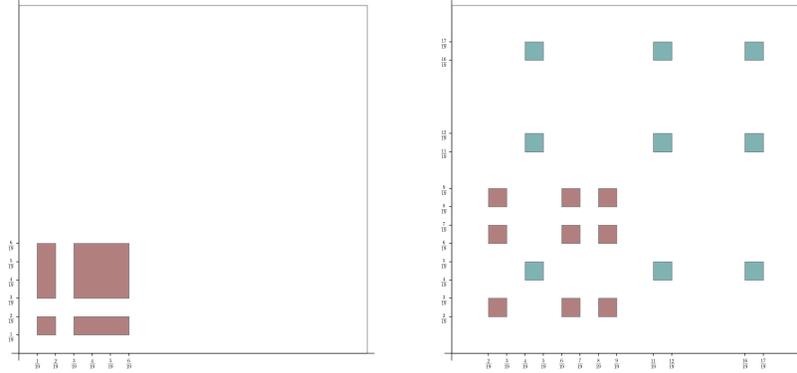


FIGURE 6. Move ensemble moves(\mathcal{C}) from connected covered components \mathcal{C} . *Left*, $\mathcal{C} = \{C_1\}$ (one component), where $C_1 = (\frac{1}{19}, \frac{2}{19}) \cup (\frac{3}{19}, \frac{6}{19})$, shown in *red*. *Right*, $\mathcal{C} = \{C_1, C_2\}$ (two components), where $C_1 = (\frac{2}{19}, \frac{3}{19}) \cup (\frac{6}{19}, \frac{7}{19}) \cup (\frac{8}{19}, \frac{9}{19})$ is shown in *red* and $C_2 = (\frac{4}{19}, \frac{5}{19}) \cup (\frac{11}{19}, \frac{12}{19}) \cup (\frac{16}{19}, \frac{17}{19})$ is shown in *cyan*.

The connected covered components of Ω^{\boxtimes} are partially ordered by set inclusion. The maximal elements in this partial order suffice to describe all covered intervals.

Corollary 6.6. *Let θ be a bounded function. Suppose θ respects a kaleidoscopic joined ensemble Ω^{\boxtimes} . Let C be a connected covered component of Ω^{\boxtimes} . Then θ is affine on all open intervals in C with a common slope.*

Proof. Let $D, I \subseteq C$ be open intervals. Then $D \times I \subseteq C \times C$, and hence θ respects moves($D \times I$). By Lemma 6.2 (4), θ is affine on D and I with the same slope. \square

(Later in section 10, we will also consider so-called connected uncovered components.)

6.4. Presentations by moves Ω^{fin} and components $\mathcal{C} = \{C_1, \dots, C_k\}$.

Now we are prepared to define a convenient finite presentation for a large class of kaleidoscopic joined ensembles, which we announced in Remark 4.10.

Definition 6.7. Take a finite list of connected covered components $\mathcal{C} = \{C_1, \dots, C_k\}$, where each C_i is a finite union of disjoint open intervals. Define

$$\begin{aligned} \text{moves}(\mathcal{C}) &= \bigcup_{i=1}^k \text{moves}(C_i \times C_i) \\ &= \{ \gamma|_D \in \Gamma^{\subseteq}(\mathbb{R}) \mid D, \gamma(D) \subseteq C_i \text{ for some } i = 1, \dots, k \}. \end{aligned}$$

The graph $\text{Gr}(\text{moves}(\mathcal{C}))$ is a union of open rectangles. See Figure 6 for a visualization. We plot the components with different colors.

Note that any ensemble of the form $\text{moves}(\mathcal{C})$ or $\Omega^{\text{fin}} \cup \text{moves}(\mathcal{C})$, where Ω^{fin} is a finite move ensemble, satisfies (restrict) and (kaleido), but is not necessarily join-closed. To make a kaleidoscopic joined ensemble, we use the following.

Definition 6.8. For any finite move ensemble Ω^{fin} and a finite list \mathcal{C} of connected covered components, define

$$\text{jmoves}(\Omega^{\text{fin}}, \mathcal{C}) = \text{join}(\Omega^{\text{fin}} \cup \text{moves}(\mathcal{C})).$$

If $\Omega^{\text{fin}} = \emptyset$, we simply write $\text{jmoves}(\mathcal{C})$.

Definition 6.9. The ordered pair $(\Omega^{\text{fin}}, \mathcal{C})$ is said to be a *finite presentation* (by moves Ω^{fin} and components \mathcal{C}) of the kaleidoscopic joined ensemble $\text{jmoves}(\Omega^{\text{fin}}, \mathcal{C})$.

Corollary 6.10. *Let θ be a bounded function. Suppose θ respects a move ensemble Ω^{\boxtimes} that has the finite presentation $(\Omega^{\text{fin}}, \mathcal{C})$. Then θ is affine on all intervals in \mathcal{C} and shares a common slope on all intervals of each component C_i of \mathcal{C} .*

Proof. This is a restatement of Corollary 6.6. □

It is clear that these presentations are not unique, which motivates the next subsection.

6.5. Finite presentation in reduced form $(\Omega^{\text{red}}, \mathcal{C})$.

Definition 6.11. A finite presentation $(\Omega^{\text{red}}, \mathcal{C})$ of a kaleidoscopic joined ensemble Ω^{\boxtimes} is said to be in *(long) reduced form* if the following holds:

$$\text{(reduce)} \quad \Omega^{\text{red}} \subseteq \text{Max}(\Omega^{\boxtimes}) \setminus \text{jmoves}(\mathcal{C}),$$

that is, each move $\gamma|_D \in \Omega^{\text{red}}$ is maximal in Ω^{\boxtimes} with respect to the restriction partial order \subseteq , and the graph $\text{Gr}(\gamma|_D)$ is not covered by the union of open rectangles $C_i \times C_i$, $C_i \in \mathcal{C}$.

Lemma 6.12. *If a kaleidoscopic joined ensemble Ω^{\boxtimes} has a finite presentation $(\Omega^{\text{fin}}, \mathcal{C})$, then there is a unique finite ensemble Ω^{red} such that $(\Omega^{\text{red}}, \mathcal{C})$ is in reduced form and $\Omega^{\boxtimes} = \text{jmoves}(\Omega^{\text{red}}, \mathcal{C})$.*

Figure 7 illustrates the operation of going from a finite presentation to a reduced presentation of the same ensemble.

Remark 6.13. As the examples in Figure 7 illustrate, the domains of moves in Ω^{fin} may be extended.

6.6. Finite presentations of generating ensembles of move semi-groups. Move ensembles have a crucial rôle as generating sets of move semi-groups. We now describe an operation that changes the generating ensemble, but preserves the move semigroup that is generated by it.

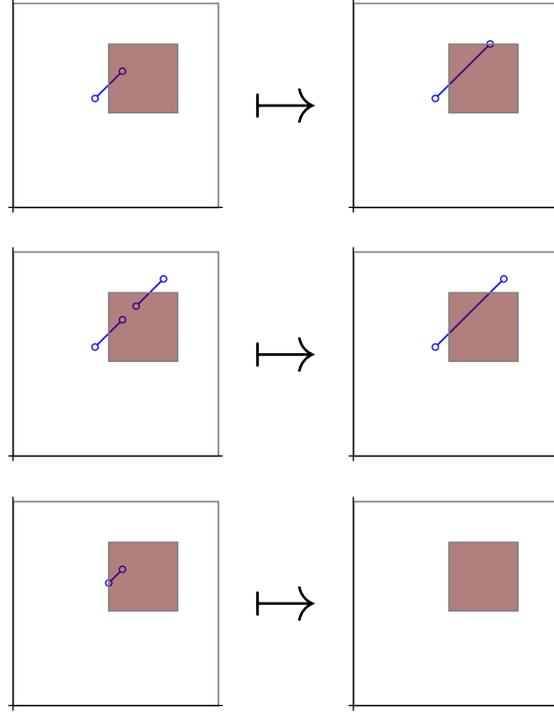


FIGURE 7. Finite presentation in reduced form. *Left*, finite presentations $(\Omega^{\text{fin}}, \mathcal{C})$ of kaleidoscopic joined ensembles Ω^{\boxtimes} . *Right*, finite presentations $(\Omega^{\text{red}}, \mathcal{C})$ in reduced form of the same ensembles. (a) A move poking into a component is extended to become a maximal move of Ω^{\boxtimes} . (b) Two restrictions of the same move are extended to become a maximal move of Ω^{\boxtimes} . (c) A move that lies completely in a component is removed.

Lemma 6.14 (Extend component by move). *Let \mathcal{C} be a list of connected components and let Ω be a move ensemble such that $\text{moves}(\mathcal{C}) \subseteq \Omega$. If $\gamma|_D \in \Omega$ and $D \subseteq C_i$ for some $C_i \in \mathcal{C}$, then $\text{moves}(\mathcal{C}') \subseteq \text{isemi}(\Omega)$, where $C'_i = C_i \cup \gamma(D)$ and all other components of \mathcal{C}' are the same as \mathcal{C} .*

See Figure 8 for an illustration.

Proof. Let $x \in C_i$, $z \in \gamma(D)$, and $y = \gamma^{-1}(z) \in D$. Since $x \in C_i$, x is in the domain of moves τ_0 and ρ_0 in Ω . Thus, we can both translate and reflect x to z by

$$x \xrightarrow{\tau_{t_{y-x}}} y \xrightarrow{\gamma} z,$$

and

$$x \xrightarrow{\rho_{r_0}} x \xrightarrow{\tau_{t_{y-x}}} y \xrightarrow{\gamma} z.$$

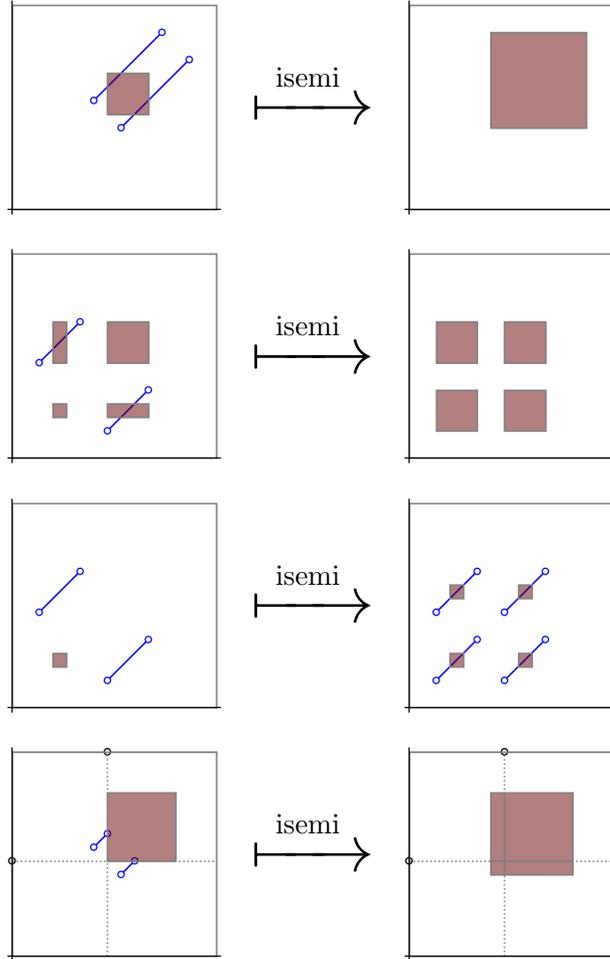


FIGURE 8. Extending components by moves, Lemma 6.14. *Left*, reduced finite presentations of a kaleidoscopic joined ensemble Ω^\boxtimes . *Right*, reduced finite presentations of $\text{isemi}(\Omega^\boxtimes)$.

Note that which one above is a translation or reflection depends on the character $\chi(\gamma)$. \square

7. LIMIT-CLOSED ENSEMBLES AND CONTINUOUS FUNCTIONS. CLOSED MOVE SEMIGROUPS

Let $A \subseteq \mathbb{R}$ be an open set. We now consider the space $\mathcal{C}_b(A)$ of bounded continuous functions on A . For $\mathcal{C}_b(A)$, some notions of convergence of moves are natural to study.

7.1. Limit-closed move ensembles $\bar{\Omega}$; closures $\text{lim}(\Omega)$, $\text{arlim}(\Omega)$.

7.1.1. *Convergence of unrestricted moves.*

Definition 7.1. A sequence $\{\gamma^i\}_{i \in \mathbb{N}} \subseteq \Gamma(\mathbb{R})$ of unrestricted moves *converges*

- (a) to an unrestricted translation $\tau_t \in \Gamma(\mathbb{R})$ if all but finitely many γ^i are translations τ_{t^i} and $t^i \rightarrow t$.
- (b) to an unrestricted reflection $\rho_r \in \Gamma(\mathbb{R})$ if all but finitely many γ^i are reflections ρ_{r^i} and $r^i \rightarrow r$.

7.1.2. *Limits closure.*

Definition 7.2. We define the *limits closure* $\lim(\Omega)$ of a moves ensemble Ω to be the smallest (by set inclusion) moves ensemble $\bar{\Omega}$ containing Ω that satisfies the following axiom.

- (lim) Let D be an open interval.
If $\gamma^i \rightarrow \gamma$ and $\gamma^i|_D \in \bar{\Omega}$ for all i , then $\gamma|_D \in \bar{\Omega}$.

We note that the domain D is fixed for all moves in the sequence. Thus, the limits closure will in general not satisfy (continuation) and (inv). Instead we can consider the following axiom.

Definition 7.3. Define $\text{arblim}(\Omega)$ to be the smallest moves ensemble $\bar{\Omega}$ containing Ω that satisfies the following axiom.

- (arblim) If $\gamma^i \rightarrow \gamma$, $l^i \rightarrow l$, $u^i \rightarrow u$ and $\gamma^i|_{(l^i, u^i)} \in \bar{\Omega}$ for all i ,
then $\gamma|_{(l, u)} \in \bar{\Omega}$.

For our purposes, when considered together with (continuation), the notions turn out to be equivalent.

Theorem 7.4. *Let Ω^\vee be a join-closed move ensemble. Then*

$$\text{join}(\lim(\Omega^\vee)) = \text{join}(\text{arblim}(\Omega^\vee)).$$

Proof. It is clear that $\lim(\Omega^\vee) \subseteq \text{arblim}(\Omega^\vee)$, so it suffices to show that

$$(7.1) \quad \text{arblim}(\Omega^\vee) \subseteq \text{join}(\lim(\Omega^\vee)).$$

Let $\tau_t|_{(l, u)} \in \text{arblim}(\Omega^\vee)$. By (arblim), there is a convergent sequence $\{\tau_{t^i}|_{(l^i, u^i)}\}_{i \in \mathbb{N}}$ of moves in Ω^\vee such that $l^i \rightarrow l$, $u^i \rightarrow u$ and $t^i \rightarrow t$. For every integer $j > \frac{2}{u-l}$, there exists a large integer n_j such that for any $i \geq n_j$, we have $l_i < l + \frac{1}{j}$ and $u - \frac{1}{j} < u_i$. Since Ω^\vee satisfies (continuation), $\tau_{t^i}|_{D_j} \in \Omega^\vee$ for any $i \geq n_j$, where $D_j := (l + \frac{1}{j}, u - \frac{1}{j})$. Since $t_i \rightarrow t$, we have $\tau_t|_{D_j} \in \lim(\Omega^\vee)$ for every j , hence $\tau_t|_{(l, u)} \in \text{join}(\lim(\Omega^\vee))$. We showed that (7.1) holds for translations. The proof for reflections is similar. \square

Theorem 7.5. *Let Ω^\vee be a join-closed move ensemble. The following are equivalent.*

- (1) Ω^\vee satisfies (lim).
- (2) Ω^\vee satisfies (arblim).

The proof is essentially the same and we omit it.

7.1.3. Respecting limits.

Lemma 7.6 (Limits). *Let D be an open interval and let θ be continuous on D . If there exists a sequence $\gamma^i \rightarrow \gamma$ such that θ respects $\gamma^i|_D$ for all i , then θ also respects $\gamma|_D$.*

Proof. We prove the lemma for a sequence $t_i \rightarrow t$ such that θ respects the translations $\tau_{t_i}|_D$ for all i . We will show that θ also respects $\tau_t|_D$.

Since θ is continuous on D , θ is also continuous on $\tau_{t_i}(D)$ for all i . Fix $\bar{x} \in D$. Since $t_i \rightarrow t$, and $\bar{x} \in \text{int}(D)$ since D is open, there exists an \hat{i} such that for all $i \geq \hat{i}$, we have $\bar{x} + t \in D + t_i$. Hence, for a neighborhood $N_{\bar{x}}$ of \bar{x} , θ is continuous in $N_{\bar{x}} + t$. Now, for all $x \in N_{\bar{x}}$,

$$\theta(x + t) - \theta(x) = \lim_{t_i \rightarrow t} \theta(x + t_i) - \theta(x) = \lim_{i \rightarrow \infty} c_{\tau_{t_i}|_D}^\theta.$$

Since the limit on the right-hand side is independent of x , we define $c_{\tau_t|_{N_{\bar{x}}}}^\theta$ to be this limit. Thus, θ respects $\tau_t|_{N_{\bar{x}}}$.

Now the connected open set D is covered by the open neighborhoods $N_{\bar{x}}$ of each $\bar{x} \in D$. It follows that $c_{\tau_t|_{N_{\bar{x}}}}^\theta = c_{\tau_t|_{N_{\bar{x}'}}}^\theta$ for all $\bar{x}, \bar{x}' \in D$. Therefore, θ respects $\tau_t|_D$. Moreover, θ is continuous on $\tau_t|_D$.

The proof for a sequence of reflections is the same. \square

7.1.4. Limit-closed move semigroups.

Lemma 7.7. *Let Γ be a move semigroup. Then $\text{arblim}(\Gamma)$ is also a move semigroup.*

Proof. It is clear that $\text{arblim}(\Gamma)$ satisfies (inv), as Γ satisfies (inv). We now show that $\text{arblim}(\Gamma)$ satisfies (composition). Let $\gamma_1|_{D_1}, \gamma_2|_{D_2} \in \text{arblim}(\Gamma)$ such that $\gamma_1|_{D_1} \circ \gamma_2|_{D_2}$ is not an empty move. $\gamma_1|_{D_1}$ and $\gamma_2|_{D_2}$ are the (arblim) of sequences of moves $\{\gamma_1^i|_{D_1^i}\}_{i \in \mathbb{N}}$ and $\{\gamma_2^i|_{D_2^i}\}_{i \in \mathbb{N}}$ in Γ . Since Γ satisfies (composition), $\gamma^i|_{D^i} := (\gamma_1^i|_{D_1^i}) \circ (\gamma_2^i|_{D_2^i}) \in \Gamma$ for every i . The (arblim) of the sequence $\{\gamma^i|_{D^i}\}_{i \in \mathbb{N}}$ is $\gamma_1|_{D_1} \circ \gamma_2|_{D_2}$. Thus, we obtain that $\gamma_1|_{D_1} \circ \gamma_2|_{D_2} \in \text{arblim}(\Gamma)$. This show that $\text{arblim}(\Gamma)$ is a semigroup. \square

Lemma 7.8. *Let Γ^\vee be a join-closed semigroup. Then $\text{join}(\text{lim}(\Gamma^\vee)) = \text{join}(\text{arblim}(\Gamma^\vee))$ is a semigroup.*

Proof. It follows from Lemma 7.7, Lemma 4.6 and Theorem 7.4. \square

Theorem 7.9 (Limits imply components). *Let Γ^\vee be a join-closed move semigroup. Assume that $\gamma|_D$ is the limit move (in the sense of lim or arblim) of a sequence $\{\gamma^i|_{D^i}\}_{i \in \mathbb{N}}$ of moves in Γ^\vee with $\gamma^i \neq \gamma$ for every i . Let $I = \gamma(D)$. Then the following holds.*

- (1) *If γ is a translation, then $\text{moves}_+((D \cup I) \times (D \cup I)) \subseteq \text{join}(\text{lim}(\Gamma^\vee))$.*
- (2) *If γ is a reflection, then*

$$\text{moves}_-((D \times I) \cup (I \times D)), \text{moves}_+((D \times D) \cup (I \times I)) \subseteq \text{join}(\text{lim}(\Gamma^\vee)).$$

Proof. Let $D = (l, u)$. If a sequence $\{\gamma^i|_{D^i}\}_{i \in \mathbb{N}}$ of moves in Γ^\vee with $\gamma^i \neq \gamma$ converges to $\gamma|_D$ in the sense of arblim, then $\gamma^i|_{D^i \cap (l+\epsilon, u-\epsilon)} \rightarrow \gamma|_{(l+\epsilon, u-\epsilon)}$ in the sense of lim for any small $\epsilon > 0$. Thus, it suffices to prove the statement for a limit move $\gamma|_D$ in the sense of lim; the statement for arblim follows from Lemma 7.8 and continuation.

We first show that $\text{moves}_+(D \times D) \subseteq \text{join}(\text{lim}(\Gamma^\vee))$. Let $\epsilon > 0$ be an arbitrary small number. Since $\gamma|_D$ is a limit move, there exist $\gamma^i|_D, \gamma^j|_D \in \Gamma^\vee$ in the convergent sequence such that the constants $\gamma - \gamma^i$ and $\gamma - \gamma^j$ have the same sign, and $0 < \gamma^j - \gamma^i < \epsilon$. Let $\delta = \gamma^j - \gamma^i$ and $D^1 = (l, u) \cap (l - \delta, u - \delta)$. We notice that $(\gamma^i|_D)^{-1} \circ \gamma^j|_D = \tau_\delta|_{D^1}$ when γ is a translation, and $(\gamma^j|_D)^{-1} \circ \gamma^i|_D = \tau_\delta|_{D^1}$ when γ is a reflection. Therefore, $\tau_\delta|_{D^1} \in \Gamma^\vee$. Let $D^k := (l, u) \cap (l - k\delta, u - k\delta)$ for $k \in \mathbb{Z}$. For $k \geq 1$, $\tau_{k\delta}|_{D^k}$ is the k times composition of $\tau_\delta|_{D^1}$, hence it is in Γ^\vee . For $k = -1$, $\tau_{-\delta}|_{D^{-1}} = (\tau_\delta|_{D^1})^{-1} \in \Gamma^\vee$. For $k \leq -2$, $\tau_{k\delta}|_{D^k}$ is the $-k$ times composition of $\tau_{-\delta}|_{D^{-1}}$, and hence is in Γ^\vee . Finally, for $k = 0$, we have $(\tau_\delta|_{D^1}) \circ (\tau_{-\delta}|_{D^{-1}}), (\tau_{-\delta}|_{D^{-1}}) \circ (\tau_\delta|_{D^1}) \in \Gamma^\vee$, so their join $\tau_0|_{D^0}$ is also in Γ^\vee . Therefore, for every $k \in \mathbb{Z}$ such that D^k is not empty, we have $\tau_{k\delta}|_{D^k} \in \Gamma^\vee$. By letting $\epsilon \rightarrow 0$, we obtain that $\text{moves}_+(D \times D) \subseteq \text{join}(\text{lim}(\Gamma^\vee))$.

Since $\gamma|_D \in \text{lim}(\Gamma^\vee) \subseteq \text{join}(\text{lim}(\Gamma^\vee))$ and $\text{join}(\text{lim}(\Gamma^\vee))$ is a semigroup by Lemma 7.8, we have that $\text{moves}_+(D \times I) \subseteq \text{join}(\text{lim}(\Gamma^\vee))$ when γ is a translation, and $\text{moves}_-(D \times I) \subseteq \text{join}(\text{lim}(\Gamma^\vee))$ when γ is a reflection.

The other two subsets follow from applying the above argument to $(\gamma|_D)^{-1}$ instead of $\gamma|_D$. \square

7.2. Continuous domain extension $\text{extend}_A(\Omega)$. Next we introduce a topological version of axiom (continuation).

7.2.1. *Extended move ensembles $\overline{\Omega}^\vee$.*

Definition 7.10. Let Ω be a move ensemble with $\text{dom}(\Omega), \text{im}(\Omega) \subseteq A$, where $A \subseteq \mathbb{R}$ is an open set. Then the *extended move ensemble* $\text{extend}_A(\Omega)$ of Ω is defined to be the smallest set $\overline{\Omega}^\vee$ containing Ω that satisfies the following axiom

(extend_A) Let $\gamma \in \Gamma(\mathbb{R})$ and D empty or an open interval.
 If there is an ensemble $\{\gamma|_{D^i}\}_{i \in \mathcal{J}} \subseteq \overline{\Omega}^\vee$ such that
 $D \subseteq \text{cl}(\bigcup_{i \in \mathcal{J}} D^i) \cap A \cap \gamma^{-1}(A)$, then $\gamma|_D \in \overline{\Omega}^\vee$.

Remark 7.11. An ensemble satisfying (extend_A) is join-closed.

The most simple application of (extend_A) allows us to join two adjacent moves across a point of continuity; see Figure 9.

Lemma 7.12. *Let $\overline{\Omega}^\vee$ be a move ensemble that satisfies (extend_A). Then we have:*

(2- extend_A) If $\gamma|_{(l,m)}, \gamma|_{(m,u)} \in \overline{\Omega}^\vee$, where $l < m < u$, and $m, \gamma(m) \in A$,
 then $\gamma|_{(l,u)} \in \overline{\Omega}^\vee$.

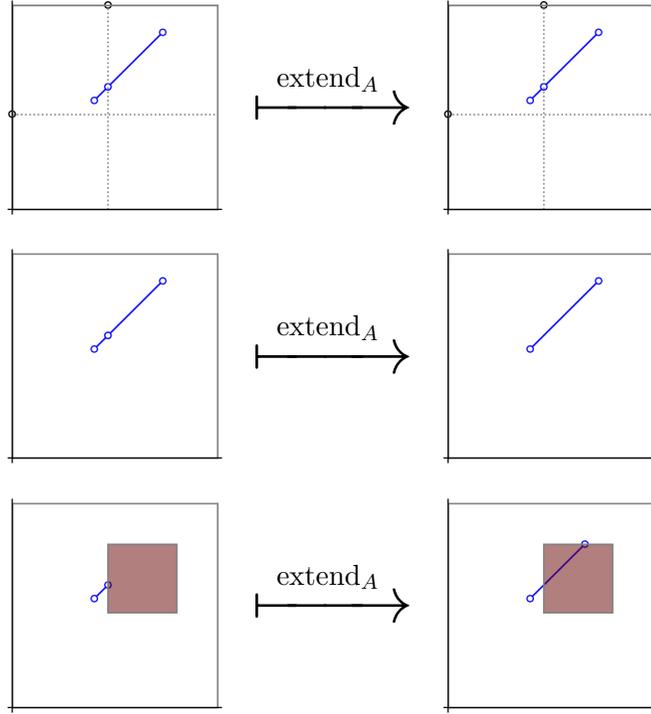


FIGURE 9. Extended move ensembles $\text{extend}_A(\Omega)$ of ensembles Ω . Points not in the continuity set A are indicated by black circles at the top and left border.

The following is clear from the definition.

Lemma 7.13. *Let Ω be a move ensemble with $\text{dom}(\Omega) = \text{im}(\Omega) \subseteq A$. Let $\overline{\Omega}^\vee = \text{extend}_A(\Omega)$. Then $\text{dom}(\overline{\Omega}^\vee) = \text{im}(\overline{\Omega}^\vee) \subseteq A$.*

Remark 7.14. If Ω^\vee is a joined ensemble with finite $\text{Max}(\Omega^\vee)$, then repeated application of (2-extend_A) , followed by applying (continuation), suffices to obtain $\text{extend}_A(\Omega^\vee)$.

However, this is not true for arbitrary joined ensembles Ω^\vee . As an example, let $A = \mathbb{R}$ and consider Ω^\vee consisting of the restrictions of a move γ to all subintervals of $(-1, 0)$ and $(\frac{1}{n+1}, \frac{1}{n})$ for $n \in \mathbb{N}$. (These maximal intervals are disjoint.) Domains of moves of $\text{extend}_A(\Omega^\vee)$ are all subintervals of $(-1, 1)$. The domains of moves of $2\text{-extend}_A(\Omega^\vee)$ are $(-1, 0)$ and its subintervals and the infinite chain $(\frac{1}{m}, 1)$ for $m \in \mathbb{N}$ and some of its subintervals; the supremum of the chain, $(0, 1)$ is not an element. Then the domains of maximal moves of $\text{join}(2\text{-extend}_A(\Omega^\vee))$ are $(-1, 0)$ and $(0, 1)$. It takes *another round* of 2-extend_A to arrive at $\text{extend}_A(\Omega^\vee)$.

We have an explicit description of the moves in the extended move ensemble $\text{extend}_A(\Omega)$, similar to Lemma 3.6 for $\text{join}(\Omega)$.

Remark 7.15. For a move ensemble Ω with $\text{dom}(\Omega), \text{im}(\Omega) \subseteq A$, where $A \subseteq \mathbb{R}$ is an open set, we have

$$(7.2) \quad \text{extend}_A(\Omega) = \{ \gamma|_D \mid \gamma \in \Gamma(\mathbb{R}), D \text{ empty or open interval,} \\ D \subseteq \text{cl}(C_\gamma) \cap A \cap \gamma^{-1}(A) \},$$

where $C_\gamma := \bigcup \{ I \mid \gamma|_I \in \Omega \}$, which is a subset of $A \cap \gamma^{-1}(A)$.

7.2.2. *Domain extension and semigroups.*

Lemma 7.16. *Let Γ be a move semigroup with $\text{dom}(\Gamma), \text{im}(\Gamma) \subseteq A$, where $A \subseteq \mathbb{R}$ is an open set. Then $\text{extend}_A(\Gamma)$ is a move semigroup that satisfies (extend_A) .*

Proof. Since Γ satisfies (inv), it is clear that $\text{extend}_A(\Gamma)$ satisfies (inv). We now show that $\text{extend}_A(\Gamma)$ satisfies (composition), too.

Let $\gamma_1|_{D_1}, \gamma_2|_{D_2} \in \text{extend}_A(\Gamma)$. Let

$$C_1 = C_{\gamma_1} = \bigcup \{ I \mid \gamma_1|_I \in \Gamma \} \quad \text{and} \quad C_2 = C_{\gamma_2} = \bigcup \{ I \mid \gamma_2|_I \in \Gamma \}.$$

By equation (7.2), the open set D_1 and D_2 satisfy that

$$D_1 \subseteq \text{cl}(C_1) \cap A \cap \gamma_1^{-1}(A) \quad \text{and} \quad D_2 \subseteq \text{cl}(C_2) \cap A \cap \gamma_2^{-1}(A).$$

Let $\gamma = \gamma_2 \circ \gamma_1$, $C = C_\gamma = \bigcup \{ I \mid \gamma|_I \in \Gamma \}$ and let $D = \gamma_1^{-1}(D_2) \cap D_1$ be a non-empty open set. We will show that

$$(7.3) \quad D \subseteq \text{cl}(C) \cap A \cap \gamma^{-1}(A).$$

It then follows again from (7.2) that $\gamma_2|_{D_2} \circ \gamma_1|_{D_1} = \gamma|_D \in \text{extend}_A(\Gamma)$, and hence $\text{extend}_A(\Gamma)$ is a move semigroup. It suffices to show (7.3) for

$$D_1 = \text{int}(\text{cl}(C_1) \cap A \cap \gamma_1^{-1}(A)) \quad \text{and} \quad D_2 = \text{int}(\text{cl}(C_2) \cap A \cap \gamma_2^{-1}(A)).$$

We have on the left hand side of (7.3)

$$\begin{aligned} D &= \gamma_1^{-1}(D_2) \cap D_1 \\ &= \text{int}(\gamma_1^{-1}(\text{cl}(C_2)) \cap \gamma_1^{-1}(A) \cap \gamma^{-1}(A)) \cap \text{int}(\text{cl}(C_1) \cap A \cap \gamma_1^{-1}(A)) \\ &= \text{int}(\text{cl}(C_1)) \cap \gamma_1^{-1}(\text{int}(\text{cl}(C_2))) \cap A \cap \gamma_1^{-1}(A) \cap \gamma^{-1}(A), \end{aligned}$$

and on the right hand side of (7.3) $\text{cl}(C) \cap A \cap \gamma^{-1}(A)$. Thus, it suffices to prove that if $x \in \text{int}(\text{cl}(C_1))$ such that $\gamma_1(x) \in \text{int}(\text{cl}(C_2))$, then $x \in \text{int}(\text{cl}(C))$. This holds since Γ satisfies (composition). \square

7.2.3. *Respecting extensions.*

Lemma 7.17 (Extend moves by continuity). *Let θ be a function that respects a move ensemble Ω with $\text{dom}(\Omega), \text{im}(\Omega) \subseteq A$. Then it respects the extended move ensemble $\text{extend}_A(\Omega)$.*

Proof. We use the characterization of $\text{extend}_A(\Omega)$ from Remark 7.15. Let $\gamma \in \Gamma(\mathbb{R})$ and let $C_\gamma \subseteq A \cap \gamma^{-1}(A)$ be as in Remark 7.15. The function $x \mapsto \theta(\gamma(x)) - \chi(\gamma)\theta(x)$ is constant on the connected components of C_γ and it is continuous on $A \cap \gamma^{-1}(A)$. Then it is constant on the connected components of $\text{cl}(C_\gamma) \cap A \cap \gamma^{-1}(A)$. \square

Applied to the simple case of Lemma 7.12, we have the following.

Corollary 7.18. *Suppose θ respects the moves $\gamma|_{(l,m)}, \gamma|_{(m,u)}$ with $l < m < u$ and suppose θ is continuous at $m, \gamma(m)$. Then θ respects $\gamma|_{(l,u)}$.*

Remark 7.19. The assumption regarding continuity at both m and $\gamma(m)$ cannot be removed, which explains why we use $A \cap \gamma^{-1}(A)$ in (extend_A) . We illustrate this by the following example. Let $A = (0, 2) \cup (2, 3)$. Let $\gamma = \tau_1$ and $\Omega = \{\gamma|_{(0,1)}, \gamma|_{(1,2)}\}$, so $\text{dom}(\Omega) = (0, 1) \cup (1, 2) \subseteq A$ and $\text{im}(\Omega) = (1, 2) \cup (2, 3) \subseteq A$. Then $1 \in A$, but $\gamma(1) = 2 \notin A$. Define $\theta = 0$ on A and $\theta(2) = 1$, so it is continuous at 1 but not at $\gamma(1) = 2$. Then θ respects Ω , but it does not respect the move $\gamma|_{(0,2)}$.

7.3. Closed move semigroups, the moves closure $\text{clsemi}_A(\Omega)$. Now all axioms that we have introduced above come together.

Definition 7.20. A *closed move semigroup* is a limits-closed extension-closed kaleidoscopic joined move semigroup, i.e., a move ensemble that satisfies all the following axioms: (composition), (inv), (continuation), (restrict), (extend_A) , (lim), and (kaleido).

Definition 7.21. Let Ω be a move ensemble with $\text{dom}(\Omega), \text{im}(\Omega) \subseteq A$. We define the *closed move semigroup* $\text{clsemi}_A(\Omega)$ generated by Ω (or just *moves closure* of Ω) to be the smallest (by set inclusion) closed move semigroup containing Ω .

Lemma 7.22. *Let \mathbb{L} be the family of closed move semigroups containing Ω . Then $\text{clsemi}_A(\Omega) = \bigcap \mathbb{L} = \bigcap_{\Omega' \in \mathbb{L}} \Omega'$.*

Proof. First of all, $\bigcap \mathbb{L}$ contains Ω . We show that $\bigcap \mathbb{L}$ is a closed move semigroup. Note that each axiom is a closure property of a set Ω' of the form: For all $(\Omega_1, \Omega_2) \in \mathbb{X}$, if $\Omega_1 \subseteq \Omega'$, then $\Omega_2 \subseteq \Omega'$. Now if $\Omega_1 \subseteq \bigcap \mathbb{L}$, then $\Omega_1 \subseteq \Omega'$ for all $\Omega' \in \mathbb{L}$, and thus $\Omega_2 \subseteq \Omega'$ for all $\Omega' \in \mathbb{L}$. This implies $\Omega_2 \subseteq \bigcap \mathbb{L}$.

On the other hand, $\bigcap \mathbb{L}$ is contained in each of the ensembles $\Omega' \in \mathbb{L}$ and is therefore the smallest closed move semigroup containing Ω . \square

Remark 7.23. In contrast to Lemma 4.6 (regarding (continuation) and (restrict) and the axioms of an inverse semigroup), we do not know whether $\text{clsemi}_A(\Omega)$ can be obtained by applying a finite sequence of closures with respect to the individual axioms.

Theorem 7.24 (Main theorem on the moves closure). *Suppose θ is bounded and continuous on A . If θ respects a move ensemble Ω with $\text{dom}(\Omega), \text{im}(\Omega) \subseteq A$, then θ respects the moves closure $\text{clsemi}_A(\Omega)$.*

Proof of Theorem 7.24. Let $\theta|_A$ denote the restriction of θ to A . We consider the ensemble $\Gamma = \Gamma^{\text{resp}}(\theta|_A)$ of moves that $\theta|_A$ respects, introduced in subsection 5.2. By definition, $\text{dom}(\Gamma), \text{im}(\Gamma) \subseteq A$. Since, by assumption, θ respects Ω , we have $\Gamma \supseteq \Omega$. By Theorem 5.5, Γ is a join-closed move semigroup. By Lemma 6.2, because $\theta|_A$ is bounded, Γ satisfies the axiom (kaleido). Because $\theta|_A$ is continuous, we can apply Lemma 7.6 to all convergent sequences $\{\gamma^i|_D\}_{i \in \mathbb{N}} \subseteq \Gamma$, and thus Γ satisfies the axiom (lim). Finally, by Lemma 7.17, it satisfies the axiom (extend $_A$). Hence, $\Gamma^{\text{resp}}(\theta)$ is a closed move semigroup. By Lemma 7.22, we conclude that θ respects $\text{clsemi}_A(\Omega)$. \square

8. THE INITIAL ADDITIVE MOVE ENSEMBLE Ω^0 OF A SUBADDITIVE FUNCTION

We will now apply the theory of the previous sections to compute the effective perturbation spaces of minimal valid functions. Let $\pi: \mathbb{R} \rightarrow \mathbb{R}$ be a minimal valid function. Recall from the introduction that π is nonnegative, \mathbb{Z} -periodic, and satisfies $\pi(0) = 0$, $\pi(f) = 1$. Its key property is subadditivity, which we express using the *subadditivity slack* function

$$(8.1) \quad \Delta\pi(x, y) = \pi(x) + \pi(y) - \pi(x + y)$$

as $\Delta\pi(x, y) \geq 0$. Moreover, the symmetry condition $\Delta\pi(x, f - x) = 0$ holds for all x . This is the characterization that appeared in the introduction as (1.3).

Since π is \mathbb{Z} -periodic, we will work with its fundamental domain $[0, 1]$. Let $A = A(\pi)$ be the maximal open subset of $(0, 1)$ on which π is continuous.

8.1. The initial move ensemble Ω^0 . We begin by defining an ensemble of *initial moves* $\Omega^0 = \Omega^0(\pi)$ that consists of *additive moves* and *limit additive moves*, together with their inverses and the empty moves. We define these moves $\gamma|_D$ on domains D that are open intervals such that the domain D and the image $\gamma(D)$ are subsets of A .

Definition 8.1. (i) An *additive move* is any translation $\tau_t|_D$, where $t \in (-1, 1)$ and $D \subseteq A$ is an open interval such that $\tau_t(D) \subseteq A$ and

$$\Delta\pi(x, t) = \pi(x) + \pi(t) - \pi(x + t) = 0 \quad \forall x \in D$$

or any reflection $\rho_r|_D$, where $r \in (0, 2)$, and $D \subseteq A$ is an open interval such that $\rho_r(D) \subseteq A$ such that

$$\Delta\pi(x, r - x) = \pi(x) + \pi(r - x) - \pi(r) = 0 \quad \forall x \in D.$$

(ii) A *limit-additive move* is any translation $\tau_{\bar{t}}|_D$, where $\bar{t} \in (-1, 1)$ and $D \subseteq A$ is an open interval such that $\tau_{\bar{t}}(D) \subseteq A$ and

$$\lim_{t \rightarrow \bar{t}^+} \Delta\pi(x, t) = 0 \quad \text{or} \quad \lim_{t \rightarrow \bar{t}^-} \Delta\pi(x, t) = 0 \quad \forall x \in D$$

or any reflection $\rho_{\bar{r}}|_D$, where $\bar{r} \in (0, 2)$, and $\rho_{\bar{r}}(D) \subseteq A$ such that

$$\lim_{r \rightarrow \bar{r}^+} \Delta\pi(x, r - x) = 0 \quad \text{or} \quad \lim_{r \rightarrow \bar{r}^-} \Delta\pi(x, r - x) = 0 \quad \forall x \in D.$$

- (iii) An *initial move* in $\Omega^0(\pi)$ is a move that is either additive or limit-additive, or an inverse of such a move, or an empty move.

Remark 8.2. The property of the moves $\gamma|_D \in \Omega^0$ that the function π is continuous on the domain D and image $\gamma(D)$ will be preserved throughout.

Remark 8.3. The initial move ensemble Ω^0 is join-closed. Therefore, by Lemma 3.8, it is equal to the restriction closure of its maximal elements. Moreover, by definition, Ω^0 satisfies (inv). However, Ω^0 in general is not a semigroup.

The function π is affinely Ω^0 -equivariant (subsection 5.1), i.e., it respects all moves in Ω^0 .

8.2. Moves from connected open sets of additivities. We now specialize our results from subsection 4.3 regarding connected open ensembles to the initial moves. We have the following corollary. Recall from subsection 6.1 the projections $p_1(x, y) = x$, $p_2(x, y) = y$, and $p_3(x, y) = x + y$ as functions from \mathbb{R}^2 to \mathbb{R} .

Corollary 8.4. *Suppose $E \subseteq \mathbb{R}^2$ be a connected open set on which π is additive, i.e., $\Delta\pi(x, y) = 0$ for $(x, y) \in E$. Let $C = p_1(E) \cup p_2(E) \cup p_3(E)$ and assume that $C \subseteq A$. Then*

$$\text{moves}(C \times C) \subseteq \text{jsemi}(\Omega^0),$$

so C is a connected covered component of $\text{jsemi}(\Omega^0)$.

See Figure 12 for an illustration.

Remark 8.5. In [16], the intervals $p_1(E), p_2(E), p_3(E)$ are referred to as *directly covered intervals*.

Proof of Corollary 8.4. Denote $\Gamma^\vee = \text{jsemi}(\Omega^0)$. By Lemma 4.6, $\Gamma^\vee = \text{isemi}(\Gamma^\vee)$. We first show that $\text{Gr}_\pm(\Gamma^\vee)$ contains E . Let $(x, y) \in E$. Since E is open, there exists an open interval $D \ni x$ such that the diagonal segment $\{(x', r - x') \mid x' \in D\} \subseteq E$, where $r = x + y$. By Definition 8.1, we have $\rho_r|_D \in \Omega^0$, with $\rho_r|_D(x) = y$. Thus, $(x, y) \in \text{Gr}_-(\Gamma^\vee)$. There exist open intervals $D_y \ni y$ and $D_x \ni x$ such that the vertical segment $\{x\} \times D_y$ and the horizontal segment $D_x \times \{y\}$ are contained in E . Again by Definition 8.1, we have $\tau_y|_{D_x}, \tau_x|_{D_y} \in \Omega^0$. Notice that

$$x \xrightarrow{\tau_y|_{D_x}} (x + y) \xrightarrow{(\tau_x|_{D_y})^{-1}} y.$$

Thus, $(x, y) \in \text{Gr}_+(\Gamma^\vee)$. We showed that $\text{Gr}_\pm(\Gamma^\vee)$ contains E . By Theorem 4.8-(3), $\text{moves}((p_1(E) \cup p_2(E)) \times (p_1(E) \cup p_2(E))) \subseteq \Gamma^\vee$.

For any point $x + y \in p_3(E)$, where $x \in p_1(E)$ and $y \in p_2(E)$, the above translation move $\tau_y|_{D_x}$ satisfies that $\tau_y|_{D_x} \in \Omega^0$ and $\tau_y|_{D_x}(x) = x + y$. By

applying Lemma 6.14 to $\mathcal{C} = \{p_1(E) \cup p_2(E)\}$ and all such moves $\tau_y|_{D_x}$, we obtain that moves $((p_1(E) \cup p_2(E) \cup p_3(E)) \times (p_1(E) \cup p_2(E) \cup p_3(E))) \subseteq \Gamma^\vee$. \square

9. PIECEWISE LINEAR FUNCTIONS, POLYHEDRAL COMPLEXES, EFFECTIVE PERTURBATIONS

We now specialize our theory to the important case of piecewise linear functions. We begin with the basic definitions and review some tools that were developed in the previous papers of the present series.

9.1. Continuous and discontinuous piecewise linear functions π , complex \mathcal{P}_B . We begin by giving a definition of \mathbb{Z} -periodic piecewise linear functions $\pi: \mathbb{R} \rightarrow \mathbb{R}$ that are allowed to be discontinuous, following [20]. [16] discusses how these functions are represented in the software [21].

Let $0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1$. Denote by

$$(9.1) \quad B = \{x_0 + t, x_1 + t, \dots, x_{n-1} + t \mid t \in \mathbb{Z}\}$$

the set of all breakpoints. The 0-dimensional faces are defined to be the singletons, $\{x\}$, $x \in B$, and the 1-dimensional faces are the closed intervals, $[x_i + t, x_{i+1} + t]$, $i = 0, \dots, n-1$, $t \in \mathbb{Z}$. The empty face, the 0-dimensional and the 1-dimensional faces form $\mathcal{P} = \mathcal{P}_B$, a locally finite polyhedral complex, periodic modulo \mathbb{Z} .

Definition 9.1. We call a function $\pi: \mathbb{R} \rightarrow \mathbb{R}$ *piecewise linear* over \mathcal{P}_B if for each face $I \in \mathcal{P}_B$, there is an affine linear function $\pi_I: \mathbb{R} \rightarrow \mathbb{R}$, $\pi_I(x) = c_I x + d_I$ such that $\pi(x) = \pi_I(x)$ for all $x \in \text{rel int}(I)$.

Under this definition, piecewise linear functions can be discontinuous. Let $I = [a, b] \in \mathcal{P}_B$ be a 1-dimensional face. The function π can be determined on $\text{int}(I) = (a, b)$ by linear interpolation of the limits $\pi(a^+) = \lim_{x \rightarrow a, x > a} \pi(x) = \pi_I(a)$ and $\pi(b^-) = \lim_{x \rightarrow b, x < b} \pi(x) = \pi_I(b)$.

9.2. Two-dimensional polyhedral complex $\Delta\mathcal{P}$ and additive faces. For a piecewise linear function (see subsection 9.1 for our notation), we now explain the structure of the initial moves. We will use the notion of the polyhedral complex $\Delta\mathcal{P}$ and its *additive faces* from [16, section 4]. $\Delta\mathcal{P}$ is a two-dimensional polyhedral complex, which expresses the domains of linearity of the subadditivity slack $\Delta\pi(x, y)$ introduced in subsection 1.2.

Definition 9.2. The polyhedral complex $\Delta\mathcal{P}$ of $\mathbb{R} \times \mathbb{R}$ consists of the faces

$$F(I, J, K) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \in I, y \in J, x + y \in K\},$$

where $I, J, K \in \mathcal{P}$, so each of I, J, K is either empty, a breakpoint of π , or a closed interval delimited by two consecutive breakpoints.

In the continuous case, since the function π is piecewise linear over \mathcal{P} , we have that $\Delta\pi$ is affine linear over each (closed) face $F \in \Delta\mathcal{P}$. We say that a

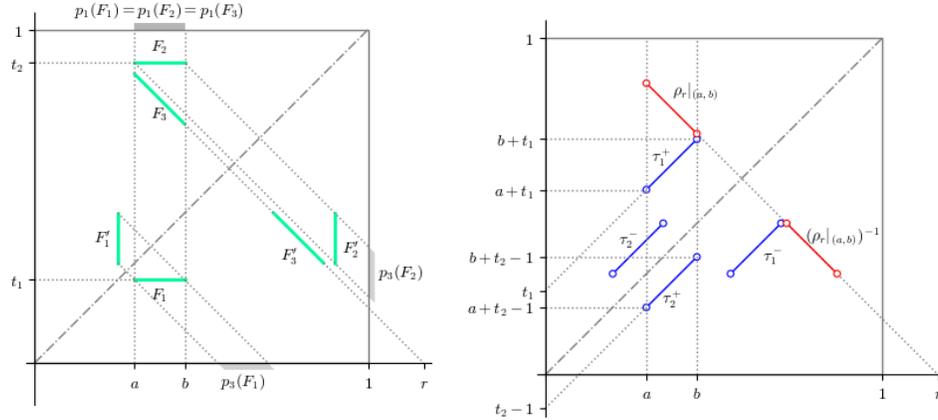


FIGURE 10. Additive edges in $\Delta\mathcal{P}_B$ and the corresponding initial moves (additive moves and their inverses) in Ω^0

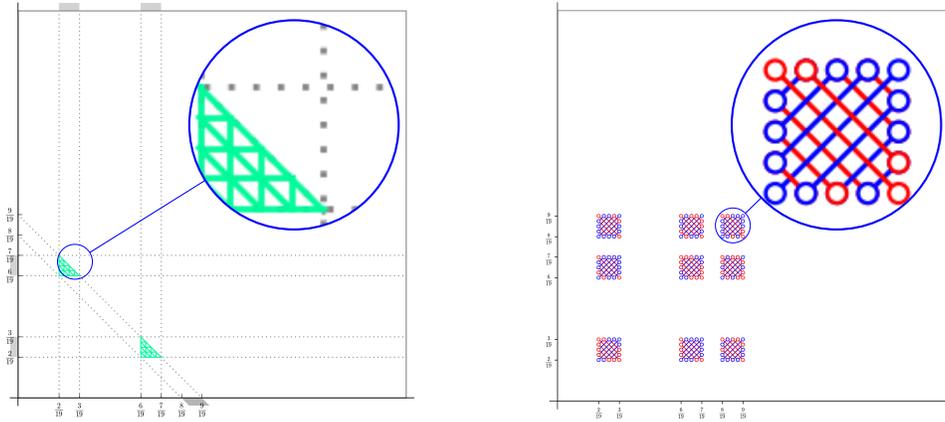


FIGURE 11. Additivities and initial moves. *Left*, additivities sampled from two-dimensional additive faces of $\Delta\mathcal{P}$. *Right*, the move semigroup $\text{jsemi}(\Omega^0)$ generated by the initial moves. The graphs $\text{Gr}_+(\Omega)$ (*blue*) and $\text{Gr}_-(\Omega)$ (*red*) are plotted on top of each other. For illustration purposes, only a finite set of additive moves is considered.

face $F \in \Delta\mathcal{P}$ is *additive* if $\Delta\pi = 0$ over all F . If π is subadditive, then the *set of additivities*

$$(9.2) \quad E(\pi) = \{(x, y) \mid \Delta\pi(x, y) = 0\}$$

is the union of all additive faces $F \in \Delta\mathcal{P}$; see [7, section 3.4].

For a discontinuous function π , the subadditivity slack $\Delta\pi$ is affine linear only over the relative interior of each face F . For additivity, beside the subadditivity slack $\Delta\pi(x, y)$ at a point (x, y) , we also consider its limits.

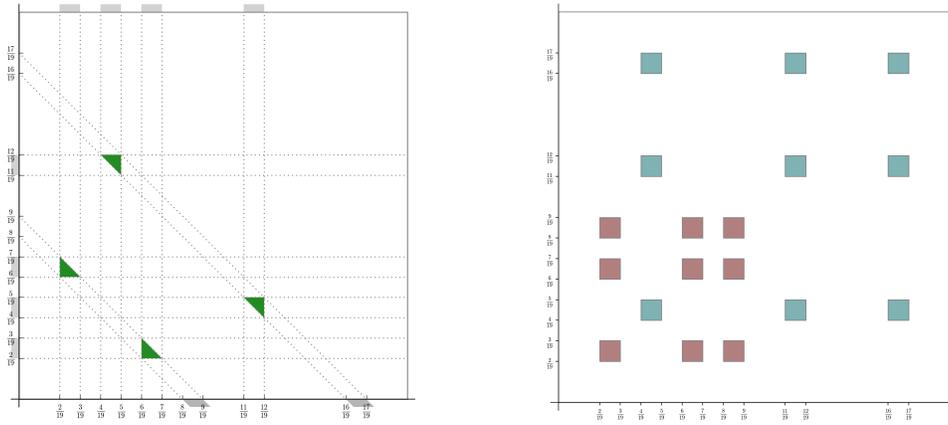


FIGURE 12. Additivities in $E(\pi)$ and the corresponding connected covered components of moves

Definition 9.3. The limit value of $\Delta\pi$ at the point (x, y) approaching from the relative interior of a face $F \in \Delta\mathcal{P}$ containing (x, y) is denoted by

$$\Delta\pi_F(x, y) = \lim_{\substack{(u,v) \rightarrow (x,y) \\ (u,v) \in \text{relint}(F)}} \Delta\pi(u, v).$$

Definition 9.4. Let $F \in \Delta\mathcal{P}$. Define the set of additivities and limit-additivities approaching from the relative interior of F as

$$(9.3) \quad E_F(\pi) = \{ (x, y) \in F \mid \Delta\pi_F(x, y) \text{ exists, and } \Delta\pi_F(x, y) = 0 \}.$$

Remark 9.5. The points $(x, y) \in E_F(\pi)$ that lie in $\text{relint}(F)$ capture all additivities of π , whereas those that lie on the relative boundary capture all limit-additivities. The set $E(\pi)$ that we introduced in the continuous case can be partitioned as

$$E(\pi) = \bigcup_{F \in \Delta\mathcal{P}} (E_F(\pi) \cap \text{relint}(F)).$$

Lemma 9.6. Let π be a subadditive function that is piecewise linear over \mathcal{P} . Let $F \in \Delta\mathcal{P}$. Let $(x_0, y_0) \in E_F(\pi) \subseteq F$ and let E be the unique face of F containing (x_0, y_0) in its relative interior. Then $E \subseteq E_F(\pi)$.

We make the following general definition, which is equivalent to the one found in [16, 20].

Definition 9.7. In the situation of Lemma 9.6, we say that the face E is *additive*.

Now the following lemma is clear from the definition. [16] only states this fact for the case of continuous π .

Lemma 9.8. Let π be a subadditive function that is piecewise linear over \mathcal{P} . Then the set of additive faces of π is a polyhedral subcomplex of $\Delta\mathcal{P}$, i.e.,

it is closed under taking subfaces. In particular, each additive face is the convex hull of some additive vertices (zero-dimensional additive faces).

For a piecewise linear function π , a finite presentation of the initial moves is easy to compute using the additive faces of the complex $\Delta\mathcal{P}$. For a detailed explanation of diagrams visualizing the additivities and limit-additivities, we refer to [16, sections 4.2–4.3]. See Figure 10 for the moves from one-dimensional additive faces (edges) and Figure 11 and Figure 12 for the moves from two-dimensional additive faces. In the forthcoming paper [14], we will give a more detailed description how to compute the finite presentation of the initial moves.

Remark 9.9. The zero-dimensional additive faces (i.e., additive vertices) of $\Delta\mathcal{P}_B$ do not give rise to moves (cf. Remark 5.2). Instead they will be considered in section 10 to determine a refinement of \mathcal{P}_B for the decomposition of perturbations.

9.3. Effective perturbations $\tilde{\pi}$. We recall the notion of effective perturbations from subsection 1.4. An effective perturbation is a function $\tilde{\pi}: \mathbb{R} \rightarrow \mathbb{R}$ for which there exists an $\epsilon > 0$ such that $\pi^\pm = \pi \pm \epsilon\tilde{\pi}$ are minimal valid functions.

Remark 9.10. Let π be a minimal valid function for $R_f(\mathbb{R}, \mathbb{Z})$. From (1.3a), (1.3c), and (1.3e) it follows that $0 \leq \pi \leq 1$, so π is a bounded function. Now if $\tilde{\pi}$ is an effective perturbation, then $\pi^\pm = \pi \pm \epsilon\tilde{\pi}$ for some $\epsilon > 0$, where also $0 \leq \pi^\pm \leq 1$, and so $\tilde{\pi}$ is a bounded function as well.

We note that the space $\tilde{\Pi}^\pi$ of effective perturbations, introduced in subsection 1.4, is a vector space.

Lemma 9.11. *Let π be a minimal valid function. The space $\tilde{\Pi}^\pi$ of effective perturbation functions is a vector space, a subspace of the space $\mathcal{B}(\mathbb{R})$ of bounded functions.*

For the case of piecewise linear functions π that are continuous from at least one side of the origin, we have the following regularity theorem for effective perturbations.

Lemma 9.12 ([16, Lemma 6.4]). *Let π be a piecewise linear minimal valid function that is continuous from the right at 0 or continuous from the left at 1. If π is continuous on a proper interval $I \subseteq [0, 1]$, then for any $\tilde{\pi} \in \tilde{\Pi}^\pi$ we have that $\tilde{\pi}$ is Lipschitz continuous on the interval I .*

(This is a strengthening of [8, Theorem 2].)

The purpose of the additive move ensemble is to infer properties of the effective perturbation functions. For additive moves $\gamma|_D$, it follows from convexity that every effective perturbation $\tilde{\pi}$ respects $\gamma|_D$. In the case of piecewise linear functions, this extends to limit-additive moves. The following lemma is shown by the proof of [16, Theorem 6.3], along with [16, Footnote 13] and also by [20, Theorem 3.3] in the case where π is two-sided discontinuous at the origin.

Lemma 9.13. *Let π be a piecewise linear minimal valid function for $R_f(\mathbb{R}, \mathbb{Z})$. Let $\gamma|_D \in \Omega^0$ be an initial move, where $D \subseteq (0, 1)$ is an open interval. Then π respects $\gamma|_D$, and every effective perturbation function $\tilde{\pi} \in \tilde{\Pi}^\pi$ respects $\gamma|_D$.*

Corollary 9.14. *Let π be a piecewise linear minimal valid function for $R_f(\mathbb{R}, \mathbb{Z})$. Then π respects the moves closure $\text{clsemi}_A(\Omega^0)$. If π is continuous from at least one side of the origin, then every effective perturbation function $\tilde{\pi} \in \tilde{\Pi}^\pi$ also respects the moves closure $\text{clsemi}_A(\Omega^0)$.*

Proof. This follows from Lemma 9.13, Remark 9.10, Lemma 9.12 and Theorem 7.24. \square

9.4. Closed move semigroup generated by Ω^0 , rational case. We have the following theorem.

Theorem 9.15 (Finite presentation of the moves closure, rational case). *Let π be a piecewise linear function whose breakpoints are rational, i.e., $B \subseteq G = \frac{1}{q}\mathbb{Z}$ for some $q \in \mathbb{N}$. Then the moves closure $\text{clsemi}_A(\Omega^0)$ has a finite presentation $(\Omega^{\text{red}}, \mathcal{C})$ in reduced form, where (i) the endpoints of all domains and the values t and r of moves $\tau_t, \rho_r|_D \in \Omega^{\text{red}}$ lie in $G \cap [0, 1]$, (ii) the endpoints of all maximal intervals of all $C_i \in \mathcal{C}$ lie in $G \cap [0, 1]$.*

Proof sketch. We can compute $\text{clsemi}_A(\Omega^0)$ in finitely many steps using a completion-type algorithm that manipulates finite presentations, maintaining properties (i) and (ii), using only the algebraic and order-theoretic axioms and (extend_A) . The initialization is provided by Corollary 4.9, noting that vertices of additive faces of $\Delta\mathcal{P}$ lie in $G \times G$. There are only finitely many finite presentations satisfying (i) and (ii); this implies the finiteness of the algorithm. \square

We defer all details about such an algorithm, as well as its generalization to non-rational input, to the forthcoming paper [14].

Instead, in the next section, we assume that a finite presentation $(\Omega^{\text{fin}}, \mathcal{C})$ of the moves closure $\text{clsemi}_A(\Omega^0)$ is given. Using the finite presentation, we can give a description of the space of effective perturbations.

10. PERTURBATION SPACE

Let $\pi: \mathbb{R} \rightarrow \mathbb{R}$ be a minimal valid function. In this section, we work with the following assumptions. (We will mention them explicitly only in statements of main theorems.)

10.1. Assumptions: Piecewise linear π , one-sided continuous at 0, finitely presented moves closure $\text{clsemi}_A(\Omega^0)$.

Assumption 10.1. The minimal valid function π is piecewise linear (subsection 9.1) and continuous from at least one side of the origin.

Assumption 10.2. The set B is minimal, i.e., \mathcal{P}_B is the coarsest polyhedral complex over which π is piecewise linear.

Let $\Omega^0 = \Omega^0(\pi)$ be the initial additive move ensemble (section 8) of π .

Assumption 10.3. The moves closure $\text{clsemi}_A(\Omega^0)$ has a finite presentation (Ω, \mathcal{C}) in reduced form (subsection 6.5).

Thus Ω has finitely many moves and \mathcal{C} has finitely many connected covered components C_1, C_2, \dots, C_k , each of which is a finite union of open intervals. Each $\gamma|_D \in \Omega$ is maximal in the restriction partial order of $\text{clsemi}_A(\Omega^0)$ and is not contained in $\text{jmoves}(\mathcal{C})$. Figures 13 (right), 15, and 16 show examples of $\text{clsemi}_A(\Omega^0)$ satisfying Assumption 10.3.

10.2. Properties of the finitely presented moves closure. Let $C := C_1 \cup C_2 \cup \dots \cup C_k$ denote the open set of points in $(0, 1)$ that are covered. We will refer to the open set $U := (0, 1) \setminus \text{cl}(C)$ as the set of points in $(0, 1)$ that are *uncovered*. Let

$$(10.1) \quad X := \{0\} \cup \partial C \cup \{1\} = \{0\} \cup \partial U \cup \{1\}$$

be the set of endpoints of all covered and uncovered intervals. Thus we have the partition $[0, 1] = C \cup X \cup U$.

Example 10.4. Consider the discontinuous minimal valid function for $f = \frac{1}{2}$, defined by

$$\pi(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } 0 < x < \frac{1}{2} \\ 2(1-x) & \text{if } \frac{1}{2} \leq x < 1. \end{cases}$$

It is provided by the software [21] as $\pi = \text{equiv7_example_1}()$. Figure 13 shows the two-dimensional polyhedral complex $\Delta\mathcal{P}$ and the moves closure $\text{clsemi}_A(\Omega^0)$. The interval $C = (\frac{1}{2}, 1)$ is covered, $U = (0, \frac{1}{2})$ is uncovered. We have $X = \{0, \frac{1}{2}, 1\}$.

Example 10.5. Consider the continuous minimal valid function π that is provided as $\text{equiv7_example_xyz_2}()$ by the software [21], shown in Figure 14. Figures 14 and 15 show the additive faces and the moves closure. We have $X = \{0, \frac{1}{24}, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{2}{3}, \frac{3}{4}, \frac{19}{24}, \frac{7}{8}, \frac{11}{12}, 1\}$.

Example 10.6. Consider the minimal valid function π that is provided as $\text{equiv7_minimal_2_covered_2_uncovered}()$ by the software [21]; see Figure 16. It has two connected covered components. The set of uncovered points is $U = (\frac{12}{49}, \frac{13}{49}) \cup (\frac{14}{49}, \frac{15}{49}) \cup \dots \cup (\frac{20}{49}, \frac{21}{49})$. Thus we have $X = \{0, \frac{12}{49}, \frac{13}{49}, \dots, \frac{20}{49}, \frac{21}{49}, 1\}$.

Recall the two-dimensional polyhedral complex $\Delta\mathcal{P}_B$ and its additive faces, introduced in subsection 9.2. Let

$$(10.2) \quad V := \{p_i(x, y) \mid (x, y) \text{ additive vertex of } \Delta\mathcal{P}_B, i = 1, 2, 3\} \cap [0, 1]$$

be the set of p_1, p_2 and p_3 projections (within the fundamental domain) of the zero-dimensional additive faces (i.e., additive vertices). Because $(x, 0)$ is an additive vertex of $\Delta\mathcal{P}_B$ for every $x \in B$, the set V contains $B \cap [0, 1]$. By Remark 8.3, the initial move ensemble Ω^0 is join-closed. We consider

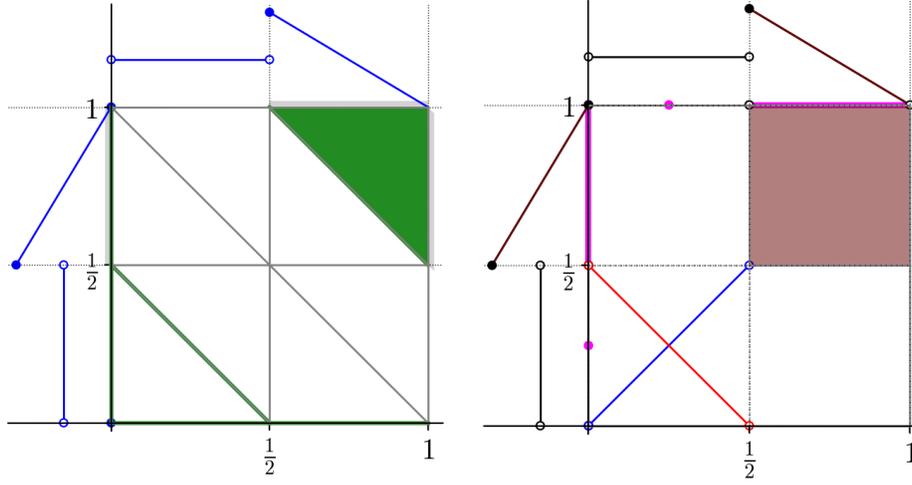


FIGURE 13. (*Left*) Two-dimensional polyhedral complex $\Delta\mathcal{P}$ of the one-sided discontinuous minimal valid function $\pi = \text{equiv7_example_1}()$ from Example 10.4 (*blue graph at the left and top borders*), where the additive faces are *colored in green*. (*Right*) The graph of the moves closure $\text{clsemi}_A(\Omega^0)$ of π . It has a finite presentation by $\Omega = \{\tau_0|_{(0,1/2)}, \rho_{1/2}|_{(0,1/2)}\}$ (*blue and red line segments of slopes ± 1*) and one (maximal) connected covered component $C = (\frac{1}{2}, 1)$ (*the brown square shows $C \times C$*). The set $C \cup X \cup Y \cup Z$ of covered points and refined breakpoints is marked in *magenta on the left and top borders*.

the ensemble $\Omega^0|_U$ of moves restricted to U , as defined in subsection 3.2. By Lemma 3.10 it is also join-closed and therefore, by Lemma 3.8, has a presentation by its maximal elements. It follows from Lemma 9.8 that its maximal elements have the following relation to the set V .

Lemma 10.7. *If $\gamma|_{(a,b)} \in \text{Max}(\Omega^0|_U)$, then the endpoints a, b lie in $V \cap U$ or ∂U .*

Next we define the set

$$(10.3) \quad Y := \Omega(V \cap U) = \{\gamma|_D(x) \mid x \in V \cap U, x \in D \text{ and } \gamma|_D \in \Omega\},$$

the orbit of $V \cap U$ under Ω , which is a finite set by Assumption 10.3.

Remark 10.8. In terms of graphs of ensembles, we have

$$(10.4) \quad Y = \{y \mid \exists x \in V \cap U \text{ such that } (x, y) \in \text{Gr}(\Omega)\}.$$

Lemma 10.9. *We have (a) $V \cap U \subseteq Y$, and (b) $Y \subseteq U$.*

Proof. (a) Let $x \in V \cap U$. Since $\Delta\pi(x, y) = 0$ for any $x \in \mathbb{R}$ when $y = 0$, we know that there is open interval D with $x \in D \subseteq U$ such that the idempotent

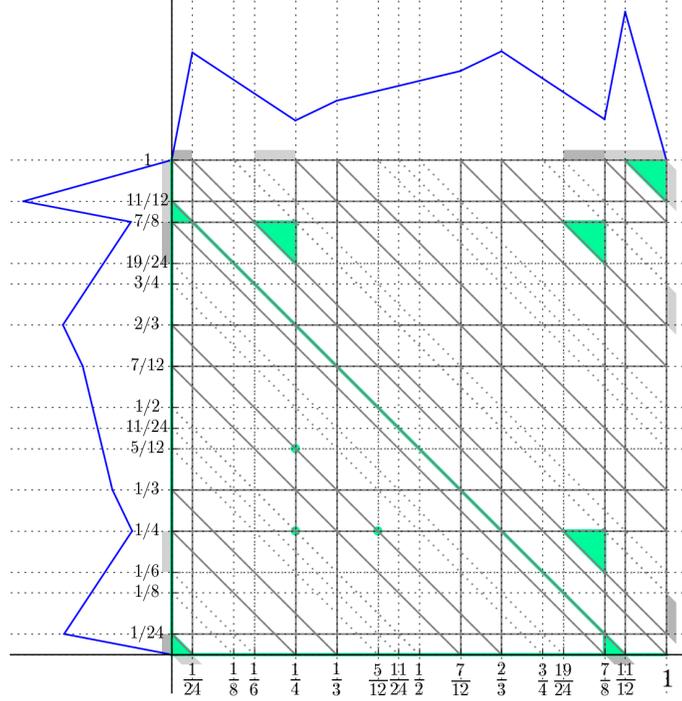


FIGURE 14. The function $\pi = \text{equiv7_example_xyz_2}()$ from Example 10.5 (blue graph at the left and top borders) and its two-dimensional polyhedral complex $\Delta\mathcal{P}$ (solid gray lines), where the additive faces are colored in green. The refined complex $\Delta\mathcal{T}$ is shown with dotted gray lines.

$\tau_0|_D$ is in Ω^0 and hence in $\text{restrict}(\Omega)$ as well. Since $x = \tau_0|_D(x)$, we obtain that $x \in Y$.

(b) Suppose for the sake of contradiction that there is $y \in Y$ but $y \in \text{cl}(C)$. We can write y as $y = \gamma|_D(x)$ where $x \in V \cap U$, $x \in D$ and $\gamma|_D \in \Omega$. Under Assumption 10.3, by Lemma 6.14 applied to C and $\text{clsemi}_A(\Omega^0)$, we have that C is invariant under the action of moves from $\text{clsemi}_A(\Omega^0)$. Since the inverse move $(\gamma|_D)^{-1} \in \Omega \subseteq \text{clsemi}_A(\Omega^0)$, we obtain that $x = (\gamma|_D)^{-1}(y) \in \text{cl}(C)$. This contradicts $x \in U$. \square

Example 10.10 (Example 10.6, continued). In the example shown in Figure 15, we have $V \cap U = \{\frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}\}$. This set is already closed under the action of Ω , as $\rho_{11/12}(\frac{1}{3}) = \frac{7}{12}$ and $\rho_{11/12}(\frac{5}{12}) = \frac{1}{2}$. Thus $Y = V \cap U$ in the example.

We consider the ensembles $\Omega|_U$ and $U|\Omega|_U$ of moves restricted and double-restricted to U , as defined in subsection 3.2. We have the following results.

Lemma 10.11. *The move ensemble $\Omega|_U$ satisfies:*

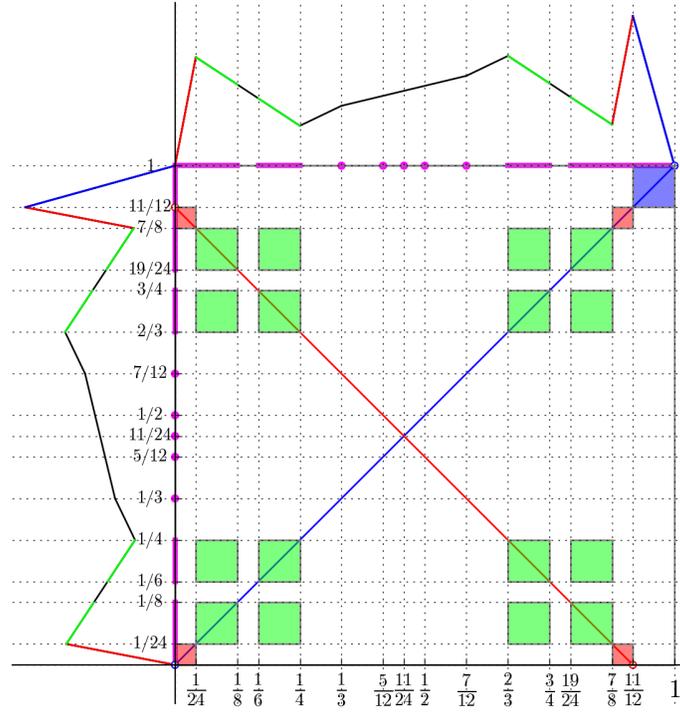


FIGURE 15. The function $\pi = \text{equiv7_example_xyz_2}()$ from Example 10.5 (colored graph at the left and top borders) and the graph of the moves closure $\text{clsemi}_A(\Omega^0)$ of π , as computed by the command `igp.equiv7_mode = True; igp.extremality_test(igp.equiv7_example_xyz_2(), True, show_all_perturbations=True)`. It has a finite presentation by $\Omega = \{\tau_0|_{(0,1)}, \rho_{11/12}|_{(0,11/12)}\}$ (blue and red line segments of slopes ± 1) and a set $\mathcal{C} = \{C_1, C_2, C_3\}$ of (maximal) connected covered components $C_1 = (\frac{11}{12}, 1)$ (the lavender square shows $C_1 \times C_1$), $C_2 = (0, \frac{1}{24}) \cup (\frac{7}{8}, \frac{11}{12})$ (coral), and $C_3 = (\frac{1}{24}, \frac{1}{8}) \cup (\frac{1}{6}, \frac{1}{4}) \cup (\frac{2}{3}, \frac{3}{4}) \cup (\frac{19}{24}, \frac{7}{8})$ (lime). The set $\mathcal{C} \cup B' = \mathcal{C} \cup X \cup Y \cup Z$ of covered points and refined breakpoints is marked in magenta on the left and top borders.

- (a) $\Omega|_U = U|\Omega|_U$.
- (b) $\Omega|_U$ is a finite move ensemble.

Proof. It follows directly from Assumption 10.3. □

Lemma 10.12 (Filtration of $\text{isemi}(\Omega^0|_U)$ by word length; maximal moves).
For $k \in \mathbb{N}$, let

$$\Omega^0|_U^k = \{ \gamma^k|_{D^k} \circ \cdots \circ \gamma^1|_{D^1} \mid \gamma^i|_{D^i} \in \Omega^0|_U \text{ for } 1 \leq i \leq k \}.$$

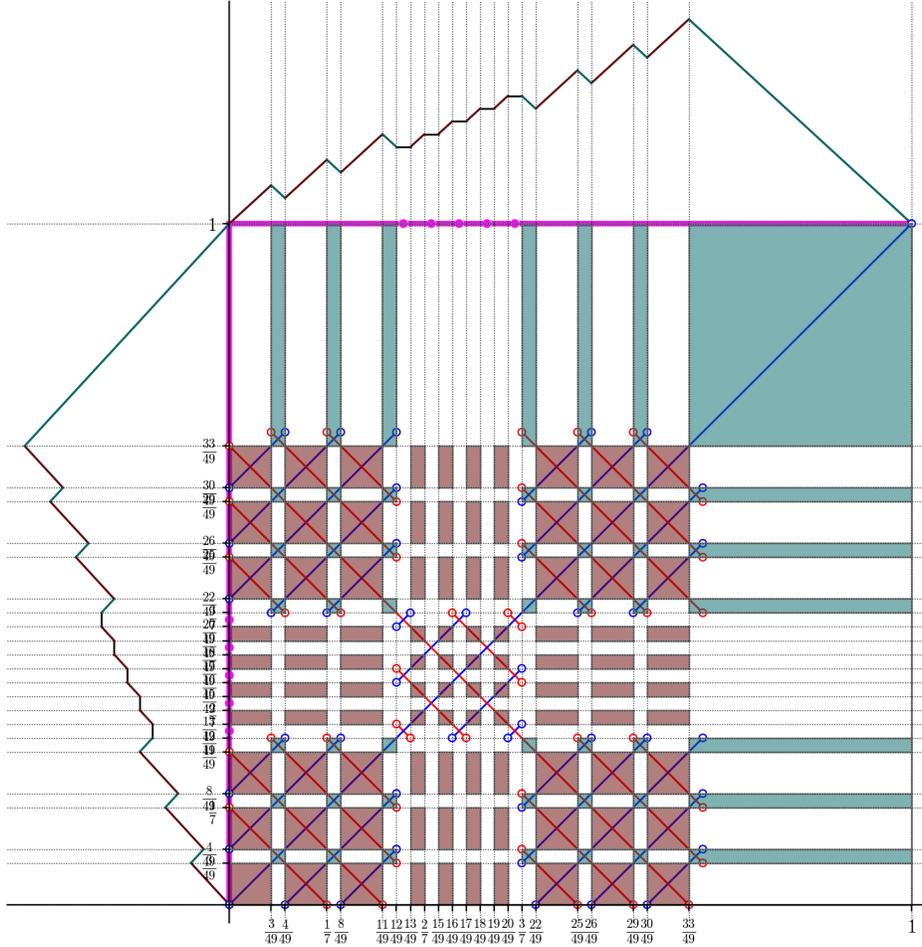


FIGURE 16. Moves closure $\text{clsemi}_A(\Omega^0)$ for the function from Example 10.6, $\pi = \text{equiv7_minimal_2_covered_2_uncovered}()$. It has two connected covered components (*cyan, brown rectangles*).

Then $\Omega^0|_{U^1} \subseteq \Omega^0|_{U^2} \subseteq \dots$ and $\text{isemi}(\Omega^0|_U) = \bigcup_{k \in \mathbb{N}} \Omega^0|_{U^k}$. For each $k \in \mathbb{N}$, the ensemble $\Omega^0|_{U^k}$ satisfies (restrict) and has a presentation by the set $\text{Max}(\Omega^0|_{U^k})$ of its maximal elements, which is a finite set. For $\gamma|_{(a,b)} \in \text{Max}(\Omega^0|_{U^k})$, we have $a, b, \gamma(a), \gamma(b) \in X \cup Y$.

Proof. Because Ω^0 satisfies (inv), (continuation), and (restrict) by 8.3, so does its double restriction $U|\Omega^0|_U$ to the uncovered set U . By Lemma 10.11-(a), we have $\text{im}(\Omega^0|_U) \subseteq \text{im}(\Omega|_U) = \text{im}(U|\Omega|_U) \subseteq U$, hence $\Omega^0|_U = U|\Omega^0|_U$. Recall that $\Omega^0|_U$ is join-closed and therefore has a presentation by its maximal elements. It follows from the definition of Ω^0 , Assumption 10.1 with Remark 9.5, and Corollary 8.4 that $\text{Max}(\Omega^0|_U)$ is finite.

Let $\text{Max}(\Omega^0|_U)^k = \{\gamma^k|_{\hat{D}^k} \circ \cdots \circ \gamma^1|_{\hat{D}^1} \mid \gamma^i|_{\hat{D}^i} \in \text{Max}(\Omega^0|_U)\}$, a finite set. Then $\text{Max}(\Omega^0|_{U^k}) \subseteq \text{Max}(\Omega^0|_U)^k$ is finite, and every element of $\Omega^0|_{U^k}$ is the restriction of an element of $\text{Max}(\Omega^0|_{U^k})$. The chain of inclusions $\Omega^0|_{U^1} \subseteq \Omega^0|_{U^2} \subseteq \dots$ holds because the idempotents $\tau_0|_D$ for intervals $D \subseteq U$ are elements of $\Omega^0|_U$.

Last, we prove the claim regarding the endpoints; we actually prove the slightly stronger claim $a, b, \gamma(a), \gamma(b) \in \partial U \cup Y$ by induction by word length k . Since each $\Omega^0|_{U^k}$ satisfies (inv), it suffices to prove $a, b \in \partial U \cup Y$. For $k = 1$, let $\gamma|_{(a,b)} \in \text{Max}(\Omega^0|_{U^1}) = \text{Max}(\Omega^0|_U)$. Then, by Lemma 10.7, each of the endpoints a, b lies in $V \cap U \subseteq Y$, or it lies in ∂U . Now we proceed by induction. Take $\gamma^1|_{(a,b)} \in \text{Max}(\Omega^0|_U)$ and $\gamma^2|_{(c,d)} \in \text{Max}(\Omega^0|_{U^{k-1}})$, so $a, b, c, d \in \partial U \cup Y$. Then, by (1.9), $\gamma^2|_{(c,d)} \circ \gamma^1|_{(a,b)}$ has domain $(a, b) \cap (\gamma^1)^{-1}((c, d))$. If the domain is nonempty, let x be an endpoint of it. If $x = a, b$, nothing is to show, so assume $x \in (a, b)$ and $x = (\gamma^1)^{-1}(y)$, where $y = c$ or $y = d$, so $y \in \partial U \cup Y$. But $y = \gamma^1|_{(a,b)}(x) \in U$, so $y \in Y$. Then it follows that also $x \in Y$. \square

By Assumption 10.3, all elements of Ω are maximal moves of the moves closure $\text{clsemi}_A(\Omega^0)$. Therefore, by Lemma 3.11, all elements of $\Omega|_U$ are maximal moves of $\text{clsemi}_A(\Omega^0)|_U$.

After these preliminaries, we are able to state the main theorem.

Theorem 10.13 (Structure and generation theorem for finitely presented moves closures). *Under Assumption 10.3, we have*

- (a) $\text{clsemi}_A(\Omega^0) = \text{extend}_A(\text{clsemi}_A(\Omega^0|_U) \cup \text{clsemi}_A(\Omega^0|_C))$.
- (b) $\Omega|_U = \text{Max}(\text{extend}_A(\text{isemi}(\Omega^0|_U)))$.
- (c) $a, b, \gamma(a), \gamma(b) \in X \cup Y$ for any $\gamma|_{(a,b)} \in \Omega|_U$.

We emphasize that the theorem does *not* depend on an algorithm to compute the moves closure.

Proof. Part (a). Let Ω' denote the right hand side of the equation in part (a). Clearly, $\Omega^0 \subseteq \Omega' \subseteq \text{clsemi}_A(\Omega^0)$. We now show that Ω' is a closed move semigroup. By Lemma 10.11-(a), we have that

$$\begin{aligned} \text{clsemi}_A(\Omega^0|_U) &\subseteq \text{restrict}(\Omega|_U) \subseteq \text{moves}(U \times U); \\ \text{clsemi}_A(\Omega^0|_C) &= \text{moves}(\mathcal{C}) \subseteq \text{moves}(C \times C), \end{aligned}$$

where the open sets U and C are disjoint. Thus, we have that $\text{clsemi}_A(\Omega^0|_U) \cup \text{clsemi}_A(\Omega^0|_C)$ is a move semigroup, under Assumption 10.3. It follows from Lemma 7.16 that Ω' is a move semigroup that satisfies (extend_A) . Note that for any open intervals D and I such that $\text{moves}(D \times I) \subseteq \text{clsemi}_A(\Omega^0)$, we have $\text{moves}(D \times I) \subseteq \text{clsemi}_A(\Omega^0|_C)$. Therefore, Ω' also satisfies (kaleido) . Moreover, (lim) holds by Theorem 7.9. We conclude that Ω' is a closed move semigroup. Hence, part (a) holds.

Part (b). By restricting the moves ensembles on both sides of the equation in part (a) to domain U , we obtain that

$$(10.5) \quad \text{restrict}(\Omega|_U) = \text{clsemi}_A(\Omega^0)|_U = \text{clsemi}_A(\Omega^0|_U)$$

Next, we show that

$$(10.6) \quad \text{clsemi}_A(\Omega^0|_U) = \text{extend}_A(\text{isemi}(\Omega^0|_U)).$$

It follows from Lemma 7.16 that $\text{extend}_A(\text{isemi}(\Omega^0|_U))$ is a move semigroup that satisfies (extend_A) (and also (continuation) and (restrict)). Since

$$(10.7) \quad \text{extend}_A(\text{isemi}(\Omega^0|_U)) \subseteq \text{clsemi}_A(\Omega^0|_U) = \text{restrict}(\Omega|_U),$$

where the equality follows from (10.5), and $\Omega|_U$ is a finite move ensemble by Lemma 10.11-(b), we obtain that the move semigroup $\text{extend}_A(\text{isemi}(\Omega^0|_U))$ also satisfies (kaleido) and (lim) . Therefore, $\text{extend}_A(\text{isemi}(\Omega^0|_U))$ is a closed move semigroup which contains $\Omega^0|_U$. Since $\text{clsemi}_A(\Omega^0|_U)$ is the smallest closed move semigroup containing $\Omega^0|_U$, we have

$$\text{clsemi}_A(\Omega^0|_U) \subseteq \text{extend}_A(\text{isemi}(\Omega^0|_U)).$$

Together with (10.7), we conclude that (10.6) holds. Since Ω has only maximal moves, (10.5) and (10.6) imply the equation in part (b).

Part (c). Let $\gamma|_{(a,b)} \in \Omega|_U$. By symmetry, it suffices to show that $a, b \in X \cup Y$. Consider $x = a$ or $x = b$. Part (b) implies that

$$\Omega|_U = \text{Max}(\text{extend}_A(\text{jsemi}(\Omega^0|_U))).$$

Together with (7.2), we know that x is the limit of a sequence $\{x^j\}_{j \in \mathbb{N}}$, where x^j is an endpoint of the domain D^j of a move $\gamma|_{D^j} \in \text{Max}(\text{jsemi}(\Omega^0|_U))$. By Lemma 10.12 and Lemma 3.6, for any $j \in \mathbb{N}$, we have that D^j is a maximal subinterval of $\bigcup \{D \mid \gamma|_D \in \bigcup_{k \in \mathbb{N}} \text{Max}(\Omega^0|_{U^k})\}$. Thus for every $j \in \mathbb{N}$, there exists a sequence $\{x_k^j\}_{k \in \mathbb{N}}$ such that each x_k^j is an endpoint of the domain of a move $\gamma|_{D_k^j} \in \text{Max}(\Omega^0|_{U^k})$, and $x_k^j \rightarrow x^j$ as $k \rightarrow \infty$. We obtain that $x_k^k \rightarrow x$ as $k \rightarrow \infty$, where each $x_k^k \in X \cup Y$ by Lemma 10.12. Since $X \cup Y$ is a finite discrete set under Assumption 10.3, we obtain that $x \in X \cup Y$. \square

10.3. Refined breakpoints B' , complex \mathcal{T} . In addition to the finite sets X and Y , we define

$$(10.8) \quad Z := \{x \mid x \in U, x = \rho|_D(x) \text{ for some reflection move } \rho|_D \in \Omega\},$$

the set of *uncovered character conflicts*.

Remark 10.14. In terms of Gr_+ and Gr_- notations, the set Z is the set of projections of the intersection of the translation and reflection moves graphs restricted to the uncovered intervals, $Z = \{x \mid x \in U, (x, x) \in \text{Gr}_\pm(\Omega)\}$.

Example 10.15 (Example 10.6, continued). In the example shown in Figure 15, we have $Z = \{\frac{11}{24}\}$.

Theorem 10.16. *Under Assumption 10.3, the sets X , Y , and Z are closed under the action of all moves from $\text{clsemi}_A(\Omega^0)$.*

Proof. Let $\gamma|_D$ be a move in $\text{clsemi}_A(\Omega^0)$.

Let $x \in X$ such that $x \in D$. Since C is invariant under the action of all moves from $\text{clsemi}_A(\Omega^0)$, we have that $\gamma|_D(x) \in X$.

Let $y \in Y$ such that $y \in D$. There exist $x \in V \cap U$ and $\gamma'|_{D'} \in \Omega$ such that $\gamma'|_{D'}(x) = y$. We have $\gamma|_D(y) = \gamma|_D \circ \gamma'|_{D'}(x)$, where $\gamma|_D \circ \gamma'|_{D'} \in \text{restrict}(\Omega)$. Therefore, $\gamma|_D(y) \in Y$.

Let $z \in Z$ such that $z \in D$. By definition, $z \in U$ and $z = \rho|_{D'}(z)$ for some reflection move $\rho|_{D'} \in \Omega$. Let $z' = \gamma|_D(z)$. We have that $z' \in U$ and $z' = \gamma|_D \circ \rho|_{D'} \circ (\gamma|_D)^{-1}(z')$, where $\gamma|_D \circ \rho|_{D'} \circ (\gamma|_D)^{-1} \in \text{restrict}(\Omega)$. Therefore, $z' = \gamma|_D(z) \in Z$. \square

Under Assumption 10.3, the sets X , Y , Z are finite. We then define B' , which is a finite set of points under Assumption 10.3, a *refined set of breakpoints*,

$$(10.9) \quad B' := (X \cup Y \cup Z) + \mathbb{Z}.$$

By Assumption 10.2, $B \cap C = \emptyset$ and thus $B \subseteq B'$. Hence, the polyhedral complex

$$(10.10) \quad \mathcal{T} := \mathcal{P}_{B'},$$

is a refinement of \mathcal{P}_B , so our function π is piecewise linear over \mathcal{T} . The following result shows that each of the p_1, p_2 and p_3 projections of any additive vertex of the two-dimensional polyhedral complex $\Delta\mathcal{T}$ is either in B' or covered by C .

Theorem 10.17 (Breakpoint stabilization theorem). *Let (x, y) be an additive vertex of $\Delta\mathcal{T}$. Let $z = x + y$. Then, $x, y, z \in B' \cup (C + \mathbb{Z})$.*

Proof. Let F be the unique face of $\Delta\mathcal{P}_B$ such that $(x, y) \in \text{relint}(F)$. Since (x, y) is an additive vertex of $\Delta\mathcal{T}$, and $\Delta\pi$ is non-negative and affine linear over F , we have that F is an additive face of $\Delta\mathcal{P}_B$. Consider $t = x, y$ or z . By \mathbb{Z} -periodicity, we can assume $t \in [0, 1]$. To show that $t \in (B' \cap [0, 1]) \cup C$, we distinguish three cases, as follows. We recall that $B' \cap [0, 1] = X \cup Y \cup Z$ and $U = (0, 1) \setminus \text{cl}(C)$.

Assume that F is a zero-dimensional additive face of $\Delta\mathcal{P}_B$. Then, (x, y) is an additive vertex of $\Delta\mathcal{P}_B$, and thus $t \in V$. If $t \in \text{cl}(C)$, then $t \in X \cup C \subseteq B' \cup C$. Otherwise, $t \in V \cap U$. Since $V \cap U \subseteq Y$ by Lemma 10.9, we obtain that $t \in Y \subseteq B'$.

Assume that F is a one-dimensional additive face (say, a horizontal additive edge) of $\Delta\mathcal{P}_B$. Then, $y \in B \subseteq B'$ and the move $\tau_y|_D$ with $x \in D := \text{int}(p_1(F))$ is in Ω^0 . Since (x, y) is a vertex of $\Delta\mathcal{T}$, at least two of x, y, z are in B' , and hence at least one of x and z is in B' . Without loss of generality, we assume that $x \in B'$. By Theorem 10.16, $z = \tau_y|_D(x) \in B'$ as well. We showed that $x, y, z \in B'$ in this case. We omit the proof of the cases where

F is a vertical or diagonal additive edge of $\Delta\mathcal{P}_B$, which are similar to the above proof.

Assume that F is a two-dimensional additive face of $\Delta\mathcal{P}_B$. Then, by Corollary 8.4, we have $t \in C$. \square

Remark 10.18. Theorem 10.17 is key to our grid-free theory. In the grid case of [3], where $B = \frac{1}{q}\mathbb{Z}$, the projections $p_1: (x, y) \mapsto x$, $p_2: (x, y) \mapsto y$, and $p_3: (x, y) \mapsto x + y$ map all vertices of $\Delta\mathcal{P}_B$ back to the set B . We have stabilization of breakpoints due to unimodularity. Going to higher dimension (minimal valid functions of several variables), the piecewise linear functions defined on a standard triangulation of \mathbb{R}^2 studied in [4, 7] also stabilize. However, the non-existence of triangulations with stabilization for \mathbb{R}^k , $k \geq 3$ [13] blocks the path for further generalizations of the approach of [3, 4, 7]. Our Theorem 10.17 depends on more detailed data of the function than the group G generated by B . This “dynamic” stabilization result could pave the way to generalizations to higher dimension.

10.4. **Connected uncovered components U_i .** Define

$$(10.11) \quad U' := U \setminus B'.$$

The interval $[0, 1]$ is partitioned into the set C of covered points, the set U' of uncovered points and the set $B' \cap [0, 1]$ of breakpoints of \mathcal{T} . We consider the ensemble $\Omega|_{U'}$ of maximal moves restricted to U' as defined in subsection 3.2. Lemma 10.11 and Theorems 10.13 and 10.16 imply the following corollary.

Corollary 10.19. *Under Assumptions 10.2 and 10.3, the move ensemble $\Omega|_{U'}$ satisfies that:*

- (a) $\Omega|_{U'} = U'|_{U'}\Omega|_{U'}$.
- (b) $\Omega|_{U'}$ is a finite move ensemble.
- (c) For any $\gamma|_D \in \Omega|_{U'}$, $\text{cl}(D)$ and $\text{cl}(\gamma(D))$ are faces of \mathcal{T} .

We partition the set of uncovered points U' into the (maximal) *connected uncovered components* $\{U_1, \dots, U_l\}$, as follows.² A connected uncovered component U_i ($1 \leq i \leq l$) is a maximal subset of U' that is the disjoint union of all the uncovered intervals $I_1, I_2, \dots, I_p \subseteq U'$ such that any pair of intervals I_j and I_k ($1 \leq j, k \leq p$) are connected by a maximal move $\gamma|_{I_k} \in \Omega|_{U'}$ with domain I_k and image $I_j = \gamma(I_k)$.

Remark 10.20. The set $\Omega|_{U'}$ only has moves $\gamma|_D$ whose domain D and image $\gamma(D)$ are both contained in the same U_i , for $i = 1, 2, \dots, l$.

Since the function π is piecewise linear over \mathcal{T} and it respects $\Omega|_{U'}$, we have that π is affine linear with the same slope on the maximal intervals I_1, I_2, \dots, I_p of the same connected uncovered component U_i . Since an effective perturbation $\tilde{\pi} \in \tilde{\Pi}^\pi$ also respects $\Omega|_{U'}$, it takes the same shape on the uncovered intervals $I_1, I_2, \dots, I_p \subseteq U_i$. We pick $D \in \{I_1, I_2, \dots, I_p\}$ arbitrarily as the *fundamental domain*, and write $I_j = \gamma_j(D)$ where $\gamma_j|_D \in \Omega|_{U'}$

²This extends the terminology of [3] where connected components are grid-based.

for $j = 1, 2, \dots, p$. Then, the connected uncovered component $U_i \subseteq U'$ can be written as $U_i = \bigcup \gamma_j(D)$.

10.5. Finite-dimensional and equivariant perturbation subspaces. Under Assumption 10.1, we define the following spaces.

Definition 10.21. Define the *space of finite-dimensional perturbations* that are piecewise linear over \mathcal{T} :

$$(10.12) \quad \tilde{\Pi}_{\mathcal{T}}^{\pi} := \{ \tilde{\pi} \in \tilde{\Pi}^{\pi} \mid \tilde{\pi} \text{ is piecewise linear over } \mathcal{T} \}.$$

Thus, functions in $\tilde{\Pi}_{\mathcal{T}}^{\pi}$ are allowed to be discontinuous.

Definition 10.22. Define the *space of equivariant perturbations* that vanish on the vertices of \mathcal{T} :

$$\tilde{\Pi}_{\text{zero}(\mathcal{T})}^{\pi} := \left\{ \tilde{\pi} \in \tilde{\Pi}^{\pi} \mid \tilde{\pi}(t) = \lim_{\substack{x \rightarrow t \\ x < t}} \tilde{\pi}(t) = \lim_{\substack{x \rightarrow t \\ x > t}} \tilde{\pi}(t) = 0, \forall t \in \text{vert}(\mathcal{T}) \right\}.$$

We will show in Theorem 10.28 that all functions in $\tilde{\Pi}_{\text{zero}(\mathcal{T})}^{\pi}$ are Lipschitz continuous. We will also show that the space is equivariant under the action of $\text{clsemi}_A(\Omega^0)$, in the sense of subsection 5.1. This will justify the name.

Remark 10.23. In Lemma 9.11 we showed that the space $\tilde{\Pi}^{\pi}$ of effective perturbations is a vector space. The space $\tilde{\Pi}_{\mathcal{T}}^{\pi}$ of finite-dimensional perturbations and the space $\tilde{\Pi}_{\text{zero}(\mathcal{T})}^{\pi}$ of equivariant perturbations are vector subspaces of it.

Remark 10.24. The vector spaces $\tilde{\Pi}_{\mathcal{T}}^{\pi}$ and $\tilde{\Pi}_{\text{zero}(\mathcal{T})}^{\pi}$ should not be confounded with the vector spaces $\bar{\Pi}_{\mathcal{T}}^E$ and $\bar{\Pi}_{\text{zero}(\mathcal{T})}^E$ with prescribed additivities $E = \{ (x, y) \mid \Delta\pi(x, y) = 0 \}$, used in [5, Lemma 3.14], where the function π is assumed to be continuous piecewise linear over \mathcal{T} with $\text{vert}(\mathcal{T}) = \frac{1}{q}\mathbb{Z}$, $q \in \mathbb{N}$.

10.6. Finite-dimensional linear algebra for $\tilde{\Pi}_{\mathcal{T}}^{\pi}$. Let $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}_{\mathcal{T}}^{\pi}$ be a finite-dimensional perturbation. Note that $\tilde{\pi}_{\mathcal{T}}$ is a piecewise linear function, and it is uniquely determined by its values $\tilde{\pi}_{\mathcal{T}}(x)$ and limits $\tilde{\pi}_{\mathcal{T}}(x^-) := \lim_{t \rightarrow x, t < x} \tilde{\pi}_{\mathcal{T}}(t)$, $\tilde{\pi}_{\mathcal{T}}(x^+) := \lim_{t \rightarrow x, t > x} \tilde{\pi}_{\mathcal{T}}(t)$ at the breakpoints $x \in B' + \mathbb{Z} = \text{vert}(\mathcal{T})$.

Lemma 10.25. *A function $\tilde{\pi}_{\mathcal{T}}: \mathbb{R} \rightarrow \mathbb{R}$ is a finite-dimensional perturbation, $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}_{\mathcal{T}}^{\pi}$, if and only if $\tilde{\pi}_{\mathcal{T}}$ is piecewise linear over \mathcal{T} and satisfies the following conditions.*

- (i) $\tilde{\pi}_{\mathcal{T}}(0) = 0$ and $\tilde{\pi}_{\mathcal{T}}(f) = 0$;
- (ii) $\tilde{\pi}_{\mathcal{T}}(x) = \tilde{\pi}_{\mathcal{T}}(x + t)$ for all $x \in \mathbb{R}$, $t \in \mathbb{Z}$;
- (iii) For any additive vertex (x, y) of $\Delta\mathcal{T}$ and any face $F \in \Delta\mathcal{T}$ such that $(x, y) \in F$, $\Delta\pi_F(x, y) = 0$ implies $\Delta(\tilde{\pi}_{\mathcal{T}})_F(x, y) = 0$.

Before we give the proof, we define another space $\bar{\Pi}^{E_\bullet(\pi, \mathcal{T})}$, following [19]. Recall from subsection 9.2 the family of sets $E_F(\pi)$, indexed by faces F of a polyhedral complex, which capture the set of additivities and limit-additivities of π . Here we use this family with the refined polyhedral complex $\Delta\mathcal{T}$, considering π as a piecewise linear function on \mathcal{T} .

Definition 10.26. For a family $E_\bullet = \{E_F\}_{F \in \Delta\mathcal{T}}$, define the *space of perturbation functions with prescribed additivities and limit-additivities* E_\bullet ,

$$\bar{\Pi}^{E_\bullet} = \left\{ \bar{\pi}: \mathbb{R} \rightarrow \mathbb{R} \left| \begin{array}{ll} \bar{\pi}(0) = \bar{\pi}(f) = 0 & \\ \Delta\bar{\pi}_F(x, y) = 0 & \text{for } (x, y) \in E_F, F \in \Delta\mathcal{T} \\ \bar{\pi}(x+t) = \bar{\pi}(x) & \text{for } x \in \mathbb{R}, t \in \mathbb{Z} \end{array} \right. \right\}.$$

Proof of Lemma 10.25. We consider π as piecewise linear over \mathcal{T} , which is a refinement of \mathcal{P}_B . Let $\tilde{\pi}_\mathcal{T} \in \tilde{\Pi}_\mathcal{T}^\pi$. Then by definition, $\tilde{\pi}_\mathcal{T}$ is also piecewise linear over \mathcal{T} . Since $\tilde{\pi}_\mathcal{T} \in \tilde{\Pi}^\pi$, we have that $\tilde{\pi}_\mathcal{T} \in \bar{\Pi}^{E_\bullet}$, where $E_\bullet = E_\bullet(\pi, \mathcal{T})$ is the family of sets $E_F(\pi)$, indexed by $F \in \Delta\mathcal{T}$. Namely, $\tilde{\pi}_\mathcal{T}$ satisfies the conditions (i), (ii) and

(iii') For any face $F \in \Delta\mathcal{T}$ and any $(x, y) \in F$, if $\Delta\pi_F(x, y) = 0$ then $\Delta(\tilde{\pi}_\mathcal{T})_F(x, y) = 0$.

The condition (iii') clearly implies (iii). Thus, we proved the “only if” direction. Now let $\tilde{\pi}_\mathcal{T}$ be a piecewise linear function over \mathcal{T} that satisfies (i)–(iii). Notice that function π is subadditive and also piecewise linear over \mathcal{T} . Hence, the condition (iii) implies (iii'). We obtain that $\tilde{\pi}_\mathcal{T} \in \bar{\Pi}^{E_\bullet}$, where $E_\bullet = E_\bullet(\pi, \mathcal{T})$. It then follows from [19, Theorem 3.1] that $\tilde{\pi}_\mathcal{T} \in \tilde{\Pi}^\pi$. Therefore, $\tilde{\pi}_\mathcal{T} \in \tilde{\Pi}_\mathcal{T}^\pi$, we proved the “if” direction. \square

Assume that $B' = \{x'_0 = 0, x'_1, \dots, x'_{n'-1}, x'_n = 1\}$ and we identify $\tilde{\pi}_\mathcal{T}(x)$ and $\tilde{\pi}_\mathcal{T}(x+t)$ for all $t \in \mathbb{Z}$. Lemma 10.25 shows that $(\tilde{\pi}_\mathcal{T}(x'_0^-), \tilde{\pi}_\mathcal{T}(x'_0), \tilde{\pi}_\mathcal{T}(x'_0^+), \tilde{\pi}_\mathcal{T}(x'_1^-), \dots, \tilde{\pi}_\mathcal{T}(x'_{n'-1}^-), \tilde{\pi}_\mathcal{T}(x'_{n'-1}), \tilde{\pi}_\mathcal{T}(x'_{n'-1}^+))$ is a solution to the finite-dimensional linear system defined by (i) and (iii). The interpolation of such a solution gives an effective perturbation function $\tilde{\pi}_\mathcal{T} \in \tilde{\Pi}_\mathcal{T}^\pi$. We know that $(0, 0, \dots, 0)$ is a trivial solution. If a nontrivial solution exists, then its interpolation $\tilde{\pi}_\mathcal{T} \not\equiv 0$, implying that the function π is not extreme.

Remark 10.27. In fact, one can reduce the number of variables in the above linear system of equations to solve, by considering the connected components, as follows. Corollary 9.14 and (10.12) imply that $\tilde{\pi}_\mathcal{T}$ is affine linear with the same slope over all the intervals from a connected covered component C_i ($i = 1, 2, \dots, k$) or from a connected uncovered component U_i ($i = 1, 2, \dots, l$). Let $\tilde{s}_1^c, \dots, \tilde{s}_k^c$ and $\tilde{s}_1^u, \dots, \tilde{s}_l^u$ denote the corresponding slope variables.

In the discontinuous case, by Lemma 9.12, using Assumption 10.1, the perturbation $\tilde{\pi}_\mathcal{T}$ can only be discontinuous at the points where π is discontinuous. Let the variables \tilde{d}_i ($i = 1, 2, \dots, m$) denote the changes of the value of $\tilde{\pi}_\mathcal{T}$ at the m discontinuity points of π . In other words, the variables \tilde{d}_i denote jumps $\tilde{\pi}_\mathcal{T}(x) - \tilde{\pi}_\mathcal{T}(x^-)$ when π is discontinuous at x on the left, or $\tilde{\pi}_\mathcal{T}(x^+) - \tilde{\pi}_\mathcal{T}(x)$ when π is discontinuous at x on the right.

Then, for any fixed $x \in \mathbb{R}$, the value $\tilde{\pi}_{\mathcal{T}}(x)$ is uniquely determined by the slope variables \tilde{s}_i^c ($i = 1, 2, \dots, k$), \tilde{s}_i^u ($i = 1, 2, \dots, l$) and the jump variables \tilde{d}_i ($i = 1, 2, \dots, m$). These $k + l + m \leq 3n'$ variables satisfy the system of linear equations given by Lemma 10.25, where $(0, 0, \dots, 0)$ is a trivial solution. See [16, Example 7.2] for a concrete example.

10.7. Equivariant perturbation space $\tilde{\Pi}_{\text{zero}(\mathcal{T})}^{\pi}$. Let $\tilde{\pi}_{\text{zero}(\mathcal{T})} \in \tilde{\Pi}_{\text{zero}(\mathcal{T})}^{\pi}$ be an equivariant perturbation of π . By Corollary 6.6 (or Corollary 6.10) and Corollary 9.14, $\tilde{\pi}_{\text{zero}(\mathcal{T})}$ is affine linear on all covered intervals. By definition, $\tilde{\pi}_{\text{zero}(\mathcal{T})}(t) = \tilde{\pi}_{\text{zero}(\mathcal{T})}(t^-) = \tilde{\pi}_{\text{zero}(\mathcal{T})}(t^+) = 0$ for every $t \in \text{vert}(\mathcal{T})$, and $\partial C \subseteq \text{vert}(\mathcal{T})$. Therefore, $\tilde{\pi}_{\text{zero}(\mathcal{T})}$ is zero on $\text{cl}(C)$. If the set of uncovered points $U' = \emptyset$, then $\tilde{\pi}_{\text{zero}(\mathcal{T})} \equiv 0$. Otherwise, recall from subsection 10.4 that U' is partitioned into connected uncovered components U_1, U_2, \dots, U_l . The following theorem gives the characterization of the projection of a perturbation $\tilde{\pi}_{\text{zero}(\mathcal{T})}$ onto the space of functions with support contained in a connected uncovered component U_i .

Theorem 10.28 (Characterization of the equivariant perturbations supported on an uncovered component). *Suppose that Assumptions 10.1, 10.2 and 10.3 hold. Let $U_i = \bigcup \gamma_j(D)$ be a connected uncovered component, where D is the fundamental domain for U_i and $\gamma_j|_D \in \Omega|_{U'}$ ($j = 1, \dots, p$). Let $\tilde{\pi}_i: \mathbb{R} \rightarrow \mathbb{R}$ be a \mathbb{Z} -periodic function such that $\tilde{\pi}_i(x) = 0$ for $x \notin U_i$. Then $\tilde{\pi}_i \in \tilde{\Pi}_{\text{zero}(\mathcal{T})}^{\pi}$ if and only if*

- (i) $\tilde{\pi}_i$ is Lipschitz continuous on D ;
- (ii) $\tilde{\pi}_i(x) = \tilde{\pi}_i(x^-) = \tilde{\pi}_i(x^+) = 0$ for $x \in \partial D$;
- (iii) $\tilde{\pi}_i(x) = \chi(\gamma_j)\tilde{\pi}_i(\gamma_j(x))$ for $x \in D$, $j = 1, \dots, p$.

Proof. Let $\tilde{\pi}_i \in \tilde{\Pi}_{\text{zero}(\mathcal{T})}^{\pi}$. Since π is continuous on D , by Lemma 9.12, $\tilde{\pi}_i$ is Lipschitz continuous on \mathbb{R} . Hence, the condition (i) holds. The condition (ii) is clearly satisfied, as $\tilde{\pi}_i(x) = \tilde{\pi}_i(x^-) = \tilde{\pi}_i(x^+) = 0$ for each $x \in \text{vert}(\mathcal{T})$. Since $\tilde{\pi}_i$ respects $\Omega|_{U'}$, the condition (iii) also holds.

Conversely, let $\tilde{\pi}_i: \mathbb{R} \rightarrow \mathbb{R}$ be a \mathbb{Z} -periodic function such that $\tilde{\pi}_i(x) = 0$ for $x \notin U_i$ and the conditions (i)–(iii) hold. It follows from (ii) that $\tilde{\pi}_i(x) = \tilde{\pi}_i(x^-) = \tilde{\pi}_i(x^+) = 0$ for $x \in \partial U_i$. Since $\tilde{\pi}_i(x) = 0$ for $x \notin U_i$, we have

$$(10.13) \quad \tilde{\pi}_i(x) = \tilde{\pi}_i(x^-) = \tilde{\pi}_i(x^+) = 0 \text{ for } x \in [0, 1] \setminus U_i \supseteq B' \cup C.$$

We claim that $\tilde{\pi}_i$ satisfies all the additivities (including the limits) that π has. Indeed, let F be a face of $\Delta\mathcal{T}$ and let $(x, y) \in F$ such that $\Delta\pi_F(x, y) = 0$. We show that $(\Delta\tilde{\pi}_i)_F(x, y) = 0$ by distinguishing the following three cases.

(a) If (x, y) is an additive vertex of $\Delta\mathcal{T}$, then by Theorem 10.17 and (10.13), we have $(\Delta\tilde{\pi}_i)_F(x, y) = 0$.

(b) If (x, y) is contained in the relative interior of an edge F' of $\Delta\mathcal{T}$, then $F' \subseteq F$ and F' is an additive face of $\Delta\mathcal{T}$. Consider the move $\gamma|_{D'}$ associated with F' . We have either D' and $\gamma(D') \subseteq (0, 1) \setminus U_i$, or D' and $\gamma(D') \subseteq U_i$. In the former case, the claim holds because of (10.13); and in the latter case,

the structure of $\Delta\mathcal{T}$ (Theorem 10.17) implies that $\gamma|_{D'} \in \Omega|_{U'}$, and thus $(\Delta\tilde{\pi}_i)_F(x, y) = 0$ by the condition (iii).

(c) If (x, y) is contained in the relative interior of a two-dimensional face F' of $\Delta\mathcal{T}$, then $F' = F$ is a two-dimensional additive face of $\Delta\mathcal{T}$. We have $x, y, (x + y) \bmod 1 \in C$, hence the claim follows from (10.13).

We showed that $\tilde{\pi}_i \in \bar{\Pi}^{E_\bullet}$, where $E_\bullet = E_\bullet(\pi, \mathcal{T})$ is the family from the proof of Lemma 10.25. Then, [19, Theorem 3.1] implies that $\tilde{\pi}_i \in \tilde{\Pi}^\pi$. Therefore, $\tilde{\pi}_i \in \tilde{\Pi}_{\text{zero}(\mathcal{T})}^\pi$. \square

For $i = 1, \dots, l$, denote the space of functions $\tilde{\pi}_i$ as in the theorem by $\tilde{\Pi}_{U_i}^\pi$. It is independent of the choice of fundamental domain.

Theorem 10.29 (Direct sum decomposition of equivariant perturbations by uncovered components). *We have the direct sum decomposition $\tilde{\Pi}_{\text{zero}(\mathcal{T})}^\pi = \tilde{\Pi}_{U_1}^\pi \oplus \dots \oplus \tilde{\Pi}_{U_l}^\pi$, i.e., if $\tilde{\pi} \in \tilde{\Pi}_{\text{zero}(\mathcal{T})}^\pi$, then it has a unique decomposition $\tilde{\pi} = \tilde{\pi}_1 + \tilde{\pi}_2 + \dots + \tilde{\pi}_l$ such that $\tilde{\pi}_i \in \tilde{\Pi}_{U_i}^\pi$ for $i = 1, \dots, l$.*

Proof. Let $\tilde{\pi} \in \tilde{\Pi}_{\text{zero}(\mathcal{T})}^\pi$. For $i = 1, 2, \dots, l$, define $\tilde{\pi}_i : \mathbb{R} \rightarrow \mathbb{R}$, $\tilde{\pi}_i(x) = \tilde{\pi}(x)$ if $x \in U_i$ and $\tilde{\pi}_i(x) = 0$ otherwise. Then

$$(10.14) \quad \tilde{\pi} = \tilde{\pi}_1 + \tilde{\pi}_2 + \dots + \tilde{\pi}_l,$$

where each $\tilde{\pi}_i$ ($i = 1, 2, \dots, l$) satisfies the conditions in Theorem 10.28. \square

Each of the component functions $\tilde{\pi}_i$ ($i = 1, 2, \dots, l$) is supported on the connected uncovered component U_i and is obtained by choosing an arbitrary Lipschitz continuous template on the fundamental domain D_i , then by extending equivariantly to the other intervals through the moves in $\Omega|_{U'}$.

10.8. Decomposition theorem for effective perturbations. The following perturbation decomposition theorem, a generalization of [5, Lemma 3.14] without assuming π is continuous and $\text{vert}(\mathcal{T}) = \frac{1}{q}\mathbb{Z}$, shows that the effective perturbation space $\tilde{\Pi}^\pi$ is the direct sum of the finite-dimensional perturbation space $\tilde{\Pi}_{\mathcal{T}}^\pi$ and the equivariant perturbation space $\tilde{\Pi}_{\text{zero}(\mathcal{T})}^\pi$.

Theorem 10.30 (Perturbation decomposition theorem). *Under Assumptions 10.1, 10.2 and 10.3, for every effective perturbation $\tilde{\pi} \in \tilde{\Pi}^\pi$, there exist a unique finite-dimensional perturbation $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}_{\mathcal{T}}^\pi$ and a unique equivariant perturbation $\tilde{\pi}_{\text{zero}(\mathcal{T})} \in \tilde{\Pi}_{\text{zero}(\mathcal{T})}^\pi$ such that*

$$\tilde{\pi} = \tilde{\pi}_{\mathcal{T}} + \tilde{\pi}_{\text{zero}(\mathcal{T})}.$$

Proof. Let $\tilde{\pi} \in \tilde{\Pi}^\pi$ be an effective perturbation. By [16, Corollary 6.5], the limits $\tilde{\pi}(t^-)$ and $\tilde{\pi}(t^+)$ exist for every $t \in \text{vert}(\mathcal{T})$. Let $\tilde{\pi}_{\mathcal{T}}$ be the unique piecewise linear function over \mathcal{T} such that $\tilde{\pi}_{\mathcal{T}}(t) = \tilde{\pi}(t)$, $\tilde{\pi}_{\mathcal{T}}(t^-) = \tilde{\pi}(t^-)$ and $\tilde{\pi}_{\mathcal{T}}(t^+) = \tilde{\pi}(t^+)$ for every $t \in \text{vert}(\mathcal{T})$. Define $\tilde{\pi}_{\text{zero}(\mathcal{T})} = \tilde{\pi} - \tilde{\pi}_{\mathcal{T}}$. Note that $\tilde{\pi}_{\mathcal{T}}$ is the unique piecewise linear function over \mathcal{T} such that $\tilde{\pi}_{\text{zero}(\mathcal{T})}(t) =$

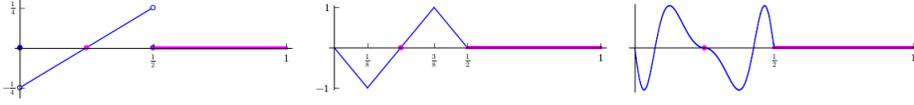


FIGURE 17. (Left) Finite-dimensional perturbation $\tilde{\pi}_{\mathcal{T}}$ of $\pi = \text{equiv7_example_1}()$ from Example 10.4. (Middle–right) Examples of equivariant perturbations $\tilde{\pi}_{\text{zero}(\mathcal{T})}$ of π .

$\tilde{\pi}_{\text{zero}(\mathcal{T})}(t^-) = \tilde{\pi}_{\text{zero}(\mathcal{T})}(t^+) = 0$ for every $t \in \text{vert}(\mathcal{T})$. It is left to show that $\tilde{\pi}_{\mathcal{T}}, \tilde{\pi}_{\text{zero}(\mathcal{T})} \in \tilde{\Pi}^{\pi}$.

We first show that $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^{\pi}$, by applying Lemma 10.25. It suffices to show that $\tilde{\pi}_{\mathcal{T}}$ satisfies condition (iii) of Lemma 10.25. Let (x, y) be an additive vertex of a face $F \in \Delta\mathcal{T}$ with $\Delta\pi_F(x, y) = 0$. By [16, Lemma 6.1], $\Delta\pi_F(x, y) = 0$ implies that $\Delta\tilde{\pi}_F(x, y) = 0$. Since (x, y) is an additive vertex of $\Delta\mathcal{T}$, Theorem 10.17 implies that $x, y, z \in B' \cup C$, where $z = (x + y) \bmod 1$. We have $\tilde{\pi}_{\mathcal{T}}(t) = \tilde{\pi}(t)$, $\tilde{\pi}_{\mathcal{T}}(t^-) = \tilde{\pi}(t^-)$ and $\tilde{\pi}_{\mathcal{T}}(t^+) = \tilde{\pi}(t^+)$ for $t = x, y$ or z , and hence $\Delta(\tilde{\pi}_{\mathcal{T}})_F(x, y) = \Delta\tilde{\pi}_F(x, y) = 0$. Therefore, $\tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^{\pi}$.

Since the vector space $\tilde{\Pi}^{\pi}$ contains both $\tilde{\pi}$ and $\tilde{\pi}_{\mathcal{T}}$, we obtain that $\tilde{\pi}_{\text{zero}(\mathcal{T})} = \tilde{\pi} - \tilde{\pi}_{\mathcal{T}} \in \tilde{\Pi}^{\pi}$. \square

Example 10.31 (Example 10.4, continued). For the function in Figure 13, $\pi = \text{equiv7_example_1}()$, the refined polyhedral complex \mathcal{T} has vertices $B' = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$. The finite-dimensional perturbation space $\tilde{\Pi}_{\mathcal{T}}^{\pi}$ has dimension 1 and is spanned by the basic perturbation

$$\tilde{\pi}_{\mathcal{T}}(x) = \begin{cases} 0 & \text{if } x = 0 \\ x - \frac{1}{4} & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq x < 1, \end{cases}$$

see Figure 17 (left). The two intervals $I_1 = (0, \frac{1}{4})$ and $I_2 = (\frac{1}{4}, \frac{1}{2})$ are uncovered, and they are connected through the move $\rho_f|_{(0,1/2)}$ in Ω . Because there is only one connected uncovered component, the equivariant perturbation space $\tilde{\Pi}_{\text{zero}(\mathcal{T})}^{\pi}$ consists of all Lipschitz continuous functions $\tilde{\pi}_{\text{zero}(\mathcal{T})}$ satisfying that $\tilde{\pi}_{\text{zero}(\mathcal{T})}(x) = 0$ for $x \in C \cup B'$ and that $\tilde{\pi}_{\text{zero}(\mathcal{T})}(x) = -\tilde{\pi}_{\text{zero}(\mathcal{T})}(f - x)$ for $x \in U'$. See Figure 17 (middle, right) for examples of such functions.

Example 10.32 (Example 10.5, continued). Figure 18 illustrates the decomposition of the space of effective perturbations.

Example 10.33 (Example 10.6, continued). Figure 19 illustrates the decomposition of the space of effective perturbations.

10.9. Relation of the moves closure to the semigroup $\Gamma^{\text{resp}}(\tilde{\Pi}^{\pi})$ of respected moves. In this section, still under the assumptions from subsection 10.1, we establish the relation between $\text{clsemi}_A(\Omega^0)$ and two other move semigroups:

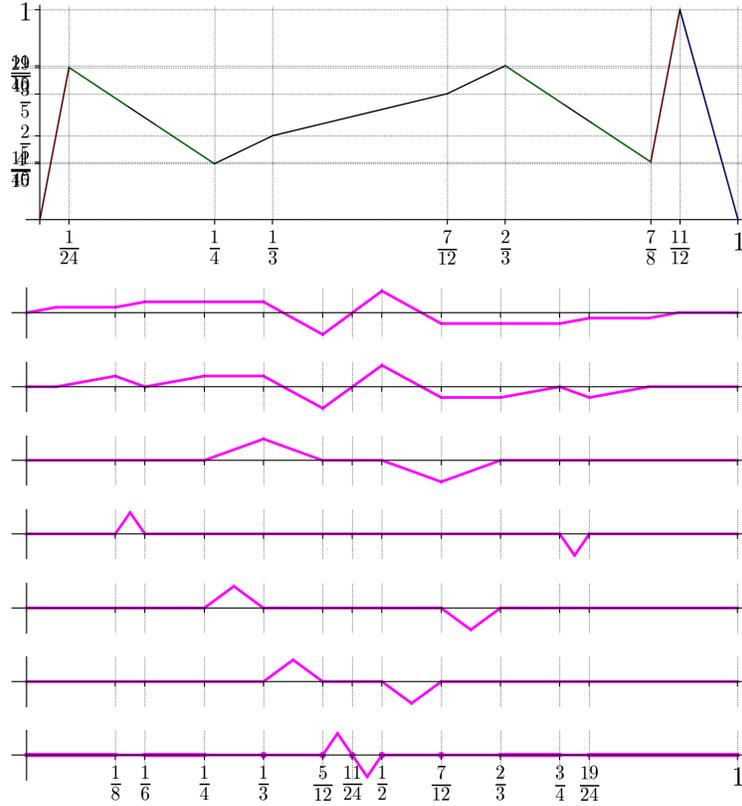


FIGURE 18. Decomposition of the space of effective perturbations for the function from Example 10.5/10.32, $\pi = \text{equiv7_example_xyz_2}()$. (a) The function π_2 . (b–d) basis of the space $\tilde{\Pi}_\mathcal{T}$ of finite-dimensional perturbations. (e–h) representatives of the equivariant perturbation spaces $\tilde{\Pi}_{U_i}^\pi$ for the 4 connected uncovered components U_i .

- (a) the semigroup $\Gamma^{\text{resp}}(\tilde{\Pi}^\pi)$ of moves respected by all effective perturbation functions $\tilde{\pi}$,
- (b) the semigroup $\Gamma^{\text{resp}}(\{\pi\} \cup \tilde{\Pi}^\pi) = \Gamma^{\text{resp}}(\pi + \tilde{\Pi}^\pi)$ of moves respected by π and its perturbations.

We already know from Corollary 9.14 that

$$(10.15) \quad \text{clsemi}_A(\Omega^0) \subseteq \Gamma^{\text{resp}}(\pi + \tilde{\Pi}^\pi) \subseteq \Gamma^{\text{resp}}(\tilde{\Pi}^\pi).$$

In the case of an extreme function π , the space $\tilde{\Pi}^\pi$ of effective perturbations is trivial; and thus, $\Gamma^{\text{resp}}(\tilde{\Pi}^\pi) = \Gamma^\subseteq(\mathbb{R})$.

More generally, whenever a function θ is affine on intervals D_1, D_2, \dots, D_k with the same slope, then $\text{moves}((D_1 \cup \dots \cup D_k) \times (D_1 \cup \dots \cup D_k)) \subseteq \Gamma^{\text{resp}}(\theta)$. Thus, we have the following:

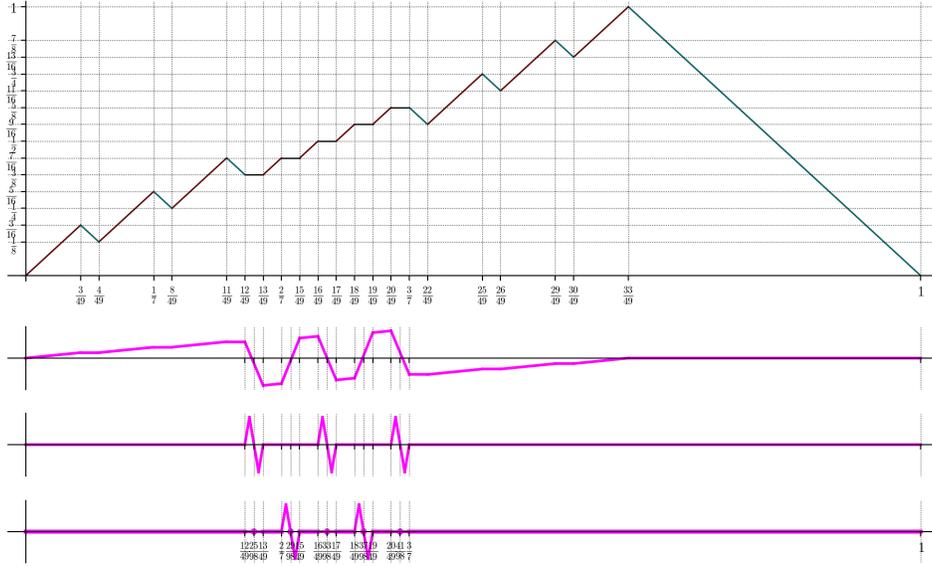


FIGURE 19. Decomposition of the space of effective perturbations for the function from Example 10.6/10.33, $\pi = \text{equiv7_minimal_2_covered_2_uncovered}()$. (a) The function π . (b) finite-dimensional perturbation $\tilde{\pi}_T$. (c), (d) examples of equivariant perturbations $\tilde{\pi}_1, \tilde{\pi}_2$ from the direct summands.

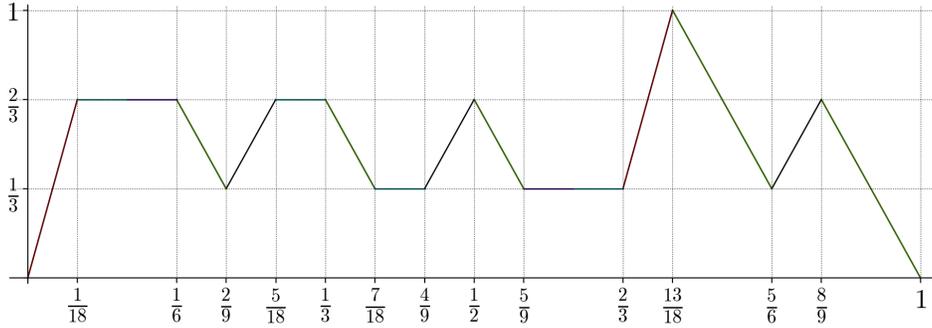


FIGURE 20. Function $\pi = \text{equiv7_example_post_3}()$ from Example 10.35.

Lemma 10.34. *Suppose the space $\tilde{\Pi}_T^\pi$ of finite-dimensional perturbations is trivial.*

- (a) *Let C be the set of covered points. Then $\text{moves}(C \times C) \subseteq \Gamma^{\text{resp}}(\tilde{\Pi}^\pi)$.*
- (b) *Let $D_1, \dots, D_k \subseteq C$ be covered intervals on which π is affine with the same slope. Then $\text{moves}((D_1 \cup \dots \cup D_k) \times (D_1 \cup \dots \cup D_k)) \subseteq \Gamma^{\text{resp}}(\pi + \tilde{\Pi}^\pi)$.*

Example 10.35. Consider the function $\pi = \text{equiv7_example_post_3}()$, shown in Figure 20. It has 4 connected covered components (colored slopes in the figure) and 2 connected uncovered components $U_1 = (\frac{2}{9}, \frac{5}{18}) \cup (\frac{4}{9}, \frac{1}{2})$ and $U_2 = (\frac{5}{6}, \frac{31}{36}) \cup (\frac{31}{36}, \frac{8}{9})$. Its finite-dimensional perturbation space is trivial.

- (a) From Lemma 10.34 (a) we see that $\text{moves}(C \times C) \subseteq \Gamma^{\text{resp}}(\tilde{\Pi}^\pi)$.
- (b) For the smaller semigroup $\Gamma^{\text{resp}}(\pi + \tilde{\Pi}^\pi)$, we observe that the function π is affine with slope 0 on the intervals $D_1 = (\frac{1}{18}, \frac{2}{18})$ and $D_2 = (\frac{2}{18}, \frac{3}{18})$, which belong to separate connected covered components (*cyan* and *lavender*). Because the finite-dimensional perturbation space is trivial, all functions in $\pi + \tilde{\Pi}^\pi$ take the same slope on D_1 and D_2 , and hence from Lemma 10.34 (b) we have $\text{moves}((D_1 \cup D_2) \times (D_1 \cup D_2)) \subseteq \Gamma^{\text{resp}}(\pi + \tilde{\Pi}^\pi)$. By continuity, we also have $\text{moves}((\frac{1}{18}, \frac{3}{18}) \times (\frac{1}{18}, \frac{3}{18})) \subseteq \Gamma^{\text{resp}}(\pi + \tilde{\Pi}^\pi)$.

Remark 10.36. Suppose the finite-dimensional perturbation space has a positive dimension. Recall its description using slope variables \tilde{s}_i^c (for the connected covered components C_i) and \tilde{s}_i^u (for the connected uncovered components U_i) from Remark 10.27. Whenever for some i, j , we have that $\tilde{s}_i^c = \tilde{s}_j^c$ holds for all solutions, then $\text{moves}((C_i \cup C_j) \times (C_i \cup C_j)) \subseteq \Gamma^{\text{resp}}(\tilde{\Pi}^\pi)$. A similar statement holds for $\Gamma^{\text{resp}}(\pi + \tilde{\Pi}^\pi)$.

Consider these move ensembles restricted to the set U of uncovered points in $(0, 1)$. We have the following theorem.

Theorem 10.37. *Under Assumptions 10.1, 10.2, and 10.3, we have that*

$$(10.16) \quad \text{clsemi}_A(\Omega^0)|_U = \Gamma^{\text{resp}}(\pi + \tilde{\Pi}^\pi)|_U = \Gamma^{\text{resp}}(\tilde{\Pi}^\pi)|_U.$$

where U is the set of uncovered points in $(0, 1)$.

Proof. We use the notations of the present section. By (10.15), it suffices to show that if the domain of a move $\gamma|_D \in \Gamma^{\text{resp}}(\tilde{\Pi}^\pi)$ is contained in U , then $\gamma|_D \in \text{clsemi}_A(\Omega^0)$.

Recall that we can write an arbitrary connected uncovered component U_i in the form of $U_i = \bigcup_{j=1}^p \gamma_j(I)$, where I is the fundamental domain for U_i , $\gamma_j|_I \in \text{clsemi}_A(\Omega^0)$, and the open intervals $\gamma_j(I)$ are disjoint. As $\text{clsemi}_A(\Omega^0)$ is join-closed and extension-closed, by taking sub-moves, it suffices to show that if a move $\gamma|_D$ satisfies that $D \subseteq I$ and the unrestricted move $\gamma \neq \gamma_j$ for all $j = 1, \dots, p$, then $\gamma|_D \notin \Gamma^{\text{resp}}(\tilde{\Pi}^\pi)$.

Consider a move $\gamma|_D$ where $D \subseteq I$ and $\gamma \neq \gamma_j$ for all $j = 1, \dots, p$. There exists an open interval $D' \subseteq D$ such that $\gamma(D') \cap \gamma_j(D') = \emptyset$ for all $j = 1, \dots, p$. We can construct a perturbation $\tilde{\pi}$ such that

- (i) $\tilde{\pi}$ is non-zero and Lipschitz continuous on D' ;
- (ii) $\tilde{\pi}(x) = \tilde{\pi}(x^-) = \tilde{\pi}(x^+) = 0$ for $x \in \partial D'$;
- (iii) $\tilde{\pi}(x) = \chi(\gamma_j)\tilde{\pi}(\gamma_j(x))$ for $x \in D'$, $j = 1, \dots, p$; and
- (iv) $\tilde{\pi}(x) = 0$ for $x \notin \bigcup_{j=1}^p \gamma_j(D')$.

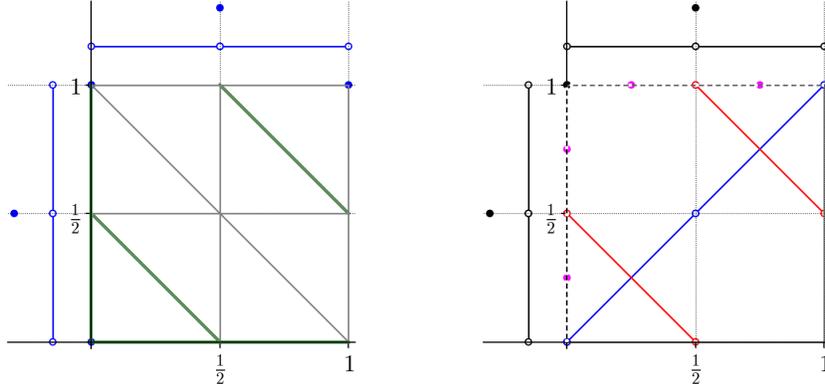


FIGURE 21. (Left) Two-dimensional polyhedral complex $\Delta\mathcal{P}$ of a two-sided discontinuous minimal valid function $\pi = \text{minimal_no_covered_interval}()$ (blue graph at the left and top borders) from Example 11.1, where the additive faces are colored in green. (Right) The graph of the move ensemble $\text{clsemi}_A(\Omega^0)$ of π , where the set $C \cup B' = [0, 1] \setminus U'$ of covered points and breakpoints are marked in magenta on the left and top borders.

Since $\tilde{\pi}|_{D'} \not\equiv 0$ but $\tilde{\pi}|_{\gamma(D')} \equiv 0$, we have that $\gamma|_{D'} \notin \Gamma^{\text{resp}}(\tilde{\pi})$, and hence $\gamma|_D \notin \Gamma^{\text{resp}}(\tilde{\pi})$. By Theorem 10.28, $\tilde{\pi} \in \tilde{\Pi}_{\text{zero}(\mathcal{T})}^{\pi} \subseteq \tilde{\Pi}^{\pi}$. Therefore, $\gamma|_D \notin \Gamma^{\text{resp}}(\tilde{\Pi}^{\pi})$. We conclude that (10.16) holds. \square

11. CONCLUSION

11.1. Forthcoming computational companion paper. In the forthcoming paper [14], part VIII of the series, we will describe a method to compute the moves closure $\text{clsemi}_A(\Omega^0)$ for a large class of piecewise linear minimal valid functions, including all functions with rational breakpoints, for which the moves closure has a finite presentation. The decomposition of the perturbation space in section 10 is already algorithmic. Thus we will obtain a grid-free extremality test.

11.2. Limits of the approach of this paper. We now discuss the limitations to the equivariant perturbation theory developed in our series of papers.

For two-sided discontinuous functions, the decomposition of the perturbation spaces breaks down. Theorem 10.28 and Theorem 10.30 do not hold when the function π is discontinuous from both sides of the origin, as the following example shows.

Example 11.1. Consider the minimal valid function $\pi = \text{minimal_no_covered_interval}()$ with $f = \frac{1}{2}$, defined by

$$\pi(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } 0 < x < \frac{1}{2} \text{ or } \frac{1}{2} < x < 1 \\ 1 & \text{if } x = \frac{1}{2}, \end{cases}$$

which is discontinuous from both sides of the origin.

Observe from Figure 21 that $C = \emptyset$, $B' = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ and the connected uncovered components are $U_1 = (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{1}{2})$ and $U_2 = (\frac{1}{2}, \frac{3}{4}) \cup (\frac{3}{4}, 1)$, where the two intervals in either U_1 or U_2 are connected through the move $\rho_f|_{(0,f)}$ or $\rho_f|_{(f,1)}$ in $\text{clsemi}_A(\Omega^0)$. Any bounded \mathbb{Z} -periodic function $\tilde{\pi}$ satisfying that $\tilde{\pi}(x) = 0$ for $x \in B'$ and $\tilde{\pi}(x) = \tilde{\pi}(\rho_f(x))$ for $x \in [0, 1]$ is an effective perturbation of π . For example, define a \mathbb{Z} -periodic function $\tilde{\pi} = \text{equiv7_example_2_crazy_perturbation}()$ by

$$\tilde{\pi}(x) = \begin{cases} 1 & \text{if } x \in (0, \frac{1}{4}) \text{ such that } x \in G, \text{ or} \\ & \text{if } x \in (\frac{1}{4}, \frac{1}{2}) \text{ such that } x - \frac{1}{4} \in G; \\ -1 & \text{if } x \in (0, \frac{1}{4}) \text{ such that } x - \frac{1}{4} \in G, \text{ or} \\ & \text{if } x \in (\frac{1}{4}, \frac{1}{2}) \text{ such that } x - \frac{1}{2} \in G; \\ 0 & \text{otherwise,} \end{cases}$$

where the group $G = \langle 1, \sqrt{2} \rangle_{\mathbb{Z}}$ is dense in \mathbb{R} . Then $\tilde{\pi}$ is an effective perturbation of π , since both $\pi \pm \epsilon \tilde{\pi}$ are minimal valid functions for $0 < \epsilon \leq \frac{1}{6}$.³ Observe that $\tilde{\pi}$ is a highly discontinuous function, which does not have a limit at any point in $(0, \frac{1}{2})$. Thus, without Assumption 10.1, an equivariant perturbation is not necessarily Lipschitz continuous; and the limits of an effective perturbation at the breakpoints might not exist. For this reason, the decomposition of perturbations does not make sense when the function π is discontinuous from both sides of the origin.

Note that in [20], though an algorithm was presented that checks the effectiveness of a given perturbation function $\tilde{\pi}$, and a perturbation was constructed for an example function, it was left as an open question how to construct perturbations in general. This is still open.

We conjecture that the equivariant perturbation theory also breaks down for the case of non-piecewise linear functions, such as the fractal functions presented in [1] and [2]. In particular we note that (1) the finite system of equations describing the space of finite-dimensional perturbations would be replaced by a system of functional equations, for which we have no lemmas available; (2) we expect that the decomposition theorem no longer holds.

³This positive ϵ is verified by calling `find_epsilon_for_crazy_perturbation($\pi, \tilde{\pi}$)`

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