ON LOCAL NON-GLOBAL MINIMIZERS OF QUADRATIC OPTIMIZATION PROBLEM WITH A SINGLE QUADRATIC CONSTRAINT

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Abstract. In this paper, we consider the nonconvex quadratic optimization problem with a single quadratic constraint. First we give a theoretical characterization of the local non-global minimizers. Then we extend the recent characterization of the global minimizer via a generalized eigenvalue problem to the local non-global minimizers. Finally, we use these results to derive an efficient algorithm that finds the global minimizer of the problem with an additional linear inequality constraint.

Key words. quadratically constrained quadratic optimization, global optimization, local minimizer, generalized eigenvalue problem

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1. Introduction.

Consider the following quadratically constrained quadratic optimization problem:

$$\begin{aligned} \min \quad & q(x) := x^T A x + 2a^T x \\ & g(x) := x^T B x + 2b^T x + \beta \le 0, \end{aligned} \tag{1.1}$$

where $A, B \in \mathbb{R}^{n\times n}$ are symmetric matrices, $a, b \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$. Computing its global optimal solution is well studied in the literature [25, 2, 20, 7, 5, 11, 21, 28, 33, 10, 36]. However, (1.1) may possess a local minimizer which is not global. This local solution will be referred to as a local non-global minimizer. The importance of the local non-global minimizers of (1.1) can be recognized by the following problem:

$$\begin{aligned} \min \quad & q(x) \\ & g(x) \le 0, \\ & h(x) \le 0, \end{aligned} \tag{1.2}$$

where $h : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, $q(x)$ and $g(x)$ are defined as in (1.1). Indeed, if x^* is a global minimizer of problem (1.2) and $h(x^*) < 0$, then x^* is a local (not necessarily global) minimizer of (1.1). In this case, x^* is the local non-global minimizer if all global minimizers of (1.1) do not satisfy the constraint $g(x) \le 0$. Several special cases of (1.2) have been studied in the literature [5, 3, 32, 35, 19, 34, 27, 17]. For instance, in the case where $h(x)$ is a strictly convex quadratic function, a necessary and sufficient condition for strong duality has been derived in [3]. The authors in [32] studied the same problem and introduced a polynomial-time algorithm which computes all Lagrange multipliers by solving a two-parameter linear eigenvalue problem, obtains the corresponding KKT points, and finds a global solution as the KKT point with the smallest objective value. However, due to the high computational costs of their algorithm, they reported numerical results for only dimension $n \le 40$. Recently, in [5], a convex relaxation which is a second order cone programming (SOCP) problem

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has been proposed for (1.2) with two quadratic constraints where the Hessian of the quadratic forms are simultaneously diagonalizable. It has been shown that if certain additional conditions are satisfied then the optimal solution of the original problem can be obtained from the optimal solution of the SOCP relaxation [5]. However, in that paper, it is also illustrated that the SOCP relaxation may not return the optimal solution of the original problem even when $B \succ 0$ and $h(x)$ is a linear constraint. The author in [23] studied the same problem and in some cases extended the results in [5], giving conditions under which the same convex relaxation returns their optimal value.

Problem (1.1) includes the classical trust-region subproblem (TRS) as a special case where $B = I$, $b = 0$ and $\beta < 0$. TRS arises in regularization or trust-region methods for unconstrained optimization [8, 37] and numerous efficient algorithms have been developed to solve it [13, 14, 29, 12, 1, 30, 26]. Specifically, it has been shown that TRS can be solved efficiently by one generalized eigenvalue problem [1]. The local non-global minimizer of TRS is well characterized in [24] where it is proved that TRS admits at most one local non-global minimizer. Recently, the authors in [35] have shown that the local non-global minimizer of TRS like its global minimizer can be computed via a generalized eigenvalue problem. They also derived an efficient algorithm for the trust-region subproblem with an additional linear inequality constraint, known as the extended trust-region subproblem (eTRS), by combining the solutions for the local non-global minimizer and global minimizer of TRS. Most recently, the authors in [2] proposed an algorithm for problem (1.1) that requires finding just one eigenpair of a generalized eigenvalue problem. Their algorithm is based on the framework established in [1] and formulates the KKT conditions as an eigenvalue problem.

In the present work, we are interested in the local non-global minimizers of (1.1). To the best of our knowledge, this is the first study on the local non-global minimizers of problem (1.1) in the literature. Under a regularity condition that there exists a $\lambda \in \mathbb{R}$ such that $A + \lambda B \succ 0$, we give a characterization of such minimizers which indeed extends the results of [24] for TRS to problem (1.1). Then we continue with the recent characterization of the global minimizer of (1.1) via a generalized eigenvalue problem [2] and extend it to the local non-global minimizers. This provides an efficient procedure for finding candidates for the local non-global minimizers of (1.1) if they exist. We further exploit the characterization of local minimizers (global and non-global) of (1.1) based on a generalized eigenvalue problem to derive an efficient algorithm for finding the global minimizer of the following problem:

$$
\begin{aligned}
\min \quad & q(x) := x^T A x + 2a^T x \\
& g(x) := x^T B x + 2b^T x + \beta \leq 0, \\
& \qquad\quad c^T x \leq \gamma,
\end{aligned}
\tag{1.3}
$$

where $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$. We assume that the regularity condition still holds for problem (1.3). When $B = I$, $b = 0$ and $\beta < 0$, problem (1.3) reduces to eTRS which can be solved efficiently via solving at most three generalized eigenvalue problems [35]. Excluding eTRS, to the best of our knowledge, no efficient algorithm has been developed to solve large-scale instances of problem (1.3) without any further assumption. It is worth to mention that our regularity condition implies that the quadratic forms in problem (1.3) are simultaneously diagonalizable and so the SOCP relaxation in [5] can be applied to this problem but this relaxation is not exact in general. Our results in this paper is a step toward solving the quadratic optimization problem with a general number of inequality constraints.

The rest of the paper is organized as follows: In Section 2, we state some results which will be useful for our analysis in the paper. In Section 3, we give our theoretical characterization of the local non-global minimizers of (1.1). The characterization using a generalized eigenvalue problem is given in Section 4. Section 5 studies problem (1.3). Finally, in Section 6, we give some numerical results to demonstrate the practical performance of our proposed algorithms for computing the local non-global minimizer of (1.1) and global optimal solution of (1.3).

Notations: Throughout this paper, for a symmetric matrix A, $A \succ 0$ ($A \succeq 0$) denotes A is positive definite (positive semidefinite), $A^\dagger$ denotes the Moore-Penrose generalized inverse of A, $\det(A)$ denotes the determinant of A, $\mathrm{Range}(A)$ and $\mathrm{Null}(A)$ denote its Range and Null spaces, respectively. For two symmetric matrices A and B, we use $\lambda_{\min}(A,B)$ and $\lambda_{\min 2}(A,B)$ to denote the smallest and second smallest generalized eigenvalue of the pencil $A-\lambda B$, respectively. Finally, for a vector $v \in \mathbb{R}^n$, v_i denotes i'th entry of v and $\mathrm{span}(v)$ denotes the space spanned by the vector v.

2. Characterization of local and global minimizers.

2.1. Assumptions. We consider the following two assumptions throughout this paper.

Assumption 1. Problem (1.1) satisfies the Slater condition, i.e., there exists $\hat{x}$ with $g(\hat{x}) < 0$.

When Assumption 1 is violated, problem (1.1) reduces to an unconstrained minimization problem [2]. Specifically, from the viewpoint of the local non-global minimizer, Assumption 1 is reasonable due to the following proposition.

PROPOSITION 2.1. *Suppose that Assumption 1 does not hold for problem (1.1). Then, there are no local non-global minimizers for (1.1).*

Proof. Assumption 1 does not hold if and only if $B \succeq 0$, $b \in \mathrm{Range}(B)$ and $g(x^*) = 0$ where $x^* = -B^\dagger b$. In this case, the feasible region of (1.1) is $x^* + \mathrm{Null}(B)$. Let the columns of $Z \in \mathbb{R}^{n\times m}$ form a basis for $\mathrm{Null}(B)$. Then problem (1.1) is equivalent to the following unconstrained minimization problem:

$$\min_{y\in\mathbb{R}^m} \quad y^T Z^T AZy + 2(Z^T Ax^* + Z^T a)^T y + q(x^*). \tag{2.1}$$

Therefore, $\bar{x}$ is a local non-global minimizer of (1.1) if and only if $\bar{x} = x^* + Zy^*$ where y^* is a local non-global minimizer of (2.1). This completes the proof since (2.1) has no local non-global minimizers. ∎

Assumption 2. There exists $\hat{\lambda} \in \mathbb{R}$ such that $A + \hat{\lambda}B \succ 0$.

In the context of global minimization, a stronger assumption such as

- (A1) there exists $\hat{\lambda} \geq 0$ such that $A + \hat{\lambda}B \succ 0$,

is considered [2, 20, 11, 7, 18, 25]. Indeed, assumption (A1) ensures that problem (1.1) has a global optimal solution [2, 25, 20]. A necessary and sufficient condition for boundedness of problem (1.1) under Assumption 1 is given below.

LEMMA 2.2 ([18]). *Suppose that Assumption 1 holds. Then problem (1.1) is bounded below if and only if there exists $\lambda \geq 0$ such that*

$$A + \lambda B \succeq 0, \quad (a + \lambda b) \in \mathit{Range}(A + \lambda B).$$

Boundedness guarantees the existence of the optimal (infimum) value for q. However, problem (1.1) may be bounded but has no optimal solution; see the example in

Subsection 2.2 of [2].

LEMMA 2.3 ([18]). *Suppose that Assumption 1 holds. Moreover, assume that problem (1.1) is bounded. The infimum of problem (1.1) is unattainable if and only if the set* $\{\lambda | A + \lambda B \succeq 0\}$ *is singleton* $\{\lambda^*\}$ *with* $\lambda^* \geq 0$*, and the following system has no solution in Null*$(A + \lambda^* B)$*:*

$$\begin{cases} g(-(A+\lambda^* B)^\dagger (a + \lambda^* b) + y) = 0, & \text{if } \lambda^* > 0, \\ g(-A^\dagger a + y) \leq 0, & \text{if } \lambda^* = 0. \end{cases}$$

Moreover, it can be shown that cases that satisfy Assumption 2 but violate assumption (A1) may be unbounded below, see Subsection 4.2. Furthermore, it is worth noting that problem (1.1) may be unbounded below but have local non-global minimizers. A problem of this type is given in the following example.
Example 1: Consider problem (1.1) with

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ a = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \beta = 1. \tag{2.2}$$

It satisfies Assumption 2 but violates assumption (A1). It is unbounded below but $x^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a local non-global minimizer following from Theorem 2.5.

2.2. Necessary and sufficient conditions. In this subsection, we provide a basis for characterization of the local non-global minimizers of (1.1) in the next section.

DEFINITION 2.4. *The feasible solution* $\bar{x}$ *of (1.1) satisfying* $g(\bar{x}) = 0$ *is called regular if* $\nabla g(\bar{x}) \neq 0$.

The following theorem states the classical necessary conditions for local minimizers of (1.1).

THEOREM 2.5. *(i) Suppose that* x^* *is a local minimizer of (1.1). If* $g(x^*) = 0$ *and* x^* *is a regular point, then there exists* $\lambda^* \geq 0$ *such that*

$$(A + \lambda^* B)x^* = -(a + \lambda^* b), \tag{2.3}$$

and

$$w^T (A + \lambda^* B) w \geq 0, \tag{2.4}$$

for all w *such that* $w^T(Bx^* + b) = 0$*. If* $g(x^*) < 0$*, then* x^* *is a global minimizer and* A *is positive semidefinite.*

(ii) Suppose that $g(x^*) = 0$*, (2.3) holds with* $\lambda^* \geq 0$ *and*

$$w^T (A + \lambda^* B) w > 0, \tag{2.5}$$

for all $w \neq 0$ *such that* $w^T(Bx^* + b) = 0$*. Then* x^* *is a strict local minimizer of (1.1).*

Proof. The assertions are a straightforward consequence from optimality conditions for constrained optimization, see [4]. ∎

The following theorem gives a set of necessary and sufficient conditions for the global optimal solution of (1.1) under Assumption 1.

THEOREM 2.6 ([25]). *Suppose that Assumption 1 holds. Then x_g^* is a global optimal solution of (1.1) if and only if there exists $\lambda_g^* \geq 0$ such that*

$$(A + \lambda_g^* B)x_g^* = -(a + \lambda_g^* b), \tag{2.6}$$
$$g(x_g^*) \leq 0, \tag{2.7}$$
$$\lambda_g^* g(x_g^*) = 0, \tag{2.8}$$
$$(A + \lambda_g^* B) \succeq 0. \tag{2.9}$$

The next lemma shows that if x^* is a local non-global minimizer of problem (1.1), then x^* is a regular point.

LEMMA 2.7. *Suppose that x^* is a local non-global minimizer of problem (1.1). Then x^* is a regular point, i.e., $Bx^* + b \neq 0$.*

Proof. Suppose by contradiction that $Bx^* + b = 0$. Let S denote the feasible region of (1.1). Since x^* is a local non-global minimizer, $g(x^*) = 0$ and there exists a neighborhood $N_r(x^*)$ around x^* such that for all $x \in N_r(x^*) \cap S$, $q(x) \geq q(x^*)$. Vector $d \neq 0$ is a feasible direction at x^* if $g(x^* + \alpha d) \leq 0$ for all $\alpha \in (0, \delta)$ for some $\delta > 0$. We have

$$g(x^* + \alpha d) = g(x^*) + 2\alpha d^T(Bx^* + b) + \alpha^2 d^T B d = \alpha^2 d^T B d,$$

implying that d is a feasible direction at x^* if and only if $d^T B d \leq 0$. Consider the following two cases:

Case 1. $B \not\succeq 0$.

Consider the value of objective function $q(x)$ along the feasible direction d inside the neighborhood $N_r(x^*)$:

$$q(x^* + \alpha d) = q(x^*) + \alpha d^T \bigtriangledown q(x^*) + \alpha^2 d^T A d, \tag{2.10}$$

where $0 \leq \alpha < r$. There are two possible subcases:

Subcase 1.1. $\nabla q(x^*) = 0$.

Since x^* is a local minimizer, $q(x^* + \alpha d) - q(x^*) \geq 0$ for all $\alpha \in (0, r)$, implying that $d^T A d \geq 0$. Therefore, by S-lemma [6], there exists nonnegative $\hat{\lambda}$ such that $A + \hat{\lambda} B \succeq 0$. Let $\Theta(\lambda)$ denote the Lagrangian dual function of problem (1.1). We have

$$\Theta(\hat{\lambda}) = \min_x \{q(x) + \hat{\lambda} g(x)\} = q(x^*),$$

due to the facts that $A + \hat{\lambda} B \succeq 0, \nabla q(x^*) = 0$ and $\nabla g(x^*) = 0$. So, it follows from the weak duality property that x^* is a global minimizer of (1.1) which contradicts our assumption that x^* is a local non-global minimizer. Therefore, $Bx^* + b \neq 0$.

Subcase 1.2. $\nabla q(x^*) \neq 0$.

Since $B \not\succeq 0$, there exists d such that $d^T \nabla q(x^*) < 0$ and $d^T B d \leq 0$. Then, it is easy to see that there also exists $r_0 < r$ such that $q(x^* + \alpha d) < q(x^*)$ for all $0 < \alpha < r_0$ which contradicts our assumption that x^* is a local minimizer. Thus $Bx^* + b \neq 0$.

Case 2. $B \succeq 0$.

In this case, $d \neq 0$ is a feasible direction at x^* if and only if $d \in \mathrm{Null}(B)$. If B is nonsingular, then there exists no feasible direction and this with convexity of feasible region imply that $S = \{x^*\}$. Therefore, x^* is a global minimizer of (1.1) which contradicts the assumption that x^* is a local non-global minimizer. Next suppose

that B is singular. Consider the following subcases:
Subcase 2.1. $d^T\nabla q(x^*) = 0$ for all $d \in \text{Null}(B)$.
In this case, since x^* is a local minimizer, it follows from (2.10) that $d^TAd \geq 0$. Let the columns of $Z \in \mathbb{R}^{n\times m}$ form a basis for Null(B). Consider the following unconstrained minimization problem:

$$\min_{y\in\mathbb{R}^m} \quad y^TZ^TAZy + 2(Z^TAx^* + Z^Ta)^Ty + {x^*}^TAx^* + 2a^Tx^*. \tag{2.11}$$

The facts that $d^TAd \geq 0$ and $d^T\nabla q(x^*) = 0$ for all $d \in \text{Null}(B)$ prove that $y = 0$ is the global minimizer of (2.11) and therefore, x^* is a global minimizer of problem (1.1) which contradicts to the fact that x^* is a local non-global minimizer.
Subcase 2.2. There exists $d \in \text{Null}(B)$ such that $d^T\nabla q(x^*) \neq 0$.
In this case, we may assume without loss of generality that $d^T\nabla q(x^*) < 0$. Then, it is easy to see that there exists $r_0 < r$ such that $q(x^* + \alpha d) < q(x^*)$ for all $0 < \alpha < r_0$ which contradicts our assumption that x^* is a local minimizer. Thus $Bx^* + b \neq 0$. ◻

The following lemma provides the first characterization of local minimizers of (1.1).

LEMMA 2.8. *Suppose that Assumption 1 holds, x^* is a local minimizer of problem (1.1) and λ^* is the corresponding Lagrange multiplier. Then matrix $A + \lambda^*B$ has at most one negative eigenvalue.*

Proof. If x^* is a global minimizer, then by Theorem 2.6, $A + \lambda^*B$ is positive semidefinite and so has no negative eigenvalues. Next, let x^* be a local non-global minimizer. It follows from Lemma 2.7 and Theorem 2.5 that $g(x^*) = 0$ and (2.4) holds. Now, suppose by contradiction that matrix $A + \lambda^*B$ has at least two negative eigenvalues. Let $A + \lambda^*B = QDQ^T$ be the eigenvalue decomposition of $A + \lambda^*B$ in which $Q = [v_1v_2\cdots v_n]$ is an orthogonal matrix and $D = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$ where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Thus we have $\lambda_1, \lambda_2 < 0$. Now consider the plane P spanned by v_1 and v_2. If $w \in P$, then $w = \alpha_1v_1 + \alpha_2v_2$ where $\alpha_1, \alpha_2 \in \mathbb{R}$. Therefore,

$$w^T(A + \lambda^*B)w = \alpha_1^2v_1^T(A + \lambda^*B)v_1 + \alpha_2^2v_2^T(A + \lambda^*B)v_2 + 2\alpha_1\alpha_2v_1^T(A + \lambda^*B)v_2.$$

Since

$$v_1^T(A + \lambda^*B)v_1 = \lambda_1,\ v_2^T(A + \lambda^*B)v_2 = \lambda_2,\ v_1^T(A + \lambda^*B)v_2 = 0,$$

then

$$w^T(A + \lambda^*B)w = \alpha_1^2\lambda_1 + \alpha_2^2\lambda_2 < 0, \tag{2.12}$$

for all $w \neq 0 \in P$. Since the dimension of P is 2 and the dimension of the orthogonal subspace to $Bx^* + b$ is $n - 1$, there exists $w \neq 0 \in P$ such that $w^T(Bx^* + b) = 0$, implying that (2.12) contradicts to (2.4). Therefore, the assumption that matrix $A + \lambda^*B$ has at least two negative eigenvalues is false and hence the assertion is proved. ◻

3. Characterizations of the local non-global minimizers. In this section, we apply the results of Section 2 to characterize the local non-global minimizers of (1.1). We provide the first characterization in the following lemma.

LEMMA 3.1. *Suppose that x^* is a local non-global minimizer of problem (1.1) and λ^* is the corresponding Lagrange multiplier. Then matrix $A + \lambda^*B$ has exactly one negative eigenvalue.*

Proof. It is a direct consequence of Lemma 2.8 and Theorem 2.6. ∎

In what follows, motivated by Lemma 3.1, we determine intervals containing λ^*, the Lagrange multiplier corresponding to the local non-global minimizers of (1.1). We proceed by making use of the fact that under Assumption 2, A and B are simultaneously diagonalizable by congruence [22], i.e., there exists an invertible matrix C and diagonal matrices $D = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_n)$ an $E = \mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_n)$ such that

$$C^T AC = D, \quad \text{and} \quad C^T BC = E. \tag{3.1}$$

Using the transformation $x = Cy$, $\eta = C^T a$ and $\phi = C^T b$, problem (1.1) is equivalent to the following separable problem:

$$\begin{aligned} \min \quad & \hat{q_1}(y) := y^T Dy + 2\eta^T y \\ & \hat{q_2}(y) := y^T Ey + 2\phi^T y + \beta \leq 0. \end{aligned} \tag{3.2}$$

Thus x^* is a local non-global minimizer of (1.1) if and only if $y^* = C^{-1}x^*$ is a local non-global minimizer of (3.2). Moreover, λ^* is also the Lagrange multiplier corresponding to y^* due to the following facts:

$$(A + \lambda^* B)x^* = -(a + \lambda^* b) \Leftrightarrow (D + \lambda^* E)y^* = -(\eta + \lambda^* \phi),$$

$$w^T(A + \lambda^* B)w \geq 0, \quad \text{for all } w \text{ such that} \quad (Bx^* + b)^T w = 0,$$

$$\Updownarrow$$

$$w^T C^{-T}(D + \lambda^* E)C^{-1}w \geq 0, \quad \text{for all } w \text{ such that} \quad (Ey^* + \phi)^T C^{-1}w = 0.$$

Therefore, to determine intervals containing λ^*, we only need to find intervals on which matrix $D + \lambda E$, for $\lambda \geq 0$, has exactly one negative eigenvalue. Now $D + \lambda E \succeq 0$ if and only if $\lambda \in [\ell_1, u_1]$ where

$$\ell_1 = \max\{-\frac{\alpha_i}{\beta_i} | \beta_i > 0\}, \qquad u_1 = \min\{-\frac{\alpha_i}{\beta_i} | \beta_i < 0\}.$$

Set $N = \{1, 2, \cdots, n\}$ and define

$$T_1 := \{i \in N | \beta_i > 0\} = \{i_1, i_2, \cdots, i_r\}, \qquad T_2 := \{i \in N | \beta_i < 0\} = \{j_1, j_2, \cdots, j_s\},$$

where $r = |T_1|$ and $s = |T_2|$. Without loss of generality, assume that

$$-\frac{\alpha_{i_r}}{\beta_{i_r}} \leq \cdots \leq -\frac{\alpha_{i_2}}{\beta_{i_2}} \leq -\frac{\alpha_{i_1}}{\beta_{i_1}} = \ell_1,$$

and

$$-\frac{\alpha_{j_s}}{\beta_{j_s}} \geq \cdots \geq -\frac{\alpha_{j_2}}{\beta_{j_2}} \geq -\frac{\alpha_{j_1}}{\beta_{j_1}} = u_1.$$

Note that $\mu_i = -\frac{\alpha_i}{\beta_i}, i \in N$ are generalized eigenvalues of the matrix pencil $A + \mu B$. It is easy to verify that matrix $D + \lambda E$ has exactly one negative eigenvalue if and only if either $\lambda \in [\ell_2, \ell_1)$ provided that the generalized eigenvalue ℓ_1 has multiplicity one or $\lambda \in (u_1, u_2]$ provided that the generalized eigenvalue u_1 has multiplicity one where $\ell_2 = -\frac{\alpha_{i_2}}{\beta_{i_2}}$ and $u_2 = -\frac{\alpha_{j_2}}{\beta_{j_2}}$. The following lemma is a direct consequence of the above discussion.

LEMMA 3.2. *Let ℓ_1, ℓ_2, u_1 and u_2 be defined as before. Moreover, suppose that x^* is a local non-global minimizer of problem (1.1) and λ^* is the corresponding Lagrange multiplier. Then either $\lambda^* \in [\ell_2, \ell_1) \cap \mathbb{R}_+$ provided that the generalized eigenvalue ℓ_1 has multiplicity one or $\lambda^* \in (u_1, u_2] \cap \mathbb{R}_+$ provided that the generalized eigenvalue u_1 has multiplicity one.*

Proof. It follows from the above discussion and $\lambda^* \geq 0$. ◻

In the following, we state situations where problem (1.1) has no local non-global minimizers.

COROLLARY 3.3. *Let ℓ_1, u_1 and u_2 be defined as before and m and p be the multiplicity of the generalized eigenvalues ℓ_1 and u_1, respectively.*

(i) If $u_2 < 0$, then there are no local non-global minimizers of (1.1).
(ii) If $\ell_1 \leq 0$, $u_2 \geq 0$ and $p > 1$, then there are no local non-global minimizers of (1.1).
(iii) If $m > 1$ and $p > 1$, then there are no local non-global minimizers of (1.1).

Proof. It follows from Lemma 3.2. ◻

The next theorem is used to describe another situation where there are no local non-global minimizers. Before stating it, we need the following lemma.

LEMMA 3.4. *Suppose that x^* is a local non-global minimizer of problem (1.1) and λ^* is the corresponding Lagrange multiplier. Moreover, let λ be a generalized eigenvalue of the pencil $A + \lambda B$, $\lambda \neq \lambda^*$, v be the corresponding eigenvector and $a + \lambda b \in Range(A + \lambda B)$. Then $v^T(Bx^* + b) = 0$.*

Proof. Since x^* is a local non-global minimizer, then by Theorem 2.5 and Lemma 2.7, $(A + \lambda^* B)x^* = -(a + \lambda^* b)$. By using the change of variable $x^* = y^* - (A + \hat{\lambda}B)^{-1}(a + \hat{\lambda}b)$, we have

$$Bx^* + b = By^* - B(A + \hat{\lambda}B)^{-1}(a + \hat{\lambda}b) + b.$$

Thus

$$v^T(Bx^* + b) = v^T By^* - v^T(B(A + \hat{\lambda}B)^{-1}(a + \hat{\lambda}b) - b). \tag{3.3}$$

On the other hand, since $(-b+B(A+\hat{\lambda}B)^{-1}(a+\hat{\lambda}b)) \in \text{Range}(A+\lambda B)$ (see Appendix, Proposition 8.2), we have

$$v^T(B(A + \hat{\lambda}B)^{-1}(a + \hat{\lambda}b) - b) = 0.$$

This together with (3.3) implies that $v^T(Bx^* + b) = v^T By^*$. We also have

$$[(A + \hat{\lambda}B) + (\lambda^* - \hat{\lambda})B]y^* = (\lambda^* - \hat{\lambda})[-b + B(A + \hat{\lambda}B)^{-1}(a + \hat{\lambda}b)].$$

Thus

$$v^T[(A + \hat{\lambda}B) + (\lambda^* - \hat{\lambda})B]y^* = (\lambda^* - \hat{\lambda})v^T[-b + B(A + \hat{\lambda}B)^{-1}(a + \hat{\lambda}b)],$$

implying

$$v^T(A + \hat{\lambda}B)y^* + (\lambda^* - \hat{\lambda})v^T By^* = 0. \tag{3.4}$$

Since v is an eigenvector corresponding to the generalized eigenvalue λ, then $(A + \hat{\lambda}B)v = -(\lambda - \hat{\lambda})Bv$. This implies that

$$v^T(A + \hat{\lambda}B)y^* = -(\lambda - \hat{\lambda})v^T By^*. \tag{3.5}$$

By substituting (3.5) into (3.4), we obtain

$$(-\lambda + \lambda^*)v^T By^* = 0.$$

Now since $\lambda \neq \lambda^*$, then $v^T By^* = 0$ and consequently $v^T(Bx^* + b) = 0$. ◻

THEOREM 3.5. *Suppose that ℓ_1 and u_1 are defined as before. Moreover, assume that x^* is a local non-global minimizer of problem (1.1), λ^* is the corresponding Lagrange multiplier, m and p are the multiplicity of ℓ_1 and u_1, respectively.*

(i) If $m > 1$ or if $m = 1$ and $(a + \ell_1 b) \in Range(A + \ell_1 B)$, then $\lambda^* \nless \ell_1$.

(ii) If $p > 1$ or if $p = 1$ and $(a + u_1 b) \in Range(A + u_1 B)$, then $\lambda^* \ngtr u_1$.

Proof. (i) If $m > 1$, then by Lemma 3.2, $\lambda^* \nless \ell_1$. Let $m = 1$ and $(a + \ell_1 b) \in \text{Range}(A + \ell_1 B)$. Suppose by contradiction that $\lambda^* < \ell_1$ and v is an eigenvector corresponding to ℓ_1 such that $v^T(A + \hat{\lambda}B)v = 1$. We show that $v^T(A + \lambda^* B)v < 0$. We have

$$\begin{aligned} v^T(A + \lambda^* B)v &= v^T((A + \hat{\lambda}B) + (\lambda^* - \hat{\lambda})B)v \\ &= v^T(A + \hat{\lambda}B)^{\frac{1}{2}}[I + (\lambda^* - \hat{\lambda})(A + \hat{\lambda}B)^{-\frac{1}{2}}B(A + \hat{\lambda}B)^{-\frac{1}{2}}](A + \hat{\lambda}B)^{\frac{1}{2}}v. \end{aligned} \tag{3.6}$$

Set $M = (A+\hat{\lambda}B)^{-\frac{1}{2}}B(A+\hat{\lambda}B)^{-\frac{1}{2}}$ and consider its eigenvalue decomposition as $M = QDQ^T$ in which Q is an orthogonal matrix containing $(A + \hat{\lambda}B)^{\frac{1}{2}}v$ as its first column and the first diagonal entry of D is $-\frac{1}{\ell_1 - \hat{\lambda}}$ (see Appendix, Proposition 8.1). Continuing (3.6) we obtain the following:

$$\begin{aligned} v^T(A + \lambda^* B)v &= v^T(A + \hat{\lambda}B)^{\frac{1}{2}}[I + (\lambda^* - \hat{\lambda})QDQ^T](A + \hat{\lambda}B)^{\frac{1}{2}}v \\ &= v^T(A + \hat{\lambda}B)^{\frac{1}{2}}Q[I + (\lambda^* - \hat{\lambda})D]Q^T(A + \hat{\lambda}B)^{\frac{1}{2}}v \\ &= \begin{bmatrix} v^T(A + \hat{\lambda}B)v & 0 \cdots 0 \end{bmatrix} (I + (\lambda^* - \hat{\lambda})D) \begin{bmatrix} v^T(A + \hat{\lambda}B)v \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= \frac{-\lambda^* + \ell_1}{\ell_1 - \hat{\lambda}}(v^T(A + \hat{\lambda}B)v)^2 < 0, \end{aligned}$$

since $\lambda^* < \ell_1$ and $\hat{\lambda} > \ell_1$. On the other hand, by Lemma 3.4, $(Bx^* + b)^T v = 0$. This implies that the necessary condition (2.4) is not satisfied for x^* and thus $\lambda^* < \ell_1$ fails.

(ii) The proof is similar to (i). ◻

The following corollary states another case where there are no local non-global minimizers for problem (1.1).

COROLLARY 3.6. *Suppose that ℓ_1 and u_1 are defined as before. If the multiplicity of ℓ_1 and u_1 is one, $(a + \ell_1 b) \in Range(A + \ell_1 B)$ and $(a + u_1 b) \in Range(A + u_1 B)$, then there are no local non-global minimizers for (1.1).*

So far, we have shown that the Lagrange multiplier λ^* corresponding to the local non-global minimizer is either in the interval $[\ell_2, \ell_1)$ or in $(u_1, u_2]$. In the following theorem we further show that λ^* must be strictly greater than ℓ_2 and strictly less that u_2.

THEOREM 3.7. *Suppose that x^* is a local non-global minimizer of (1.1) and λ^* is the corresponding Lagrange multiplier. Then, $\lambda^* > \ell_2$ and $\lambda^* < u_2$.*

Proof. From Lemma 3.2, we have $\lambda^* \leq u_2$. We show that $\lambda^* \neq u_2$. To this end, suppose by contradiction that $\lambda^* = u_2$. Since x^* is a local non-global minimizer of (1.1), then $g(x^*) = 0$ and hence it is also a local minimizer of

$$\begin{aligned} \min \quad & x^T Ax + 2a^T x \\ & x^T Bx + 2b^T x + \beta = 0. \end{aligned} \tag{3.7}$$

By the change of variables $y := C^{-1}x$, $a := C^T a$ and $b := C^T b$, where C and the diagonal matrices $D = \operatorname{diag}(\alpha_1, \alpha_2, ..., \alpha_n)$ and $E = \operatorname{diag}(\beta_1, \beta_2, ..., \beta_n)$ are defined as before, problem (3.7) is equivalent to the following problem

$$\begin{aligned} \min \quad & y^T Dy + 2a^T y \\ & y^T Ey + 2b^T y + \beta = 0. \end{aligned} \tag{3.8}$$

So, $y^* = C^{-1}x^*$ is a local minimizer of (3.8) and $\lambda^* = u_2$ is the corresponding Lagrange multiplier. The first-order necessary conditions imply that

$$(D + u_2 E)y^* = -(a + u_2 b). \tag{3.9}$$

By permuting the rows of matrix $D + u_2E$ and the right hand side vector $a + u_2b$ if necessary, we may assume without loss of generality that the diagonal entries of $D+u_2E$ are in ascending order. We also know that $D+u_2E$ has exactly one negative and one zero diagonal entry, thus

$$\alpha_1 + u_2\beta_1 < 0, \tag{3.10}$$

and

$$\alpha_2 + u_2\beta_2 = 0. \tag{3.11}$$

Moreover, since $(a+u_2b) \in \operatorname{Range}(D+u_2E)$, the second entry of $(a+u_2b)$, $(a+u_2b)_2 = 0$. Now let P be the plane spanned by the unit vectors e_1 and e_2. If $v \in P$, then $v^T(D + u_2E)v < 0$ provided that $v \notin \operatorname{Span}\{e_2\}$. Consider $v \neq 0 \in P$ such that $v^T(Ey^* + b) = 0$. Since u_2 is the Lagrange multiplier corresponding to y^*, it follows from (2.4) that $v^T(D+u_2E)v \geq 0$. This implies that $v \in \operatorname{Span}\{e_2\}$ and consequently $e_2^T(Ey^* + b) = 0$. Therefore, $\beta_2 y_2^* + b_2 = 0$. We note that $\beta_2 \neq 0$ because if $\beta_2 = 0$, then by (3.11), $\alpha_2 = 0$ which is in contradiction with Assumption 2. So, we conclude that $y_2^* = -\frac{b_2}{\beta_2}$. Furthermore, it is easy to see that problem (3.8) is equivalent to

$$\begin{aligned} \min \quad & y^T(D + u_2E)y + 2(a + u_2b)^T y + u_2\beta \\ & y^T Ey + 2b^T y + \beta = 0, \end{aligned} \tag{3.12}$$

which can be rewritten as

$$\begin{aligned} \min \quad & \sum_{i=1}^n (\alpha_i + u_2\beta_i)y_i^2 + 2\sum_{i=1}^n (a + u_2b)_i y_i + u_2\beta \\ & \beta_1 y_1^2 + 2b_1y_1 + \sum_{i=2}^n \beta_i y_i^2 + 2\sum_{i=2}^n b_i y_i + \beta = 0. \end{aligned} \tag{3.13}$$

By (3.11) and the fact that $(a+u_2b)_2=0$, problem (3.13) reduces to

$$\begin{aligned}\min \quad & (\alpha_1+u_2\beta_1)y_1^2+2(a+u_2b)_1y_1+\sum_{i=3}^{n}(\alpha_i+u_2\beta_i)y_i^2+2\sum_{i=3}^{n}(a+u_2b)_iy_i+u_2\beta\\ \text{(3.14)}\quad & \beta_1y_1^2+2b_1y_1+\sum_{i=2}^{n}\beta_iy_i^2+2\sum_{i=2}^{n}b_iy_i+\beta=0.\end{aligned}$$

By dividing the objective function of (3.14) into $-(\alpha_1+u_2\beta_1)>0$, we obtain the following equivalent problem:

$$\begin{aligned}\min \quad & -y_1^2-2\frac{(a+u_2b)_1}{\alpha_1+u_2\beta_1}y_1-\sum_{i=3}^{n}\frac{(\alpha_i+u_2\beta_i)}{\alpha_1+u_2\beta_1}y_i^2-2\sum_{i=3}^{n}\frac{(a+u_2b)_i}{\alpha_1+u_2\beta_1}y_i-\frac{u_2\beta}{\alpha_1+u_2\beta_1}\\ \text{(3.15)}\quad & \beta_1y_1^2+2b_1y_1+\sum_{i=2}^{n}\beta_iy_i^2+2\sum_{i=2}^{n}b_iy_i+\beta=0.\end{aligned}$$

Next, we note that $\beta_1\neq 0$ because if $\beta_1=0$, then by (3.10), $\alpha_1<0$ which is in contradiction with Assumption 2. Thus from the constraint of problem (3.15), we obtain

$$(y_1+\frac{b_1}{\beta_1})^2-\frac{b_1^2}{\beta_1^2}+\sum_{i=2}^{2}\frac{\beta_i}{\beta_1}y_i^2+2\sum_{i=2}^{n}\frac{b_i}{\beta_1}y_i+\frac{\beta}{\beta_1}=0,$$

which results in

$$y_1=\pm h(y_2,\cdots,y_n)-\frac{b_1}{\beta_1},$$

where

$$h(y_2,\cdots,y_n)=\sqrt{\frac{b_1^2}{\beta_1^2}-\frac{\beta_2}{\beta_1}y_2^2-\sum_{i=3}^{n}\frac{\beta_i}{\beta_1}y_i^2-2\frac{b_2}{\beta_1}y_2-2\sum_{i=3}^{n}\frac{b_i}{\beta_1}y_i-\frac{\beta}{\beta_1}}.$$

First suppose that $y_1=h(y_2,\cdots,y_n)-\frac{b_1}{\beta_1}$. By substituting y_1 into the objective function of (3.15), we obtain the following equivalent problem:

$$\min \quad F(y_2,\cdots,y_n), \tag{3.16}$$

where

$$\begin{aligned}F(y_2,\cdots,y_n)=&\frac{\beta_2}{\beta_1}y_2^2+2\frac{b_2}{\beta_1}y_2+\left(2\frac{b_1}{\beta_1}-2\frac{(a+u_2b)_1}{\alpha_1+u_2\beta_1}\right)h(y_2,\cdots,y_n)\\ &+\sum_{i=3}^{n}\left(\frac{\beta_i}{\beta_1}-\frac{\alpha_i+u_2\beta_i}{\alpha_1+u_2\beta_1}\right)y_i^2+2\sum_{i=3}^{n}\left(\frac{b_i}{\beta_1}-\frac{(a_i+u_2b_i)}{\alpha_1+u_2\beta_1}\right)y_i.\end{aligned}$$

On the other hand, by (3.9), we have $y_1^*=-\frac{(a+u_2b)_1}{\alpha_1+u_2\beta_1}$. This together with (3.16) imply that y_2^* is a local minimizer of the function $f(y)$ given by

$$f(y)=\frac{\beta_2}{\beta_1}y^2+2\frac{b_2}{\beta_1}y+2(\frac{b_1}{\beta_1}+y_1^*)g(y),$$

where

$$g(y)=\sqrt{\frac{b_1^2}{\beta_1^2}-\frac{\beta_2}{\beta_1}y^2-\sum_{i=3}^{n}\frac{\beta_i}{\beta_1}{y_i^*}^2-2\frac{b_2}{\beta_1}y-2\sum_{i=3}^{n}\frac{b_i}{\beta_1}y_i^*-\frac{\beta}{\beta_1}}.$$

Noting that $y_2^* = -\frac{b_2}{\beta_2}$ and $g(y_2^*) = \frac{b_1}{\beta_1} + y_1^*$, we obtain $f'(y_2^*) = f''(y_2^*) = f^{(3)}(y_2^*) = 0$ and $f^{(4)}(y_2^*) = -2\frac{\beta_2^2 g(y_2^*)}{\beta_1^2} < 0$. This contradicts the assumption that y_2^* is a local minimizer of $f(y)$. The contradiction comes from the initial assumption that x^* is a local minimizer of (1.1) with the Lagrange multiplier $\lambda^* = u_2$. We obtain the same result if we set $y_1 = -h(y_2, \cdots, y_n) - \frac{b_1}{\beta_1}$. Similarly, we can prove that $\lambda^* > \ell_2$. □

Now we are ready to state the main result of this section. Set $I = (\ell_2, \ell_1) \cup (u_1, u_2)$. For $\lambda \in I$,

$$\varphi(\lambda) := g(x(\lambda)) = x(\lambda)^T Bx(\lambda) + 2b^T x(\lambda) + \beta, \tag{3.17}$$

where $x(\lambda) = -(A + \lambda B)^{-1}(a + \lambda b)$. Due to the previous results, if x^* is a local non-global minimizer of (1.1), then $g(x^*) = 0$, $x^* = -(A + \lambda^* B)^{-1}(a + \lambda^* b)$ and $\lambda^* \in I$. Thus the local non-global minimizers can be characterized by the equation $\varphi(\lambda) = 0$ on I. By the equivalence between problem (1.1) and (3.2), we have

$$\varphi(\lambda) = \sum_{i=1}^{n} \left(\beta_i \frac{(\eta_i + \lambda\phi_i)^2}{(\alpha_i + \lambda\beta_i)^2} - 2\phi_i \frac{(\eta_i + \lambda\phi_i)}{(\alpha_i + \lambda\beta_i)} \right) + \beta, \tag{3.18}$$

$$\varphi'(\lambda) = \sum_{i=1}^{n} -2\frac{(\eta_i\beta_i - \phi_i\alpha_i)^2}{(\alpha_i + \lambda\beta_i)^3}, \tag{3.19}$$

$$\varphi''(\lambda) = \sum_{i=1}^{n} 6\beta_i \frac{(\eta_i\beta_i - \phi_i\alpha_i)^2}{(\alpha_i + \lambda\beta_i)^4}. \tag{3.20}$$

In the special case of TRS, we have $I = (\ell_2, \ell_1)$ and $\beta_i = 1, \phi_i = 0, i = 1, ..., n$. If TRS admits a local non-global minimizer, then by (3.20), φ is a strictly convex function on I. Therefore, the equation $\varphi(\lambda) = 0$ has at most two roots in the interval I. In [24], it has been shown that, for a local non-global minimizer of TRS, $\varphi'(\lambda^*) \geq 0$. This means that in the case where two roots exist, only the largest root can be the Lagrange multiplier of a local non-global minimizer of TRS. As a consequence, TRS has at most one local non-global minimizer. The following theorem specifies which root of equation $\varphi(\lambda) = 0$ may be the Lagrange multiplier of a local non-global minimizer of problem (1.1).

THEOREM 3.8. *Suppose that x^* is a local non-global minimizer of problem (1.1) and λ^* is the corresponding Lagrange multiplier. Then $\varphi'(\lambda^*) \geq 0$.*

Proof. Let x^* be a local non-global minimizer of (1.1) with the Lagrange multiplier λ^*. Then $y^* = C^{-1}x^*$ is a local non-global minimizer of problem (3.2) and λ^* is the corresponding Lagrange multiplier. We know that $y^* = -(D + \lambda^* E)^{-1}(\eta + \lambda^*\phi)$, implying that $Ey^* + \phi = (c_1, c_2, ..., c_n)^T$ where

$$c_i = -\beta_i \frac{\eta_i + \lambda^*\phi_i}{\alpha_i + \lambda^*\beta_i} + \phi_i, \quad i = 1, ..., n.$$

By Lemma 2.7, $Ey^* + \phi \neq 0$. Without loss of generality, let $c_1 \neq 0$ and define

$$W = \begin{bmatrix} c_2 & c_3 & \cdots & c_n \\ -c_1 & 0 & \cdots & 0 \\ 0 & -c1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & -c_1 \end{bmatrix} \in \mathbb{R}^{n\times(n-1)}.$$

Then it is easy to verify that W is a basis for $\mathrm{Null}((Ey^* + \phi)^T)$. Set $H = W^T(D + \lambda^* E)W$. We have $H = G + (\alpha_1 + \lambda^*\beta_1)gg^T$ where $g = (c_2, ..., c_n)^T$ and

$$G = \begin{bmatrix} c_1^2(\alpha_2+\lambda^*\beta_2) & & & 0 \\ & c_1^2(\alpha_3+\lambda^*\beta_3) & & \\ & & \ddots & \\ 0 & & & c_1^2(\alpha_n+\lambda^*\beta_n) \end{bmatrix}.$$

Since G is nonsingular, then $H = G\left(I + (\alpha_1 + \lambda^*\beta_1)G^{-1}gg^T\right)$ and consequently

$$\det(H) = \det(G)\left(1 + (\alpha_1 + \lambda^*\beta_1)g^T G^{-1} g\right). \tag{3.21}$$

Obviously,

$$\det(G) = c_1^{2n-2}\,(\alpha_2 + \lambda^*\beta_2) \times \cdots \times (\alpha_n + \lambda^*\beta_n)\,,$$

and

$$1 + (\alpha_1 + \lambda^*\beta_1)g^T G^{-1} g = 1 + \frac{c_2^2(\alpha_1+\lambda^*\beta_1)}{c_1^2(\alpha_2+\lambda^*\beta_2)} + \frac{c_3^2(\alpha_1+\lambda^*\beta_1)}{c_1^2(\alpha_3+\lambda^*\beta_3)} + \cdots + \frac{c_n^2(\alpha_1+\lambda^*\beta_1)}{c_1^2(\alpha_n+\lambda^*\beta_n)}.$$

Substituting these into (3.21) results in

$$\begin{aligned}\det(H) =& c_1^{2n-2}(\alpha_2+\lambda^*\beta_2)\times\cdots\times(\alpha_n+\lambda^*\beta_n)\\ &\times\left[1 + \frac{c_2^2(\alpha_1+\lambda^*\beta_1)}{c_1^2(\alpha_2+\lambda^*\beta_2)} + \frac{c_3^2(\alpha_1+\lambda^*\beta_1)}{c_1^2(\alpha_3+\lambda^*\beta_3)} + \cdots + \frac{c_n^2(\alpha_1+\lambda^*\beta_1)}{c_1^2(\alpha_n+\lambda^*\beta_n)}\right]\\ =& c_1^{2n-2}(\alpha_2+\lambda^*\beta_2)\times\cdots\times(\alpha_n+\lambda^*\beta_n)(\alpha_1+\lambda^*\beta_1)\\ &\times\left[\frac{1}{(\alpha_1+\lambda^*\beta_1)} + \frac{c_2^2}{c_1^2(\alpha_2+\lambda^*\beta_2)} + \frac{c_3^2}{c_1^2(\alpha_3+\lambda^*\beta_3)} + \cdots + \frac{c_n^2}{c_1^2(\alpha_n+\lambda^*\beta_n)}\right]\\ =& -\frac{1}{2}c_1^{2n-4}(\alpha_2+\lambda^*\beta_2)\times\cdots\times(\alpha_n+\lambda^*\beta_n)(\alpha_1+\lambda^*\beta_1)\\ &\times\left[\frac{-2c_1^2}{(\alpha_1+\lambda^*\beta_1)} + \frac{-2c_2^2}{(\alpha_2+\lambda^*\beta_2)} + \frac{-2c_3^2}{(\alpha_3+\lambda^*\beta_3)} + \cdots + \frac{-2c_n^2}{(\alpha_n+\lambda^*\beta_n)}\right].\end{aligned}$$

Thus by (3.19), we obtain

$$\det(H) = -\frac{1}{2}c_1^{2n-4}(\alpha_2+\lambda^*\beta_2)\times\cdots\times(\alpha_n+\lambda^*\beta_n)(\alpha_1+\lambda^*\beta_1)\phi'(\lambda^*).$$

It follows from Theorem 2.5 that H is positive semidefinite and thus $\det(H) \geq 0$. This together with the facts that $c_1 \neq 0$ and $(D + \lambda^* E)$ is nonsingular but has exactly one negative eigenvalue imply that $\phi'(\lambda^*) \geq 0$. □

For a general B, there is no evidence that how many roots of $\varphi(\lambda)$ may be in the interval I. Let $r_k \leq \cdots \leq r_2 \leq r_1$ and $t_1 \leq t_2 \leq \cdots \leq t_s$ be the roots of this equation in the intervals (ℓ_2, ℓ_1) and (u_1, u_2), respectively (counting multiplicities). If $\lambda^* \in (\ell_2, \ell_1)$, then $(\eta + \ell_1\phi) \notin \mathrm{Range}(D + \ell_1 E)$. Moreover, by the definition, $\ell_1 = -\frac{\alpha_{i_1}}{\beta_{i_1}}$ where $i_1 \in T_1$ and $\beta_{i_1} > 0$. Therefore, it follows from (3.18) that

$$\lim_{\lambda\to\ell_1} \varphi(\lambda) = +\infty. \tag{3.22}$$

Similarly, it is easy to see that if $\lambda^* \in (u_1, u_2)$, then

$$\lim_{\lambda \to u_1} \varphi(\lambda) = -\infty. \tag{3.23}$$

Now (3.22) and (3.23) together with Theorem 3.8 imply that only r_i and t_i with odd subscript may be the Lagrange multiplier of a local non-global minimizer of problem (1.1).

4. Characterization using a generalized eigenvalue problem. The aim of this section is to characterize the local (global and non-global) minimizers using a generalized eigenvalue problem. Throughout this section, ℓ_1, ℓ_2, u_1, u_2, matrices $D = \mathrm{diag}(\alpha_1, \alpha_2, ..., \alpha_n)$, $E = \mathrm{diag}(\beta_1, \beta_2, ..., \beta_n)$ and C are defined as before.

4.1. Computing the interval containing the Lagrange multipliers. Here we consider the problem of efficiently computing the intervals where the Lagrange multiplier for a local minimizer may be found. Recall that the Lagrange multiplier associated with a global minimizer lies in the interval $[\ell_1, u_1]$ while the one for a local non-global minimizer is either in (ℓ_2, ℓ_1) or in (u_1, u_2). Note that the value of ℓ_2 and u_2 are required when we look for the local non-global minimizers. Hence, to compute ℓ_2 and u_2, it is reasonable to assume that ℓ_1 and u_1 have multiplicity one otherwise the are no local non-global minimizers for (1.1) and any information about ℓ_2 and u_2 is unnecessary. The next lemma shows that ℓ_1, ℓ_2, u_1 and u_2 can be computed via finding a few generalized eigenvalues. We remark that the results for ℓ_1 and u_1 can also be found in [20, 28].

LEMMA 4.1. *(i) If $B \succ 0$, then $\ell_1 = -\lambda_{min}(A, B)$, $\ell_2 = -\lambda_{min2}(A, B)$ and $u_1 = u_2 = +\infty$.*

(ii) If $A \succ 0$, then

$$\ell_1 = \begin{cases} \frac{1}{\lambda_{min}(-B,A)} & if\ \lambda_{min}(-B, A) < 0, \\ -\infty & otherwise, \end{cases} \qquad \ell_2 = \begin{cases} \frac{1}{\lambda_{min2}(-B,A)} & if \lambda_{min2}(-B, A) < 0, \\ -\infty & otherwise. \end{cases}$$

and

$$u_1 = \begin{cases} -\frac{1}{\lambda_{min}(B,A)} & if\ \lambda_{min}(B, A) < 0, \\ \infty & otherwise. \end{cases} \qquad u_2 = \begin{cases} -\frac{1}{\lambda_{min2}(B,A)} & if \lambda_{min2}(B, A) < 0, \\ \infty & otherwise. \end{cases}$$

(iii) If $A \not\succ 0$ and $B \not\succ 0$, then $\ell_1 = \bar{\ell_1} + \hat{\lambda}$, $\ell_2 = \bar{\ell_2} + \hat{\lambda}$, $u_1 = \bar{u_1} + \hat{\lambda}$ and $u_2 = \bar{u_2} + \hat{\lambda}$ where $\hat{\lambda}$ is the same as in Assumption 2 and

$$\bar{\ell_1} = \begin{cases} \frac{1}{\lambda_{min}(-B,A+\hat{\lambda}B)} & if \quad \lambda_{min}(-B, A + \hat{\lambda}B) < 0, \\ -\infty & otherwise, \end{cases}$$

$$\bar{\ell_2} = \begin{cases} \frac{1}{\lambda_{min2}(-B,A+\hat{\lambda}B)} & if \quad \lambda_{min2}(-B, A + \hat{\lambda}B) < 0, \\ -\infty & otherwise, \end{cases}$$

$$\bar{u_1} = \begin{cases} -\frac{1}{\lambda_{min}(B,A+\hat{\lambda}B)} & if \quad \lambda_{min}(B, A + \hat{\lambda}B) < 0, \\ \infty & otherwise, \end{cases}$$

$$\bar{u_2} = \begin{cases} -\frac{1}{\lambda_{min2}(B,A+\hat{\lambda}B)} & if \quad \lambda_{min2}(B, A + \hat{\lambda}B) < 0, \\ \infty & otherwise. \end{cases}$$

Proof. (i) By the definition of ℓ_1 and using the fact that $E \succ 0$, we have

$$\ell_1 = \max\{-\frac{\alpha_i}{\beta_i} | \beta_i > 0\} = -\min\{\frac{\alpha_i}{\beta_i} | i \in N\} = -\lambda_{\min}(A, B),$$

and since $T_1 = N$,

$$\ell_2 = -\frac{\alpha_{i_2}}{\beta_{i_2}} = \max\{-\frac{\alpha_{i_k}}{\beta_{i_k}}|k = 2, \cdots, n\} = -\min\{\frac{\alpha_{i_k}}{\beta_{i_k}}|k = 2, \cdots, n\}$$
$$= -\lambda_{\min 2}(A, B).$$

Next, since $B \succ 0$, then T_2 is empty and hence $u_1 = u_2 = +\infty$.

(ii) Since $A \succ 0$, then $0 \in (\ell_1, u_1)$. For given $\lambda \in (0, u_1]$, $(B + \frac{1}{\lambda}A) \succeq 0$, and so by part (i), we have

$$\frac{1}{\lambda} \geq -\lambda_{\min}(B, A). \tag{4.1}$$

If $\lambda \in [\ell_1, 0)$, then $-B - \frac{1}{\lambda}A \succeq 0$ and hence, by part (i), the following holds:

$$-\frac{1}{\lambda} \geq -\lambda_{\min}(-B, A) \Leftrightarrow \frac{1}{\lambda} \leq \lambda_{\min}(-B, A). \tag{4.2}$$

Now it follows from (4.1) and (4.2) that

$$\ell_1 = \begin{cases} \frac{1}{\lambda_{\min}(-B,A)} & if\, \lambda_{\min}(-B, A) < 0, \\ -\infty & otherwise, \end{cases}$$

and

$$u_1 = \begin{cases} -\frac{1}{\lambda_{\min}(B,A)} & if\, \lambda_{\min}(B, A) < 0, \\ \infty & otherwise. \end{cases}$$

Recall that we have assumed u_1 and ℓ_1 have multiplicity one and $D \succ 0$. To compute u_2, we note that for $\lambda > 0$, the matrix $D + \lambda E$ has exactly one negative eigenvalue if and only if $E + \frac{1}{\lambda}D$ does and $E \not\succeq 0$. Therefore, by part (i), the following must hold for $\lambda > 0$:

$$-\lambda_{\min 2}(E, D) \leq \frac{1}{\lambda} < -\lambda_{\min}(E, D). \tag{4.3}$$

Since $E \not\succeq 0$ and $D \succ 0$, then $\lambda_{\min}(E, D) < 0$ and hence it follows from the right hand side inequality in (4.3) that

$$\lambda > -\frac{1}{\lambda_{\min}(E, D)}. \tag{4.4}$$

If $\lambda_{\min 2}(E, D) \geq 0$, then the left hand side inequality in (4.3) holds for all $\lambda > 0$. If $\lambda_{\min 2}(E, D) < 0$, then we must have

$$\lambda \leq -\frac{1}{\lambda_{\min 2}(E, D)}. \tag{4.5}$$

Therefore, it follows from (4.4), (4.5) and the fact that the eigenvalues of the pencils $B - \lambda A$ and $E - \lambda D$ are the same,

$$u_2 = \begin{cases} -\frac{1}{\lambda_{\min 2}(B,A)} & if \lambda_{\min 2}(B, A) < 0, \\ \infty & otherwise. \end{cases}$$

Next suppose that $\lambda < 0$, then matrix $D + \lambda E$ has exactly one negative eigenvalue if and only if $-E - \frac{1}{\lambda}D$ does and $E \nsucceq 0$. Thus, by part (i), the following must hold:

$$\lambda_{\min}(-E, D) < \frac{1}{\lambda} \leq \lambda_{\min 2}(-E, D). \tag{4.6}$$

Since $D \succ 0$ and $E \nsucceq 0$, then $\lambda_{\min}(-E, D) < 0$ and thus it follows from the left hand side inequality in (4.6) that

$$\lambda < \frac{1}{\lambda_{\min}(-E, D)}. \tag{4.7}$$

On the other hand, if $\lambda_{\min 2}(-E, D) \geq 0$, then the right hand side inequality in (4.6) holds for all $\lambda < 0$. If $\lambda_{\min 2}(-E, D) < 0$, then we must have

$$\lambda \geq \frac{1}{\lambda_{\min 2}(-E, D)}. \tag{4.8}$$

Finally, it follows from (4.7), (4.8) and the fact that the eigenvalues of the pencils $-B - \lambda A$ and $-E - \lambda D$ are the same,

$$\ell_2 = \begin{cases} \frac{1}{\lambda_{\min 2}(-B, A)} & if \lambda_{\min 2}(-B, A) < 0, \\ -\infty & otherwise. \end{cases}$$

(iii) Using Assumption 2, we have $A + \lambda B = (A + \hat{\lambda} B) + (\lambda - \hat{\lambda})B$. Now, the results follows from part (ii). □

4.2. Computing the global minimizer. Our approach to compute a possible local non-global minimizer in the next subsection is inspired by the approach by Adachi and Nakatsukasa for computing a global minimizer of (1.1) [2]. Before describing their approach, we turn to the optimality conditions (2.6) to (2.9) in order to provide the basis of the analysis below. Suppose that problem (1.1) has a global minimizer x_g^*. It follows from (2.9) that the optimal Lagrange multiplier λ_g^* lies in the interval $[\ell_1, u_1]$. Now consider the function $\varphi(\lambda)$ given by (3.17). This function has the following property.

Lemma 4.2 ([25]). *$\varphi(\lambda)$ is strictly decreasing on (ℓ_1, u_1) unless $x(\lambda)$ is constant with*

$$Ax(\lambda) = -a, \quad Bx(\lambda) = -b, \tag{4.9}$$

for all $\lambda \in (\ell_1, u_1)$.

Unless $A \succ 0$ and $g(-A^{-1}a) < 0$ which implies $\lambda_g^* = 0$ and $x_g^* = -A^{-1}a$, problem (1.1) has a boundary solution with $g(x_g^*) = 0$. Problem (1.1) is classified into easy case and hard case (case 1 and 2) instances in the literature [28]. The situation where there exists λ_g^* in the interval (ℓ_1, u_1) such that $\varphi(\lambda_g^*) = 0$ is commonly referred as the easy case or hard case 1. In this case, $x_g^* = -(A + \lambda_g^* B)^{-1}(a + \lambda_g^* b)$. Hard case 2 corresponds to the case where equation $\varphi(\lambda) = 0$ has no roots in the interval (ℓ_1, u_1). In this case, λ_g^* is either ℓ_1 or u_1.

Recently, the authors in [2] designed an efficient algorithm for (1.1) which solves the problem via computing one eigenpair of a generalized eigenvalue problem. Defining

$M_0, M_1 \in \mathbb{R}^{(2n+1)\times(2n+1)}$ as

$$M_0 = \begin{bmatrix} \beta & b^T & -a^T \\ b & B & -A \\ -a & -A & O_{n\times n} \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & O_{1\times n} & -b^T \\ O_{n\times 1} & O_{n\times n} & -B \\ -b & -B & O_{n\times n} \end{bmatrix},$$

they showed that any KKT multiplier $\lambda^* \geq 0$ satisfying

$$(A + \lambda^* B)x^* = -(a + \lambda^* b), \tag{4.10}$$

$$g(x^*) = 0, \tag{4.11}$$

is an eigenvalue of the matrix pencil $M_0 + \lambda M_1$. Under assumption (A1), they further showed that the Lagrange multiplier corresponding to the global optimal solution of (1.1) is $\lambda_g^* = \hat{\lambda} + \frac{1}{\eta^*}$ where η^* is the largest (or smallest) real eigenvalue of $M_1 + \eta \hat{M}$ where $\hat{M} = M_0 + \hat{\lambda} M_1$. Here, we rework and introduce a matrix pencil, different from the one in [2], whose eigenvalues include the KKT multipliers satisfying (4.10) and (4.11) shifted by $\hat{\lambda}$. Our discussion is based on Assumption 2. We further assume that $u_1 > 0$ due to the following proposition. It is worth to mention that, by Lemma 4.1, when $\hat{\lambda}$ is available, u_1 can be computed efficiently via finding the smallest generalized eigenvalue of a matrix pencil.

Proposition 4.3. *If $u_1 < 0$, then problem (1.1) is unbounded below. If $u_1 = 0$, then $A \succeq 0$, furthermore, if $a \in Range(A)$, then $\lambda_g^* = 0$ otherwise problem (1.1) is unbounded below.*

Proof. If $u_1 < 0$, then by Lemma 2.2, problem (1.1) is unbounded below. Next let $u_1 = 0$. This implies that $A \succeq 0$ and $\{\lambda \geq 0 | A + \lambda B \succeq 0\} = \{0\}$. If $a \notin \text{Range}(A)$, then by Lemma 2.2, problem (1.1) is unbounded. If $a \in \text{Range}(A)$, then problem (1.1) is bounded and we further show that $\lambda_g^* = 0$. To see this, set $\bar{x} = -A^\dagger a$. If $g(\bar{x}) \leq 0$, then $\bar{x}$ with $\lambda_g^* = 0$ satisfy the optimality conditions (2.6) to (2.9) and so $\bar{x}$ is the global minimizer. Next, let $g(\bar{x}) > 0$ and $v \in \text{Null(A)}$. Since $\hat{\lambda} < 0$, then $v^T B v < 0$ and hence, there exists $\alpha \in \mathbb{R}$ such that $g(\bar{x} + \alpha v) \leq 0$. Therefore, $x_g^* := \bar{x} + \alpha v$ with $\lambda_g^* = 0$ satisfy the optimality conditions. ∎

Set $x(\hat{\lambda}) = -(A + \hat{\lambda} B)^{-1}(a + \hat{\lambda} b)$. By the change of variable $y^* := x^* - x(\hat{\lambda})$, the first-order necessary conditions (4.10) and (4.11) can be written as

$$((A + \hat{\lambda} B) + \mu^* B)y^* = \mu^* d, \tag{4.12}$$

$${y^*}^T B y^* - 2d^T y^* + \varphi(\hat{\lambda}) = 0, \tag{4.13}$$

where $\mu^* = \lambda^* - \hat{\lambda}$, $d = -Bx(\hat{\lambda}) - b$. Define $\tilde{M}_0$ and $\tilde{M}_1$ as

$$\tilde{M}_0 = \begin{bmatrix} \varphi(\hat{\lambda}) & d^T & O_{1\times n} \\ d & B & -(A + \hat{\lambda} B) \\ O_{n\times 1} & -(A + \hat{\lambda} B) & O_{n\times n} \end{bmatrix}, \quad \tilde{M}_1 = \begin{bmatrix} 0 & O_{1\times n} & -d^T \\ O_{n\times 1} & O_{n\times n} & -B \\ -d & -B & O_{n\times n} \end{bmatrix}.$$

We have the following result.

Theorem 4.4. *For any μ^* satisfying (4.12) and (4.13), we have* $\det(\tilde{M}_0 + \mu^* \tilde{M}_1) = 0$.

Proof. We have

$$\tilde{M}_0 + \mu^* \tilde{M}_1 = \begin{bmatrix} \varphi(\hat{\lambda}) & d^T & -\mu^* d^T \\ d & B & -(A+\hat{\lambda}B) - \mu^* B \\ -\mu^* d & -(A+\hat{\lambda}B) - \mu^* B & O_{n\times n} \end{bmatrix}. \tag{4.14}$$

First let $\det((A+\hat{\lambda}B)+\mu^* B) \neq 0$. Define

$$C = \begin{bmatrix} B & -(A+\hat{\lambda}B) - \mu^* B \\ -(A+\hat{\lambda}B) - \mu^* B & O_{n\times n} \end{bmatrix}.$$

Then by the Schur complement of the block C of $\tilde{M}_0 + \mu^* \tilde{M}_1$ and using the fact that $y^* = \mu^*((A+\hat{\lambda}B)+\mu^* B)^{-1} d$, we obtain

$$\begin{aligned} \det(\tilde{M}_0 + \mu^* \tilde{M}_1) &= \det(C) \det\left(\varphi(\hat{\lambda}) - \begin{bmatrix} d \\ -\mu^* d \end{bmatrix}^T C^{-1} \begin{bmatrix} d \\ -\mu^* d \end{bmatrix} \right) \\ &= (-1)^n \det((A+\hat{\lambda}B)+\mu^* B)^2 \left({y^*}^T B y^* - 2d^T y^* + \varphi(\hat{\lambda}) \right). \end{aligned}$$

It follows from (4.13) that $({y^*}^T B y^* - 2d^T y^* + \varphi(\hat{\lambda})) = 0$ and hence $\det(\tilde{M}_0 + \mu^* \tilde{M}_1) = 0$. Next suppose that $\det((A+\hat{\lambda}B)+\mu^* B) = 0$. By (4.12), we have $d \in \mathrm{Range}((A+\hat{\lambda}B)+\mu^* B)$. Hence the bottom n rows of $\tilde{M}_0 + \mu^* \tilde{M}_1$ have rank $(n-1)$ or less and consequently $\det(\tilde{M}_0 + \mu^* \tilde{M}_1) = 0$. ∎

4.2.1. Computing the optimal Lagrange multiplier. In what follows, we show that the results in Subsection 3.3 of [2] also hold for the pencil $\tilde{M}_0 + \mu \tilde{M}_1$. Following [2], we consider different cases depending on the sign of $\varphi(\hat{\lambda})$.

Case 1: $\varphi(\hat{\lambda}) = 0$.

LEMMA 4.5. *If $\hat{\lambda} \geq 0$, then $\lambda_g^* = \hat{\lambda}$, otherwise $\lambda_g^* = 0$.*

Proof. If $\hat{\lambda} \geq 0$, then obviously $x(\hat{\lambda})$ with $\lambda_g^* = \hat{\lambda}$ satisfy the optimality conditions (2.6) to (2.9). Now let $\hat{\lambda} < 0$. Recall that we have assumed $u_1 > 0$. This means that $0 \in (\ell_1, u_1)$. On the other hand, by Lemma 4.2, the function $\varphi(\lambda)$ is decreasing on (ℓ_1, u_1) and thus $\varphi(0) \leq 0$, implying that $x_g^* = -A^{-1}a$ with $\lambda_g^* = 0$ satisfy the optimality conditions (2.6) to (2.9). ∎

Case 2. $\varphi(\hat{\lambda}) \neq 0$.

We have $\det(\tilde{M}_0) = \det(A+\hat{\lambda}B)^2 \varphi(\hat{\lambda})$. By the assumption that $\varphi(\hat{\lambda}) \neq 0$, we have $\det(\tilde{M}_0) \neq 0$ and hence μ is an eigenvalue of the pencil $\tilde{M}_0 + \mu \tilde{M}_1$ if and only if $\eta = \frac{1}{\mu}$ is an eigenvalue of

$$\tilde{M}_1 + \eta \tilde{M}_0. \tag{4.15}$$

We have the following theorems.

THEOREM 4.6. *Suppose that $\varphi(\hat{\lambda}) > 0$. Moreover, let η^* be the largest real eigenvalue of the pencil (4.15), then $\eta^* > 0$ and $\lambda_g^* = \frac{1}{\eta^*} + \hat{\lambda}$.*

Proof. Consider the following possible cases:

Case 1: $(a + u_1 b) \notin \mathrm{Range}(A + u_1 B)$.

In this case, the optimal Lagrange multiplier λ_g^* is the unique root of the equation $\varphi(\lambda) = 0$ in the interval $(\hat{\lambda}, u_1)$. Therefore, $\eta^* = \frac{1}{\lambda_g^* - \hat{\lambda}}$ is an eigenvalue of (4.15).

Since $\lambda_g^* > \hat{\lambda}$, then $\eta^* > 0$. Next, we show that η^* is the largest real eigenvalue of (4.15). To this end, suppose by contradiction that there exists a real eigenvalue η of (4.15) such that $\eta^* < \eta$. Set $\lambda = \frac{1}{\eta} + \hat{\lambda}$. We have $\hat{\lambda} < \lambda < \lambda_g^*$ and $\varphi(\lambda) = 0$ which is in contradiction to the fact that the function $\varphi(\lambda)$ has a unique root in the interval $(\hat{\lambda}, u_1)$. So, η^* is the largest real eigenvalue.

Case 2: $(a + u_1 b) \in \text{Range}(A + u_1 B)$.

If there exists λ_g^* in the interval $(\hat{\lambda}, u_1)$ such that $\varphi(\lambda_g^*) = 0$, then the proof is the same as in Case 1. Otherwise, problem (1.1) is hard case 2 and $\lambda_g^* = u_1$. It follows from the proof of Theorem 4.4 that $\eta^* = \frac{1}{u_1 - \hat{\lambda}}$ is an eigenvalue of (4.15). Moreover, a discussion similar to Case 1 shows that η^* is the largest real eigenvalue. □

THEOREM 4.7. *If $\varphi(\hat{\lambda}) > 0$, then the rightmost finite eigenvalue of (4.15) is real.*

Proof. The proof is identical to Theorem 5 of [2]. □

THEOREM 4.8. *Suppose that $\varphi(\hat{\lambda}) < 0$. Let η^* be the smallest real eigenvalue of the pencil (4.15), then $\eta^* < 0$. If $\hat{\lambda} = 0$ or $\hat{\lambda} + \frac{1}{\eta^*} \le 0$, then $\lambda_g^* = 0$, otherwise $\lambda_g^* = \hat{\lambda} + \frac{1}{\eta^*}$.*

Proof. First note that if $\hat{\lambda} = 0$, then $x(\hat{\lambda})$ with $\lambda_g^* = 0$ satisfy the optimality conditions (2.6) to (2.9). Next, consider the following possible cases:

Case 1: $(a + \ell_1 b) \notin \text{Range}(A + \ell_1 B)$.

In this case, there exists a unique $\lambda^* \in (\ell_1, \hat{\lambda})$ such that $\varphi(\lambda^*) = 0$. Therefore, $\eta^* = \frac{1}{\lambda^* - \hat{\lambda}}$ is an eigenvalue of (4.15). Since $\lambda^* < \hat{\lambda}$, then $\eta^* < 0$. We next show that η^* is the smallest real eigenvalue of (4.15). To this end, suppose by contradiction that η is a real eigenvalue of (4.15) such that $\eta < \eta^*$. Set $\lambda = \frac{1}{\eta} + \hat{\lambda}$. Then $\lambda^* < \lambda < \hat{\lambda}$ and $\varphi(\lambda) = 0$ which is a contradiction. Now, let $\lambda^* = \hat{\lambda} + \frac{1}{\eta^*} \le 0$. This together with $u_1 > 0$ imply that $0 \in (\ell_1, u_1)$ and $\varphi(0) \le 0$. This means that $(-A^{-1}a, 0)$ satisfies the optimality conditions (2.6) to (2.9) and so $\lambda_g^* = 0$. If $\lambda^* > 0$, then obviously $x(\lambda^*)$ with $\lambda_g^* = \lambda^*$ satisfy the optimality conditions.

Case 2: $(a + \ell_1 b) \in \text{Range}(A + \ell_1 B)$.

If equation $\varphi(\lambda) = 0$ has a root in the interval $(\ell_1, \hat{\lambda})$, then the proof is the same as in Case 1. Otherwise, from the proof of Theorem 4.4 and a discussion similar to Case 1, it follows that $\eta^* = \frac{1}{\ell_1 - \hat{\lambda}}$ is the smallest real eigenvalue of (4.15). Next let $\ell_1 = \hat{\lambda} + \frac{1}{\eta^*} < 0$. Then $0 \in (\ell_1, u_1)$ and $\varphi(0) < 0$. This means that $(-A^{-1}a, 0)$ satisfies the optimality conditions (2.6) to (2.9) and so $\lambda_g^* = 0$. If $\ell_1 = \hat{\lambda} + \frac{1}{\eta^*} \ge 0$, then problem (1.1) is hard case 2 with $\lambda_g^* = \ell_1$ or equivalently $\lambda_g^* = \hat{\lambda} + \frac{1}{\eta^*}$. □

THEOREM 4.9. *Suppose that $\varphi(\hat{\lambda}) < 0$ and η^* is the smallest real eigenvalue of the pencil (4.15). If $\eta = s + it$ with $s \le \eta^*$ and $t \ne 0$ is a complex eigenvalue of (4.15), then $\lambda_g^* = 0$ and $Re(\hat{\lambda} + \frac{1}{\eta}) \le 0$.*

Proof. The proof is similar to Theorem 7 of [2]. □

Finally, we remark that the whole process for computing λ_g^* is the same as in Algorithm 3.1 of [2] but applying the matrix pencil (4.15) and Lemma 4.5.

4.2.2. Computing the optimal solution. After the discussion about the optimal Lagrange multiplier λ_g^*, we turn to finding the optimal solution x_g^*. The discussion here is similar to Subsection 3.4 of [2].

Case 1: $A+\lambda_g^* B$ is nonsingular. If $\lambda_g^*=0$, then $A \succ 0$ and x_g^* is the solution of linear system $Ax_g^*=-a$. Next, let $\lambda_g^*>0$. Then $\eta^*=\frac{1}{\lambda_g^*-\hat{\lambda}}$ is an eigenvalue of the pencil (4.15) and so $\mu_g^*=\lambda_g^*-\hat{\lambda}$ is an eigenvalue of $\tilde{M}_0+\mu\tilde{M}_1$. Suppose that $v=[\theta, z_1^T, z_2^T]^T$ is the corresponding eigenvector. We have

$$\begin{bmatrix} \varphi(\hat{\lambda}) & d^T & -\mu_g^* d^T \\ d & B & -(A+\lambda_g^* B) \\ -\mu_g^* d & -(A+\lambda_g^* B) & O \end{bmatrix} \begin{bmatrix} \theta \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ O \\ O \end{bmatrix}. \tag{4.16}$$

This gives

$$\theta\varphi(\hat{\lambda}) + d^T z_1 - \mu_g^* d^T z_2 = 0, \tag{4.17}$$

$$\theta d + B z_1 - (A+\lambda_g^* B) z_2 = 0, \tag{4.18}$$

$$-\mu_g^* \theta d - (A+\lambda_g^* B) z_1 = 0. \tag{4.19}$$

We first show that $\theta \neq 0$. If $\theta = 0$, since $A+\lambda_g^* B$ is nonsingular, then it follows from (4.18) and (4.19) that $z_1 = z_2 = 0$ which is a contradiction to the fact that v is nonzero. Set $y^* = -\frac{z_1}{\theta}$. We obtain from (4.19) that

$$(A+\lambda_g^* B) y^* = \mu_g^* d. \tag{4.20}$$

It follows from (4.18) that

$$z_2 = (A+\lambda_g^* B)^{-1}(\theta d + B z_1). \tag{4.21}$$

By substituting (4.21) into (4.17) and using (4.20) we obtain

$$\varphi(\hat{\lambda}) - 2d^T y^* + {y^*}^T B y^* = 0,$$

which is (4.13). This together with (4.20) imply that $x_g^* = y^* + x(\hat{\lambda})$.

Case 2: $A+\lambda_g^* B$ is singular. This case corresponds to hard case 2. In this case, x_g^* can be computed via the approach explained in Subsection 3.4.2 of [2]. The following theorem is the basis for the construction of x_g^*.

THEOREM 4.10 ([2]). *Suppose that $A+\lambda_g^* B$ is singular. Let $v_1, ..., v_d$ be a basis for Null$(A+\lambda_g^* B)$ and let w be the solution of the linear system $\tilde{A}w = -\tilde{a}$ where*

$$\tilde{A} = A + \lambda_g^* B + \alpha \sum_{i=1}^{d} B v_i v_i^T B, \quad \tilde{a} = a + \lambda_g^* b + \alpha B \sum_{i=1}^{d} v_i v_i^T b,$$

in which $\alpha > 0$ is an arbitrary positive number. Then the following hold:

1. *$\tilde{A}$ is positive definite.*
2. *$(A+\lambda_g^* B) w = -(a + \lambda_g^* b)$.*
3. *$(Bw+b)^T v = 0$ for every $v \in$ Null$(A+\lambda_g^* B)$.*

4.3. Computing the local non-global minimizers. Here we present efficiently computing a possible local non-global minimizer of (1.1). Recall that there exists $\hat{\lambda}$ such that $A+\hat{\lambda}B \succ 0$. As discussed in Subsection 4.1, when $\hat{\lambda}$ is known, the interval (ℓ_1, u_1) can be determined efficiently. Since any value in the interval (ℓ_1, u_1) is allowed to be $\hat{\lambda}$, in what follows, without loss of generality, we further assume that $\varphi(\hat{\lambda}) \neq 0$. If $\varphi(\lambda) = 0$ for all $\lambda \in (\ell_1, u_1)$, then due to the following proposition, there are no local non-global minimizers.

PROPOSITION 4.11. *Suppose that $\varphi(\lambda) = 0$ for all $\lambda \in (\ell_1, u_1)$. Then there are no local non-global minimizers for problem (1.1).*

Proof. If $\varphi(\lambda) = 0$ on (ℓ_1, u_1), then by Lemma 4.2, $x(\lambda)$ is constant and $Ax(\lambda) = -a$ and $Bx(\lambda) = -b$ for all $\lambda \in (\ell_1, u_1)$. This also implies that $x(\lambda)$ is the global minimizer of (1.1). Next suppose by contradiction that x^* is a local non-global minimizer of (1.1) and λ^* is the corresponding Lagrange multiplier. Then $(A+\lambda^* B)x^* = -(a+\lambda^* b)$, implying that $(A+\lambda^* B)(x^* - x(\lambda)) = 0$. Since $A+\lambda^* B$ is nonsingular, then $x^* = x(\lambda)$, a contradiction. ∎

Suppose that x^* is a local non-global minimizer and λ^* is the corresponding Lagrange multiplier. Due to the results in Section 3, $\varphi(\lambda^*) = 0$, $\varphi'(\lambda^*) \geq 0$ and $\lambda^* \geq 0$ is either in (ℓ_2, ℓ_1) or in (u_1, u_2). On the other hand, by Theorem 4.4, $\mu^* = \lambda^* - \hat{\lambda}$ is an eigenvalue of the pencil $\tilde{M}_0 + \mu \tilde{M}_1$ or equivalently $\lambda^* = \hat{\lambda} + \frac{1}{\eta}$ where η is an eigenvalue of (4.15). Moreover, the real roots of the equation $\varphi(\lambda) = 0$ in the intervals (ℓ_2, ℓ_1) and (u_1, u_2), which are candidates for λ^*, correspond one-to-one to the real eigenvalues of the pencil $\tilde{M}_0 + \mu \tilde{M}_1$ in the intervals $(\ell_2 - \hat{\lambda}, \ell_1 - \hat{\lambda})$ and $(u_1 - \hat{\lambda}, u_2 - \hat{\lambda})$ via the transformation $\mu = \lambda - \hat{\lambda}$. These results suggest Algorithm 1 to compute the candidates for λ^* that uses the following two lemmas.

LEMMA 4.12. *Suppose that $\varphi(\hat{\lambda}) > 0$, u_1 has multiplicity one, $(a+u_1 b) \notin$ Range$(A+u_1 B)$, ℓ_1 has multiplicity one and $(a+\ell_1 b) \notin$ Range$(A+\ell_1 B)$.*

- *(i) Let $\lambda_2^* \leq \lambda_3^* \leq \cdots \leq \lambda_k^*$ be the real roots of the equation $\varphi(\lambda) = 0$ in the interval (u_1, u_2). Then $\lambda_i^* = \hat{\lambda} + \frac{1}{\eta_i^*}$ where η_i^* is the i'th largest real eigenvalue of (4.15) and $\eta_i^* > 0$ for all $i = 2, ..., k$.*
- *(ii) Let $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_k^*$ be the roots of the equation $\varphi(\lambda) = 0$ in the interval (ℓ_2, ℓ_1). Then $\lambda_i^* = \hat{\lambda} + \frac{1}{\eta_i^*}$ where η_i^* is the i'th smallest real eigenvalue of (4.15) and $\eta_i^* < 0$ for all $i = 2, ..., k$.*

Proof. (i) We note that $\lambda_i^*, i = 2, ..., k$ satisfy (4.10) and (4.11) and hence, $\lambda_i^* = \hat{\lambda} + \frac{1}{\eta_i^*}$ where η_i^* is an eigenvalue of (4.15). Furthermore, since $\hat{\lambda} < \lambda_i^*$ for all $i = 2, ..., k$, then $\eta_i^* > 0$ and consequently if $\lambda_i^* < \lambda_j^*$, then $\eta_i^* > \eta_j^*$. On the other hand, because $\varphi(\hat{\lambda}) > 0$ and $(a + u_1 b) \notin$ Range$(A + u_1 B)$, it follows from the proof of Theorem 4.6 that $\lambda_1^* = \hat{\lambda} + \frac{1}{\eta_1^*}$ is the unique Lagrange multiplier in the interval $(\hat{\lambda}, u_1)$ where η_1^* is the largest real eigenvalue of (4.15). Now, let $\eta_2^* > 0$ be the second largest real eigenvalue of (4.15). We know that $\hat{\lambda} + \frac{1}{\eta_2^*} \geq u_1$. To complete the proof, it is sufficient to show that $u_1 \neq \hat{\lambda} + \frac{1}{\eta_2^*}$. To this end, suppose by contradiction that $u_1 = \hat{\lambda} + \frac{1}{\eta_2^*}$. This implies that $\det(\tilde{M}_0 + \mu^* \tilde{M}_1) = 0$ for $\mu^* = u_1 - \hat{\lambda}$, which is a contradiction since $(a + u_1 b) \notin$ Range$(A + u_1 B)$ and u_1 has multiplicity one, implying $\det(\tilde{M}_0 + \mu^* \tilde{M}_1) \neq 0$.

(ii) The proof is similar to part (i) but it should be noted that in this case, there is no Lagrange multiplier in the interval $(\ell_1, \hat{\lambda})$ and hence the smallest real eigenvalue of (4.15) is strictly smaller than ℓ_1. ∎

LEMMA 4.13. *Suppose that $\varphi(\hat{\lambda}) < 0$, u_1 has multiplicity one, $(a+u_1 b) \notin$ Range$(A+u_1 B)$, ℓ_1 has multiplicity one and $(a+\ell_1 b) \notin$ Range$(A+\ell_1 B)$.*

- *(i) Let $\lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_k^*$ be the real roots of the equation $\varphi(\lambda) = 0$ in the interval (u_1, u_2), then $\lambda_i^* = \hat{\lambda} + \frac{1}{\eta_i^*}$ where η_i^* is the i'th largest real eigenvalue* *of (4.15) and* $\eta_i^* > 0$ *for all* $i = 2, ..., k$.

(ii) Let $\lambda_2^* \geq \lambda_3^* \geq \cdots \geq \lambda_k^*$ *be the real roots of the equation* $\varphi(\lambda) = 0$ *in the interval* (ℓ_2, ℓ_1)*, then* $\lambda_i^* = \hat{\lambda} + \frac{1}{\eta_i^*}$ *where* η_i^* *is the i'th smallest real eigenvalue of (4.15) and* $\eta_i^* < 0$ *for all* $i = 2, ..., k$.

Proof. The proof is similar to Lemma 4.12. □

Algorithm 4.1 Compute the candidates for λ^*, the Lagrange multiplier corresponding to a local non-global minimizer

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INPUT: $\hat{\lambda}$ with $A + \hat{\lambda}B \succ 0$ and $\varphi(\hat{\lambda}) \neq 0$

% Candidates for λ^* in (u_1, u_2)
if $u_2 \leq 0$ or $u_1 = u_2$ or $(a + u_1 b) \in \text{Range}(A + u_1 B)$ **then**
There are no candidates for λ^* in (u_1, u_2).
else if $\varphi(\hat{\lambda}) > 0$ **then**
Set $\lambda_{i-1} = \hat{\lambda} + \frac{1}{\eta_i}$ for $i = 2, 3, ...$ where η_i is i'th largest real eigenvalue of (4.15) that is in the interval $(\frac{1}{u_2 - \hat{\lambda}}, \frac{1}{u_1 - \hat{\lambda}})$. Save $\lambda_i \geq 0$ with odd subscript as candidates for λ^*.
else
Set $\lambda_i = \hat{\lambda} + \frac{1}{\eta_i}$ for $i = 1, 2, ...$ where η_i is i'th largest real eigenvalue of (4.15) that is in the interval $(\frac{1}{u_2 - \hat{\lambda}}, \frac{1}{u_1 - \hat{\lambda}})$. Save $\lambda_i \geq 0$ with odd subscript as candidates for λ^*.
end if
% Candidates for λ^* in (ℓ_2, ℓ_1)
if $\ell_1 \leq 0$ or $\ell_1 = \ell_2$ or $(a + \ell_1 b) \in \text{Range}(A + \ell_1 B)$ **then**
There are no candidates for λ^* in (ℓ_2, ℓ_1)
else if $\varphi(\hat{\lambda}) > 0$ and $\ell_2 > 0$ **then**
. Set $\lambda_i = \hat{\lambda} + \frac{1}{\eta_i}$ for $i = 1, 2, ...$ where η_i is i'th smallest real eigenvalue of (4.15) that is in the interval $(\frac{1}{\ell_1 - \hat{\lambda}}, \frac{1}{\ell_2 - \hat{\lambda}})$. Save λ_i with odd subscript as candidates for λ^*. **else if** $\varphi(\hat{\lambda}) > 0$ and $\ell_2 \leq 0$ **then** .
Set $\lambda_i = \hat{\lambda} + \frac{1}{\eta_i}$ for $i = 1, 2, ...$ where η_i is i'th smallest real eigenvalue of (4.15) that is in the interval $(\frac{1}{\ell_1 - \hat{\lambda}}, -\frac{1}{\hat{\lambda}}]$. Save λ_i with odd subscript as candidates for λ^*.
else if $\varphi(\hat{\lambda}) < 0$ and $\ell_2 > 0$ **then**
Set $\lambda_i = \hat{\lambda} + \frac{1}{\eta_i}$ for $i = 2, 3, ...$ where η_i is i'th smallest real eigenvalue of (4.15) that is in the interval $(\frac{1}{\ell_1 - \hat{\lambda}}, \frac{1}{\ell_2 - \hat{\lambda}})$. Save λ_i with odd subscript as candidates for λ^*.
else
Set $\lambda_i = \hat{\lambda} + \frac{1}{\eta_i}$ for $i = 2, 3, ...$ where η_i is i'th smallest real eigenvalue of (4.15) that is in the interval $(\frac{1}{\ell_1 - \hat{\lambda}}, -\frac{1}{\hat{\lambda}}]$. Save λ_i with odd subscript as candidates for λ^*.
end if

4.3.1. Computing x^***.** Suppose that $\lambda = \hat{\lambda} + \frac{1}{\eta}$ is the output of Algorithm 4.1 and $v = [\theta, z_1, z_2]^T$ is the eigenvector corresponding to η. Since $A + \lambda B$ is nonsingular, the same discussion as in Subsection 4.2.2 shows that $x^* = -\frac{z_1}{\theta} + x(\hat{\lambda})$ is a candidate for a local non-global minimizer of (1.1).

4.3.2. Complexity analysis. Here we show that the overall complexity of Algorithm 4.1 is at most $O(n^4)$. To this end, we discuss the computational complexity of Algorithm 4.1 in finding the candidates for λ^* in the interval (u_1, u_2), the result for the interval (ℓ_2, ℓ_1) is the same. We examine the computational costs of the following steps.

1. Computing u_1 and u_2: By Lemma 4.1, both u_1 and u_2 are computed in $O(n^3)$ time as they just require finding the smallest and the second smallest generalized eigenvalue of a matrix pencil of size $n \times n$. If $u_1 = u_2$, then the procedure for finding the candidates for λ^* in (u_1, u_2) terminates.
2. Verifying the condition $(a + u_1 b) \in \text{Range}(A + u_1 B)$: When $u_1 \neq u_2$, the multiplicity of u_1 is one and hence this step is done by checking whether $(a+u_1 b)^T v_1 = 0$ where v_1 is the eigenvector corresponding to the generalized eigenvalue u_1 which is already computed. Therefore, this step requires $O(n)$ operations, once the eigenpair (u_1, v_1) is computed.
3. Finding the real eigenvalues of the pencil (4.15) in the interval $(\frac{1}{u_2-\hat{\lambda}}, \frac{1}{u_1-\hat{\lambda}})$: To this end, we need to compute all eigenvalues of the pencil (4.15) whose real parts are in the interval $(\frac{1}{u_2-\hat{\lambda}}, \frac{1}{u_1-\hat{\lambda}})$. This can be done by the deflation procedure for the generalized eigenvalue problem as follows [31]. Assume we have computed the i'th rightmost eigenpair (η_i, v_i) of the pencil (4.15). Then the $(i+1)$'th rightmost eigenvalue of (4.15) can be computed by finding the rightmost eigenvalue of the matrix pencil $\bar{M} + \eta \tilde{M}_0$ where $\bar{M} = \tilde{M}_1 + \eta_i \tilde{M}_0 v_i d^H$ and d is an arbitrary vector such that $d^H v_i = 1$. If the number of eigenvalues of (4.15) whose real part are in the interval $(\frac{1}{u_2-\hat{\lambda}}, \frac{1}{u_1-\hat{\lambda}})$ is equal to k, then the deflation procedure terminates after k iteration, each iteration is of $O(n^3)$, which means the computational cost required in this step is of $O(kn^3)$. Since k is bounded by $2n+1$, then the computational cost required in this step is at most $O(n^4)$.
4. Computing the candidates for x^*: Once a candidate for λ^* is found, the corresponding x^* can be computed in $O(n^3)$ time via finding the corresponding eigenvector as discussed before.

5. Application. Consider problem (1.3). It includes eTRS as a special case where $B = I$, $b = 0$ and $\beta < 0$. The eTRS can be solved efficiently by the approach of [35] which exploits the characterization of local (global and non-global) minimizers of TRS based on a generalized eigenvalue problem. The paper [32] studies a more general form of problem (1.3) when both constraints are quadratic and one of them is strictly convex. Problem (1.3) with positive definite B is a special case of the latter problem and hence, in this case, the proposed algorithm in [32] (Algorithm 6.1 of [32]) can be applied to solve it. When matrices A and B are simultaneously diagonalizable and certain additional conditions are satisfied, the SOCP relaxation of [5] can be applied to solve (1.3). In this section, using the results in the previous sections, we develop an efficient algorithm based on a generalized eigenvalue problem to solve (1.3). This is an extension of the method proposed in [35] for eTRS. We assume that Assumption 2 holds and consider the following:

Assumption 3. Problem (1.3) satisfies the Slater condition, i.e., there exists $\hat{x}$ such that $g(\hat{x}) < 0$ and $c^T \hat{x} \leq \gamma$.

We show in the next lemma that the Slater condition can be assumed without loss of generality for the feasible instances of (1.3).

LEMMA 5.1. *(i) Problem (1.3) is feasible unless one of the following holds:*

(a) $B \succeq 0$, $c, b \in$ *Range*(B), $c^T(-B^\dagger b) \leq \gamma$ *and* $g(-B^\dagger b) > 0$.

(b) $B \succeq 0$, $c, b \in$ *Range*(B), $c^T(-B^\dagger b) > \gamma$ *and* $g(\hat{x} + Wy^*) > 0$.

(c) $B \succeq 0$, $b \in$ *Range*(B), $c \notin$ *Range*(B) *and* $g(-B^\dagger b) > 0$.

(d) $B \succeq 0$, $b \notin$ *Range*(B), $c^T d > 0$ *for all* $d \in$ *Null*(B) *satisfying* $b^T d < 0$ *and* $g(\hat{x} + Wy^*) > 0$,

where $y^* = -(W^T BW)^\dagger W^T(B\hat{x} + b)$, W *is a basis for Null*(c^T) *and* $\hat{x}$ *is a point satisfying* $c^T x = \gamma$. *Moreover, for a feasible instance of (1.3), the Slater condition holds unless (a) and (c) hold with* $g(-B^\dagger b) = 0$, *and (b) and (d) hold with* $g(\hat{x} + Wy^*) = 0$.

(ii) If the Slater condition fails for a feasible instance of (1.3), then problem (1.3) reduces to an unconstrained minimization problem.

Proof. (i) First let $B \not\succeq 0$. Then there exists $x \in \mathbb{R}^n$ such that $x^T Bx < 0$ and $c^T x \neq 0$. This implies that there also exists $\alpha \in \mathbb{R}$ such that $g(\alpha x) < 0$ and $c^T(\alpha x) < \gamma$, showing that problem (1.3) satisfies the Slater condition. Next suppose that $B \succeq 0$ and consider the following problem:

$$p^* := \inf \quad g(x) = x^T Bx + 2b^T x + \beta \qquad c^T x \leq \gamma. \tag{5.1}$$

Let us discuss the following possible cases:

Case 1. $B \succeq 0$ and $b, c \in \mathrm{Range}(B)$.

Note that, in this case, the infimum of problem (5.1) is attainable. Therefore, problem (1.3) is feasible if and only if $p^* \leq 0$ and satisfies the Slater condition if and only if $p^* < 0$. First suppose that B is singular. Let x^* be an optimal solution of (5.1). Then x^* is either an unconstrained minimizer of $g(x)$ or an optimal solution of the following problem:

$$\min \quad g(x) = x^T Bx + 2b^T x + \beta \qquad c^T x = \gamma. \tag{5.2}$$

Consider the set of unconstrained minimizers of $g(x)$, $S = \{x_c | x_c = -B^\dagger b + Zy,\ y \in \mathbb{R}^r\}$ where $Z \in \mathbb{R}^{n\times r}$ is a basis for $\mathrm{Null}(B)$. Since $c \in \mathrm{Range}(B)$, then $c^T x_c = c^T(-B^\dagger b)$ for all $x_c \in S$. If $c^T(-B^\dagger b) \leq \gamma$, then $p^* = g(-B^\dagger b)$. Therefore, in this case, problem (1.3) is feasible unless $g(-B^\dagger b) > 0$. Furthermore, for a feasible one, the Slater condition holds unless $g(-B^\dagger b) = 0$. If $c^T(-B^\dagger b) > \gamma$, then x^* is an optimal solution of (5.2). Let $W \in \mathbb{R}^{n\times(n-1)}$ be a basis of $\mathrm{Null}(c^T)$ and $\hat{x}$ be a feasible solution of (5.2). Via the transformation $x = \hat{x} + Wy$, problem (5.2) is equivalent to the following unconstrained minimization:

$$\min \quad y^T W^T BWy + 2(W^T B\hat{x} + W^T b)^T y + g(\hat{x}). \tag{5.3}$$

An optimal solution of (5.3) is $y^* = -(W^T BW)^\dagger(W^T(B\hat{x} + b))$ and consequently, $p^* = g(\hat{x} + Wy^*)$. Therefore, in this case, problem (1.3) is feasible unless $g(\hat{x} + Wy^*) > 0$. Furthermore, for a feasible one, the Slater condition holds unless $g(\hat{x} + Wy^*) = 0$. When B is nonsingular, $S = \{-B^{-1}b\}$ and we obtain the same results following the above discussion.

Case 2. $B \succeq 0$, $b \in \mathrm{Range}(B)$ and $c \notin \mathrm{Range}(B)$.

Since $c \notin \mathrm{Range}(B)$, then there exists $x_c \in S$ such that $c^T x_c \leq \gamma$, implying

that $x^* = -B^\dagger b$ is an optimal solution of (5.1). Therefore, in this case, problem (1.3) is feasible unless $g(-B^\dagger b) > 0$. Moreover, for a feasible one, the Slater condition holds unless $g(-B^\dagger b) = 0$.
Case 3. $B \succeq 0$, $b \notin \text{Range}(B)$.
Since $b \notin \text{Range}(B)$, then there exists $d \in \text{Null}(B)$ such that $b^T d < 0$. If there also exists $d \in \text{Null}(B)$ such that $b^T d < 0$ and $c^T d \leq 0$, then problem (5.1) is unbounded below. To see this, take $\hat{x}$ with $c^T \hat{x} < \gamma$. Then $\hat{x} + \alpha d$ is feasible for (5.1) for all $\alpha > 0$. Moreover, $\lim_{\alpha \to \infty} g(\hat{x} + \alpha d) = -\infty$. This implies that problem (5.1) is unbounded below and consequently problem (1.3) satisfies the Slater condition. Next, suppose that for all $d \in \text{Null}(B)$ satisfying $b^T d < 0$, we have $c^T d > 0$. In this case, problem (5.1) is bounded and reduces to (5.2). The same discussion as in Case 1 shows that $p^* = g(\hat{x} + Wy^*)$ where y^*, $\hat{x}$ and W are defined as before. Therefore, in this case, problem (1.3) is feasible unless $g(\hat{x} + Wy^*) > 0$. Furthermore, for a feasible one, the Slater condition holds unless $g(\hat{x} + Wy^*) = 0$.

(ii) By part (i), when the Slater condition fails for a feasible instance of (1.3), one of the following cases hold:
Case 1. (a) holds with $g(-B^\dagger b) = 0$.
First suppose that B is singular and set S is defined as in part (i). It follows from the discussion in Case 1 of part (i) that, in this case, the feasible region of (1.3) is the set S and hence problem (1.3) is equivalent to

$$\begin{aligned} \min \quad & x^T A x + 2a^T x \\ & x \in S. \end{aligned}$$

It is easy to see that this is also equivalent to an unconstrained minimization problem. When B is nonsingular, the set S is singleton and so problem (1.3) has the unique feasible (and so optimal) point $x^* = -B^{-1} b$.
Case 2. (b) or (d) holds with $g(\hat{x} + Wy^*) = 0$, where $\hat{x}$, W and y^* are defined as in part (i).
First suppose that B is singular. It follows from the discussion in Cases 1 and 3 of part (i) that, in this case, problem (1.3) reduces to the following problem:

$$\begin{aligned} \min \quad & x^T A x + 2a^T x \\ & x = \hat{x} + Wy, \\ & y \in \hat{S}, \end{aligned}$$

where $\hat{S} = \{y^* + Vh | h \in \mathbb{R}^r\}$ and $V \in \mathbb{R}^{(n-1)\times r}$ is a basis for $\text{Null}(W^T B W)$. Now it is easy to see that this problem is equivalent to an unconstrained minimization problem. When B is nonsingular, the set $\hat{S}$ is singleton and so problem (1.3) has the unique feasible (and so optimal) point $x^* = \hat{x} - W(W^T B W)^{-1}(W^T(B\hat{x} + b))$.
Case 3. (c) holds with $g(-B^\dagger b) = 0$.
We obtain from the discussion in Case 2 of part (i) that, in this case, problem

(1.3) is equivalent to the following problem:

$$\begin{aligned} \min \quad & x^T Ax + 2a^T x \\ & x \in S, \\ & c^T x \le \gamma. \end{aligned}$$

This is also equivalent to the following problem:

$$\begin{aligned} \min \quad & y^T Z^T AZy + 2(-Z^T AB^\dagger b + Z^T a)^T y + q(-B^\dagger b) \\ & c^T Zy \le \gamma + B^\dagger b. \end{aligned} \tag{5.4}$$

Problem (5.4) is of the form (5.1) which can be easily treated as done in part (i). It is either unbounded below or can be solved via solving at most two unconstrained minimization problems. ∎

According to Subsection 4.2, under Assumption 2, problem (1.1) is unbounded below if and only if $u_1 < 0$ or $u_1 = 0$ and $a \notin \text{Range}(A)$. The following proposition shows that in these cases, problem (1.3) may be also unbounded.

Proposition 5.2. *Suppose that Assumption 2 holds. If $u_1 < 0$, then problem (1.3) is unbounded below. If $u_1 = 0$ and $a \notin$ Range(A), then problem (1.3) is unbounded if and only if there exists $d \in$ Null(A) such that $a^T d < 0$ and $c^T d \le 0$.*

Proof. First let $u_1 < 0$. Then it follows from the definition of u_1 that there exists i such that $\alpha_i < 0$ and $\beta_i < 0$. This means that $A \not\succeq 0$ and $B \not\succeq 0$. Since $A \not\succeq 0$, then there exists x such that $x^T Ax < 0$ and $c^T x < 0$. On the other hand, since $u_1 < 0$, then Assumption 2 holds with $\hat{\lambda} < 0$, implying that $x^T Bx < 0$. We see that αx will become feasible for (1.3) if α is increased to infinity. This together with $\lim_{\alpha\to\infty} q(\alpha x) = -\infty$ prove that problem (1.3) is unbounded below. Next let $u_1 = 0$ and $a \notin \text{Range}(A)$, then $A \succeq 0$. Now it is easy to verify that when problem (1.3) is unbounded below, the following system must have a solution:

$$\begin{aligned} & Ad = 0, \\ & a^T d < 0, \\ & c^T d \le 0. \end{aligned} \tag{5.5}$$

To prove the reverse, let d be a solution of (5.5). Since $u_1 = 0$, then Assumption 2 holds with $\hat{\lambda} < 0$, implying that $d^T Bd < 0$. Take $\hat{x}$ with $c^T \hat{x} < \gamma$. Then $\hat{x} + \alpha d$ will become feasible for (1.3) if α is increased to infinity. This together with $\lim_{\alpha\to\infty} q(\hat{x} + \alpha d) = -\infty$ prove that problem (1.3) is unbounded below. ∎

Based on Proposition 5.2, when $u_1 = 0$ and $a \notin \text{Range}(A)$, problem (1.3) is unbounded if and only if system (5.5) has a solution. Algorithm 5.1 declares whether such a vector exists or not.

Now we turn to solve problem (1.3) under Assumptions 2 and 3. Let x^* be a global optimal solution of (1.3), then either $c^T x^* < \gamma$ or $c^T x^* = \gamma$. In the former case, it is known that x^* must be a local (not necessarily global) minimizer of (1.1). This fact suggests Algorithm 5.2 for solving (1.3).

In what follows, we discuss the steps of Algorithm 5.2.

Step 1: When problem (1.1) is bounded, an optimal solution can be efficiently computed by the approach discussed in Subsection 4.2. The optimal solution of problem (1.1) may not be unique when hard case 2 occurs. In this case, either $\lambda_g^* = \ell_1$ or

Algorithm 5.1 Check the feasibility of (5.5).

OUTPUT: $flag = 1$ if the system (5.5) has a solution otherwise $flag = 0$.

Step 1. Find a basis $V = \{v_1, v_2, ..., v_k\}$ for Null(A) such that $a^T v_i \le 0$ for all $i = 1, ..., k$.
Step 2. If there exists $v_i \in V$ such that $a^T v_i < 0$ and $c^T v_i \le 0$, then set $flag = 1$ and stop. Otherwise, go to Step 3.
Step 3. Set $M = [s_1\ s_2]^T$ where $s_1 = [a^T v_1\ a^T v_2\ ...\ a^T v_k]^T$ and $s_2 = [c^T v_1\ c^T v_2\ ...\ c^T v_k]^T$. If Rank$(M) = 1$, then set $flag = 0$ otherwise set $flag = 1$.

Algorithm 5.2 Solve problem (1.3)

Step 1. If $u_1 < 0$ or if $u_1 = 0$, $a \notin$ Range(A) and system (5.5) has a solution, then stop, problem (1.3) is unbounded below. Otherwise, solve problem (1.1) if it is bounded. If there is an optimal solution satisfying $c^T x \le \gamma$, then it is also optimal for (1.3) and stop. Otherwise go to Step 2.
Step 2. Find the candidates for local non-global minimizers of (1.1) which are feasible for $c^T x \le \gamma$.
Step 3. Solve problem (1.3) with the equality constraint $c^T x = \gamma$.
Step 4. The optimal solution of (1.3) is either one of the solutions in Step 2 or the solution in Step 3 whichever gives the smaller objective value.

$\lambda_g^* = u_1$. Let $V := [v_1, ..., v_d]$ be a basis for Null$(A + \lambda_g^* B)$. According to Theorem 4.10, the set of optimal solutions is $S = \{w + Vy | g(w + Vy) = 0, y \in \mathbb{R}^d\}$ where w is the same as in Theorem 4.10. Therefore, to see whether there exists an optimal solution of (1.1) that is feasible for $c^T x \le \gamma$, it is sufficient to solve the following problem:

$$\begin{aligned} \min \quad & c^T V y + c^T w \\ & y^T V^T B V y = -g(w). \end{aligned} \tag{5.6}$$

If $\lambda_g^* = u_1$, then $V^T BV \prec 0$ and if $\lambda_g^* = \ell_1$, then $V^T BV \succ 0$ (see Appendix, Proposition 7). Therefore, in both cases, problem (5.6) is a TRS which can be solved efficiently by the approach of [1].
Step 2: This step can be done efficiently by Algorithm 4.1.
Step 3: Consider the following problem:

$$\begin{aligned} \min \quad & x^T A x + 2a^T x \\ & x^T B x + 2b^T x + \beta \le 0, \\ & c^T x = \gamma. \end{aligned} \tag{5.7}$$

We can transform problem (5.7) to a problem of the form (1.1) by eliminating the equality constraint $c^T x = \gamma$. Let $|c_r| = \max\{|c_i| | i = 1, ..., n\}$. Define $\bar{c} = [\bar{c}_1^T, \bar{c}_2^T]^T$ where $\bar{c}_1 = [c_1, ..., c_{r-1}]^T$ and $\bar{c}_2 = [c_{r+1}, ..., c_n]^T$. Then the following matrix is a basis of Null(c^T):

$$W = \begin{bmatrix} c_r I_{r-1} & O_{(r-1)\times(n-r)} \\ -\bar{c}_1^T & -\bar{c}_2^T \\ O_{(n-r)\times(r-1)} & c_r I_{n-r} \end{bmatrix} \in \mathbb{R}^{n\times(n-1)}.$$

Choose $\hat{x}$ satisfying $c^T x = \gamma$ as

$$\hat{x} = \begin{cases} 0 & if \quad \gamma = 0, \\ \frac{\gamma}{||c||^2} c & if \quad \gamma \neq 0. \end{cases}$$

Then it is clear that

$$c^T x = \gamma \Leftrightarrow x = \hat{x} + Wy \quad \text{for some} \quad y \in \mathbb{R}^{n-1}.$$

Now after substituting x into (5.7), we obtain the following equivalent problem:

$$\begin{aligned} \min \quad & y^T \tilde{A} y + 2\tilde{a}^T y + q(\hat{x}) \\ & y^T \tilde{B} y + 2\tilde{b}^T y + \tilde{\beta} \leq 0, \end{aligned} \tag{5.8}$$

where $\tilde{A} = W^T A W$, $\tilde{a} = W^T(A\hat{x} + a)$, $\tilde{B} = W^T B W$, $\tilde{b} = W^T(B\hat{x} + b)$ and $\tilde{\beta} = g(\hat{x})$. This is a problem of the form (1.1) and so can be solved efficiently as discussed in Subsection 4.2.

6. Numerical experiments. In this section, we present numerical experiments to show the effectiveness of Algorithm 4.1 in finding candidates for the local non-global minimizers of problem (1.1) and Algorithm 5.2 to solve problem (1.3) in comparison with the SOCP relaxation of [5] and the Algorithm 6.1 from [32]. All experiments are performed in MATLAB R2016a on a machine with 3.3 GHz CPU and 8 GB of RAM. Moreover, all the generalized eigenvalue problems in Algorithms 4.1 and 5.2 are solved by the `eigs` command in MATLAB. We also use the MATLAB command `A=sprandsym(n,density)` and `B = sprandsym(n,density,0.1,1)` to generate sparse indefinite matrix A and positive definite matrix B, respectively. Throughout the paper, we have assumed that there exists $\hat{\lambda}$ such that $A + \hat{\lambda}B \succ 0$. In practice $\hat{\lambda}$ is usually unknown in advance, and in that case our algorithm starts by computing $\hat{\lambda}$. To this end, the algorithms in [9, 16, 25] can be applied to find $\hat{\lambda}$. Since we aim to asses the effectiveness of Algorithm 5.2 in finding a global optimal solution of (1.3), we assume that $\hat{\lambda}$ is known and skip its computation. The following lemma is useful in generating test problems.

LEMMA 6.1. *Let $A, B \in \mathbb{S}^n$, u_1, u_2, ℓ_1 and ℓ_2 be defined as before. Moreover, suppose that Assumption 2 holds.*

(i) If u_1 has multiplicity one, $(a + u_1 b) \notin$ Range$(A + u_1 B)$ and $u_2 > 0$, then there exist $a, b \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ for which the eigenvector v associated with u_1 is the local non-global minimizer of (1.1) and the corresponding Lagrange multiplier is in the interval (u_1, u_2). Moreover, if $u_1 \geq 0$, for this problem, $-v$ is the global optimal solution.

(ii) If $\ell_1 > 0$ has multiplicity one, $(a + \ell_1 b) \notin$ Range$(A + \ell_1 B)$, then there exist $a, b \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ for which the eigenvector v associated with ℓ_1 is the local non-global minimizer of (1.1) and the corresponding Lagrange multiplier is in the interval (ℓ_2, ℓ_1). Moreover, for this problem, $-v$ is the global optimal solution.

Proof. (i) The proof is constructive. Let $\mu \in (u_1, u_1 + \min\{u_1, u_1 - \ell_1, u_2 - u_1\})$ if $u_1 \geq 0$ else $\mu \in (\max\{0, u_1\}, u_2)$. Moreover, suppose that $D = \text{diag}(\alpha_1, \alpha_2, ..., \alpha_n)$, $E = \text{diag}(\beta_1, \beta_2, ..., \beta_n)$ and C are defined as in Section 3. Without loss of generality, we may assume that $u_1 = -\frac{\alpha_1}{\beta_1}$. Let v be an eigenvector associated with u_1. Set $b = 0$, $a = -(A + \mu B)v$ and $\beta = -v^T B v$.

Since $g(v) = 0$, to prove that v is a local non-global minimizer of (1.1), it is sufficient to show that $w^T(A + \mu B)w > 0$ for all $w \neq 0$ such that $w^T Bv = 0$. First note that since v is an eigenvector corresponding to u_1, then $(A + u_1 B)v = 0$. This implies that $(D + u_1 E)C^{-1}v = 0$ and so $C^{-1}v = \gamma e_1$ for some $\gamma \in \mathbb{R}$ where e_1 is the unit vector. This together with the facts that $\beta_1 \neq 0$ and $w^T Bv = w^T C^{-T} E C^{-1} v = 0$ imply that the first element of the vector $C^{-1}w$ is zero. Therefore, $w^T(A+\mu B)w = w^T C^{-T}(D+\mu E)C^{-1}w > 0$ which completes the proof. Next, suppose that $u_1 \geq 0$. Set $\lambda_g^* = 2u_1 - \mu$. Then $\lambda_g^* > 0$, $\ell_1 < \lambda_g^* < u_1$ and so $A + \lambda_g^* B \succeq 0$. Furthermore, $x^* = -v$ with λ_g^* satisfy the optimality conditions (2.6) to (2.8) which proves that $-v$ is the global optimal solution:

$$g(x^*) = 0,$$
$$(A + \lambda_g^* B)x^* + a = -(A + (2u_1 - \mu)B)v - (A + \mu B)v = -2(A + u_1 B)v = 0.$$

(ii) The proof is the same as part (i) but we let $\mu \in (\ell_1 - \min\{\ell_1, u_1 - \ell_1, \ell_1 - \ell_2\}, \ell_1)$ and set $\lambda_g^* = 2\ell_1 - \mu$. □

6.1. Computing the local non-global minimizers of problem (1.1). Here we generate some random instances of problem (1.1) having a local non-global minimizer based on Lemma 6.1. Then we apply Algorithm 4.1 to find their local non-global minimizers. For each dimension, we generate 10 instances and the corresponding numerical results are adjusted in Tables 1 and 2 where we report the dimension of problem (n), the run time in second (Time), $|g(x^*)|$ denoted by "KKT1", the first order necessary condition (2.3) denoted by "KKT2" and $||x^* - x_{exact}||_\infty$ averaged over the 10 random instances where x^* and x_{exact} are the computed local non-global minimizer and the exact local non-global minimizer from Lemma 6.1, respectively. We see from Tables 1 and 2 that Algorithm 4.1 is always successful in finding the local non-global minimizers.

TABLE 1
Computational results for positive definite A and indefinite B with density=0.01

n	Time(s)	KKT1	KKT2	$\|\|x^* - x_{exact}\|\|_\infty$
1000	0.98	1.1612e-12	7.2717e-17	5.9776e-13
2000	6.97	3.5720e-12	2.2258e-16	1.3611e-12
3000	26.84	4.8516e-12	1.7870e-16	1.9262e-12
4000	60.83	6.4981e-12	1.0310e-16	2.4420e-12
5000	122.22	2.2809e-11	1.6961e-16	6.5016e-12

TABLE 2
Computational results for indefinite A and indefinite B with density=0.01

n	Time(s)	KKT1	KKT2	$\|\|x^* - x_{exact}\|\|_\infty$
1000	1.04	4.5405e-13	6.0552e-16	2.2527e-12
2000	9.33	6.4067e-14	2.1506e-15	3.0130e-13
3000	26.58	1.0775e-13	1.5397e-15	4.7618e-13
4000	63.29	4.8446e-13	1.9081e-15	1.7986e-12
5000	127.13	4.7556e-13	3.0353e-15	1.5506e-12

6.2. Comparison of Algorithm 5.2 with Algorithm 6.1 of [32]. Here we generate some random instances of problem (1.3) for which a local non-global minimizer of (1.1) is a good candidate for the global optimal solution. To this end, we first construct problem (1.1) having a local non-global minimizer based on Lemma 6.1. Then we set $c = x_g^* - x_\ell$ and $\gamma = c^T(0.9x_\ell + 0.1x_g^*)$ to cut off x_g^* but leaves x_ℓ feasible where x_g^* and x_ℓ are the global optimal solution and the local non-global minimizer of problem (1.1), respectively. To compare with Algorithm 6.1 of [32], we consider problem (1.3) with indefinite A and positive definite B. Since B is positive definite, we can initialize Algorithm 5.2 with $\hat{\lambda} = \ell_1 + 2$. The comparison with Algorithm 6.1 of [32] is done for dimension up to 25 as it needs longer time to solve larger dimensions. For each dimension, we generate 10 instances and the corresponding numerical results are adjusted in Table 3 where we report the dimension of problem (n), run time of Algorithm 5.2 in second (TimeAlg5.2), run time of Algorithm 6.1 of [32] in second (TimeAlg6.1) and $|f_1^* - f_2^*|$ averaged over the 10 random instances where f_1^* and f_2^* are the optimal objective value obtained by Algorithm 5.2 and Algorithm 6.1, respectively. Table 3 shows that Algorithm 5.2 is much faster than Algorithm 6.1 of [32] as expected since Algorithm 6.1 requires $O(n^6)$ time while Algorithm 5.2 requires at most $O(n^4)$.

Table 3
Comparison with Algorithm 6.1 of [32]

n	TimeAlg5.2	TimeAlg6.1	$\lvert f_1^* - f_2^* \rvert$
10	0.06	0.62	4.0430e-13
15	0.04	6.40	6.0307e-13
20	0.04	33.09	7.4607e-13
25	0.04	128.05	1.4101e-12

Now we turn to solve large sparse instances of problem (1.3). To this end, we consider problem (1.3) with indefinite B. For this class of test problems, we just report the results of Algorithm 5.2 since there are currently no algorithms in the literature specialized for solving such instances of (1.3). To generate the desirable random instances, we follow the same procedure as above and consider the following two cases:

(i) A **is positive definite and** B **is indefinite.**
In this case, we set $\hat{\lambda} = 0$.

(ii) **Both** A **and** B **are indefinite.**
In this case, we first generate randomly a sparse positive definite matrix C and a sparse indefinite matrix B. Next, we set $A = C - B$. In this case, obviously, we can set $\hat{\lambda} = 1$.

For each dimension, we generate 10 instances and the corresponding numerical results are adjusted in Tables 4 and 5. Let x^* be a global optimal solution of problem (1.3). Then x^* is either the local no-global minimizer of (1.1) or is obtained from the optimal solution of (5.7). When x^* is the local non-global minimizer of (1.1), we use "KKT1" and "KKT2" to denote $\lambda^* g(x^*)$ and $||(A + \lambda^* B)x^* + a + \lambda^* b||_\infty$, respectively, where λ^* is the corresponding Lagrange multiplier. If $x^* = \hat{x} + Wy^*$ where y^* is a global optimal solution of (5.7), then "KKT1" and "KKT2" denote $\lambda^*({y^*}^T \bar{B} y^* + 2\bar{b}^T y^* + \bar{\beta})$ and $||(\bar{A} + \lambda^* \bar{B})y^* + \bar{a} + \lambda^* \bar{b}||_\infty$, respectively, where λ^* is the corresponding Lagrange multiplier.

TABLE 4
Computational results of Algorithm 5.2 for positive definite A and indefinite B with density=0.01

n	Time(s)	KKT1	KKT2
1000	2.64	7.1945e-13	4.0853e-17
2000	15.06	1.3647e-11	4.9868e-17
3000	43.40	-7.2271e-12	4.4635e-17
4000	101.20	1.2634e-11	7.2544e-17
5000	212.72	1.8033e-11	5.8602e-17

TABLE 5
Computational results of Algorithm 5.2 for indefinite A and indefinite B with density=0.01

n	Time (s)	KKT1	KKT2
1000	3.50	-4.0224e-13	2.0730e-16
2000	16.01	-8.7736e-13	1.7625e-16
3000	44.29	4.3445e-10	2.0304e-16
4000	99.46	4.9663e-11	1.5112e-16
5000	200.39	4.8836e-11	1.8334e-16

6.3. Comparison of Algorithm 5.2 with the SOCP relaxation from [5]. In this subsection, we compare Algorithm 5.2 with the SOCP relaxation in [5] on some randomly generated instances of problem (1.3) showing that the SOCP relaxation is not exact in general. To this end, we consider problem (1.3) with positive definite A and indefinite B. In this case, A and B are simultaneously diagonalizable and hence we can construct the SOCP relaxation for problem (1.3). To generate the desirable random instances, we follow the same procedure as in the previous subsection but we set $\gamma = c^T(0.2x_\ell + 0.8x_g^*)$. We initialize Algorithm 5.2 with $\hat{\lambda} = 0$ and to solve the SOCP relaxation we use CVX 2.1 [15]. For each dimension, we generate 10 instances and the corresponding numerical results are adjusted in Table 6 where we report the dimension of problem (n), algorithm run time in second (Time), feasibility of x^* including $|{x^*}^T Bx^* + 2b^T x^* + \beta|$ denoted by "Feas1" and $|c^T x^* - \gamma|$ denoted by "Feas2" averaged over the 10 random instances where x^* is the computed solution by each method. Moreover, we use "ndif" to denote the number of test problems among the 10 instances for which $f_1^* - f_2^* < -0.01$ where f_1^* and f_2^* are the optimal objective value obtained by Algorithm 5.2 and the SOCP relaxation, respectively. The "ndif" column of Table 6 shows that the SOCP relaxation is not exact for this class of test problems as Algorithm 5.2 always obtained a better minimum.

TABLE 6
Comparison with the SOCP relaxation for density=0.01

	Algorithm 5.2			SOCP relaxation			
n	Time	Feas1	Feas2	Time	Feas1	Feas2	ndif
1000	4.15	1.1546e-14	3.1086e-15	7.02	6.6827e-12	3.1086e-15	10
2000	15.76	1.0658e-14	3.5527e-15	18.47	7.5584e-12	3.5527e-15	10
3000	48.69	1.0747e-13	8.8818e-15	38.05	1.9032e-11	8.8818e-15	8
4000	101.99	1.2675e-10	9.7700e-15	101.32	2.0391e-11	9.7700e-15	7
5000	194.26	2.1842e-10	8.6597e-15	182.18	2.9830e-11	8.6597e-15	8

7. Conclusions. In this paper, we have considered the problem of minimizing a general quadratic function subject to one general quadratic constraint. Under a regularity condition, we gave a characterization of the local non-global minimizers which has been used to derive a new necessary condition for the local non-global minimizers that is based on the real generalized eigenvalues of a matrix pencil in certain intervals. This is then used to derive an efficient algorithm for finding the global minimizer of the problem with an additional linear inequality constraint. The effectiveness of the proposed method has been demonstrated by numerical experiments on several randomly generated test problems.

It is worth to mention that most of the discussions hold for characterization of the local non-global minimizers of problem (1.1) with equality constraint:

$$\begin{aligned} \min \quad & q(x) := x^T A x + 2a^T x \\ & g(x) := x^T B x + 2b^T x + \beta = 0. \end{aligned} \tag{7.1}$$

Suppose that Assumption 2 holds, $g(x)$ takes positive and negative values and $B \neq 0$. Moreover, let x^* be a local non-global minimizer of problem (7.1) and λ^* be the corresponding Lagrange multiplier. If x^* is a regular point, i.e., $Bx^* + b \neq 0$, then the whole discussions in Sections 3 and 4 hold for x^* without the nonnegativity requirement of λ^*.

One of the future research direction would be to investigate how many local non-global minimizers problems (1.1) and (7.1) may have. It is also interesting to extend the results for characterization of the local non-global minimizers of a quadratic optimization problem with several quadratic constraints.

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Appendix.

PROPOSITION 8.1. *Suppose that Assumption 2 holds. Then (λ, v) is an eigenpair of the matrix pencil $(A, -B)$ if and only if $(-\frac{1}{\lambda-\hat{\lambda}}, (A+\hat{\lambda}B)^{\frac{1}{2}}v)$ is an eigenpair of matrix $(A+\hat{\lambda}B)^{-\frac{1}{2}}B(A+\hat{\lambda}B)^{-\frac{1}{2}}$.*

Proof. (λ, v) is an eigenpair of the pencil $(A, -B)$ if and only if $(A+\lambda B)v = 0$, implying that $(A + \hat{\lambda}B + (\lambda - \hat{\lambda})B)v = 0$. Since $(A + \hat{\lambda}B) \succ 0$, it follows that $(I + (\lambda - \hat{\lambda})(A+\hat{\lambda}B)^{-\frac{1}{2}}B(A+\hat{\lambda}B)^{-\frac{1}{2}})(A+\hat{\lambda}B)^{\frac{1}{2}}v = 0$. Note that $\lambda \neq \hat{\lambda}$ because $\det(A+\hat{\lambda}B) \neq 0$. This implies that $((A+\hat{\lambda}B)^{-\frac{1}{2}}B(A+\hat{\lambda}B)^{-\frac{1}{2}} + \frac{1}{\lambda-\hat{\lambda}}I)(A+\hat{\lambda}B)^{\frac{1}{2}}v = 0$ which completes the proof. ∎

PROPOSITION 8.2. *Suppose that $\lambda \neq \hat{\lambda}$. Then $(a+\lambda b) \in Range(A+\lambda B)$ if and only if $(-b + B(A+\hat{\lambda}B)^{-1}(a+\hat{\lambda}b)) \in Range(A+\lambda B)$.*

Proof. $(a+\lambda b) \in \text{Range}(A+\lambda B)$ if and only if there exists $\bar{x}$ such that $(A+\lambda B)\bar{x} = -(a+\lambda b)$. This implies that

$$[(A+\hat{\lambda}B) + (\lambda-\hat{\lambda})B]\bar{x} = -[(a+\hat{\lambda}b) + (\lambda-\hat{\lambda})b].$$

Next, using the change of variable $\bar{x} = \bar{y} - (A+\hat{\lambda}B)^{-1}(a+\hat{\lambda}b)$ gives

$$(A+\lambda b)\bar{y} = (\lambda-\hat{\lambda})(-b + B(A+\hat{\lambda}B)^{-1}(a+\hat{\lambda}b)), \tag{8.1}$$

implying that $(-b + B(A+\hat{\lambda}B)^{-1}(a+\hat{\lambda}b)) \in \text{Range}(A+\lambda B)$. ∎

PROPOSITION 8.3 ([25]). *If u_1 is finite, then $v^T Bv < 0$ for all nonzero $v \in$ Null$(A + u_1 B)$. If ℓ_1 is finite, then $v^T Bv > 0$ for all nonzero $v \in$ Null$(A + \ell_1 B)$.*

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