Multi-step discrete-time Zhang neural networks with application to time-varying nonlinear optimization

Min Sun[1,3], Maoying Tian[2], Yiju Wang[3]

1. School of Mathematics and Statistics, Zaozhuang University, Shandong, 277160, P. R. China, E-mail: ziyouxiaodou@163.com

2. Department of Physiology, Shandong Coal Mining Health School, Shandong, 277011, P. R. China, E-mail: hfmaoying@163.com

3. School of Management, Qufu Normal University, Shandong, 276826, P. R. China, E-mail: wangyiju@163.com

Abstract. As a special kind of recurrent neural networks, Zhang neural network (ZNN) has been successfully applied to various time-variant problems solving. In this paper, we first propose a special two-step Zhang et al. discretization (ZeaD) formula and a general two-step ZeaD formula, whose truncation errors are $O(\tau^3)$ and $O(\tau^2)$, respectively, and $\tau > 0$ denotes the sampling gap. We also propose a general five-step ZeaD formula with truncation error $O(\tau^5)$, and prove that the special and general two-step ZdaD formulas is convergent but the general five-step ZeaD formula is not zero-stable, thus is not convergent. Then, to solve the time-varying nonlinear optimization (TVNO) in real time, based on the Taylor series expansion and the above two convergent two-step ZeaD formulas, we discrete the continuous-time ZNN (CTZNN) model of TVNO proposed in the literature, and thus get a special two-step discrete-time ZNN (DTZNN) model and a general two-step DTZNN model, which contains a free parameter $a_1 \in (-1/2, +\infty)$. Theoretical analyses indicate that the sequence generated by the first DTZNN model is not convergent, and for any $a_1 \in (-1/2, +\infty)$ and the step-size $h \in (0, (2+4a_1)/(1+a_1))$, the sequence generated by the second DTZNN model converges to zero in an $\mathcal{O}(\tau^2)$ manner, where $\mathcal{O}(\tau^2)$ denotes a vector with every entries being $O(\tau^2)$. Furthermore, we prove that for any fixed $a_1 \in (-1/2, +\infty)$, the constant $(2+4a_1)/(1+a_1)$ is the tight upper bound of the step-size h and the constant $(1 + 2a_1)/(1 + a_1)$ is the optimal step-size. Finally, some numerical results and comparisons are provided and analyzed to substantiate the efficacy of the proposed DTZNN models.

Keywords: Time-varying nonlinear optimization; Zhang et al. discretization; discrete-time Zhang neural network.

1 Introduction

As a subcase of nonlinear programming, nonlinear optimization has been widely encountered in a variety of scientific and engineering applications, and many applications can be modeled or reformulated as nonlinear optimization, e.g., the Markowitz mean-variance model in finance, the transportation problem in management, the shortest path in network model, and the (non)convex separable optimization in image denoising etc [1, 2]. Due to its fundamental roles, nonlinear optimization has been extensively studied by many researchers during the last several centuries, and many efficient algorithms have been developed and investigated in the literature to solve it [3, 4, 5], which can be classified into two categories. The first category includes the first-order iteration methods, which only use the first derivative information of the objective function, such as the steepest descent method, the conjugate gradient method and the memory gradient method, etc. These methods are suitable to solve large-scale nonlinear optimization due to their simple structure and low storage. Numerically the conjugate gradient method and the memory gradient method usually performs better than the steepest descent method [6]. The second category includes the Newton method and its variants, e.g., the quasi-Newton BFGS and DFP methods, which need to compute the second derivative of the objective function or an approximation of it. Therefore they are not suitable to solve large scale nonlinear optimization though they possess locally fast convergent rate. Then, to overcome the drawback of the quasi-Newton method, Nocedal [7] designed a limited memory BFGS method (L-BFGS) for nonlinear optimization, and numerical results indicate that the L-BFGS method is very efficient due to its low storage. In addition, some neural networks have been developed to solve nonlinear optimization during the last decades [8, 9, 11].

Most of the above algorithms are designed intrinsically for solving static nonlinear optimization, therefore they might not be effective enough for solving time-varying nonlinear optimization (TVNO), whose objective function, denoted by $f(x(t), t)$, is a multivariate function with respect to the decision variable x and the time variable t. Obviously, at each single time instant, TVNO can be viewed as a static nonlinear optimization, thus can be solved by the above mentioned algorithms. However this treatment is not advisable due to its inefficiency and low precision[12]. What's more, in the online solution process of discrete-time TVNO, whose objective function is $f(x_k, t_k)(t_k = k\tau, k = 0, 1, \ldots)$ and $\tau > 0$ denoting the sampling gap, the present and/or previous data with respect to $x_i(i \leq k)$ should be used sufficiently to generate the unknown decision variable x_{k+1}. Thus, the information of the time derivative should be taken into consideration to get fast and accurate solution of TVNO. During the last decades, neural network has drawn extensive attention of researchers and practitioners due to its nice properties, e.g., distributed storage, high-speed parallel-processing, hardware applications and superior performance in large-scale online applications [13]. As a special recurrent neural network, Zhang neural network (ZNN), named after Chinese scholar Zhang Yunong, serves as a unified approach to solve various online time-varying problems, such as time-varying quadratic function minimization [14], future minimization [15], time-varying matrix pseudoinversion [13], TVNO [10, 12, 17], etc. For example, based on ZNN, Jin et al. [10] presented a one-step discrete-time ZNN (DTZNN) model for TVNO, whose maximal residual error is theoretically $\mathcal{O}(\tau^2)$. Subsequently, Guo et al. [12] proposed two DTZNN models for TVNO, which belong to three-step DTZNN with steady-state residual error (SSRE) changing in an $\mathcal{O}(\tau^3)$ manner. Then, quite recently Zhang et al. [17] presented a general four-step discrete-time derivative dynamics model and a general four-step DTZNN model for TVNO, both models contain a free parameter which can take any values of the interval $(1/12, 1/6)$ and convergent with truncation error of $\mathcal{O}(\tau^4)$.

Generally speaking, in the real-time solution process of TVNO, the more past data utilized, the smaller of the truncation error of the corresponding Zhang et al. discretization (ZeaD) formula is. For example, the truncation errors of the one-step DTZNN model in [10], the three-step DTZNN models in [12], the general three-step DTZNN models in [16] and the four-step DTZNN models in [17] are $\mathcal{O}(\tau^2)$, $\mathcal{O}(\tau^3)$, $\mathcal{O}(\tau^3)$ and $\mathcal{O}(\tau^4)$, respectively. The corresponding ZeaD formulas are summarized in Table 1, in which the effective domains of the parameter a_1 in [16] and [17] are $(-\infty, 0)$ and $(1/12, 1/6)$, respectively. Obviously, when $a_1 = -1/2$ or $-1/10$, the ZeaD formula in [16] reduces the two ZeaD formulas in [12].

Table 1: ZeaD formulas used in [10, 12, 16, 17].

	Formula	n-step	General form	Truncation error
[10]	$\dot{f}_k = \frac{f_{k+1}-f_k}{\tau}$	1	No	$O(\tau^2)$
[12]	$\dot{f}_k = \frac{2f_{k+1}-3f_k+2f_{k-1}-f_{k-2}}{2\tau}$	3	No	$O(\tau^3)$
[12]	$\dot{f}_k = \frac{6f_{k+1}-3f_k-2f_{k-1}-f_{k-2}}{10\tau}$	3	No	$O(\tau^3)$
[16]	$\dot{f}_k = \frac{(-a_1+1/2)f_{k+1}+3a_1f_k-(3a_1+1/2)f_{k-1}+a_1f_{k-2}}{\tau}$	3	Yes	$O(\tau^3)$
[17]	$\dot{f}_k = \frac{(a_1+1/3)f_{k+1}-(4a_1-1/2)f_k+(6a_1-1)f_{k-1}-(4a_1-1/6)f_{k-2}+a_1f_{k-3}}{\tau}$	4	Yes	$O(\tau^4)$

The motivation of the paper can be summarized as:

The expression of the general two-step ZeaD formula or five-step ZeaD formula have not been investigated in the literature.

Then, in this paper, we are going to present a special two-step ZeaD formula with truncation error $O(\tau^3)$ and a general two-step ZeaD formula with truncation error $O(\tau^2)$, and prove that they are convergent. In particular, the general ZeaD formula contains a free parameter a_1, whose feasible region is $(-1/2, +\infty)$, and when $a_1 = 0$, this ZeaD formula reduces to the ZeaD formula in [10]. We also study the general five-step ZeaD formula with truncation error $O(\tau^5)$, and prove that it is not zero-stable, and thus is not convergent. Then, based on the Taylor series expansion and the above two convergent ZeaD formulas, we discrete the continuous-time ZNN (CTZNN) model for TVNO, and thus get a special two-step DTZNN model and a general two-step DTZNN model for TVNO. Theoretical analyses indicate that the first DTZNN model is not convergent, while the second DTZNN model is convergent for any $a_1 \in (-1/2, +\infty)$ and the step-size $h \in (0, (2+4a_1)/(1+a_1))$. In addition, the tight upper bound of h and the optimal step-size are discussed.

The rest of the paper is organized as follows. We first recall some basic definitions and results in Section 2, including problem formulation of TVNO, continuous-time ZNN (CTZNN) model for TVNO and the general n-step ZeaD formula. In Section 3, a special two-step ZeaD formula with truncation error of $O(\tau^3)$ and a general two-step ZeaD formula with truncation error of $O(\tau^2)$ are presented and analyzed, and we also prove that the five-step ZeaD formula with truncation error of $O(\tau^5)$ or $O(\tau^6)$ is not convergent in this section. Furthermore, based on the two convergent ZeaD formulas, two discrete-time ZNN (DTZNN) models for TVNO are presented, and we prove that the first DTZNN model is not convergent and the second DTZNN is convergent. Later, in Section 4, some numerical experiments are presented in order to illustrate and compare the performances of the convergent two-step DTZNN model with other variants. Finally, a concluding remark with future research direction is given in Section 5.

2 Preliminaries

In this section, the results in [12, 17] are summarized for the foundation of further discussion, including the problem formulation of TVNO, the CTZNN model for TVNO and the general ZeaD formula.

Firstly, the problem formulation of the TVNO is as follows [17]:

$$\min_{x(t)\in\mathcal{R}^n} f(x(t),t) \in \mathcal{R}, \quad t \in [0, t_f], \tag{1}$$

where the time-varying nonlinear function $f(\cdot,\cdot) : \mathcal{R}^n \times [0, t_f] \to \mathcal{R}$ is second-order differentiable and bounded. Problem (1) aims to find $x(t) \in \mathcal{R}^n$ such that the function $f(x(t), t)$ achieves its minimum at any instant $t = k\tau$ with $k = 0, 1, 2, \ldots$. Thus in the sequent analyse, we assume that the solution of problem (1) exists at any time instant $t = k\tau$.

It is well known that it is often hard to find the global optimum solution of time-invariant nonlinear optimization by traditional numerical algorithms [3, 6]. Therefore, researchers have resorted to find the stationary point of time-invariant nonlinear optimization. Similarly, we transform problem (1) into finding its stationary point $x(t)$, which satisfies the following nonlinear equations:

$$g(x(t),t) \doteq \frac{\partial f(x(t),t)}{\partial x(t)} = 0 \in \mathcal{R}^n, \quad \forall t \in [0, t_f], \tag{2}$$

where symbol $\doteq$ denotes the computational assignment operation. In the following, we aim to find the solutions of problem (2) at any time instant $t = k\tau$. Generally speaking, the solutions of problem (1) are the solutions of problem (2), but the inverse maybe not, and if $f(x(t),t)$ is convex with respect to $x(t)$, the inverse also holds [3].

Setting $e(t)=g(x(t),t)$ in Zhang neural network (ZNN) design formula [18]

$$\dot{e}(t) \doteq \frac{de(t)}{dt} = -\gamma e(t), \quad \gamma > 0,$$

we get the following continuous-time ZNN (CTZNN) model of problem (1) [10]:

$$\dot{x}(t) = -H(x(t),t)^{-1}(\gamma g(x(t),t) + g_t(x(t),t)), \tag{3}$$

where $g_t(x(t),t)$ is the partial derivative of the mapping $g(x(t),t)$ with respect to its second variable t, i.e.,

$$g_t(x(t),t) = \frac{\partial g(x(t),t)}{\partial t} = \frac{\partial^2 f(x(t),t)}{\partial x(t)\partial t} = \Big[\frac{\partial^2 f}{\partial x_1 \partial t}, \frac{\partial^2 f}{\partial x_2 \partial t}, \dots, \frac{\partial^2 f}{\partial x_n \partial t}\Big]^{\top} \in \mathcal{R}^n,$$

and $H(x(t),t)$ is the Hessian matrix of problem (1), i.e.,

$$H(x(t),t) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix} \in \mathcal{R}^{n\times n},$$

which is assumed to be positive definite throughout the paper to ensure that the stationary point $x(t)$ of problem (1) is also its solution.

The general n-step ZeaD formula is defined as follows [19]:

$$\dot{f}_k = \frac{1}{\tau}\Big(\sum_{i=1}^{n+1} a_i f_{k-n+i}\Big) + O(\tau^p), \tag{4}$$

where n is the amount of the steps of ZeaD formula (4); $a_i \in \mathcal{R}$ $(i=1,2,\dots,n+1)$ denotes the coefficients; $\mathcal{O}(\tau^p)$ denotes the truncation error; f_k is the value of $f(t)$ at time instant $t_k = k\tau$, i.e., $f_k = f(t_k)$; k denotes the updating index. Equation (4) with $a_1 \neq 0, a_{n+1} \neq 0$ is termed as n-step pth-order ZeaD formula.

3 Multi-step ZeaD formulas and discrete-time models

In this section, we first propose two two-step ZeaD formulas with truncation error of $O(\tau^3)$ and $O(\tau^2)$, and prove that the five-step ZeaD formula with truncation error of $O(\tau^5)$ or $O(\tau^6)$ is not convergent. Then, two DTZNN models for TVNO are presented and analyzed subsequently.

3.1 Concepts of convergence of discrete-time models

The following concepts about zero-stability and consistency are used to analyze the theoretical results of our proposed discrete-time models [20].

Concept 3.1. The zero-stability of an n-step discrete-time method

$$x_{k+1}+\sum_{i=1}^{n}\alpha_i x_{k+1-i}=\tau\sum_{i=0}^{n}\beta_i v_{k+1-i}$$

can be checked by determining the roots of the characteristic polynomial $P(\upsilon)=\upsilon^n+\sum_{i=1}^{n}\alpha_k\upsilon^{n-i}$. If the roots of $P(\upsilon)=0$ are such that

- all roots lie in the unit disk, i.e., $|\upsilon|\leq 1$; and,
- any roots on the unit circle (i.e., $|\upsilon|=1$) are simple (i.e., not multiple);

then, the n-step discrete-time method zero-stable.

Concept 3.2. An n-step discrete-time method is said to be consistency with order p, if its truncation error is $\mathcal{O}(\tau^p)$ with $p>0$ for the smooth exact solution.

Concept 3.3. For an n-step discrete-time method, it is convergent, i.e., $x_{[(t-t_0)/\tau]}\rightarrow x^*(t)$ for all $t\in[t_0,t_f]$, as $\tau\rightarrow 0$, if and only if such an algorithm is zero-stable and consistent (see Concepts 3.1 and 3.2). That is, zero-stability and consistency result in convergence. In particular, a zero-stable and consistent method converges with the order of its truncation error.

3.2 Multi-step ZeaD formulas

Based on the Taylor series expansion

$$f_{k+1}=f_k+\tau\dot{f}_k+\frac{\tau^2}{2}\ddot{f}_k+\frac{\tau^3}{3!}\dddot{f}_k+\ldots+\frac{\tau^m}{m!}f_k^{(m)}+O(\tau^{m+1}), \tag{5}$$

where m is a nonnegative integer, we can derive the two-step ZeaD formula, which is presented in the following theorem.

Theorem 3.1. The two-step ZeaD formula with truncation error of $O(\tau^3)$ can be expressed as

$$\dot{f}_k=\frac{f_{k+1}-f_{k-1}}{2\tau} \tag{6}$$

which is convergent, and the general two-step ZeaD formula with truncation error of $O(\tau^2)$ can be expressed as

$$\dot{f}_k=\frac{(1+a_1)f_{k+1}-(1+2a_1)f_k+a_1f_{k-1}}{\tau}, \tag{7}$$

which is convergent for any $a_1\in(-1/2,+\infty)\backslash\{0\}$.

Proof. Based on (5), the Taylor series expansions of f_{k+1} and f_{k-1} at x_k are given as

$$f_{k+1}=f_k+\tau\dot{f}_k+\frac{\tau^2}{2}\ddot{f}_k+O(\tau^3), \tag{8}$$

$$f_{k-1}=f_k-\tau\dot{f}_k+\frac{\tau^2}{2}\ddot{f}_k+O(\tau^3). \tag{9}$$

Substituting (8) and (9) into (4) with $n=2$, i.e.,

$$\dot{f}_k=\frac{1}{\tau}(a_3f_{k+1}+a_2f_k+a_1f_{k-1})+O(\tau^p), \tag{10}$$

we get the following equation:

$$b_0 f_k + b_1 \tau \dot{f}_k + b_2 \tau^2 \ddot{f}_k + O(\tau^3) = O(\tau^{p+1}), \tag{11}$$

where $b_0 = a_3 + a_2 + a_1$, $b_1 = a_3 - a_1 - 1$, $b_2 = a_3/2 + a_1/2$. If $p = 2$, from Concept 3.2, to ensure that the two-step ZeaD formula (10) has a truncation error of $O(\tau^3)$, we only need to ensure that the following conclusions are satisfied:

$$\begin{cases} b_0 = a_3 + a_2 + a_1 = 0, \\ b_1 = a_3 - a_1 - 1 = 0, \\ b_2 = a_3/2 + a_1/2 = 0. \end{cases}$$

Solving the above three linear equations, we get

$$a_1 = -1/2, a_2 = 0, a_3 = 1/2. \tag{12}$$

Substituting (12) into (10), we get the two-step ZeaD formula (6) with truncation error of $O(\tau^3)$. Then, the characteristic polynomial of (6) is

$$\rho(\gamma) = \gamma^2 - 1 = 0,$$

whose two roots are -1 and 1. By Concept 3.1, the two-step ZeaD formula (6) is zero-stable, and thus is convergent by Concept 3.3.

Similarly, if $p = 1$, to ensure that the two-step ZeaD formula (10) has a truncation error of $O(\tau^2)$, we only need to ensure that the following conclusions are satisfied:

$$\begin{cases} b_0 = a_3 + a_2 + a_1 = 0, \\ b_1 = a_3 - a_1 - 1 = 0. \end{cases}$$

Solving the above two linear equations and let a_1 be a free parameter, we get

$$a_2 = -1 - 2a_1, a_3 = 1 + a_1. \tag{13}$$

In this case, $b_2 = a_1 + 1/2$. Then, Equation (11) is reformulated as

$$\left(a_1 + \frac{1}{2}\right)\tau^2 \ddot{f}_k + O(\tau^3) = O(\tau^2), \tag{14}$$

which is obviously true. Substituting (13) into (10), we get the two-step ZeaD formula (7) with truncation error of $O(\tau^2)$. Then, the characteristic polynomial of (7) is

$$\rho(\gamma) = (1 + a_1)\gamma^2 - (1 + 2a_1)\gamma + a_1.$$

By adopting bilinear transform $\gamma = (1 + \omega\tau/2)/(1 - \omega\tau/2)$ [21], we get the following equation:

$$c_2\left(\frac{\omega\tau}{2}\right)^2 + c_1\left(\frac{\omega\tau}{2}\right) + c_0 = 0,$$

where $c_2 = 2 + 4a_1$, $c_1 = 2$, $c_0 = 0$. Then, according the Routh's stability criterion [22], the general two-step ZeaD formula is zero-stable if and only if

$$c_2 > 0, \quad \text{i.e.,} \quad a_1 > -1/2.$$

Therefore the general two-step ZeaD formula (7) is convergent with truncation error of $O(\tau^2)$ if $a_1 > -1/2$. Since $a_1 \neq 0$, then the effective domain of a_1 is $(-1/2, +\infty)\backslash\{0\}$. This completes the proof.

Remark 3.1. If $a_1 = 0$, then the general two-step ZeaD formula (7) reduces to the one-step ZeaD formula in [10]:

$$x_{k+1} = x_k - H(x_k, t_k)^{-1}(hg(x_k, t_k) + \tau g_t(x_k, t_k)). \tag{15}$$

Then, in the following the effective domain of a_1 is set as $(-1/2, +\infty)$.

Corollary 3.1. For any fixed and sufficiently small sampling gap $\tau > 0$, the truncation error of the general two-step ZeaD formula (8) becomes smaller as the parameter $a_1 \to -1/2^+$.

Proof. For any fixed and sufficiently small sampling gap $\tau > 0$, from (14), the truncation error is dominated by the term

$$\Big(a_1 + \frac{1}{2}\Big)\tau^2 \ddot{f}_k,$$

which obviously becomes smaller as $a_1 \to -1/2^+$. The proof is completed.

The following theorem reveals that five-step ZeaD formula with truncation error of $O(\tau^5)$ or $O(\tau^6)$ is not convergent.

Theorem 3.2. Five-step ZeaD formula with truncation error of $O(\tau^5)$ or $O(\tau^6)$ is not convergent.

Proof. Based on (5), the Taylor series expansions of f_{k+1}, f_{k-1}, f_{k-2}, f_{k-3} and f_{k-4} at x_k are given as

$$f_{k+1} = f_k + \tau \dot{f}_k + \frac{\tau^2}{2}\ddot{f}_k + \frac{\tau^3}{6}f_k^{(3)} + \frac{\tau^4}{24}f_k^{(4)} + \frac{\tau^5}{120}f_k^{(5)} + O(\tau^6), \tag{16}$$

$$f_{k-1} = f_k - \tau \dot{f}_k + \frac{\tau^2}{2}\ddot{f}_k - \frac{\tau^3}{6}f_k^{(3)} + \frac{\tau^4}{24}f_k^{(4)} - \frac{\tau^5}{120}f_k^{(5)} + O(\tau^6), \tag{17}$$

$$f_{k-2} = f_k - 2\tau \dot{f}_k + 2\tau^2\ddot{f}_k - \frac{4\tau^3}{3}f_k^{(3)} + \frac{2\tau^4}{3}f_k^{(4)} - \frac{4\tau^5}{15}f_k^{(5)} + O(\tau^6), \tag{18}$$

$$f_{k-3} = f_k - 3\tau \dot{f}_k + 2\frac{9\tau^2}{2}\ddot{f}_k - \frac{9\tau^3}{2}f_k^{(3)} + \frac{27\tau^4}{8}f_k^{(4)} - \frac{81\tau^5}{40}f_k^{(5)} + O(\tau^6), \tag{19}$$

$$f_{k-4} = f_k - 4\tau \dot{f}_k + 8\tau^2\ddot{f}_k - \frac{32\tau^3}{3}f_k^{(3)} + \frac{32\tau^4}{3}f_k^{(4)} - \frac{128\tau^5}{15}f_k^{(5)} + O(\tau^6). \tag{20}$$

Substituting (16)-(20) into (4) with $n = 5$, i.e.,

$$\dot{f}_k = \frac{1}{\tau}(a_6 f_{k+1} + a_5 f_k + a_4 f_{k-1} + a_3 f_{k-2} + a_2 f_{k-3} + a_1 f_{k-4}) + O(\tau^p), \tag{21}$$

we get the following equation:

$$b_0 f_k + b_1 \tau \dot{f}_k + b_2 \tau^2 \ddot{f}_k + b_3 \tau^3 f_k^{(3)} + b_4 \tau^4 f_k^{(4)} + b_5 \tau^5 f_k^{(5)} + O(\tau^6) = O(\tau^{p+1}), \tag{22}$$

where

$$\begin{cases} b_0 = a_6 + a_5 + a_4 + a_3 + a_2 + a_1, \\ b_1 = a_6 - a_4 - 2a_3 - 3a_2 - 4a_1 - 1, \\ b_2 = \frac{a_6}{2} + \frac{a_4}{2} + 2a_3 + \frac{9a_2}{2} + 8a_1, \\ b_3 = \frac{a_6}{6} - \frac{a_4}{6} - \frac{4a_3}{3} - \frac{9a_2}{2} - \frac{32a_1}{3}, \\ b_4 = \frac{a_6}{24} + \frac{a_4}{24} + \frac{2a_3}{3} + \frac{27a_2}{8} + \frac{32a_1}{3}, \\ b_5 = \frac{a_6}{120} - \frac{a_4}{120} - \frac{4a_3}{15} - \frac{81a_2}{40} - \frac{128a_1}{15}. \end{cases}$$

If $p = 5$, from Concept 3.2, to ensure that the five-step ZeaD formula (21) has a truncation error of $O(\tau^5)$, we only need to ensure that the following conclusions are satisfied:

$$\begin{cases} b_0 = a_6 + a_5 + a_4 + a_3 + a_2 + a_1 = 0, \\ b_1 = a_6 - a_4 - 2a_3 - 3a_2 - 4a_1 - 1 = 0, \\ b_2 = \frac{a_6}{2} + \frac{a_4}{2} + 2a_3 + \frac{9a_2}{2} + 8a_1 = 0, \\ b_3 = \frac{a_6}{6} - \frac{a_4}{6} - \frac{4a_3}{3} - \frac{9a_2}{2} - \frac{32a_1}{3} = 0, \\ b_4 = \frac{a_6}{24} + \frac{a_4}{24} + \frac{2a_3}{3} + \frac{27a_2}{8} + \frac{32a_1}{3} = 0, \\ b_5 = \frac{a_6}{120} - \frac{a_4}{120} - \frac{4a_3}{15} - \frac{81a_2}{40} - \frac{128a_1}{15} = 0. \end{cases} \tag{23}$$

Solving the above six linear equations with respect to $a_i (i = 1, 2, \ldots, 6)$, we get

$$a_1 = 1/20, a_2 = -1/3, a_3 = 1, a_4 = -2, a_5 = 13/12, a_6 = 1/5. \tag{24}$$

Substituting (24) into (21), we get a five-step ZeaD formula with truncation error of $O(\tau^6)$, whose characteristic polynomial is

$$\rho(\gamma) = 12\gamma^5 + 65\gamma^4 - 120\gamma^3 + 60\gamma^2 - 20\gamma + 3 = 0,$$

of which the roots are 1, -6.9614, 0.26698, 0.13887 - 0.33945i and 0.13887 + 0.33945i with i denoting imaginary unit. By Concept 3.1, the resulting five-step ZeaD formula is not zero-stable since the root -6.9614 lies outside unit disk. Therefore the five-step ZeaD formula with truncation error of $O(\tau^6)$ is not zero-stable, and thus is not convergent.

Similarly, if $p = 4$, to ensure that the five-step ZeaD formula (21) has a truncation error of $O(\tau^5)$, we only need to ensure that the first five linear equations of (23) holds, i.e., the following conclusions are satisfied:

$$\begin{cases} b_0 = a_6 + a_5 + a_4 + a_3 + a_2 + a_1 = 0, \\ b_1 = a_6 - a_4 - 2a_3 - 3a_2 - 4a_1 - 1 = 0, \\ b_2 = \frac{a_6}{2} + \frac{a_4}{2} + 2a_3 + \frac{9a_2}{2} + 8a_1 = 0, \\ b_3 = \frac{a_6}{6} - \frac{a_4}{6} - \frac{4a_3}{3} - \frac{9a_2}{2} - \frac{32a_1}{3} = 0, \\ b_4 = \frac{a_6}{24} + \frac{a_4}{24} + \frac{2a_3}{3} + \frac{27a_2}{8} + \frac{32a_1}{3} = 0. \end{cases}$$

Solving the above five linear equations and let a_6 be a free parameter, we have

$$\begin{cases} a_1 = \frac{1}{4} - a_6, \\ a_2 = -\frac{3}{4} + 5a_6, \\ a_3 = 3 - 10a_6, \\ a_4 = -4 + 10a_6, \\ a_5 = \frac{25}{12} - 5a_6. \end{cases} \tag{25}$$

In this case, $b_5 = -\frac{1}{5} + a_6$. Then, Eq. (22) is reformulated as

$$\Big(-\frac{1}{5} + a_6\Big)\tau^5 f_k^{(5)} + O(\tau^6) = O(\tau^5),$$

which is obviously true. Substituting (25) into (21), we can get a general five-step ZeaD formula with truncation error of $O(\tau^5)$, whose characteristic polynomial is

$$\rho(\gamma) = a_6\,\gamma^5 + \left(\frac{25}{12} - 5\,a_6\right)\gamma^4 + (10\,a_6 - 4)\,\gamma^3 + (3 - 10\,a_6)\,\gamma^2 + \left(5\,a_6 - \frac{4}{3}\right)\gamma - a_6 + \frac{1}{4}. \tag{26}$$

Then, by adopting bilinear transform $\gamma = (1+\omega\tau/2)/(1-\omega\tau/2)$ again, we get the following equation:

$$c_5\Big(\frac{\omega\tau}{2}\Big)^5 + c_4\Big(\frac{\omega\tau}{2}\Big)^4 + c_3\Big(\frac{\omega\tau}{2}\Big)^3 + c_2\Big(\frac{\omega\tau}{2}\Big)^2 + c_1\Big(\frac{\omega\tau}{2}\Big) + c_0 = 0,$$

where $c_5 = 32a_6 - \frac{32}{3}, c_4 = -2, c_3 = \frac{14}{3}, c_2 = 6,\ c_1 = 2,\ c_0 = 0$. Unfortunately, $c_4 = -2 < 0$. Then, according the Routh's stability criterion again, the general five-step ZeaD formula with truncation error of $O(\tau^5)$ is not zero-stable, and thus is not convergent. This completes the proof.

Remark 3.2. The polynomial (26) has five roots, which is denoted by $\gamma_i (i = 1, 2, \ldots, 5)$. Let

$$Q = \max\{|\gamma_i||i = 1, 2, \ldots, 5\}.$$

Figure 1 illustrates the relationship of a_6 and Q, from which we can see that Q is always bigger than 1 except to $a_6 = 0$. However, from the definition of the general n-step ZeaD formula, we have that $a_6 \neq 0$.

3.3 Discrete-time ZNN models

In this subsection, two discrete-time ZNN (DTZNN) models are presented for TVNO based on the two two-step ZeaD formulas (6) and (7).

Firstly, applying the two-step 2th-order ZeaD formula (6) to discretize the CTZNN model (3), we get the following DTZNN model for TVNO:

$$x_{k+1} = x_{k-1} - H(x_k, t_k)^{-1}(hg(x_k, t_k) + 2\tau g_t(x_k, t_k)), \tag{27}$$

where the step-size $h = 2\tau\gamma > 0$.

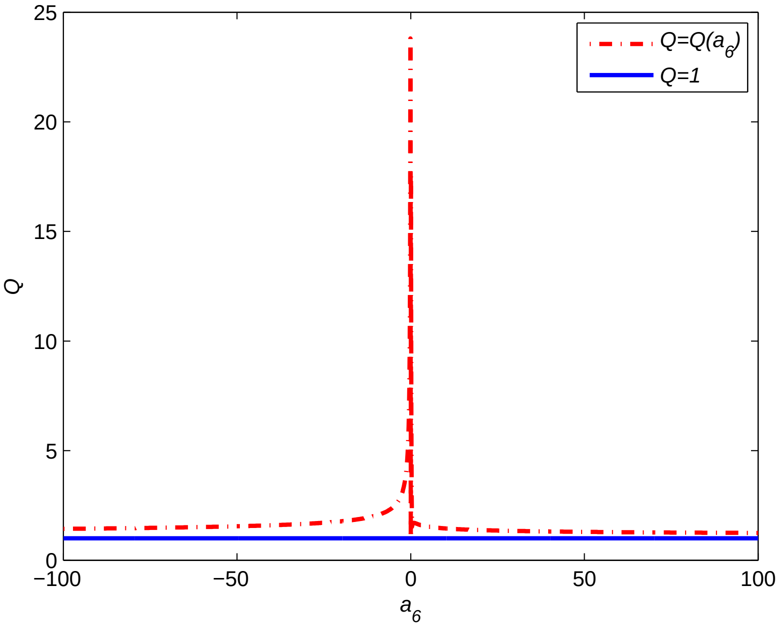

Figure 1: The curve of Q with respect to a_6.

Similarly, applying the general two-step 1th-order ZeaD formula (7) to discretize the CTZNN model (3), we get the following DTZNN model for TVNO:

$$x_{k+1} = \frac{1+2a_1}{1+a_1}x_k - \frac{a_1}{1+a_1}x_{k-1} - H(x_k,t_k)^{-1}\Big(hg(x_k,t_k) + \frac{\tau}{1+a_1}g_t(x_k,t_k)\Big), \tag{28}$$

where the step-size $h = \frac{\tau\gamma}{1+a_1} > 0$ and $a_1 \in (-1/2, +\infty)$.

To analyze the theoretical results of the two-step DTZNN models (27) and (28), we need the following lemma.

Lemma 3.1. [23] Both roots of the real quadratic equation $\lambda^2 - b\lambda + c = 0$ are less than one in modulus if and only if $|c| < 1$ and $|b| < 1 + c$.

Theorem 3.3. Suppose $\{(x_k, t_k)\}$ be the sequence generated by the two-step DTZNN model (27), and let $\|g(x_k, t_k)\|_2$ be the generated steady-state residual error (SSRE). Then the sequence $\{\|g(x_k, t_k)\|_2\}$ is not convergent.

Proof. Let $\Delta x_k = x_{k+1} - x_k$ and $\Delta x_{k-1} = x_k - x_{k-1}$. Then, the proposed two-step DTZNN model (27) can be reformulated as

$$\frac{1}{2}\Delta x_k + \frac{1}{2}\Delta x_{k-1} = -H(x_k,t_k)^{-1}\big(\frac{h}{2}g(x_k,t_k) + \tau g_t(x_k,t_k)\big). \tag{29}$$

On the other hand, by the Taylor series expansion, we have

$$g(x_{k+1},t_{k+1}) = g(x_k,t_k) + H(x_k,t_k)\Delta x_k + \tau g_t(x_k,t_k) + \mathcal{O}(\tau^2), \tag{30}$$

and

$$g(x_{k-1},t_{k-1}) = g(x_k,t_k) - H(x_k,t_k)\Delta x_{k-1} - \tau g_t(x_k,t_k) + \mathcal{O}(\tau^2), \tag{31}$$

where $\mathcal{O}(\Delta x_k^2)$ and $\mathcal{O}(\Delta x_{k-1}^2)$ is absorbed into $\mathcal{O}(\tau^2)$ as they are assumed to be of the same order of magnitude [20]. By the algebraic manipulation "$\frac{1}{2}\times(30) - \frac{1}{2}\times(31)$", the following results can be obtained

$$\frac{1}{2}g(x_{k+1},t_{k+1}) - \frac{1}{2}g(x_{k-1},t_{k-1}) = \frac{1}{2}H(x_k,t_k)(\Delta x_k + \Delta x_{k-1}) + \tau g_t(x_k,t_k) + \mathcal{O}(\tau^2),$$

which together with (29) implies

$$\frac{1}{2}g(x_{k+1},t_{k+1}) - \frac{1}{2}g(x_{k-1},t_{k-1}) = -\frac{h}{2}g(x_k,t_k) + \mathcal{O}(\tau^2),$$

i.e.,

$$(g(x_{k+1},t_{k+1}) - \mathcal{O}(\tau^2)) - (g(x_{k-1},t_{k-1}) - \mathcal{O}(\tau^2))) = -h(g(x_k,t_k) - \mathcal{O}(\tau^2))). \tag{32}$$

Letting $G_k = g(x_{k+1},t_{k+1}) - \mathcal{O}(\tau^2)$, then the equation (32) can be written as

$$G_{k+1} + hG_k - G_{k-1} = 0. \tag{33}$$

The characteristic equation of the difference equation (33) is

$$\lambda^2 + h\lambda - 1 = 0, \tag{34}$$

which has two different real roots from the discriminant $\Delta = h^2 + 4 > 0$. By Lemma 3.1, at least one root of the real quadratic equation (34) is greater than or equal to one in modulus. Thus, the sequence $\{G_k\}$ is not convergent, so is the sequence $\{g(x_k,t_k)\}$. The proof is completed.

Theorem 3.4. Suppose $\{(x_k,t_k)\}$ be the sequence generated by the two-step DTZNN model (28), and let $\|g(x_k,t_k)\|_2$ be the generated steady-state residual error (SSRE). Then for any $a_1 \in (-1/2,+\infty)$ and the step-size $h \in (0,(2+4a_1)/(1+a_1))$, we have $\lim_{k\to\infty} \|g(x_k,t_k)\|_2\}$ is of order $O(\tau^2)$, thus the sequence $\{\|g(x_k,t_k)\|_2\}$ convergence of order $O(\tau^2)$ to zero.

Proof. Let $\Delta x_k = x_{k+1} - x_k$ and $\Delta x_{k-1} = x_k - x_{k-1}$. Then, the proposed two-step DTZNN model (28) can be reformulated as

$$(1+a_1)\Delta x_k - a_1\Delta x_{k-1} = -H(x_k,t_k)^{-1}\big((1+a_1)hg(x_k,t_k) + \tau g_t(x_k,t_k)\big). \tag{35}$$

By the algebraic manipulation "$(1+a_1)\times(30) + a_1\times(31)$", the following results can be obtained

$$\begin{aligned}&(1+a_1)g(x_{k+1},t_{k+1}) + a_1 g(x_{k-1},t_{k-1})\\&= (1+2a_1)g(x_k,t_k) + H(x_k,t_k)((1+a_1)\Delta x_k - a_1\Delta x_{k-1}) + \tau g_t(x_k,t_k) + \mathcal{O}(\tau^2),\end{aligned}$$

which together with (35) implies

$$(1+a_1)g(x_{k+1},t_{k+1}) + a_1 g(x_{k-1},t_{k-1}) = \big(1+2a_1-(1+a_1)h\big)g(x_k,t_k) + \mathcal{O}(\tau^2),$$

i.e.,

$$(g(x_{k+1},t_{k+1})-\mathcal{O}(\tau^2))+\frac{a_1}{1+a_1}(g(x_{k-1},t_{k-1})-\mathcal{O}(\tau^2)))=\frac{1+2a_1-(1+a_1)h}{1+a_1}(g(x_k,t_k)-\mathcal{O}(\tau^2))). \tag{36}$$

Similarly, letting $G_k = g(x_{k+1}, t_{k+1}) - \mathcal{O}(\tau^2)$, then the equation (36) can be written as

$$G_{k+1} - \frac{1+2a_1-(1+a_1)h}{1+a_1}G_k + \frac{a_1}{1+a_1}G_{k-1} = 0. \tag{37}$$

The characteristic equation of the difference equation (37) is

$$\lambda^2 - \frac{1+2a_1-(1+a_1)h}{1+a_1}\lambda + \frac{a_1}{1+a_1} = 0, \tag{38}$$

By Lemma 3.1, two roots of (38) are less than one in modulus if and only if

$$\left|\frac{a_1}{1+a_1}\right| < 1, \quad \left|\frac{1+2a_1-(1+a_1)h}{1+a_1}\right| < 1 + \frac{a_1}{1+a_1}.$$

Obviously, the first inequality always holds for any $a_1 > -1/2$, therefore we only to analyze the second inequality, which is equivalent to

$$|1+2a_1-(1+a_1)h| < |1+a_1| + a_1,$$

thus

$$|1+2a_1-(1+a_1)h| < 1+2a_1.$$

So,

$$0 < h < \frac{2+4a_1}{1+a_1}.$$

Then, the sequence $\{G_k\}$ is convergent for any $a_1 \in (-1/2, +\infty)$ and step-size $h \in (0, (2+4a_1)/(1+a_1))$, so is the sequence $\{g(x_k, t_k)\}$. The proof is completed.

Obviously, two initial states (i.e.,x_0, x_1) are needed to start the iteration of the DTZNN model (28). We use the following DTZNN model [10] to initiate the iterative computation:

$$x_1 = x_0 - H(x_0,t_0)^{-1}(hg(x_0,t_0) + \tau g_t(x_0,t_0)).$$

Remark 3.3. The upper bound of the step-size h, that is $\frac{2+4a_1}{1+a_1}$, is an increasing function with respect to the parameter a_1, and when $a_1 \to +\infty$, it converges to 4.

The following theorem shows that the upper bound of the step-size h, that is $\frac{2+4a_1}{1+a_1}$, is tight.

Theorem 3.5. For any $a_1 > -1/2$, if $h \geq \frac{2+4a_1}{1+a_1}$, then the sequence $\{\|g(x_k, t_k)\|_2\}$ generated by the DTZNN model (28) does not converge to zero.

Proof. If $h \geq \frac{2+4a_1}{1+a_1}$, then the characteristic equation of the difference equation (37) reduces to

$$\lambda^2 + \frac{1+2a_1}{1+a_1}\lambda + \frac{a_1}{1+a_1} = 0, \tag{39}$$

which has two different real roots:

$$\lambda_1 = -1, \quad \lambda_2 = -\frac{a_1}{1+a_1}.$$

Thus the general solution of the difference equation (37) is

$$G_k = c_1(-1)^k + c_2\Big(\frac{a_1}{1+a_1}\Big)^k,$$

where c_1, c_2 are two arbitrary constant which are determined by two initial states x_0, x_1. So, the limit of the sequence $\{G_k\}$ general does not exist except $c_1 = 0$, which indicates that the sequence $\{\|g(x_k, t_k)\|_2\}$ generally does not converge to zero. This completes the proof.

In the remainder of this section, let us investigate the optimal choice of the step-size h for given $a_1 \in (-1/2, +\infty)$. Corollary 3.1 indicates that when $a_1 \to -1/2^+$, the truncation error of the general two-step ZeaD formula (8) becomes smaller and smaller, and the DTZZN model (28) is based on the ZeaD formula (8), so in the following analyse, we only consider $a_1 \in (-1/2, 0)$. Then, the discriminant of Eq. (38) is

$$\Delta = \frac{1}{(1+a_1)^2}\Big(\bar{h}^2 - 2(1+2a_1)\bar{h} + 1\Big), \tag{40}$$

where $\bar{h} = (1+a_1)h \in (0, 2+4a_1)$. Set $\Delta_1 = \bar{h}^2 - 2(1+2a_1)\bar{h} + 1$, which is a quadratic function with respect to $\bar{h}$, and its discriminant is

$$\bar{\Delta} = 4a_1(1+a_1),$$

which is less than or equal to zero for any $a_1 \in (-1/2, 0)$. So $\Delta \geq 0$ defined in (40) for any $a_1 \in (-1/2, 0)$, which shows that Eq. (38) has two different real roots, denoted by λ_1, λ_2. Thus

$$\lambda_1 + \lambda_2 = -\frac{1+2a_1-(1+a_1)h}{1+a_1}, \quad \lambda_1\lambda_2 = \frac{a_1}{1+a_1}, \tag{41}$$

and the second equation indicates that $\lambda_i (i = 1, 2)$ have contrary sign; then we assume that $\lambda_1 > 0, \lambda_2 < 0$ without loss of generality. Furthermore, for any $a_1 \in (-1/2, 0)$ and the step-size $h \in (0, (2+4a_1)/(1+a_1))$, from Theorem 3.4, we have $0 \leq |\lambda_i| < 1 (i = 1, 2)$. The general solutions of Eq. (37) can be written as:

$$G_k = c_1\lambda_1^k + c_2\lambda_2^k,$$

which together with (41) results in the following model to determine the optimal step-size:

$$\begin{aligned}
&\min\{c_1\lambda_1^k + c_2\lambda_2^k\} \\
&\text{s.t.} \lambda_1 + \lambda_2 = -\tfrac{1+2a_1-(1+a_1)h}{1+a_1}, \\
&\quad \lambda_1\lambda_2 = \tfrac{a_1}{1+a_1}, \\
&\quad 0 < \lambda_1 < 1, -1 < \lambda_2 < 0, 0 < h < (2+4a_1)/(1+a_1).
\end{aligned} \tag{42}$$

which is often difficult to solve, and in the following we give an intuitive analyse about the optimal value of h. Obviously, the smaller $\max\{|\lambda_1|, |\lambda_2|\}$ is, the smaller the term G_k is. Thus, under the constraint conditions of (42), we aim to minimize the term $\lambda_1 - \lambda_2$, and equivalently minimize the term $(\lambda_1 - \lambda_2)^2$, which can be written as

$$\begin{aligned}
&(\lambda_1 - \lambda_2)^2 \\
&= (\lambda_1 - \lambda_2)^2 - 4\lambda_1\lambda_2, \\
&= \Delta = \frac{1}{(1+a_1)^2}\Big((1+a_1)^2 h^2 - 2(1+a_1)(1+2a_1)\bar{h} + 1\Big),
\end{aligned}$$

which obtains the minimum value at $h^* = (1+2a_1)/(1+a_1) \in (0, (2+4a_1)/(1+a_1))$. Thus, we get the following theorem.

Theorem 3.6. For any given $a_1 \in (-1/2, 0)$, the optimal step-size of the DTZNN model (28) is $h^* = (1+2a_1)/(1+a_1)$.

4 Numerical results

In this section, we present some numerical results to substantiate the efficiency and superiority of the proposed DTZNN model (28) (denoted by DTZNN-I) for TVNO and compared with the one-step DTZNN model (15) (denoted by DTZNN-II) in [10]. All the numerical experiments are performed on an Thinkpad laptop with Intel Core 2 CPU 2.10 GHZ and RAM 4.00 GM. All the programs are written in Matlab R2014a.

Consider the following TVNO [17]:

$$\begin{aligned}
&\min_{x\in\mathcal{R}^4} f(x(t), t) \\
&= \Big(x_1(t) + 10\sin\big(\tfrac{\pi t}{40}\big)\Big)^2 + \Big(x_2(t) + \tfrac{t}{4}\Big)^2 \\
&\quad + \Big(x_3(t) - \exp\big(-\tfrac{t}{4}\big)\Big)^2 + 0.025(t-1)x_3(t)x_4(t) \\
&\quad + \Big(x_1(t) + \ln\big(\sin\big(\tfrac{\pi t}{40}\big) + 1\big)\Big)^2\Big(x_2(t) + \sin\big(\tfrac{t}{8}\big) + \cos\big(\tfrac{t}{8}\big)\Big)^2 \\
&\quad - (x_1(t) + (x_1(t) + \sin(t)))x_3(t) + \Big(x_4(t) + \exp\big(-\tfrac{t}{4}\big)\Big)^2,
\end{aligned} \tag{43}$$

and we can get its stationary point by Matlab, which is omitted due to its complicated expression. Now, we use DTZNN-I and DTZNN-II to solve problem (43), and the parameters are set as follows: $\tau = 0.01$s or 0.001s, $h = 0.1$ and $a_1 = -1/3$. The initial state vector $x_0 = x(0) = [1, 2, 3, 4]^\top$ with time duration being 10s. The trajectories of SSRE $\|g(x_k, t_k)\|_2$ of TVNO problem (43) generated by the three tested DTZNN models are depicted in Figure 2.

Figure 2 illustrates that the performance of the DTZNN model (28) with $a_1 = -1/3$ is better than that of the DTZNN model (15), and both generated SSREs converge to zero in an $O(\tau^2)$ manner. So when the sampling gap $\tau > 0$ decreases, both SSREs can be made sufficiently small.

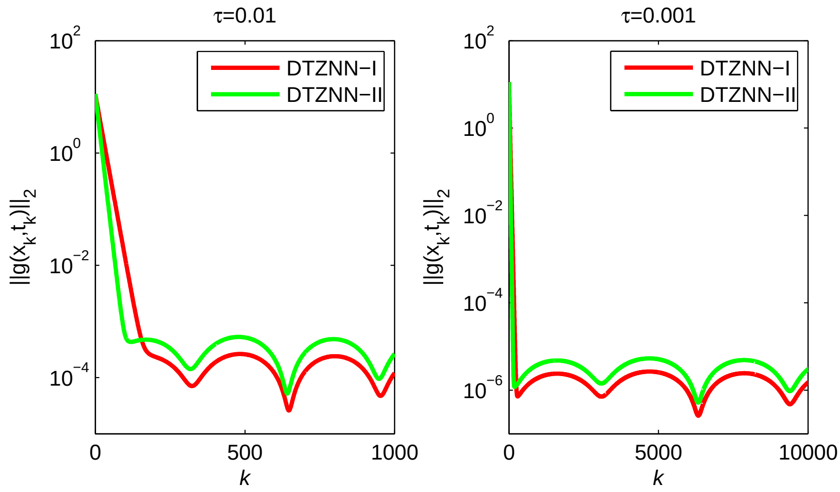

Figure 2: Trajectories of SSRE $\|g(x_k, t_k)\|_2$ of problem (43) generated by the DTZNN models (15) and (28), left $\tau = 0.01$, right $\tau = 0.001$.

Figure 3 depicts the trajectories of the theoretical solution $x_i^*(t_k)(i = 1, 2, 3, 4)$ of problem (43) and $x_i(t_k)(i = 1, 2, 3, 4)$ generated by the DTZNN model (28) with $a_1 = -1/3, \tau = 0.01$, which shows that the numerical results generated by the DTZNN model (28) approximate the theoretical solution $x_i^*(t_k)(i = 1, 2, 3, 4)$ with high accuracy.

Figure 4 shows the trajectories of $f(x_k, t_k)$ generated by the two tested models and their differences when $\tau = 0.01$ or 0.001. From which we can find that $f(x_k, t_k)$ generated by the DTZNN-I model is general smaller than that generated by the DTZNN-II model, which means that the former is more accurate than the latter.

Now, let us verify Theorem 3.3 with $\tau = 0.01, a_1 = -1/2, h = 0.1$ and Theorem 3.5 with $\tau = 0.01, a_1 = -1/3, h = (2 + 4a_1)/(1 + a_1) = 1$. The numerical results are depicted in Figure 5, from which we find that the generated sequence $\{\|g(x_k, t_k)\|_2\}$ does not converge to zero when $a_1 = -1/2$ or h equals to the upper bound $(2 + 4a_1)/(1 + a_1)$, and these are consistent to Theorems 3.3 and 3.5.

In the remainder of this section, let us verify Theorem 3.6 with $\tau = 0.01, a_1 = -1/3, h = 0.1$ and $h = 0.5$, the optimal value of the step-size from Theorem 3.5 for fixed $a_1 = -1/3$. The numerical results are depicted in Figure 6, which shows the performance of the DTZNN model with $h = 0.5$ is better than that of the DTZNN model with $h = 0.1$, and this is consistent to Theorem 3.6.

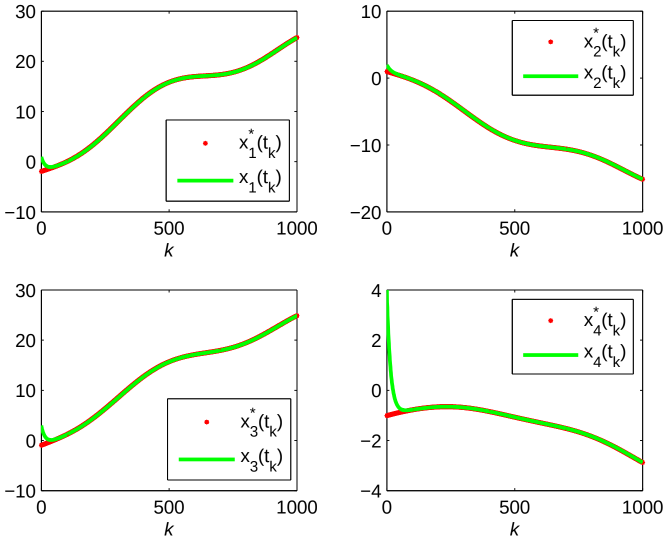

Figure 3: Trajectories of $x_i^*(t_k)(i = 1, 2, 3, 4)$ and $x_i(t_k)(i = 1, 2, 3, 4)$ generated by the DTZNN models (15) and (28) and their difference for problem (43).

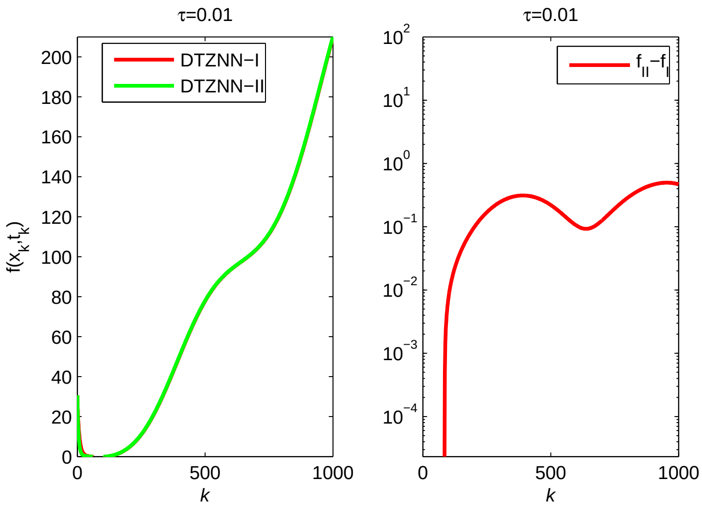

Figure 4: Trajectories of $f(x_l, t_k)$ generated by the DTZNNs model (28) for problem (43).

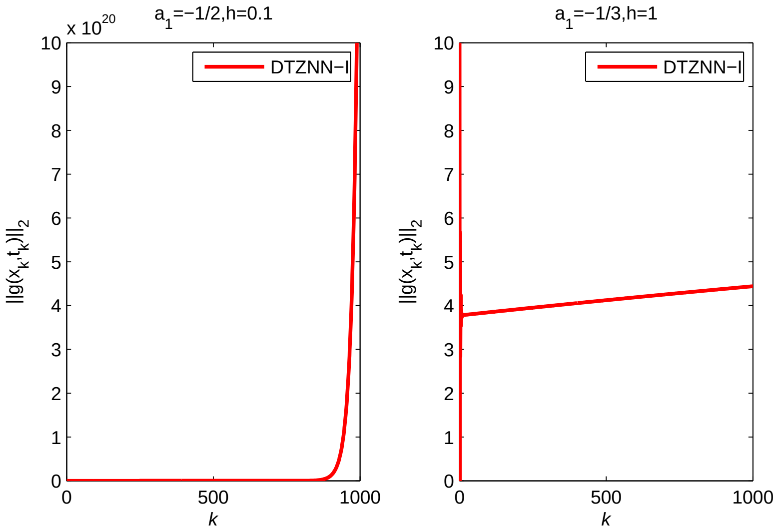

Figure 5: Trajectories of SSRE $\|g(x_k, t_k)\|_2$ generated by the DTZNN model (28) for problem (43).

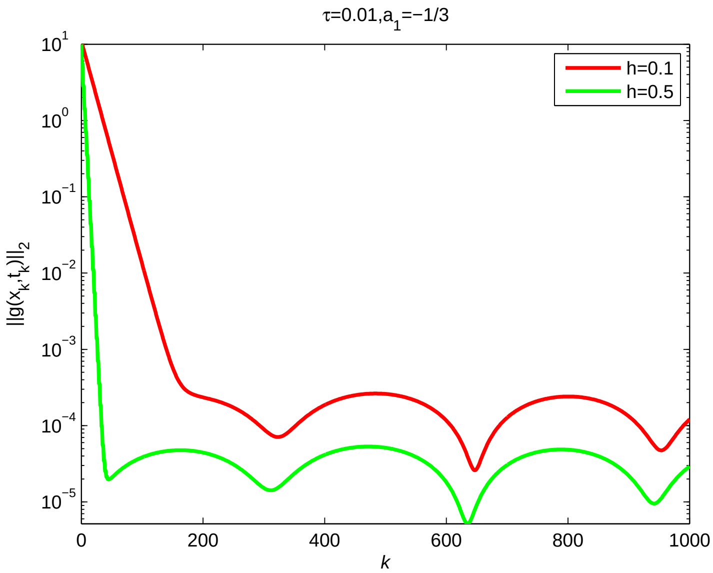

Figure 6: Trajectories of SSRE $\|g(x_k, t_k)\|_2$ generated by the DTZNN model (28) with different step-size for problem (43).

5 Conclusions

In this paper, we have investigated a convergent two-step ZeaD formulas with truncation error of $O(\tau^3)$, a convergent general two-step ZeaD formula with truncation error of $O(\tau^2)$ and a general five-step ZeaD formula with truncation error of $O(\tau^5)$, which is not convergent. Then, based on the two convergent ZeaD formulas, we presented two DTZNN models for TVNO, and proved that one is divergent and the other with the free parameter $a_1 \in (-1/2, +\infty)$ and step-size $0 < h < (2 + 4a_1)/(1 + a_1)$ is convergent. We also proved that $(2 + 4a_1)/(1 + a_1)$ is tight upper bound of h and $(1 + 2a_1)/(1 + a_1)$ is the optimal step-size. Numerical results illustrates that the proposed DTZNN model is efficient for solving TVNO.

In the future the following two issues related to this paper deserve further studying: (i) Theorem 3.6 only considers the optimal step-size for any given $a_1 \in (-1/2, 0)$, therefore we need to study the optimal step-size for any given $a_1 \in (0, +\infty)$; (ii) the general three-step DTZNN model and the general four-step DTZNN model proposed in [16, 17] both do not give the relationship of the free parameter a_1 and step-size h, therefore we are going to extend the technique used in Theorems 3.3 and 3.4 to study the two general multi-step DTZNN models, and explore the relationship of a_1 and h.

Acknowledgements

Not applicable.

Funding

This research was partially supported by the National Natural Science Foundation of China and Shandong Province (No. 11671228, 11601475, ZR2016AL05).

Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors have made the same contribution and finalized the current version of this article. They read and approved the final manuscript.

References

[1] Xiong W., Operations Research, Machinery Industry Press, 2014.

[2] Sun M., Wang Y.J., Liu J., Generalized Peaceman-Rachford splitting method for multiple-block separable convex programming with applications to robust PCA, Calcolo, 54(1), 77-94, 2017.

[3] Wang Y.J., Xiu N.H., Nonlinear Programming Theory and Algorithm, Shaanxi Science and Technology Press, 2008.

[4] Sun M., Liu J., A modified Hestenes-Stiefel projection method for constrained nonlinear equations and its linear convergence rate, Journal of Applied Mathematics and Computating, 49(1-2): 145-156, 2015.

[5] Sun M., Liu J., New hybrid conjugate gradient projection method for the convex constrained equations, Calcolo, 53: 399-411, 2016.

[6] Sun M., Bai Q.G., A new descent memory gradient method and its global convergence, Journal of Systems Science and Complexity, 24(4): 784-794, 2011.

[7] Nocedal J., Updating quasi-Newton matrixes with limited storage, Mathematics of Computation, 35, 773-782, 1980.

[8] He B.S., Yang H., A neural network model for monotone linear asymmetric variational inequalities, IEEE Transactions on Neural Networks, 11(1), 3-16, 2000.

[9] Xia Y.S., New cooperative projection neural network for nonlinearly constrained variational inequality, Science in China Series F: Information Sciences, 52(10), 1766-1777, 2009.

[10] Jin L., Zhang Y.N., Discrete-time Zhang neural network for online time-varying nonlinear optimization with application to manipulator motion generation, IEEE Transactions on Neural Networks and Learning Systems, 26(7), 1525-1531, 2015.

[11] Xu M.H., Li M, Yang C.C, Neural networks for a class of bi-level variational inequalities, Journal of Global Optimization, 44(4), 535-552, 2009.

[12] Guo D.S., Lin X.J., Su Z.Z., Sun S.B., Huang Z.J., Design and analysis of two discrete-time ZD algorithms for time-varying nonlinear minimization, Numerical Algorithms, 77(1), 23-36, 2018.

[13] Jin L., Zhang Y.N., Discrete-time Zhang neural network of $\mathcal{O}(\tau^3)$ pattern for time-varying matrix pseudoinversion with application to manipulator motion generation, Neurocomputing, 142, 165-173, 2014.

[14] Zhang Y.N., Li Z., Yi C.F., Chen K., Zhang neural network versus gradient neural network for online time-varying quadratic function minimization, ICIC 2008, LNAI 5227, 807-814, 2008.

[15] Zhang Y.N., Yang M., Li J., He L., Wu S., ZFD formula 4IgSFD_Y applied to future minimization, Physics Letters A, 381, 1677-1681, 2017.

[16] Hu C.W., Kang X.G., Zhang Y.N., Three-step general discrete-time Zhang neural network design and application to time-variant matrix inversion, Neurocomputing, 306, 108-118, 2018.

[17] Zhang Y.N., He L., Hu C.W., Guo J.J., Li J., Shi Y., General four-step discrete-time zeroing and derivative dynamics applied to time-varying nonlinear optimization, Journal of Computational and Applied Mathematics, 347, 314-329, 2019.

[18] Zhang Y.N., Yi C., Zhang Neural Networks and Neural-Dynamic Method. New York, NY, USA: Nova, 2011.

[19] Zhang Y.N., Jin J., Guo D.S., Yin Y.H., Chou Y., Taylor-type 1-step-ahead numerical differentiation rule for first-order derivative approximation and ZNN discretization, Journal of Computational and Applied Mathematics, 273, 29-40, 2015.

[20] Griffiths, D.F., Higham, D.J., Numerical methods for ordinary differential equations: Initial value problems. Springer, England, 2010.

[21] Oppenheim A.V., Discrete-Time Signal Processing, Pearson Higher Education Inc., New Jersey, NJ, 2010.

[22] Ogata K., Modern Control Engineering, Prentice-Hall, Inc., Englewood Cliffs, NJ, 2001.

[23] Golub G.H., Wu X., Yuan J.Y., SOR-like methods for augmented systems, BIT Numerical Mathematics, 41, 71-85, 2001.