On prime and minimal representations of a face of a polyhedron

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Abstract: In this paper, a new method for determining all minimal representations of a face of a polyhedron is proposed. A main difficulty for determining prime and minimal representations of a face is that the deletion of one redundant constraint can change the redundancy of other constraints. To reduce computational efforts in finding all minimal representations of a face, we prove and use properties that deleting strong redundant inequality constraints does not change the redundancy of other constraints and all minimal representations of the face can be found in only the set of all prime representations of the face corresponding to the maximal descriptor index set for it. An algorithm based on a top-down search method is given for finding all minimal representations of a face. Numerical examples are given to illustrate the performance of the algorithm.

Keywords: Faces of a polyhedron; degeneracy degrees of faces; prime and minimal representations of a face; the maximal descriptor index set.

1. Introduction

A convex polyhedral set can be stated in the following form:

$$a_i x \le b_i, i = 1,...,m, \qquad (1)$$

where $x \in R^n$, $b_i \in R$ and a_i is a row vector in R^n. For brevity of presentation we shall use the following notation: For two vectors $y = (y_1,...,y_n)$ and $z = (z_1,...,z_n)$, $y \le z$ if and only if $y_i \le z_i$ for all $i = 1,...,n$; for two subsets Ω^1 and Ω^2 of a set, $\Omega^1 \subset \Omega^2$ if and only if $\Omega^1 \subseteq \Omega^2$ and $\Omega^1 \ne \Omega^2$. Let *P* be polyhedron (1), $\overline{I} = \{1,...,m\} \setminus I$ and

$$S(I,J) = \left\{ x \in R^n \,\middle|\, \begin{array}{l} a_i x = b_i, i \in I \\ a_i x \le b_i, i \in J \end{array} \right\}. \qquad (2)$$

A nonempty subset F of P is said to be a face of it if there is a subset I of $\{1,\dots,m\}$ such that $F = S(I,\bar{I})$. Such a set I is called a *descriptor index set* for F and $S(I,\bar{I})$ is called *a descriptor set* for F corresponding to I. An index pair (I,J) satisfying conditions that $I,J \subseteq \{1,...,m\}$ and $I \cap J = \varnothing$ is called a *descriptor index pair* for a face F if $S(I,J) = F$. An index set $I \in RE(F)$ is said to be a *maximal descriptor index set* for a face F, denoted by $I_{\max}$, if there is no $J \in RE(F)$ such that $I \subset J$, where

$$RE(F) = \left\{ J \subseteq \{1,...,m\} \middle| S(J,\bar{J}) = F \right\}. \tag{3}$$

A face F is said to be *degenerate* if $|RE(F)| \geq 2$, where $|.|$ denotes the number of elements of a set. An index $i \in I$ or a constraint $a_i x = b_i$ is called a *redundant equality* for $S(I,J)$ if $S(I,J) = S(I \setminus \{i\}, J)$ and an index $j \in J$ or a constraint $a_j x \leq b_j$ is a *redundant inequality* for $S(I,J)$ if $S(I,J) = S(I, J \setminus \{j\})$. We say that an index pair (I,J) contains a redundant index for $S(I,J)$ if I contains at least one redundant equality index or J contains at least one redundant inequality index for $S(I,J)$. An index pair (I^1, J^1) is called a *weak reduction* of (I,J) if $S(I^1, J^1) = S(I,J)$, $I^1 \subseteq I$ and $J^1 \subseteq J$. An index pair (I,J) is called a prime representation of a face F corresponding to a descriptor index set K for F if (I,J) is a weak reduction of $(K,\bar{K})$ and contains no redundant indices for $S(I,J)$. An index pair (I,J) is called a *prime representation* of a face F if there is an element $K \in RE(F)$ such that (I,J) is a prime representation of it corresponding to K. A descriptor index pair (I,J) for a face F is called a *minimal representation* of it if

$$|I \cup J| = \min\left\{ |K \cup M| \,\middle|\, (K,M) \in T(F) \right\}, \tag{4}$$

where $T(F)$ is the set of all descriptor index pairs for the face F and is stated by

$$T(F) = \left\{ (M,N) \middle| S(M,N) = F,\ M,N \subseteq \{1,...,m\} \text{ and } M \cap N = \varnothing \right\}. \tag{5}$$

A minimal representation of a face also is a prime representation of this face and can be not unique (see Property 2.2 or Remark 3.1 or Example 5.2 later).

There are differences between the concept of minimal representations of a face and the concept of minimal representations of a polyhedron presented in Telgen [8]. Minimal representations of a face of polyhedron (1) are defined and determined on the basis of only the set $\{a_i x \le b_i, i = 1,...,m\}$. The concept of minimal representations of a polyhedron in Telgen [8] can be applied to a face of polyhedron (1) but is defined on the basis of a larger set of linear equalities and inequalities that contains the set $\{a_i x \le b_i, i = 1,...,m\}$. This leads to that a minimal representation of a face of polyhedron (1) in the concept of Telgen [8] can contain other linear constraints not belonging to the set $\{a_i x \le b_i, i = 1,...,m\}$. Our concept has valuable practical applications. We consider finding preferred solutions of a practical problem stated by a mathematical model whose feasible set is a convex polyhedral set. In our concept, minimal representations of faces of the polyhedron can be found from the constraints of this mathematical model and therefore they can be utilized to reduce difficulties in solving and analysing the mathematical problem by reducing the degeneracy degree of the constraint collection describing the constraint polyhedron and the size of the mathematical model (see Section 6 for more details).

In order to find a prime or minimal representation of a face, a removal of redundant indices must be done. A main difficulty for determining prime and minimal representations of a face is that the deletion of one redundant constraint can change the redundancy of other constraints. A removal of some redundant indices from the special index pair $(\varnothing, \{1,...,m\})$ is dealt with in Greenberg [4] but only few results are found. Prime representations of the special face $S(\varnothing, \{1,...,m\})$ also is dealt with in Boneh et al. [2], but only the difference between the cardinalities of any two prime representations of the special face is mainly investigated. A method for determining all prime representations of a face corresponding to a given descriptor index set for it has not been given. For a descriptor index set J for a face, a necessary and sufficient condition for the index pair $(J, \overline{J})$ to be a minimal representation of this face can be found in Telgen [8]. A parameterization of all minimal representations of a polyhedron in the concept of Telgen is also dealt with. Theoretically, this way can be applied to find minimal representations of a face but is very difficult.

In this paper, first we propose a new method for determining all minimal representations of a given face of a convex polyhedral set. To reduce computational efforts for finding all minimal representations of a face, we prove and use properties that deleting strong redundant inequality constraints does not change the redundancy of other constraints and all minimal representations of the face can be obtained by determining only the set of all prime representations of the face corresponding to the maximal descriptor index set for it. In addition, the set of all prime representations of a face corresponding to the maximal descriptor index set for it is found by a top-down search method. This method is simple, easy to implement and has many computational advantages. For applications of minimal representations, we deal with a reduction of the number of constraints used to represent it, a reduction of degeneracy degrees of sub-faces of the face and ideas to improve some known methods for finding all maximal efficient faces in multiple objective linear programming and some known methods for optimizing a function over the efficient set.

This paper is organized as follows: Some properties of minimal representations of a face of a polyhedron are presented in Section 2. Determining all prime representations of a face corresponding to the maximal descriptor index set for it is dealt with in Section 3. An algorithm for determining all minimal representations of a given face and examples to illustrate the working of the algorithm are presented in Sections 4 and 5. Some applications of minimal representations of a face are considered in Section 6.

2. Some properties of minimal representations of a face of a polyhedron

Through this paper, let F be the face described by a descriptor index set I. A point x^0 of the face F is called an *inner point* of it if there is $J \in RE(F)$ such that $a_i x^0 < b_i$ for all $i \in \overline{J}$. From Property 2.6 in Tu [9] it follows that every face has at least one inner point. An index $j \in K$ is called an *implicit equality index* for $S(J,K)$ if $a_j x = b_j$ for all $x \in S(J,K)$. From the definition of minimal representations of a face and Theorem 4.1 in Telgen [8] the following property is easily obtained:

Property 2.1: A pair index $(J,K) \in T(F)$ is a minimal representation of the face F if and only if (J,K) contains no redundant indices and K does not contain implicit equality indices for $S(J,K)$.

Thus, a minimal representation of a face also is a prime representation of this face. Another relation between minimal representations and prime representations of a face is considered in the following:

Property 2.2: $RE_{\min}(F) \subseteq RE_{\text{prim}}(F, I_{\max})$,

where $RE_{\min}(F)$ and $RE_{\text{prim}}(F, I_{\max})$ are the set of all minimal representations of the face F and the set of all prime representations of the face F corresponding to $I_{\max}$, respectively.

Proof: We consider an arbitrary element $(J,K) \in RE_{\min}(F)$. From the definition of a minimal representation of a face and Property 2.1 it is easily seen that $J \cap K = \emptyset$, *J* contains no redundant equality indices, *K* contains no redundant inequality indices and implicit equality indices for $S(J,K)$. If there is $j^0 \in J \setminus I_{\max}$, then $j^0 \in \bar{I}_{\max}$ and $a_{j^0} x^0 < b_{j^0}$, where x^0 is an inner point of *F*. Thus $x^0 \notin S(J,K)$. This contradicts to that $S(J,K) = F$. Therefore $J \subseteq I_{\max}$. Assume that there is $k^0 \in K \setminus \bar{I}_{\max}$. It is easily seen that $k^0 \in I_{\max}$. Hence $a_{k^0} x = b_{k^0}$ for all $x \in S(I_{\max}, \bar{I}_{\max})$. Since $S(I_{\max}, \bar{I}_{\max}) = S(J,K)$, k^0 is an implicit equality index for $S(J,K)$. This contradicts to that K contains no implicit equality indices for $S(J,K)$ (Property 2.1). Thus, we also have $K \subseteq \bar{I}_{\max}$. Therefore, from the definition of a prime representation of a face it follows that $(J,K) \in RE_{\text{prim}}(F, I_{\max})$. □

The set $RE_{\min}(F)$ can be obtained by determining all prime representations of *F* but this method requires many computational efforts. Property 2.2 shows that only the set $RE_{\text{prim}}(F, I_{\max})$ is needed to find for determining the set $RE_{\min}(F)$. Now we deal with another important property of the set $RE_{\text{prim}}(F, I_{\max})$.

Property 2.3: For an arbitrary element $(J,K) \in RE_{\text{prim}}(F, I_{\max})$, the index pair (J,K) contains no redundant indices and the index set *K* does not contain implicit equality indices for $S(J,K)$.

Proof: From the definition of a prime representations of a face it is clear that (J,K) contains no redundant indices for $S(J,K)$. Assume that there is $j \in K$ such that *j* is an implicit equality index for $S(J,K)$. Hence, we have $\min\left\{a_j x \middle| x \in S(J,K)\right\} = b_j$. Since $S(J,K) = S(I_{\max}, \bar{I}_{\max})$ and $K \subseteq \bar{I}_{\max}$ (Property 2.2), $\min\left\{a_j x \middle| x \in S(I_{\max}, \bar{I}_{\max})\right\} = b_j$ and $j \in \bar{I}_{\max}$. Thus, *j* is an implicit equality index for $S(I_{\max}, \bar{I}_{\max})$. Hence, $F = S(I_{\max}, \bar{I}_{\max}) = S(I_{\max}, \bar{I}_{\max}) \cap \left\{x \in R^n \middle| a_j x = b_j\right\} = S(I_{\max} \cup \{j\}, \bar{I}_{\max} \setminus \{j\})$. Thus $\{I_{\max} \cup \{j\}\} \in RE(F)$. This contradicts to the definition of the maximal descriptor index set for the face *F*. Therefore, *K* does not contain implicit equality indices for $S(J,K)$. □

Based on Property 2.3 we can obtain a result stronger than that in Property 2.2.

Theorem 2.4: $RE_{\min}(F) = RE_{\text{prim}}(F, I_{\max})$.

Proof: From Properties 2.1 and 2.3 it follows that $RE_{\min}(F) \supseteq RE_{\text{prim}}(F, I_{\max})$. Therefore, from Property 2.2 we have $RE_{\min}(F) = RE_{\text{prim}}(F, I_{\max})$. The proof is complete. □

It is clear that $RE_{\min}(F) \subseteq RE_{\text{prim}}(F)$, where $RE_{\text{prim}}(F)$ is the set of all prime representations of the face *F*. The following corollary shows a condition for equality in this inequality:

Corollary 2.5: If the face *F* is not degenerate, then $RE_{\min}(F) = RE_{\text{prim}}(F)$.

Proof: Since *F* is not degenerate, $RE(F) = \{I_{\max}\}$. Therefore, from Theorem 2.4 it follows that $RE_{\text{prim}}(F)$ = $\bigcup_{J \in RE(F)} RE_{\text{prim}}(F, J) =$ $RE_{\text{prim}}(F, I_{\max})$ = $RE_{\min}(F)$. □

Based on Theorem 2.4 we only need to find the set $RE_{\text{prim}}(F, I_{\max})$ for determining all minimal representations of the face *F*.

3. Determining all prime representations of a face corresponding to the maximal descriptor index set for it

Let $iq(J, K)$ be the set of all implicit equality indices for $S(J, K)$. The set $I_{\max}$ can be determined on the basis of an index set $iq(I, \bar{I})$ found by solving $|\bar{I}|$ linear programming (LP) problems

$$\min\left\{a_i x \,\middle|\, x \in S(I, \bar{I})\right\}, \qquad (6)$$

and $iq(I, \bar{I}) = \left\{i \in \bar{I} \,\middle|\, o^i = b_i\right\}$, where $i \in \bar{I}$ and o^i is the optimal value of (6). Another method for determining the sets $iq(I, \bar{I})$ and $I_{\max}$ is shown in Tu [10] by solving only one LP problem:

Property 3.1: If $I \in RE(F)$ and $\left(x^o, z^o(\bar{I}), y^o(I), y^o(\bar{I}), \alpha^o\right)^T$ is an arbitrary feasible solution of $PM(I)$ with $\alpha^0 > 0$, then

(i) $iq(I, \bar{I}) = IQ(I, x^0)$,

(ii) $I_{\max} = I \cup iq(I, \bar{I})$,

where $IQ(I, x^0) = \left\{i \in \bar{I} \,\middle|\, z_i^0(\bar{I}) = 0\right\}$, $PM(I)$ is the linear problem:

$$\max \alpha \qquad (7)$$

$$A(I)x = b(I), \qquad (8)$$

$$A(\bar{I})x + z(\bar{I}) = b(\bar{I}), \qquad (9)$$

$$y^T(I)A(I) + y^T(\bar{I})A(\bar{I}) = 0, \qquad (10)$$

$$y^T(I)b(I) + y^T(\bar{I})b(\bar{I}) = 0, \qquad (11)$$

$$z(\bar{I}) + y(\bar{I}) - \alpha e(\bar{I}) \geq 0, \qquad (12)$$

$$z(\bar{I}) \geq 0,\ y(\bar{I}) \geq 0, \qquad (13)$$

$A(J)$ is a matrix obtained from the left side matrix A defined in (1) by deleting rows whose indices are not in J, $b(J)$ is a vector obtained from the right side vector b defined in (1) by deleting components whose indices are not in J, $y(J)$, $z(J)$ and $e(J)$ are similarly obtained from vectors $y \in R^m$, $z \in R^m$ and $e = (1,...,1)^T \in R^m$, respectively. An efficient algorithm for determining the set $I_{\max}$ is presented in Subroutine INDEXFACE(I,α,β) in [10].

Let $RE_{\text{prim}}(F,J)$ be the set of all prime representations of a face F corresponding to a descriptor index set $J \in RE(F)$, $r(J) = rank\{a_i | i \in J\}$,

$P(j,K,L)$ be the problem $\max\{a_j x | x \in S(K,L)\}$, (14)

$$a_j^*(K,L) = \begin{cases} (b_j + 1) \text{ if there is } x^0 \in S(K,L) \text{ such that } a_j x^0 > b_j, \\ a_j^{opt}(K,L) \text{ in the other case,} \end{cases}$$

where $a_j^{opt}(K,L)$ is the optimal value of $P(j,K,L)$. It is clear that if (K,L) is a weak reduction of the index pair $(J,\bar{J})$ corresponding to an element $J \in RE(F)$, then $S(K,L) \neq \varnothing$ and $a_j^*(K,L)$ exists. In addition, the problem $P(j,K,L)$ need not be solved to optimality for determining $a_j^*(K,L)$ if there is $x^0 \in S(K,L)$ such that $a_j x^0 > b_j$.

Some simple conditions for the redundancy of an index pair are given in the following property whose proof is easily obtained from the definitions of redundant indices and the Gaussian elimination:

Property 3.2: (i) An index $k \in K$ is redundant for $S(K,L)$ if and only if $r(K) = r(K \setminus \{k\})$.

(ii) An index $j \in L$ is redundant for $S(K,L)$ if and only if $a_j^*(K, L \setminus \{j\}) \le b_j$.

From the definition of prime representations of a face and Property 3.2 we easily have the following property:

Property 3.3: If $J \in RE(F)$, then the index pair $(J, \bar{J})$ is a prime representation of the face F if and only if $r(J) = |J|$ and $a_j^*(J, \bar{J} \setminus \{j\}) > b_j$ for all $j \in \bar{J}$.

For an element $J \in RE(F)$ we define the following sets: $\bar{J}^{11} = \left\{ j \in \bar{J} \,\middle|\, a_j^*(J, \bar{J} \setminus \{j\}) < b_j \right\}$,

$\bar{J}^{12} = \left\{ j \in \bar{J} \,\middle|\, a_j^*(J, \bar{J} \setminus \{j\}) = b_j \right\}$, $\bar{J}^1 = \bar{J}^{11} \cup \bar{J}^{12}$, $\bar{J}^2 = \left\{ j \in \bar{J} \,\middle|\, a_j^*(J, \bar{J} \setminus \{j\}) > b_j \right\}$.

$$T_1(F,J) = \left\{ K \subseteq J \,\middle|\, |K| = r(K) \text{ and } r(K) = r(J) \right\},$$

$$T_2(F,J) = \left\{ G \subseteq \bar{J}^1 \,\middle|\, a_j^*(J, \bar{J} \setminus G) \le b_j \text{ for all } j \in G \right\},$$

$$T_3(F,J) = \left\{ G \in T_2(F,J) \,\middle|\, \overline{\exists} G^1 \in T_2(F,J) : G \subset G^1 \right\}.$$

In order to determine the set $RE_{\min}(F)$, based on Theorem 2.4 it is enough to find the set $RE_{\text{prim}}(F, I_{\max})$. Now we consider a formula to compute the set $RE_{\text{prim}}(F, I_{\max})$.

Theorem 3.4: $RE_{\text{prim}}(F, I_{\max}) = \left\{ (K, \bar{I}_{\max} \setminus G) \,\middle|\, K \in T_1(F, I_{\max}), G \in T_3(F, I_{\max}) \right\}$.

Proof: We will show that $(K, \bar{I}_{\max} \setminus G)$ is a prime representation of F corresponding to $I_{\max}$ for every element $(K, \bar{I}_{\max} \setminus G) \in RE^1(F, I_{\max})$, where $RE^1(F, I_{\max}) = \left\{ (K, \bar{I}_{\max} \setminus G) \,\middle|\, K \in T_1(F, I_{\max}), G \in T_3(F, I_{\max}) \right\}$. First, we will show that $S(K, \bar{I}_{\max} \setminus G) = F$. It is clear that $F = S(I_{\max}, \bar{I}_{\max}) = S(I_{\max}, \bar{I}_{\max} \setminus G) \cap S(\varnothing, G)$. From the definition of the set $T_2(F, I_{\max})$ it follows that $S(I_{\max}, \bar{I}_{\max} \setminus G) \subseteq S(\varnothing, G)$. Thus $S(I_{\max}, \bar{I}_{\max} \setminus G) = F$. By a proof similar to that of Property 5.2 in [5] it can be easily obtained that $S(K, \bar{I}_{\max} \setminus G) = F$. Consequently, from Property 3.2 it follows that K does not contain any redundant equality indices for $S(K, \bar{I}_{\max} \setminus G)$. From the definition of $T_3(F,I_{\max})$ it is clear that $\bar{I}_{\max}\setminus G$ does not contain any redundant inequality indices for $S\left(K,\bar{I}_{\max}\setminus G\right)$. Thus, $\left(K,\bar{I}_{\max}\setminus G\right)\in RE_{\text{prim}}(F,I_{\max})$ and $RE^1(F,I_{\max})\subseteq RE_{\text{prim}}(F,I_{\max})$.

Conversely, we consider an arbitrary prime representation $\left(I^1,I^2\right)\in RE_{\text{prim}}(F,I_{\max})$ and will show that $\left(I^1,I^2\right)\in RE^1(F,I_{\max})$. Since $\left(I^1,I^2\right)\in RE_{\text{prim}}(F,I_{\max})$, $I^1\subseteq I_{\max}$, $I^2\subseteq \bar{I}_{\max}$, I^1 does not contain any redundant equality indices and I^2 does not contain any redundant inequality indices for $S\left(I^1,I^2\right)$. From Property 3.2 it follows that $r\left(I^1\right)=\left|I^1\right|$. Since $I^1\subseteq I_{\max}$, $r\left(I^1\right)\le r(I_{\max})$. We will show that $r\left(I^1\right)=r(I_{\max})$. Assume that $r\left(I^1\right)<r(I_{\max})$. It is clear that there is $i^0\in I_{\max}\setminus I^1$ such that $r\left(I^1\cup\left\{i^0\right\}\right)=\left|I^1\cup\left\{i^0\right\}\right|$. Since $r(I_{\max})\le n$, $\left|I^1\right|+1\le n$. Let SM_{J^1} be a $\left(\left|I^1\right|+1\right)\times\left(\left|I^1\right|+1\right)$ square nonsingular submatrix of $\begin{pmatrix} A\left(I^1\right) \\ a_{i^0}\end{pmatrix}$, where J^1 is the set of indices of all columns of the submatrix and $d=\left(d_1,\dots,d_{\left|I^1\right|+1}\right)^T$ be the $\left(\left|I^1\right|+1\right)-th$ column of the inverse matrix $SM_{J^1}^{-1}$. For convenience of presentation and without loss of generality, we can assume that $J^1=\left\{1,\dots,\left|I^1\right|+1\right\}$. We consider the point $x^1=x^0+x^2$, where x^0 is an inner point of *F*, $x^2=\left(d_1\beta^0,\dots,d_{\left|I^1\right|+1}\beta^0,0,\dots,0\right)^T$,

$N\left(J^1\right)=\left\{i\in\bar{I}_{\max}\,\middle|\,\sum_{j\in J^1}a_{ij}d_j<0\right\}$, $\beta^0=\max_{i\in N\left(J^1\right)}\left(b_i-a_ix^0\right)\Big/\sum_{i\in J^1}a_{ij}d_j$ if $N\left(J^1\right)\ne\varnothing$ and β^0 is a negative number if $N\left(J^1\right)=\varnothing$. Since x^0 is an inner point of *F*, $\beta^0<0$. It can be easily seen that x^1 satisfies the conditions

$$\begin{cases} A\left(I^1\right)x^1=b\left(I^1\right), \\ a_{i^0}x^1=\beta^0+b_{i^0}, \\ A\left(\bar{I}_{\max}\right)x^1\le b\left(\bar{I}_{\max}\right), \end{cases}$$

. Hence it easily follows that $x^1\in S\left(I^1,I^2\right)$ and $x^1\notin S\left(I_{\max},\bar{I}_{\max}\right)$. This is a contradiction because $F=S\left(I^1,I^2\right)$ and $F=S\left(I_{\max},\bar{I}_{\max}\right)$. Therefore, $r\left(I^1\right)=r(I_{\max})$ and $I^1\in T_1(F,I_{\max})$.

Consequently, since $\left(I^1, I^2\right) \in RE_{\text{prim}}\left(F, I_{\max}\right)$, there is $G^1 \subseteq \bar{I}_{\max}$ such that $I^2 = \bar{I}_{\max} \setminus G^1$. It is clear that $S\left(I_{\max}, \bar{I}_{\max}\right) = S\left(I_{\max}, \bar{I}_{\max} \setminus G^1\right)$ and $\bar{I}_{\max} \setminus G^1$ does not contain any redundant inequality indices for $S\left(I_{\max}, \bar{I}_{\max} \setminus G^1\right)$. If there are $j^0 \in G^1$ and $x^0 \in S\left(I_{\max}, \bar{I}_{\max} \setminus G^1\right)$ such that $a_{j^0} x^0 > b_{j^0}$, then $x^0 \notin S\left(I_{\max}, \bar{I}_{\max} \setminus \left\{G^1 \setminus \left\{j^0\right\}\right\}\right)$. Thus, it is clear that $S\left(I_{\max}, \bar{I}_{\max}\right) \subseteq S\left(I_{\max}, \bar{I}_{\max} \setminus \left\{G^1 \setminus \left\{j^0\right\}\right\}\right) \subset S\left(I_{\max}, \bar{I}_{\max} \setminus G^1\right)$. This is a contradiction because $S\left(I_{\max}, \bar{I}_{\max}\right) = S\left(I_{\max}, \bar{I}_{\max} \setminus G^1\right)$. Therefore, we have $a_j^{\max}\left(I_{\max}, \bar{I}_{\max} \setminus G^1\right) \le b_j$ for all $j \in G^1$ and $G^1 \in T_2\left(F, I_{\max}\right)$. From the definition of $T_3\left(F, I_{\max}\right)$ and $\bar{I}_{\max} \setminus G^1$ does not contain any redundant inequality indices for $S\left(I_{\max}, \bar{I}_{\max} \setminus G^1\right)$ it follows that $G^1 \in T_3\left(F, I_{\max}\right)$. Therefore $RE^1\left(F, I_{\max}\right) \supseteq RE_{\text{prim}}\left(F, I_{\max}\right)$. The proof is complete. □

From the proof of Theorem 3.4 the following property is easily obtained:

Property 3.5: If $G \subseteq K$, then $S(J, K) = S(J, K \setminus G)$ if and only if $a_j^*(J, K \setminus G) \le b_j$ for all $j \in G$.

Remark 3.1: From Theorems 2.4 and 3.4 it can be seen that a face can have many minimal representations.

An index $j \in L$ is called a *strong redundant index* for $S(K, L)$ if $a_j^*\left(K, L \setminus \{j\}\right) < b_j$. Based on Theorems 2.4 and 3.4 a method for determining all minimal representations of a face can be established. In order to increase the usefulness of the method, we will prove and utilize a property that deleting strong redundant indices does not change the redundancy of other indices for $S\left(J, \bar{J}\right)$, where *J* is an element of $RE(F)$.

Remark 3.2: From the definition of redundant indices and Property 3.2 it follows that $S\left(J, \bar{J} \setminus \{j\}\right) = F$ for all $j \in \bar{J}^1$. Therefore $\max\left\{a_j x \middle| x \in F\right\} < b_j$ for all $j \in \bar{J}^{11}$ and $\max\left\{a_j x \middle| x \in F\right\} = b_j$ for all $j \in \bar{J}^{12}$.

We consider the following property:

Property 3.6: (i) If $\bar{J}^{11} \ne \varnothing$, then $S\left(J, \bar{J} \setminus \{i, j\}\right) = F$ for all $i \in \bar{J}^{11}$ and $j \in \bar{J}^1$.

(ii) $S\left(J, \bar{J} \setminus \left\{\bar{J}^{11} \cup \{j\}\right\}\right) = F$ for every $j \in \bar{J}^{12}$.

Proof: It can easily be seen that $S\left(J,\bar{J}\setminus\{i,j\}\right)$ = $\left\{x\in S\left(J,\bar{J}\setminus\{i,j\}\right)\middle|a_i x\le b_i\right\}$ $\cup$ $\left\{x\in S\left(J,\bar{J}\setminus\{i,j\}\right)\middle|a_i x> b_i\right\}$ =

$\left\{x\in S\left(J,\bar{J}\setminus\{j\}\right)\right\}\cup\left\{x\in S\left(J,\bar{J}\setminus\{i,j\}\right)\middle|a_i x> b_i\right\}$ $=F\cup$ $\left\{x\in S\left(J,\bar{J}\setminus\{i,j\}\right)\middle|a_i x> b_i\right\}$.

Noting that $F=\left\{x\in R^n\middle|\begin{array}{l} a_t x=b_t, t\in J,\\ a_t x\le b_t, t\in \bar{J}\setminus\{i,j\},\\ a_i x\le b_i\end{array}\right\}$, $\max\left\{a_i x\middle|x\in F\right\}<b_i$ (Remark 3.2) and $a_i x$ is a continuous

function on $S\left(J,\bar{J}\setminus\{i,j\}\right)$, we easily have $S\left(J,\bar{J}\setminus\{i,j\}\right)$ $\subseteq$ $\left\{x\in R^n\middle|a_i x\le b_i\right\}$. Therefore

$\left\{x\in S\left(J,\bar{J}\setminus\{i,j\}\right)\middle|a_i x> b_i\right\}=\varnothing$ and $S\left(J,\bar{J}\setminus\{i,j\}\right)$ $=F$.

(ii) If $\bar{J}^{11}=\varnothing$, then the proof is obvious. If $\bar{J}^{11}\ne\varnothing$ and $\left|\bar{J}^1\right|\in\{1,2\}$, then the proof can be easily obtained from part (i). In the case when $\bar{J}^{11}\ne\varnothing$ and $\left|\bar{J}^1\right|\ge 3$, we consider a subset $G\subseteq\bar{J}^1$ with $|G|\ge 3$ and $\left|G\cap\bar{J}^{12}\right|\le 1$. It can be written that $S\left(J,\bar{J}\setminus G\right)$ = $\left\{x\in S\left(J,\bar{J}\setminus G\right)\middle|a_i x\le b_i\right\}\cup\left\{x\in S\left(J,\bar{J}\setminus G\right)\middle|a_i x> b_i\right\}$, where $i\in G\cap\bar{J}^{11}$. By an argument similar to that presented in part (i) and the induction method the proof of part (ii) can be easily obtained. □

From Property 3.6 and Theorem 3.4 we can easily obtain the following corollaries:

Corollary 3.7: If $\bar{J}^{12}=\varnothing$, then $S\left(J,\bar{J}\setminus\bar{J}^{11}\right)$ $=F$.

Corollary 3.8: If $\left|\bar{I}^1_{\max}\right|\le 2$ and $\bar{I}^{11}_{\max}\ne\varnothing$, then $RE_{\text{prim}}(F,I_{\max})=\left\{\left(J,\bar{I}^2_{\max}\right)\middle|J\in T_1(F,I_{\max})\right\}$.

4. An algorithm for determining all minimal representations of a given face

Let

$SO(j,I_{\max},G)$ be the set of all new feasible solutions of the problem $P\left(j,I_{\max},\bar{I}_{\max}\setminus G\right)$ that have been found in determining $a_j^*\left(I_{\max},\bar{I}_{\max}\setminus G\right)$;

$SO(I_{\max})=\ \cup\left\{SO(j,I_{\max},G)\middle|a_j^*\left(I_{\max},\bar{I}_{\max}\setminus G\right)\text{ has been determined}\right\}$;

$SO(I_{\max},G)=\left\{y\in SO(I_{\max})\middle|a_j y\le b_j,\forall j\in\bar{I}_{\max}\setminus G\right\}$;

$\Omega^1=\left\{G\subseteq\bar{I}^1_{\max}\middle|\exists j\in G:\ a_j^*\left(I_{\max},\bar{I}^1_{\max}\setminus G\right)\text{ has been determined and is larger than } b_j\right\}$.

An algorithm for finding all minimal representations of the face F described by an index set I is stated as follows:

Step 1. Determine the sets $I_{\max}$, $T_1(F, I_{\max})$.

If $\bar{I}_{\max} = \varnothing$, then set $T_4(F, I_{\max}) = \varnothing$ and go to Step 9.

Set $t = 1$, $TG^1 = \{S \subseteq \bar{I}_{\max} \| S| = 1\}$, $T_4(F, I_{\max}) = \varnothing$, $\Omega^1 = \varnothing$, $\bar{I}^{11}_{\max} = \varnothing$, $\bar{I}^{12}_{\max} = \varnothing$, $SO(I_{\max}) = \varnothing$

and go to Step 3.

Step 2. Determine $\bar{I}^1_{\max}$.

If $\bar{I}^1_{\max} = \varnothing$, then set $T_4(F, I_{\max}) = \{\{\varnothing\}\}$ and go to Step 9.

Set $T_4(F, I_{\max}) = \left\{\bar{I}^{11}_{\max} \cup \{j\} \middle| j \in \bar{I}^{12}_{\max}\right\}$.

If $\left|\bar{I}^1_{\max}\right| = 1$ or $\left|\bar{I}^1_{\max}\right| \le 2$ and $\bar{I}^{11}_{\max} \ne \varnothing$, then go to Step 9.

Set $t = 2$, $TG^{t+1} = \varnothing$ and $TG^t = \left\{S \subseteq \bar{I}^1_{\max} \| S| = t\right\}$.

Step 3. Take $G \in TG^t$.

If there is $\Omega \in T_4(F, I_{\max})$ such that $G \subseteq \Omega$, then go to Step 7. (a1)

Set $H = G$.

Step 4. If $SO(I_{\max}, G) = \varnothing$, then take an arbitrary element $j^0 \in H$, determine $a^*_{j^0}(I_{\max}, \bar{I}_{\max} \setminus G)$ based on

the problem $P(j^0, I_{\max}, \bar{I}_{\max} \setminus G)$ and go to Step 5.

Find an index $j^0 \in H$ and a feasible solution x^* determined by

$$a_{j^0} x^* = \max_{j \in H} \max_{x \in SO(I_{\max}, G)} a_j x. \tag{a2}$$

Determine $a^*_{j^0}(I_{\max}, \bar{I}_{\max} \setminus G)$ based on solving the problem $P(j^0, I_{\max}, \bar{I}_{\max} \setminus G)$ starting from x^*.

Step 5. If $a^*_{j^0}(I_{\max}, \bar{I}_{\max} \setminus G) > b_{j^0}$, then set $\Omega^1 = \Omega^1 \cup \{G\}$ and go to Step 8. (a3)

If $t = 1$ and $a^*_{j^0}(I_{\max}, \bar{I}_{\max} \setminus G) < b_{j^0}$, then set $\bar{I}^{11}_{\max} = \bar{I}^{11}_{\max} \cup \{j^0\}$.

If $t=1$ and $a^*_{j^0}\left(I_{\max},\bar{I}_{\max}\setminus G\right)=b_{j^0}$, then set $\bar{I}^{12}_{\max}=\bar{I}^{12}_{\max}\cup\left\{j^0\right\}$.

If $SO\left(j^0,I_{\max},G\right)\neq\varnothing$, then set $SO\left(I_{\max},G\right)=SO\left(I_{\max},G\right)\cup SO\left(j^0,I_{\max},G\right)$. (a4)

Step 6. Set $H=H\setminus\left\{j^0\right\}$.

If $H\neq\varnothing$, then go to Step 4.

If $SO\left(I_{\max},G\right)\neq\varnothing$, then set $SO\left(I_{\max}\right)=SO\left(I_{\max}\right)\cup SO\left(I_{\max},G\right)$.

If $t=1$, then go to Step 8.

Let $T^1_4\left(F,I_{\max},G\right)=\left\{\Omega\in T_4\left(F,I_{\max}\right)\middle|\Omega\subseteq G\right\}$ and set

$$T_4\left(F,I_{\max}\right)=T_4\left(F,I_{\max}\right)\setminus T^1_4\left(F,I_{\max},G\right)\cup\left\{G\right\}. \quad \text{(a5)}$$

If $t=\left|\bar{I}^1_{\max}\right|$, then go to Step 8.

Step 7. Let $TG^{t+1}_1\left(G\right)=\left\{G\cup\left\{i\right\}\middle|i\in\bar{I}^1_{\max}\setminus G\right\}$, $TG^{t+1}_2\left(G\right)=\left\{\Omega\in TG^{t+1}\cup TG^{t+1}_1\left(G\right)\middle|\exists S\in\Omega^1:S\subseteq\Omega\right\}$ and set

$$TG^{t+1}=TG^{t+1}\cup TG^{t+1}_1\left(G\right)\setminus TG^{t+1}_2\left(G\right). \quad \text{(a6)}$$

Step 8. $TG^t=TG^t\setminus\left\{G\right\}$.

If $TG^t\neq\varnothing$, go to Step 3.

If $t=1$, then go to Step 2.

If $TG^{t+1}\neq\varnothing$, then set $t=t+1$, $TG^{t+1}=\varnothing$ and go to Step 3.

Step 9. $RE_{\min}\left(F\right)=\left\{\left(K,\bar{I}_{\max}\setminus G\right)\middle|K\in T_1\left(F,I_{\max}\right),G\in T_4\left(F,I_{\max}\right)\right\}$.

Step 10. Stop.

The validity of the algorithm is dealt with in the following property:

Property 4.3: The set $RE_{\min}\left(F\right)$ has been obtained after the final iteration of the algorithm.

Proof: From Theorems 2.4 and 3.4 it is enough to prove that $T_4\left(F,I_{\max}\right)=T_3\left(F,I_{\max}\right)$. If $\bar{I}^1_{\max}=\varnothing$, then the poof is obvious. Now we present the proof in the case when $\bar{I}^1_{\max}\neq\varnothing$. Let t^1 be a maximal integer number such that $TG^{t^1} \neq \varnothing$. We consider an arbitrary subset G of $\bar{I}^1_{\max}$ and will show that if $G \notin \bigcup_{t=1}^{t^1} TG^t$, then $G \notin T_2\left(F, \bar{I}_{\max}\right)$.

Since $G \notin \bigcup_{t=1}^{t^1} TG^t$, there is $S \in \Omega^1$ such that $S \subseteq G$. Noting that $S \notin T_2\left(F, \bar{I}_{\max}\right)$, it is easily seen that

$G \notin T_2\left(F, \bar{I}_{\max}\right)$. Thus, we have $T_2\left(F, \bar{I}_{\max}\right) \subseteq \bigcup_{t=1}^{t^1} TG^t$. It can be easily seen that all elements of $T_2\left(F, I_{\max}\right)$ are found by the algorithm and $T_4\left(F, \bar{I}_{\max}\right) \subseteq T_2\left(F, \bar{I}_{\max}\right)$. In addition, from rule (a5) it follows that the set $T_4\left(F, I_{\max}\right)$ consists of all maximal elements of $T_2\left(F, I_{\max}\right)$ ordered by the inclusion. Therefore, from the definition of $T_3\left(F, \bar{I}_{\max}\right)$ we have $T_4\left(F, \bar{I}_{\max}\right) = T_3\left(F, \bar{I}_{\max}\right)$. □

Properties and advantages of the above algorithm are presented in detail in [15].

5. Examples

Example 5.1: Determine the set of all minimal representations of the face *F* described by index set $I = \varnothing$ of polyhedron (1) when $A = \begin{pmatrix} -1 & -2 & -3 & 0 & -1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 & -1 \\ 1 & 2 & 3 & 1 & 1 & 1 & 1 \end{pmatrix}^T$ and $b = \begin{pmatrix} 1 & 2 & 3 & 2 & 1 & 1 & 1 \end{pmatrix}^T$.

To illustrate the working of the algorithm, in this example the simplex method is used to solve problems of type (14).

Step 1. $I_{\max} = \varnothing$, $T_1\left(F, I_{\max}\right) = \varnothing$.

$t = 1$, $TG^1 = \left\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\}\right\}$, $T_4\left(F, I_{\max}\right) = \varnothing$, $\Omega^1 = \varnothing$, $\bar{I}^{11}_{\max} = \varnothing$, $\bar{I}^{12}_{\max} = \varnothing$, $SO\left(I_{\max}\right) = \varnothing$.

Step 3. Take $G = \{1\}$. $H = \{1\}$.

Step 4. $SO\left(I_{\max}, \{1\}\right) = \varnothing$, take $j^0 = 1$. $SO\left(1, I_{\max}, \{1\}\right) = \left\{(0,0,0)^T, (0,0,1)^T, (-.3333, -1.3333, 0)\right\}$,

$a_1^*\left(I_{\max}, \bar{I}_{\max} \setminus \{1\}\right) = 2$.

Step 5. $a_1^*\left(I_{\max}, \bar{I}_{\max} \setminus \{1\}\right) > 1$, $\Omega^1 = \left\{\{1\}\right\}$.

Step 6. $H = \varnothing$.

$SO\left(I_{\max}\right) = \left\{(0,0,0)^T, (0,0,1)^T, (-.3333, -1.3333, 0)^T\right\}$.

Step 8. $TG^1 = \{\{2\},\{3\},\{4\},\{5\},\{6\},\{7\}\}$.

Step 3. Take $G = \{2\}$. $H = \{2\}$.

Step 4. $SO(I_{\max},\{2\}) = \{(0,0,0)^T, (0,0,1)^T\}$, $j^0 = 2$, $x^* = (0,0,1)^T$, x^* is an optimal solution of problem $P(2, I_{\max}, \bar{I}_{\max} \setminus \{2\})$, $a_2^*(I_{\max}, \bar{I}_{\max} \setminus \{2\}) = 2$. $\bar{I}_{\max}^{12} = \{\{2\}\}$. $SO(2, I_{\max}, \{2\}) = \varnothing$.

Step 6. $H = \varnothing$.

Step 8. $TG^1 = \{\{3\},\{4\},\{5\},\{6\},\{7\}\}$.

Step 3. Take $G = \{3\}$. $H = \{3\}$.

Step 4. $SO(I_{\max},\{3\}) = \{(0,0,0)^T, (0,0,1)^T\}$, $j^0 = 3$, $x^* = (0,0,1)^T$, x^* is an optimal solution of problem $P(3, I_{\max}, \bar{I}_{\max} \setminus \{3\})$, $a_3^*(I_{\max}, \bar{I}_{\max} \setminus \{3\}) = 3$, $\bar{I}_{\max}^{12} = \{\{2\},\{3\}\}$. $SO(3, I_{\max}, \{3\}) = \varnothing$.

Step 6. $H = \varnothing$.

Step 8. $TG^1 = \{\{4\},\{5\},\{6\},\{7\}\}$.

Step 3. Take $G = \{4\}$. $H = \{4\}$.

Step 4. $SO(I_{\max},\{4\}) = \{(0,0,0)^T, (0,0,1)^T\}$, $j^0 = 4$, $x^* = (0,0,1)^T$, x^* is an optimal solution of problem $P(4, I_{\max}, \bar{I}_{\max} \setminus \{4\})$, $a_4^*(I_{\max}, \bar{I}_{\max} \setminus \{4\}) = 1$, $\bar{I}_{\max}^{11} = \{\{4\}\}$. $SO(4, I_{\max}, \{4\}) = \varnothing$.

Step 6. $H = \varnothing$.

Step 8. $TG^1 = \{\{5\},\{6\},\{7\}\}$.

Step 3. Take $G = \{5\}$. $H = \{5\}$.

Step 4. $SO(I_{\max},\{5\}) = \{(0,0,0)^T, (0,0,1)^T\}$, $j^0 = 5$, $x^* = (0,0,1)^T$, $SO(5, I_{\max}, \{5\}) = \{(-2,3,0)^T\}$, $a_5^*(I_{\max}, \bar{I}_{\max} \setminus \{5\}) = 2$.

Step 5. $a_5^*(I_{\max}, \bar{I}_{\max} \setminus \{5\}) > 1$, $\Omega^1 = \{\{1\},\{5\}\}$.

Step 6. $H = \varnothing$.

$SO(I_{\max}) = \{(0,0,0)^T, (0,0,1)^T, (-.3333, -1.3333, 0)^T, (-2,3,0)^T\}$.

Step 8. $TG^1 = \{\{6\},\{7\}\}$.

Step 3. Take $G = \{6\}$. $H = \{6\}$.

Step 4. $SO(I_{\max}, \{6\}) = \{(0,0,0)^T, (0,0,1)^T\}$, $x^* = (0,0,1)^T$, $j^0 = 6$, $P(6, I_{\max}, \bar{I}_{\max} \setminus \{6\})$ is unbounded from above, $a_6^*(I_{\max}, \bar{I}_{\max} \setminus \{6\}) = 2$.

Step 5. $a_6^*(I_{\max}, \bar{I}_{\max} \setminus \{6\}) > 1$, $\Omega^1 = \{\{1\},\{5\},\{6\}\}$.

Step 6. $H = \varnothing$.

Step 8. $TG^1 = \{\{7\}\}$.

Step 3. Take $G = \{7\}$. $H = \{7\}$.

Step 4. $SO(I_{\max}, \{7\}) = \{(0,0,0)^T, (0,0,1)^T\}$, $j^0 = 7$, $x^* = (0,0,1)^T$, $P(7, I_{\max}, \bar{I}_{\max} \setminus \{7\})$ is unbounded from above, $a_7^*(I_{\max}, \bar{I}_{\max} \setminus \{7\}) = 2$.

Step 5. $a_7^*(I_{\max}, \bar{I}_{\max} \setminus \{7\}) > 1$, $\Omega^1 = \{\{1\},\{5\},\{6\},\{7\}\}$.

Step 6. $H = \varnothing$.

Step 8. $TG^1 = \varnothing$.

Step 2. $\bar{I}_{\max}^1 = \{\{2\},\{3\},\{4\}\}$, $SO(I_{\max}) = \{(0,0,0)^T, (0,0,1)^T, (-.3333, -1.3333, 0)^T, (-2,3,0)^T\}$.

$T_4(F, I_{\max}) = \{\{2,4\},\{3,4\}\}$.

$t = 2$, $TG^3 = \varnothing$, $\Omega^1 = \varnothing$, $TG^2 = \{\{2,3\},\{2,4\},\{3,4\}\}$.

Step 3. Take $G = \{2,3\}$. $H = \{2,3\}$.

Step 4. $SO(I_{\max}, \{2,3\}) = \{(0,0,0)^T, (0,0,1)^T\}$, $j^0 = 3$, $x^* = (0,0,1)^T$, x^* is an optimal solution of problem $P(3, I_{\max}, \bar{I}_{\max} \setminus \{2,3\})$, $a_3^*(I_{\max}, \bar{I}_{\max} \setminus \{2,3\}) = 3$.

Step 6. $H = \{2\}$.

Step 4. $SO(I_{\max},\{2,3\})=\left\{(0,0,0)^T,(0,0,1)^T\right\}$, $j^0=2$, $x^*=(0,0,1)^T$, x^* is an optimal solution of problem

$P(2,I_{\max},\bar{I}_{\max}\setminus\{2,3\})$, $a_2^*(I_{\max},\bar{I}_{\max}\setminus\{2,3\})=2$.

Step 6. $H=\varnothing$.

$T_4^1(F,I_{\max},\{2,3\})=\varnothing$, $T_4(F,I_{\max})=\{\{2,3\},\{2,4\},\{3,4\}\}$.

Step 7. $TG_1^3(\{2,3\})=\{\{2,3,4\}\}$, $TG_2^3(\{2,3\})=\varnothing$, $TG^3=\{\{2,3,4\}\}$.

Step 8. $TG^2=\{\{2,4\},\{3,4\}\}$.

Step 3. Take $G=\{2,4\}$. $G\in T_4(F,I_{\max})$.

Step 7. $TG_1^3(\{2,4\})=\{\{2,3,4\}\}$, $TG_2^3(\{2,4\})=\varnothing$, $TG^3=\{\{2,3,4\}\}$.

Step 8. $TG^2=\{\{3,4\}\}$.

Step 3. Take $G=\{3,4\}$. $G\in T_4(F,I_{\max})$.

Step 7. $TG_1^3(\{3,4\})=\{\{2,3,4\}\}$, $TG_2^3(\{3,4\})=\varnothing$, $TG^3=\{\{2,3,4\}\}$.

Step 8. $TG^2=\varnothing$.

$TG^3=\{\{2,3,4\}\}$, $t=3$, $TG^4=\varnothing$.

Go on the same way, we obtain that $T_4(F,I_{\max})=\{\{2,3,4\}\}$, $TG^4=\varnothing$.

Step 9. $RE_{\min}(F)=\{(\varnothing,\{1,5,6,7\})\}$.

Step 10. Stop.

Example 5.2: Find all minimal representations of the face *F* described by index set $I=\varnothing$ of polyhedron (1) when

$A=\begin{pmatrix}1 & -1 & 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1 & -1 & 1\end{pmatrix}^T$ and $b=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T$.

This example is also considered in Boneh et al. [2].

Step 1. $I_{\max}=\{1,2,3,4,5,6\}$, $r(I_{\max})=2$, $T_1(F,I_{\max})=\{\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\}\}\cup\ \{\{2,5\},\{2,6\}\}$

$\cup\{\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\}$, $\bar{I}_{\max}=\varnothing$, $T_4(F,I_{\max})=\varnothing$.

Step 9. $RE_{\min}(F) = \{(\{1,3\},\varnothing),(\{1,4\},\varnothing),(\{1,5\},\varnothing),(\{1,6\},\varnothing),(\{2,3\},\varnothing),(\{2,4\},\varnothing)\}$

$\cup \{(\{2,5\},\varnothing),(\{2,6\},\varnothing),(\{3,5\},\varnothing),(\{3,6\},\varnothing)\} \cup \{(\{4,5\},\varnothing),(\{4,6\},\varnothing),(\{5,6\},\varnothing)\}$.

Remark 5.1: In Example 5.2, the face F has 13 minimal representations in which only one minimal representation $(\{1,3\},\varnothing)$ is shown in Boneh et al. [2].

6. On applications of minimal representations of a face

Most of known methods for finding all maximal efficient faces for a multiple objective linear programming (MOLP) problem are top-down search methods or bottom-up search methods. These methods have certain advantages but still have many drawbacks. A combined top-down and bottom-up search method for determining all maximal efficient faces for an MOLP problem proposed in [14] has all advantages and can improve drawbacks of known bottom-up and top-down search methods. This method is based on a new test for the efficiency of faces and some results given in our recent papers [9] – [13]. The efficiency test helps us construct an efficient combination of a top-down search method and a bottom-up search method for finding all maximal efficient faces and can simultaneously check the efficiency of many faces. By using simple conditions for comparing faces, properties of the new efficiency test and an idea that a combination generated to find all maximal efficient faces emanating from every efficient extreme point, the method in [14] also has many advantages over all known methods for determining all maximal efficient faces of an MOLP problem.

By using a minimal representation of a face to represent it, the sizes and the degeneracy degrees of sub-faces of the face (see [15]) can be reduced (a notion of the degeneracy degrees of sub-faces of a face is a generalization of that of a polyhedron introduced by Sierksma and Tijssen [7]). Therefore, difficulties and the sizes of problems (the size of a problem is the number of the objectives and constraints used to state the problem) can be reduced in solving them. Using a minimal representation of a face to represent it gives us special advantages in many methods, for example, in face search methods, face decomposition based methods, descriptor set based methods (methods are based on descriptor sets for faces), etc.. By using a minimal representation of a face to represent it, we can improve methods for solving a problem for optimizing a function over the efficient set of a multiple objective linear programming (MOLP) problem, for example, Benson and Sayin [1], Sayin [5], Tu [11] and methods for finding the efficient set or determining all maximal efficient faces of an MOLP problem, for example, Ecker et al. [3], Yu and Zeleny [16], Sayin [6], Tu [9]- [14] (see [15] for more detail).

7. Conclusions

A new method for determining all minimal representations of a face of a polyhedron is proposed. To reduce computational efforts in finding all minimal representations of a face, we prove and use properties that deleting strong redundant inequality constraints does not change the redundancy of other constraints and all minimal representations of the face can be determined by finding only the set of all prime representations of the face corresponding to the maximal descriptor index set for it. An algorithm based on a top-down search method is given for finding all minimal representations of a face. This method is simple, easy to implement and has many computational advantages. Based on minimal representations of a face, a reduction of the degeneracy degrees of sub-faces of the face and ideas to improve some known methods for solving a problem for optimizing a function over the efficient set and for finding all maximal efficient faces in multiple objective linear programming are presented.

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