A New Extended Formulation with Valid Inequalities for the Capacitated Concentrator Location Problem

Massimo Di Francesco
Dipartimento di Matematica e Informatica, Università di Cagliari, 09124 Cagliari, Italy
Email: mdifrance@unica.it

Manlio Gaudioso
Dipartimento di Ingegneria Informatica, Modellistica, Elettronica e Sistemistica,
Università della Calabria, 87036 Rende, Italy
Email: manlio.gaudioso@unical.it

Enrico Gorgone
Dipartimento di Matematica e Informatica, Università di Cagliari, 09124 Cagliari, Italy
Indian Institute of Management Bangalore, 560076 Bangalore, India
Email: egorgone@unica.it

Ishwar Murthy
Indian Institute of Management Bangalore, 560076 Bangalore, India,
Email: ishwar@iimb.ac.in

In this paper, we first present a new extended formulation of the Capacitated Concentrator Location Problem (CCLP) using the notion of cardinality of terminals assigned to a concentrator location. The disaggregated formulation consists of O(mn^2) variables and constraints, where m denotes the number of concentrators and n the number of terminals. An immediate benefit of the extended formulation is that it is stronger than the traditional formulation consisting of O(mn) variables and constraints. We also present two classes of inequalities which exploit the cardinality effect of the extended formulation. The first class of inequalities are generalizations of the well-known Cover and (1, k)-Configuration inequalities which collectively are stronger than the original Cover and (1, k)-Configuration inequalities. The second class of inequalities, called the 2-Facility Cardinality Matching Inequality is a facet of the un-capacitated version of the Concentrator Location Problem, which can be lifted to become a strong inequality for CCLP. In our solution approach, we first solve the LP relaxation of the extended formulation. We then use separation heuristics to identify and sequentially add valid inequalities described above to further improve the lower bound. This approach is embedded in a branch-and-bound to obtain the optimal solution. We test our solution approach on a large set of bench-mark problems. We were able to demonstrate clearly that this approach was able to identify the optimal solution at the root node itself in most of the reasonable sized instances. For the much larger sized test problems, the proposed branch-and-cut procedure using the disaggregated formulation outperforms the branch-and-cut procedure applied to the traditional formulation by a significant order both in terms of CPU and the number of branches required to solve the problem to optimality.

Keywords: Integer Programming, Concentrator Location, Valid Inequalities.

1. Introduction

The Capacitated Concentrator Location Problem (CCLP) is a classic problem in network design and has relevant applications in computer networks. The problem is best described as the optimal design of a layered network where a central node is to be connected to a set of terminals through a set of satellite nodes or concentrators [1]. This problem takes on greater importance due to its equivalence to the Single Source Capacitated Facility Location Problem (SSCFLP), which has applications is logistics and supply chain. Given a set $M = \{1,\ldots, m\}$ of potential concentrator locations and a set $N = \{1,\ldots, n\}$ of terminals, a subset of locations in M have to be selected where concentrators will be opened with each terminal in N being assigned to exactly one of the concentrators that are set up. The cost of setting up a concentrator in location i is f_i, while its associated capacity to provide computing resource to terminals is C_i. The cost of serving each terminal j from concentrator location i is c_{ij}, while the terminal's computing demands is d_j. Without loss of generality, one can assume $d_j \leq C_i$. The CCLP is a decision problem in which a set of concentrators in M are opened and each terminal in N is assigned to one of the open concentrators. Such a decision must ensure that terminals are assigned to the opened concentrators such that their capacity is respected. The problem then is to determine the decision that minimizes the sum of the cost of setting up concentrators and the cost of assignment of terminals to concentrators. Let $y_i = 1$ if concentrator $i \in M$ is set up, 0 otherwise. Similarly, let $x_{ij} = 1$ if terminal j is assigned to concentrator i, 0 otherwise. The standard integer programming formulation of CCLP is

($\mathbf{P_{x\text{-}y}}$) *Minimize* $F(x, y) = \sum_{i\in M}\sum_{j\in N} c_{ij}x_{ij} + \sum_{i\in M} f_i y_i$

s.t.

$$\sum_{i\in M} x_{ij} = 1 \qquad \forall j \in N \tag{1}$$

$$x_{ij} \leq y_i \qquad \forall i \in M, j \in N \tag{2}$$

$$\sum_{j\in N} d_j x_{ij} \leq C_i y_i \qquad \forall i \in M \tag{3}$$

$$x_{ij}, y_i \in \{0,1\} \tag{4}$$

In **($\mathbf{P_{x\text{-}y}}$)**, (1) (known as *semi-assignment constraints*) specifies that each terminal in N is to be assigned to exactly one concentrator. Constraints (2), known as Variable Upper Bound (VUB) constraints, ensure that if a concentrator i is not opened, terminal j cannot be assigned to it. Knapsack constraints (3) enforce the capacity restriction on each concentrator i. CCLP is known to be NP-Hard and therefore there has been of considerable interest in developing workable solutions that can solve large instances of it in reasonable time.

1.1 Literature Review

While several articles have been published on solution procedures to solve CCLP, they broadly fall under two categories. The first and perhaps the most prevalent has been the use of Lagrangian relaxation. Boffey [1], Pirkul [13] and Lo and Kershenbaum [11] were one of the earliest to address the solution to CCLP and their variants. Pirkul [13], followed by Pirkul and Nagarajan [14], Sridharan

[16], Holmberg et. Al. [8] and Cortinhal and Captivo [6], all approached this problem by using Lagrangian relaxation. Starting with the formulation (**P$_{x\text{-}y}$**), constraints (1) was dualized, giving rise to a sub-problem that involves solving a series of knapsack problem. While solving knapsack problems are not "easy", it is well known that dualizing (1) instead of (2), provides tighter lower bounds, and therefore overall more beneficial. There exist algorithms that solve the knapsack problems optimally whose computational effort is pseudo-polynomial in the worst case. The papers listed above all used sub-gradient optimization to solve the Lagrangian dual. Their papers differed in the primal heuristic methods used to generate good upper bounds. However, all the approaches essentially used information provided by the Lagrangian dual to obtain good feasible solutions. Here, it is worth mentioning the work of Celani et al. [4], who developed a fast dual-ascent procedure to solve the Lagrangian dual, instead of employing sub-gradient optimization. Chen and Ting [5] supplemented the Lagrangian relaxation approach with Ant Colony method to generate upper bounds. Similarly, Barcelo et al [2], combined Column generation with Lagrangian relaxation to solve CCLP.

Another category of papers focused on the polyhedral structure of formulations of CCLP, introducing strong formulations along with their attendant valid inequalities. These then result in a branch-and-cut approach to find an exact solution to CCLP. Shima [15] considered a generalization of CCLP where in each location i, K concentrator options are possible, each with different costs and capacities. Starting with a formulation similar to (**P$_{x\text{-}y}$**), they introduced z variables as complements of the y variables. This results in constraints similar to (3) into knapsack constraints with the right-hand-side being constant. They then introduced a form of lifted cover inequalities, which are incorporated into a branch-and-cut framework. Labbè and Yaman [9] introduced the Quadratic Capacitated-Concentrator Location (QCL) Problem, whose formulation involves constraints which are quadratic. They studied the polytope of the resulting formulation and developed strong inequalities for it. They then incorporated these inequalities as cuts in a branch-and-cut methodology. Yang et a. [17] considered the formulation (**P$_{x\text{-}y}$**) for CCLP. In their approach, they introduced Lifted Cover Inequalities (LCI) and Fenchel cutting planes (FCI) that arise from (3). They implemented exact separation algorithms for both. Further, they implemented a cut-and-solve approach, wherein the branching is done on a sum of variables, akin to a GUB constraint. Thus, a sub-problem is partitioned into a *sparse* problem and a *dense* problem. The sparse problem being small is solved exactly and therefore fathomed. Subsequent branching, if needed, is only done on the *dense* problem. Gouveia and Saldanha-da-Gama [7] considered a different variant of CCLP in which the demands $d_j = 1$ for all $j \in N$. Thus, constraint (3) amounts to each concentrator i being capable of handling at most C_i terminals. They further considered an extension wherein several capacity options can be chosen at each concentrator location. For this problem, they presented an extended formulation that disaggregates the y variables in (**P$_{x\text{-}y}$**) into various cardinalities, each representing the number of terminals assigned to it. They also presented "≤" and "≥" inequalities for their extended formulation. Finally, Ahuja et al. [1] focus entirely on reliable heuristics to find a good feasible solution to CCPL.

1.2 Contributions of this Paper

In this paper, we first present a new extended formulation of CCLP that induces cardinality of terminals assigned to concentrator locations by appropriately disaggregating x and y variables in (**P$_{x\text{-}y}$**)

(similar to but not the same as in [7]). The disaggregated formulation consists of O(mn^2) variables and constraints. While the disaggregated formulation is larger than that in **(P_{x-y})**, it is stronger, i.e., its LP relaxation provides a tighter lower bound than that provided by **(P_{x-y})**. We next present two classes of inequalities that are specific to the disaggregated formulation. First, generalizations of the well-known *Cover* and (1-*k*) *Configuration* inequalities for the disaggregated formulation are presented. In an accompanying working paper [6], it is shown that these generalizations when added to the disaggregated formulation, provide a tighter lower bound than when their counterparts are added to **(P_{x-y})**. This is because several of the *Cover* and (1-*k*) *Configuration* inequalities in the disaggregated formulation imply stronger knapsack inequalities in **(P_{x-y})**. The second class of inequalities, called the 2-*Facility Cardinality Matching* inequality, is a facet of the un-capacitated version of the Concentrator Location Problem, which can be lifted to become a strong inequality for CCLP. We provide details on how they can be lifted easily. At the root node, first the disaggregated formulation is solved, followed by the addition of violated valid inequalities that are identified by appropriate separation heuristics. If no more violated inequalities can be identified and the LP solution is still fractional, then branching takes place. Thus, our procedure can be characterized as a branch-and-cut approach, wherein all the cuts are added at the root node.

The rest of this paper is organized as follows. In section 2, the disaggregated formulation is formally presented. We also show why the LP relaxation of the disaggregated formulation is tighter than that of **(P_{x-y})**. In section 3, we present the two classes of valid inequalities alluded to earlier: a) generalizations of Cover and (1, *k*) Configuration inequalities, and b) 2-Facility Cardinality Matching inequalities. We also describe briefly, separation heuristics for each. In section 4, we present a detailed computational study that assesses the effectiveness of our approach in terms of strength of the bounds and time required to solve the problems to optimality. We provide concluding remarks in section 5.

2. Extended Formulation of CCLP

Associated with each concentrator location $i \in M$, let K_i denote the maximum number of terminals that the concentrator at location i has the capacity to handle. We disaggregate the model **(P_{x-y})** by separating each location into K_i location-cardinality combinations. Note that for each $i \in M$, K_i can be determined as follows. The demands d_j are first sorted from smallest to largest. K_i then represents the maximum number of terminals that can be accommodated so that the accumulation does not exceed C_i. The following binary variables are now defined:

$$y_{ik_i} = \begin{Bmatrix} 1, \ if\ k_i\ terminals\ are\ assigned\ to\ concentrator\ i, \\ 0,\ otherwise \end{Bmatrix},$$

$$z_{ijk_i} = \begin{Bmatrix} 1, \ if\ terminal\ j\ is\ assigned\ to\ concentrator\ i\ with\ cardinality\ k, \\ 0,\ otherwise. \end{Bmatrix}$$

The extended formulation is:

(P_{z-y})

$$Minimize \qquad F(z,y) = \sum_{i\in M}\sum_{j\in N}\sum_{k_i=1}^{K_i} c_{ij} z_{ijk_i} + \sum_{i\in M}\sum_{k_i=1}^{K_i} f_i y_{ik_i}$$

s.t.

$$\sum_{i\in M}\sum_{k_i=1}^{K_i} z_{ijk_i} = 1 \qquad \forall j \in N \qquad (5)$$

$$z_{ijk_i} \le y_{ik_i} \qquad \forall i \in M, j \in N, k_i = 1, \dots\dots, K_i \qquad (6)$$

$$\sum_{j\in N} d_j z_{ijk_i} \le C_i y_{ik_i} \qquad \forall i \in M, k_i = 1, \dots\dots, K_i \qquad (7)$$

$$\sum_{j\in N} z_{ijk_i} = k_i y_{ik_i} \qquad \forall i \in M, k_i = 1, \dots\dots, K_i \qquad (8)$$

$$\sum_{k_i=1}^{K_i} y_{ik_i} \le 1 \qquad \forall i \in M, \qquad (9)$$

$$z_{ijk_i}, y_{ik_i} \in \{0,1\} \qquad \forall i \in M, j \in N, k_i = 1, \dots\dots, K_i \qquad (10)$$

In **($P_{z\text{-}y}$)**, (5) represents the *semi-assignment* constraints across concentrator locations and cardinality. Constraints (6), known as *variable upper bound* (VUB) constraints ensure that if a concentrator i with designated cardinality k_i is not setup, then terminal j associated with cardinality k_i cannot be assigned to it. Constraints (7) represents the knapsack constraints, defined for each concentrator and cardinality. *Cardinality* constraints (8) enforce the requirement that if a concentrator of a specified cardinality k_i is used, then the number of terminals assigned to it must equal the cardinality. Constraints (9) ensure that at most one cardinality type is used for each concentrator.

We first show that **($P_{z\text{-}y}$)** and **($P_{x\text{-}y}$)** are equivalent. Let $(x^+, y^+)\in IP(x, y) = \{(x, y)\in R^{mn+m}|$ (1)-(3) *are satisfied*$\}$. Further, for some $(x^+, y^+)\in IP(x, y)$, if $y_i^+ = 1$, then let $J(i) = \{j\in N|\ x_{ij}^+ = 1\}$, else $J(i) = \varnothing$. By definition, (i) $J(i_1)\cap J(i_2) = \varnothing$ for all $i_1, i_2 \in M$, $i_1\neq i_2$, (ii) $\cup_{i\in M} J(i) = N$, and (iii) $\sum_{j\in J(i)} d_j \le C_i$ for all $i\in M$. For each $(x^+, y^+)\in IP(x, y)$, a solution $(z^+, y^+)\in IP(z, y) = \{(z,y)\in R^{mn^2+mn} \mid (4)-(9)\ \textit{are satisfied}\}$ can be obtained as follows. For each $i\in M$, if $y_i = 1$, then for $k(i) = |J(i)|$, $y_{ik(i)} = 1$ and $z_{ijk(i)} = 1$ for all $j\in J(i)$, $y_{ik_i} = z_{ijk_i} = 0$ for all $k\neq k(i)$. Clearly, $F(z^+, y^+) = F(x^+, y^+)$. Conversely, for each $(z^+, y^+)\in IP(z, y)$, a solution $(x^+, y^+)\in IP(x, y)$ can be constructed by aggregating (z^+, y^+) as follows:

$$x_{ij}^+ = \sum_{k_i=1}^{K_i} z_{ijk_i}^+ \quad and \quad y_i^+ = \sum_{k_i=1}^{K_i} y_{ik_i}^+, \qquad for\ each\ i \in M, j \in N. \qquad (11)$$

Here as well, $F(x^+, y^+) = F(z^+, y^+)$. We now show that the LP relaxation of **($P_{z\text{-}y}$)** provides a tighter lower bound than the LP relaxation of **($P_{x\text{-}y}$)**. Let,

$LP(x, y) = \{(x, y)\in R^{mn+m} \mid (1)\text{-}(3),\ x \geq 0,\ y \geq 0, \text{ are satisfied}\}$, (12)

$LP(z, y) = \{(z, y)\in R^{mn^2+mn} \mid (5)\text{-}(9),\ z \geq 0,\ y \geq 0, \text{ are satisfied}\}$, (13)

$LP_a(x, y) = \{(x, y)\in R^{mn+m} \mid (11) \text{ are satisfied for each } (z, y)\in LP(z, y)\}$. (14)

Proposition 1. Let $\upsilon(LP(x, y)) = Min\ \{F(x, y) \mid x\in LP(x, y)\}$ and $\upsilon(LP(z, y)) = Min\ \{F(z, y) \mid (z, y)\in LP(z, y)\}$. Then $\upsilon(LP(x, y)) \leq \upsilon(LP(z, y))$.

Proof: To begin with, it is easy to show that $LP_a(x, y) \subseteq LP(x, y)$. Consider a solution $(z', y')\in LP(z, y)$. By aggregating z' and y' as described in (11) we obtain $(x', y')\in LP_a(x, y)$. As well, constraints (5)-(7) get aggregated to become constraints (1)-(3), respectively, which along with $x' \geq 0$ and $y' \geq 0$, are satisfied. Hence, $(x', y')\in LP(x, y)$ and $LP_a(x, y) \subseteq LP(x, y)$. In addition, since the cost coefficients of z_{ijk_i} and y_{ik_i} in **($P_{z\text{-}y}$)** are the same for all indices k_i, it follows that $F(z', y') = F(x', y')$. Hence, $\upsilon(LP(x, y)) \leq \upsilon(LP(z, y))$.

□

It is easy to find instances where $(x, y)\in LP(x, y)$, but $(x, y)\notin LP_a(x, y)$. Consider a $(\hat{x}, \hat{y}) \in LP(x, y)$, where for some pair $(i_1, i_2) \in M$, $\hat{y}_{i_1} = \hat{y}_{i_2} = 1$. Consider sets $J(i_1) \subset N$, $J(i_2) \subset N$, with $J(i_1)\cap J(i_2) = \emptyset$, such that, i) $\sum_{j\in J(i_1)} d_j < C_{i_1}$, ii) $\sum_{j\in J(i_2)} d_j < C_{i_2}$, but there exists a $j_1\in\{N\text{-}J(i_1)\text{-}J(i_2)\}$ such that $\sum_{j\in J(i_1)} d_j + d_{j_1} > C_{i_1}$ but $\sum_{j\in J(i_2)} d_j + d_{j_1} \leq C_{i_2}$. To complete the solution $(\hat{x}, \hat{y})$, $\hat{x}_{i_1 j} = 1$ for $j\in J(i_1)$, $\hat{x}_{i_1 j_1} = \Delta = (C_{i_1} - \sum_{j\in J(i_1)} d_j)/d_{j_1}$, $\hat{x}_{i_2 j} = 1$ for $j\in J(i_2)$ and $\hat{x}_{i_2 j_1} = 1 - \Delta$.

By contradiction, suppose that $(\hat{x}, \hat{y})\in LP_a(x, y)$. Then, $\hat{y}_{i_1} = \sum_{k_i=1}^{K_{i1}} \hat{y}_{i_1 k_i} = 1$. Let $|J(i_1)| = k_1$. Since the number of positive $\hat{x}_{i_1 j}$ variables in $(\hat{x}, \hat{y})$ is k_1+1, it follows from (6) and (8), that $\sum_{k_i=k_1+2}^{K_{i1}} \hat{y}_{i_1 k_i} = 0$. Observe that with regards to i_1, $(\hat{x}, \hat{y})$ satisfies knapsack constraint (3) exactly. Therefore,

$$\sum_{j\in J(i_1)} d_j \left(\sum_{k_i=1}^{k_1+1} \hat{z}_{i_1 j k_i}\right) + d_{j_1}\left(\sum_{k_i=1}^{k_1+1} \hat{z}_{i_1 j_1 k_i}\right) = C_{i_1}\left(\sum_{k_i=1}^{k_1+1} \hat{y}_{i_1 k_i}\right). \tag{15}$$

It is clear from (6) that the only way (15) would hold is if each knapsack constraint (7) associated with i_1 and each $k_i=1,\ldots k_1+1$ is satisfied exactly. However, since $\sum_{j\in J(i_1)} d_j + d_{j_1} > C_{i_1}$, knapsack constraint (7) associated with $k_i = k_1+1$ is violated. Hence $(\hat{x}, \hat{y})\notin LP_a(x, y)$.

3. Valid Inequalities for ($P_{z\text{-}y}$)

In this section, we present three classes of inequalities, i) Cardinality Constrained Cover and (1, $\hat{p}_k$)-Configuration inequalities, ii) Bar-and-Handle inequalities, and iii) 2-Facility Cardinality Matching inequalities. We also present separation heuristics for each one of them.

3.1. Cardinality Constrained Cover and (1, $\hat{p}_k$)-Configuration inequalities

We first present a generalization of the Cover inequality for **($P_{z\text{-}y}$)**, followed by a generalization of the (1, *k*)-Configuration inequality. Let

$$H(x, y) = Conv\{(x, y) \in R^{mn+m} \mid (1) - (4)\} \text{ and } \quad (16)$$

$$H(z, y) = Conv\{(z, y) \in R^{p} \mid (5) - (10), p = \sum_{k=1}^{K_i} m(K_i + 1)\}. \quad (17)$$

A cover inequality that is valid for $H(x, y)$ is defined as follows. Consider a subset $N_i \subseteq N$ with $|N_i| = n_i$ such that,

(i) for all $R_i \subset N_i$, with $|R_i| = r_i$, $\sum_{j \in R_i} d_j \le C_i$,

(ii) however, for all $R_{i+1} \subseteq N_i$, $|R_{i+1}| = r_i+1$, $\sum_{j \in R_{i+1}} d_j > C_i$.

Given the above conditions, a (n_i, r_i)-cover inequality that is valid for $H(x, y)$ is:

$$\sum_{j \in N_i} x_{ij} \le r_i y_i. \quad (18)$$

Definition 1. Consider a knapsack inequality, $\sum_{j \in N} d_j x_j \le C$. For some $D \subseteq N$, and $0 \le k \le |D|$, $V^*(D, k) = Min\{\sum_{j \epsilon D} d_j x_j \mid \sum_{j \epsilon D} x_j = k, x \epsilon B^{|D|}\}$.

Given that *k* items have to be selected, $V^*(D, k)$ represents the cumulative 'demand' if *k* smallest items are selected out of set *D*. Now consider a set N_i as defined in (18) and $k \ge 2$. For some integer $\hat{r}_{ik} < Min\ \{k, n_i\}$ that satisfies the conditions

a) for some $\hat{R}_{ik_i} \subset N_i$, $|\hat{R}_{ik_i}| = \hat{r}_{ik_i}$, $\sum_{j \in \hat{R}_{ik_i}} d_j + V^*(N - \hat{R}_{ik_i}, k_i - \hat{r}_{ik_i}) \le C_i$, (19)

b) but for all $\hat{R}_{ik_i+1} \subseteq N_i$, $|\hat{R}_{ik_i+1}| = \hat{r}_{ik_i}+1$, $\sum_{j \in \hat{R}_{ik_i+1}} d_j + V^*(N - \hat{R}_{ik_i+1}, k_i - \hat{r}_{ik_i} - 1) > C_i$, (20)

the following $(n_i, k_i, \hat{r}_{ik_i})$-*cover* inequality is valid for $H(z, y)$:

$$\sum_{j \in N_i} z_{ijk_i} \le \hat{r}_{ik_i} y_{ik_i}. \quad (21)$$

In an accompanying paper [10], the strength of the $(n_i, k_i, \hat{r}_{ik_i})$-*cover* inequality is discussed in depth, including conditions under which (21) is a facet of $H(z, y)$. Suffice it to say, two properties of the $(n_i, k_i, \hat{r}_{ik_i})$-*cover* inequality stand out vis-à-vis the (n_i, r_i)-cover inequality. First, for a given N_i there is one (n_i, r_i)-cover inequality, while several $(n_i, k_i, \hat{r}_{ik_i})$-*cover* inequalities can be constructed, one for each $k_i \ge 2$. Thus, $(n_i, k_i, \hat{r}_{ik_i})$-*cover* inequalities are a lot more ubiquitous. Second, given that $V^*(N - \hat{R}_{ik_i+1}, k_i - \hat{r}_{ik_i} - 1) \ge 0$, it follows that $\hat{r}_{ik_i} \le r_i$ and (21) is stronger than (18). Consequently, the addition of the $(n_i, k_i, \hat{r}_{ik_i})$-*cover* inequalities to $LP(z, y)$ will provide a tighter lower bound than the addition of (n_i, r_i)-cover inequalities to $LP(x, y)$. The following example compares the relative strength of the two.

Example 1. Consider a concentrator *i* with $C_i = 60$ with demands d_j in sorted order being [15, 15, 14, 14, 13, 12, 8, 8, 7, 5, 4, 4, 4, 3, 3]. For $N_i = \{1,\ldots, 6\}$, a (6, 4)-cover inequality is $\sum_{j \in N_i} x_{ij} \le 4y_i$. For a given N_i, several (n_i, k_i, $\hat{r}_{ik_i}$)-*cover* distinct inequalities can be configured. Observe that since $V^*(N-N_i,4)$= 14, $V^*(N-N_i,5)$= 18 and $V^*(N-N_i,6)$= 23, it follows that *i)* $\sum_{j\in N_i} z_{ij8} \leq 3y_{i8}$ and *ii)* $\sum_{j\in N_i} z_{ij9} \leq 2y_{i9}$. Similarly, when N_i = {1,….,4}, then $\sum_{j\in N_i} z_{ij8} \leq 2y_{i8}$and $\sum_{j\in N_i} z_{ij9} \leq 2y_{i9}$ are valid. We now illustrate possible solutions $(z, y)\in LP(z, y)$ solutions that violate one or more of the (n_i, k_i, $\hat{r}_{ik_i}$)-*cover* inequalities above. More importantly, the corresponding solution $(x, y)\in LP_a(x, y)$ satisfies all the pertinent (n_i, r_i)-cover inequalities. Consider the partial (z, y) solution: y_{i8} = 0.5, $z_{i3,8}$ = 0.1, $z_{i4,8} = z_{i5,8} = z_{i6,8}$ = 0.5, $z_{i11,8}$ = 0.4, $z_{i12,8} = z_{i13,8} = z_{i14,8} = z_{i15,8}$ = 0.5, y_{i9} = 0.5, $z_{i2,9}$ = 0.125, $z_{i3,9} = z_{i4,9}$ = 0.5, $z_{i,9,9}$ = 0.375, $z_{i,10,9} = z_{i,11,9} = z_{i,12,9} = z_{i,13,9} = z_{i,14,9} = z_{i,15,9}$ = 0.5. Clearly, this LP solution satisfies (6)-(9). However inequalities, *i)* $\sum_{j\in N_i} z_{ij8} \leq 3y_{i8}$, where N_i = {1,…, 6} and *ii)* $\sum_{j\in N_i} z_{ij9} \leq 2y_{i9}$, where N_i = {1,…,4}, are both violated. The corresponding $(x, y)\in LP_a(x, y)$ is: y_i = 1, x_{i2} = 0.125, x_{i3} = 0.6, x_{i4} = 1.0, $x_{i5} = x_{i6}$ = 0.5, x_{i9} = 0.375, x_{i10} = 0.5, x_{i11} = 0.9, $x_{i12} = x_{i13} = x_{i14} = x_{i15}$ = 1.0. This solution satisfies the (6, 4)-cover inequality $\sum_{j\in N_i} x_{ij} \leq 4y_i$, where N_i = {1,…, 6}.

□

An additional point is worth noting about the (n_i, k_i, $\hat{r}_{ik_i}$)-*cover* inequalities described above. First, a special case of the (n_i, k_i, $\hat{r}_{ik_i}$)-*cover* inequality is when $\hat{r}_{ik_i}$= 0. In Example 1 above, observe that for $N_i = \{j\in N: d_j \geq 23\}$, $\hat{r}_{i9} = 0$, implying that $z_{ij9} = 0$ for all $j\in N_i$. This is because, $V^*(N-N_i,8) = 38$.

To identify a (n_i, k_i, $\hat{r}_{ik_i}$)-*cover* inequality that a solution to the LP relaxation of (**P$_{z\text{-}y}$**) violates, the following separation heuristic is proposed. First, a *i-k$_i$* combination for which $y_{ik_i} > 0$, along with its support, $S_{ik_i} = \{j \in N | z_{ijk_i} > 0\}$ such that $|S_{ik_i}| = s_{ik_i}$ is identified. Next, items in S_{ik_i} are sorted in increasing order, in terms of the function value, $g_j(y_{ik_i}, z_{ijk_i}, d_j)$ = $(y_{ik} - z_{ijk} + 0.1)/d_j$. This function captures the idea that larger the value of z_{ijk} and d_j, more preferred is it to include j in N_i. A linear search for $N_i \subseteq S_{ik_i}$ begins with n_i = 2. In the first iteration, the most 'preferred' items 1 to n_i are chosen from S_{ik_i} and added to N_i. If (19) or (20) are not satisfied with $\hat{r}_{ik_i}$ = n_i-1, then in the second iteration, items 2 to n_i+1 are chosen. Thus, in each subsequent iteration l, items l to n_i+l-1 are tested for (19) and (20) till n_i+l-1 equals s_{ik_i}. When this occurs, n_i increments by 1 and the search continues. The search terminates when n_i = s_{ik_i}. Whenever both (19) and (20) are satisfied for a given N_i, then $\hat{r}_{ik_i}$ = n_i-1. At this point the cover inequality (21) is lifted by expanding N_i to selectively include items in $\{S_{ik_i}\text{-}N_i\}$ in the following way. First, all items $j\in\{S_{ik_i}\text{-}N_i\}$ whose $d_j \geq Max\ \{d_j | j\in N_i\}$ are included. Next, if for $j\in\{S_{ik_i}\text{-}N_i\}$ such that $d_j < Max\ \{d_j| j\in N_i\}$, $V^*(N_i, \hat{r}_{ik_i}) + d_j + V^*(S_{ik_i}\text{-}N_i, k_i\text{-}\hat{r}_{ik_i}\text{+}1) > C_i$, then j is included in N_i. After this lifting procedure, if the resulting cover inequality (21) is violated, it is added as a cut.

The (1, k)-Configuration inequality is a well-known inequality credited to Padberg [12] that is used to strengthen knapsack constraints. Since in this paper, k refers to cardinality, to avoid confusion, the same inequality is henceforth referred to as (1, p)-configuration inequality. For some concentrator i, let $N_i \subset N$, with $|N_i| = n_i$ and $q\in N\backslash N_i$, be such that

(i) $\sum_{j\in N_i} d_j \leq C_i$, (22)

(ii) for all sets $P \subseteq N_i$, with $|P| = p$ and $2 \leq p \leq n_i$, $\sum_{j\in P\cup q} d_j > C_i$, but (23)

(iii) for all sets $P(p\text{-}1) \subseteq N_i$, with $|P(p\text{-}1)| = p\text{-}1$, $\sum_{j\in P(p-1)\cup q} d_j \leq C_i$. (24)

For some $R_{ik}(r_i) \subseteq N_i$, with $|R_{ik}(r_i)| = r_i$ satisfying $p \le r_i \le n_i$, the (1, p)-configuration inequality that is valid for $H(x, y)$ is,

$$(r_i - p + 1)x_{iq} + \sum_{j \in R_{ik}(r_i)} x_{ij} \le r_i y_i. \tag{25}$$

The (1, p)-configuration inequality (25) essentially enforces the condition that in the absence of q, concentrator i can accommodate all terminals in $R_{ik}(r_i)$. However, in the presence of terminal q, at most (p-1) terminals in $R_{ik}(r_i)$ can be accommodated. This inequality can be generalized into several inequalities by enforcing cardinality requirements in **($\mathbf{P_{z\text{-}y}}$)** as follows. For some terminal-cardinality combination i-k, consider $N_i \subset N$ such that $|N_i| = n_i \le k$ and a $q \in N\text{-}N_i$ that satisfies

i) $\sum_{j \in N_i} d_j + V^*(N - N_i, k - n_i) \le C_i,$ (26)

ii) for every $\hat{P}_k \subset N_i$, with $|\hat{P}_k| = \hat{p}_k$, $\sum_{j \in \hat{P}_k \cup q} d_j + V^*(N\text{-}\hat{P}_k\text{-}q, k\text{-}\hat{p}_k\text{-}1) > C_i$, (27)

iii) but for all $\hat{P}_k(\hat{p}_k - 1) \subset N_i$, $|\hat{P}_k(\hat{p}_k - 1)| = \hat{p}_k$-1,
$\sum_{j \in \hat{P}_k(\hat{p}_k - 1) \cup q} d_j + V^*(N\text{-}\hat{P}_k(\hat{p}_k - 1)\text{-}q,\ k\text{-}\hat{p}_k) \le C_i.$ (28)

Given that (26), (27) and (28) are satisfied by N_i and q, for every $R_{ik}(r_i) \subseteq N_i$ with $\hat{p}_k \le r_i \le n_i$, the following (1, $\hat{p}_k$)-configuration inequality is valid for $H(z, y)$:

$$(r_i - \hat{p}_k + 1)z_{iqk} + \sum_{j \in R_{ik}(r_i)} z_{ijk} \le r_i y_{ik}. \tag{29}$$

The validity of (29) is established by the fact that all the following feasible solutions, i) $y_{ik} = 0$ and $z_{ijk} = 0$ for all $j \in N$ satisfies (29), ii) $y_{ik} = 1$, $z_{ijk} = 1$, $j \in R_{ik}(r_i)$, $z_{iqk} = 0$ satisfies (26) and therefore satisfies (29), and iii) $y_{ik} = 1$, $z_{ijk} = 1$ for $j \in \hat{P}_k(\hat{p}_k - 1)$ for every $\hat{P}_k(\hat{p}_k - 1) \subset N_i$ and $z_{iqk} = 1$ satisfies (28) and therefore (29).

It is worth observing that since $V^*(N\text{-}\hat{P}_k\text{-}q, k\text{-}\hat{p}_k\text{-}1) \ge 0$, $\hat{p}_k \le p$. Alternately, with $V^*(N - R_{ik}(r_i), k - r_i) \ge 0$, the r_i in (29) could be less than the r_i in (25). Thus, (29) dominates (25). Equally important, several distinct (1, $\hat{p}_k$)-configuration inequalities can be constructed from the same set $R_{ik}(r_i) \cup q$, one for each $k \ge r_i$. In [10], the details on conditions under which the (1, $\hat{p}_k$)-configuration inequality is a facet are discussed. The following example best illustrates the strength of the (1, $\hat{p}_k$)-configuration inequality vis-à-vis the (1, p)-configuration inequality.

Example 2. Consider a concentrator i with $C_i = 60$ and terminal demands in sorted order as [22, 15, 15, 14, 14, 13, 12, 8, 8, 7, 5, 4, 4, 4, 3, 3]. For $N_i = \{2,\ldots,6\}$ and $q = 1$, one (1, p)-configuration inequality is: $2x_{i1}+x_{i3}+x_{i4}+x_{i5}+x_{i6} \le 4y_i$, which after sequential lifting becomes $2x_{i1}+x_{i2}+x_{i3}+x_{i4}+x_{i5}+x_{i6}+x_{i7} \le 4y_i$. Automatically, the corresponding (1, $\hat{p}_k$)-configuration inequality for $k = 4$ is, $2z_{i1k}+z_{i2k}+z_{i3k}+z_{i4k}+z_{i5k}+z_{i6k}+z_{i7k} \le 4y_{ik}$. In addition, for the same N_i, several more (1, $\hat{p}_k$)-configuration inequalities can be obtained. For $k = 6$, the (1, $\hat{p}_k$)-configuration inequality is $z_{i1k}+z_{i2k}+z_{i3k}+z_{i4k}+z_{i5k}+z_{i6k} \le 3y_{ik}$, while for $k = 7$, the lifted inequality is $2z_{i1k}+z_{i2k}+z_{i3k}+z_{i4k}+z_{i5k}+z_{i6k}+z_{i7k} \le 3y_{ik}$. The partial LP solution (z, y): $y_{i6} = 0.5$, $z_{i2,6} = z_{i3,6} = 0.25$, $z_{i4,6} = 0.5$, $z_{i5,6} = 0.4$, $z_{i6,6} = 0.5$, $z_{i13,6} = 0.1$, $z_{i14,6} = z_{i15,6} = 0.5$, $y_{i7} = 0.5$, $z_{i1,7} = z_{i2,7} = 0.25$, $z_{i5,7} = z_{i6,7} = z_{i12,7} = z_{i13,7} = z_{i14,7} = z_{i15,7} = 0.5$, satisfies constraints (5)-(9). However, the (1, $\hat{p}_k$)-configuration inequalities listed above for $k = 6$ and for $k = 7$, are both violated. At the same time, the (x, y) solution obtained by aggregating as described in (11) is: $y_i = 1.0$, $x_{i1} = 0.25$, $x_{i2} = 0.5$, $x_{i3} = 0.25$, $x_{i4} = 0.5$, $x_{i5} = 0.9$, $x_{i6} = 1.0$, $x_{i12} = 0.5$, $x_{i13} = 0.6$, $x_{i14} = x_{i15} = 1.0$, which satisfies the (1, p)-configuration inequality listed above.

The separation heuristic proposed to identify violated (1, $\hat{p}_k$)-configuration inequalities is briefly described next. An i-k combination for which $y_{ik} > 0$, along with its support, $S_{ik} = \{j \in N | z_{ijk} > 0\}$ such that $|S_{ik}| = s_{ik} > k$ is first identified. The terminal q is obtained as $d_q = Max\ \{d_j | j \in S_{ik}\}$. Let, R_{ik}

$= S_{ik}$-q. The set N_i and $\hat{p}_k$ in (29) determined from R_{ik} as follows. First, R_{ik} sorted in increasing order, in terms of $g_j(y_{ik_i}, z_{ijk_i}, d_j) = (y_{ik} - z_{ijk} + 0.1)/d_j$. To obtain $\hat{p}_k$, the smallest subset $R_{ik}(\hat{p}_k) \subset R_{ik}$ consisting of the first $\hat{p}_k$ elements in R_{ik} that satisfy $\sum_{j\in R_{ik}(\hat{p}_k)} d_j + V^*(N - R_{ik}(\hat{p}_k) - q, k - \hat{p}_k - 1) > C_i - d_q$ is obtained. Having identified $R_{ik}(\hat{p}_k)$, $N_i = R_{ik}(\hat{p}_k)$. Next, N_i is expanded to include those items $j^* \in \{R_{ik}\text{-}R_{ik}(\hat{p}_k)\}$ that satisfy $V^*(N_i, \hat{p}_k - 1) + d_{j^*} + V^*(\mathrm{N} - R_{ik}(\hat{p}_k - 1) - j^* - q, k - \hat{p}_k - 1) > C_i - d_q$, as well as $\sum_{j\in N_i} d_j + d_{j^*} + V^*(N - N_i - j^*, k - n_i - 1) \le C_i$. Whenever this happens, $N_i = N_i \cup j^*$ and $n_i = n_i + 1$. This process of expanding N_i continues till all items in $\{R_{ik}\text{-}R_{ik}(\hat{p}_k)\}$ have been explored or that $n_i = k$. Having obtained N_i and $\hat{p}_k$, several $(1, \hat{p}_k)$-configuration inequalities (29) can be constructed, each associated with a unique set $R_{ik}(r_i) \subseteq N_i$, for all $\hat{p}_k \le r_i \le n_i$, where $|R_{ik}(r_i)| = r_i$. Amongst these, those that the LP solution violates are added as cuts.

3.2 2-Facility Cardinality Matching Inequality

The 2-Facility Cardinality Matching inequality presented in this section is derived for the un-capacitated version of **($\mathbf{P_{z\text{-}y}}$)**, i.e., one without knapsack constraints (7). Of course, by definition, this inequality is valid for **($\mathbf{P_{z\text{-}y}}$)** as well, albeit weaker. However, as will be shown later, it can be lifted to become a strong inequality for **($\mathbf{P_{z\text{-}y}}$),** and the lifting procedure is quick. Intuitively, this inequality accounts for how terminals are assigned to concentrators (or facilities) in a way that the cardinalities associated with terminals are matched to that of concentrators.

A 2-Facility Cardinality Matching inequality is constructed around a pair of concentrators $W = \{i_1, i_2\}$ and a set of 4 terminals $H_q = \{j_{q1}, j_{q2}, j_{q3}, j_{q4}\}$. We now define a cardinality set for each $i \in M$ to be in the range $[k_i\text{-}r_{il}, k_i\text{+}r_{iu}]$. Ideally, $k_i\text{-}r_{il} = 1$ and $k_i\text{+}r_{iu} = K_i$. However, to ensure that the problem formulation does not become excessively large, the range of cardinality is shortened, wherein the values of r_{il} and r_{iu} are kept reasonably small while still ensuring that the optimal solution is revealed upon solving it. Further, for each $i \in M$, k_i is determined such that when the LP relaxation of **($\mathbf{P_{z\text{-}y}}$)** is solved initially, $0 < y_{ik_i} < 1$. For i_1 such a cardinality is denoted as k_1, while for i_2 the same cardinality is referred to as k_2. Also, let $k_{il} = k_i\text{-}r_{il}$ and $k_{iu} = k_i\text{+}r_{iu}$. The construction of the 2-Facility Cardinality Matching inequality begins by aggregating constraints in (6) over i and k, resulting in

$$\sum_{i\in M}\sum_{j\in N}\sum_{k_i=k_{il}}^{k_{iu}} z_{ijk_i} = \sum_{i\in M}\sum_{k_i=k_{il}}^{k_{iu}} k_i y_{ik_i}. \tag{30}$$

First, a set of z variables are selectively removed from left-hand-side of (30). They are: a) $z_{ij_{q1}k_i}$ with $i=i_1$ and all $k_{i_1l} \le k_{i_1} \le k_1 - 1$, $i = i_2$ with $k_2 + 1 \le k_{i_2} \le k_{i_2u}$, with $i \in M\text{-}W$ and all $k_{il} \le k_i \le k_{iu}$, b) $z_{ij_{q2}k_i}$ with $i=i_2$ and all $k_{i_2l} \le k_{i_2} \le k_2$, $i = i_1$ with $k_1 \le k_{i_1} \le k_{i_1u}$, with $i \in M\text{-}W$ and all $k_{il} \le k_i \le k_{iu}$, c) $z_{ij_{q3}k_i}$ with $i=i_1$ and all $k_{i_1l} \le k_{i_1} \le k_1 - 1$, $i=i_2$ with $k_{i_2l} \le k_{i_2} \le k_2$, with $i \in M\text{-}W$ and all $k_{il} \le k_i \le k_{iu}$, d) $z_{ij_{q4}k_i}$ with $i=i_1$ and all $k_1 \le k_{i_1} \le k_{i_1u}$, $i=i_2$ with $k_2 + 1 \le k_{i_2} \le k_{i_2u}$, with $i \in M\text{-}W$ and all $k_{il} \le k_i \le k_{iu}$. These missing z variables represent 'hidden' assignments. Next, the right-hand-side of (30) is modified wherein the coefficients of variables $y_{i_1k_{i_1}}$ and $y_{i_2k_{i_2}}$ are decreased by 1, for all $k_{i_1l} \le k_{i_1} \le k_{i_1u}$ and $k_{i_2l} \le k_{i_2} \le k_{i_2u}$. Finally, a constant 1 is added to the right-hand-side. This results in the following 2-Facility Cardinality Matching inequality:

$$\sum_{j\in N-\{j_{q1},j_{q3}\}}\sum_{k_{i1}=k_{i_1 l}}^{k_1-1} z_{i_1 j k_{i1}} + \sum_{j\in N-\{j_{q2},j_{q4}\}}\sum_{k_{i1}=k_1}^{k_{i_1 u}} z_{i_1 j k_{i1}} + \sum_{j\in N-\{j_{q2},j_{q3}\}}\sum_{k_{i2}=k_{i_2 l}}^{k_2} z_{i_2 j k_{i2}}$$
$$+ \sum_{j\in N-\{j_{q2},j_{q4}\}}\sum_{k_{i2}=k_2+1}^{k_{i_2 u}} z_{i_2 j k_{i2}} + \sum_{i\in M-W}\sum_{j\in\{N-H_q\}}\sum_{k_i=k_{il}}^{k_{iu}} z_{ijk_i}$$
$$\leq \sum_{k_{i1}=k_{i_1 l}}^{k_{i_1 u}} (k_{i1}-1) y_{i_1 k_{i1}} + \sum_{k_{i2}=k_{i_2 l}}^{k_{i_2 u}} (k_{i2}-1) y_{i_2 k_{i2}} + \sum_{i\in M-W}\sum_{k_i=k_{il}}^{k_{iu}} k_i y_{ik_i} + 1 \quad (31)$$

Figure 1 below illustrates some of those hidden assignments.

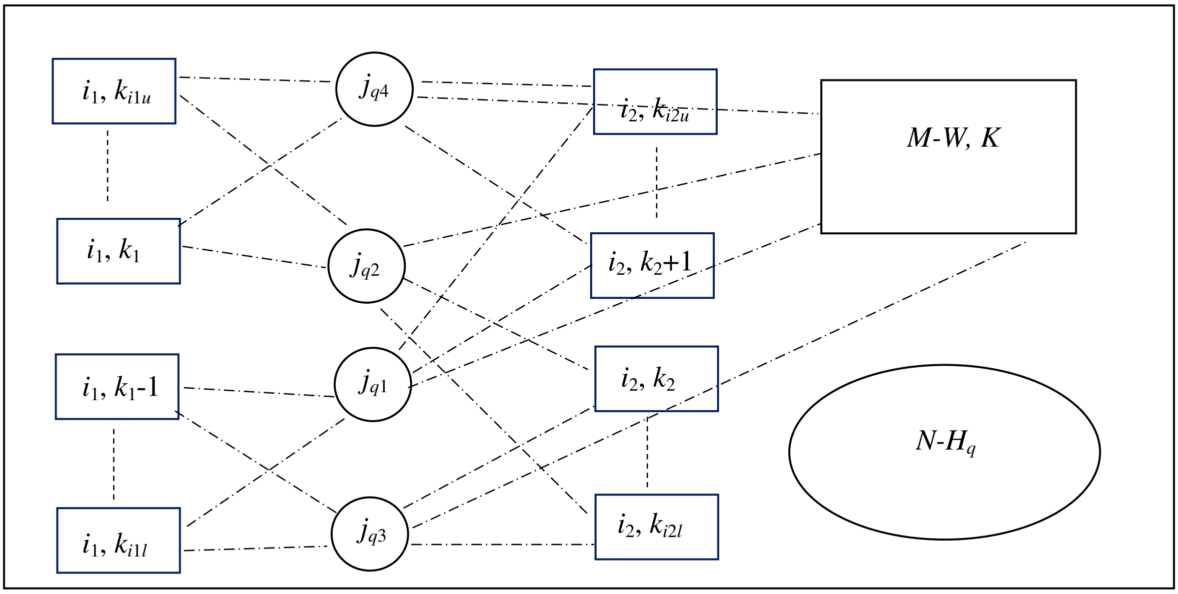


Figure 1. Illustration of hidden assignments in 2-Facility Cardinality Matching Inequality

Proposition 1. Given a pair of concentrators $W = \{i_1, i_2\}$ whose respective, designated cardinalities $\{k_1, k_2\}$, a subset $H_q = \{j_{q1}, j_{q2}, j_{q3}, j_{q4}\}$ of terminals whose assignments are hidden as described above, the 2-Facility Cardinality Matching inequality (31) is valid for $H(z, y)$.

Proof: We simply show that (31) is satisfied by all categories of feasible solutions to (**P**$_{\mathbf{z\text{-}y}}$).

Case I: Consider feasible solutions where no concentrator in W is used. Here, (31) reduces to $\sum_{i\in M-W}\sum_{j\in\{N-H_q\}}\sum_{k_i=k_{il}}^{k_{iu}} z_{ijk_i} \leq \sum_{i\in M-W}\sum_{k_i=k_{il}}^{k_{iu}} k_i y_{ik_i} + 1$. Due to (6), (8) and (9), all such feasible solutions satisfy it.

Case II: Consider feasible solutions in which $y_{i_1 k_{i1}} = y_{i_2 k_{i2}} = 1$, where $k_{i1l} \leq k_{i1} \leq k_1$-1 and k_2+1 $\leq k_{i2} \leq k_{i2u}$. Observe that in this instance, all possible assignments of j_{q1} is hidden. Hence, the left-hand-side value of (31) is at most n-1, with n-k_{i1}-k_{i2} terminals assigned to concentrators in M-W. The right-hand-side is (k_{i1}-1)+(k_{i2}-1)+(n-k_{i1}-k_{i2})+1 = n-1. Hence, (31) is satisfied.

Case III: Consider instances where $y_{i_1 k_{i1}} = y_{i_2 k_{i2}} = 1$, with $k_1 \leq k_{i1} \leq k_{i1u}$ and $k_{i2l} \leq k_{i2} \leq k_2$. Here, all z variables that describe the assignment of j_{q2} to any concentrator is missing. Therefore, here as well, the left-hand-side value of (31) is at most n-1, while the right-hand-side is $(k_{i1}-1)+(k_{i2}-1)+(n-k_{i1}-k_{i2})+1 = n-1$. Therefore, (31) is satisfied.

Case IV: Consider the case of $y_{i_1 k_{i1}} = y_{i_2 k_{i2}} = 1$, with $k_{i1l} \leq k_{i1} \leq k_1-1$ and $k_{i2l} \leq k_{i2} \leq k_2$. Here, j_{q3} is the terminal whose assignment is missing and therefore the left-hand-side of (31) is at most n-1. However, as with Case II and Case III, the right-hand-side of (31) will still be $(k_{i1}-1)+(k_{i2}-1)+(n-k_{i1}-k_{i2})+1 = n-1$ and therefore satisfy (31).

Case V: The next case is when $y_{i_1 k_{i1}} = y_{i_2 k_{i2}} = 1$, with $k_1 \leq k_{i1} \leq k_{i1u}$ and $k_2+1 \leq k_{i2} \leq k_{i2u}$. Here, j_{q4} is the terminal whose assignment is missing and therefore the left-hand-side of (31) is at most n-1, while the right-hand-side will be equal to n-1 and therefore (31) is satisfied.

Case VI: The last possible case is when exactly either $y_{i_1 k_{i1}} = 1$ or $y_{i_2 k_{i2}} = 1$, not both. Observe that in this case, regardless of the value of k_{i1} or k_{i2}, the assignment of exactly two terminals in H_q are hidden. For instance, when $y_{i_1 k_{i1}} = 1$ with $k_{i1l} \leq k_{i1} \leq k_1-1$, then the assignment of j_{q1} and j_{q3} are hidden, while if $k_1 \leq k_{i1} \leq k_{i1u}$ then the assignment of j_{q2} and j_{q4} are hidden. When $y_{i_2 k_{i2}} = 1$, then if $k_{i2l} \leq k_{i2} \leq k_2$, then the assignment of j_{q2} and j_{q3} are hidden, while if $k_2+1 \leq k_{i2} \leq k_{i2u}$, then the assignment of j_{q1} and j_{q4} are hidden. Thus, in all such instances the left-hand-side is at most n-2, while the right-hand-side is n. Hence (31) is satisfied.

□

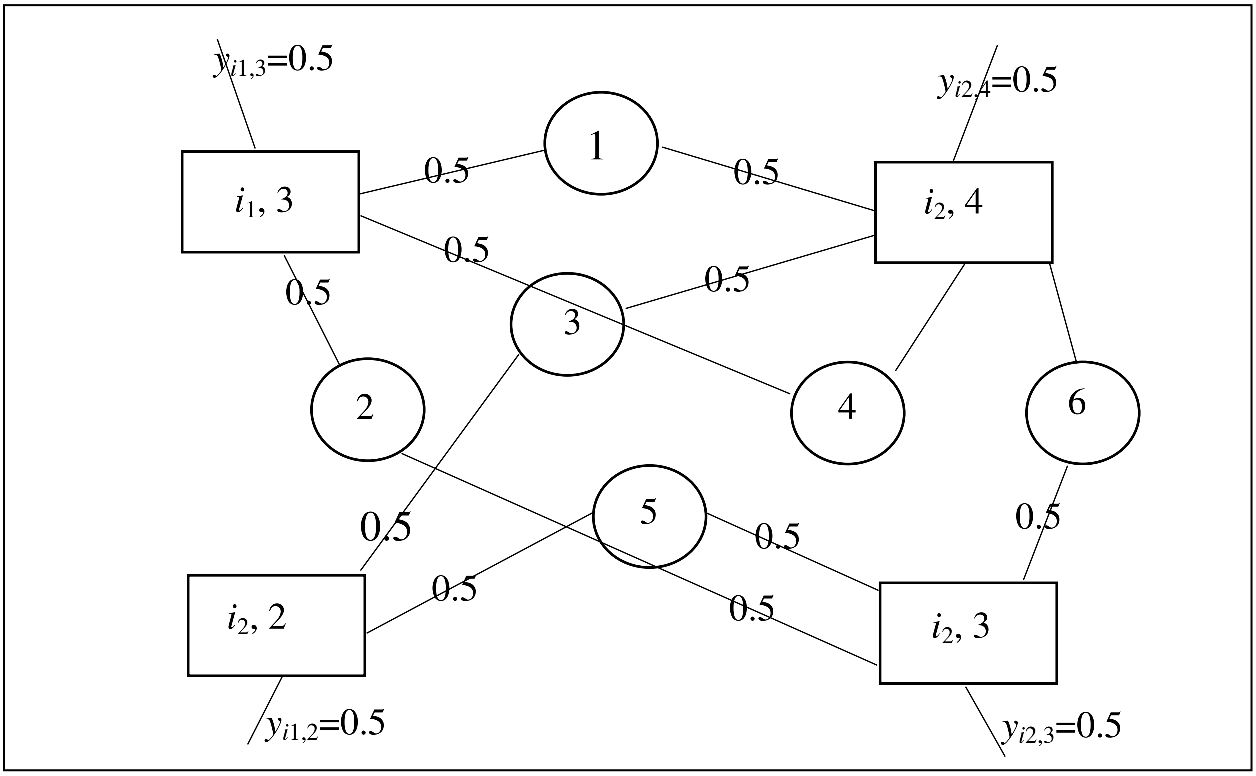
Example 3. Consider the problem instance – $W = \{i_1, i_2\}$, $k_1 = 3$, $k_2 = 3$, $k_{i1l} = 1$, $k_{i1u} = 4$, $k_{i2l} = 2$, $k_{i2u} = 5$, $H_q = \{j_{q1}, j_{q2}, j_{q3}, j_{q4}\} = \{2, 3, 4, 5\}$. The partial LP solution that satisfies the un-capacitated version of $(\mathbf{P_{z\text{-}y}})$ is: $y_{i_1,3} = y_{i_1 2} = y_{i_2 3} = y_{i_2 4} = 0.5$, $z_{i_1,2,3} = z_{i_1,4,3} = z_{i_1,3,2} = z_{i_1,5,2} = z_{i_1 1,3} = 0.5$, $z_{i_2,1,4} =$

$z_{i_2,3,4} = z_{i_2,4,4} = z_{i_2,6,4} = z_{i_2,2,3} = z_{i_2,5,3} = z_{i_2,6,3} = 0.5$. This partial solution is illustrated in Figure 3 below. Observe that this solution satisfies constraints (5), (6), (8) and (9). However, it violates the 2-Facility Cardinality Matching inequality where only the variables that are not zero are listed below:

$$z_{i_1 1,3} + z_{i_1,2,3} + z_{i_1,4,3} + z_{i_1,3,2} + z_{i_1,5,2} + z_{i_2,1,4} + z_{i_2,3,4} + z_{i_2,4,4} + z_{i_2,6,4} + z_{i_2,2,3} + z_{i_2,5,3} + z_{i_2,6,3} + \sum_{i \in M-W} \sum_{j \in \{N-H_q\}} \sum_{k_i=k_{il}}^{k_{iu}} z_{ijk_i} \leq 2y_{i_1,3} + y_{i_1 2} + 2y_{i_2 3} + 3y_{i_2 4} + \sum_{i \in M-W} \sum_{k_i=k_{il}}^{k_{iu}} k_i y_{ik_i} + 1. \quad (32)$$

It is clear from Figure 2 below that the fractional solution listed above violates (32) by 1.

In an accompanying paper [10], we describe in detail the strength of the 2-Facility Cardinality Matching inequality. In that paper we show that the 2-Facility Cardinality Matching inequality is a



facet of the polytope defined by the convex hull of all feasible solutions to the un-capacitated version of **($P_{z\text{-}y}$)**. In addition, we show that for the special case of m = 2, the 2-Facility Cardinality Matching inequalities, along with constraints (5), (7), (8) and (9), completely describe the convex hull of all feasible solutions to the un-capacitated version of **($P_{z\text{-}y}$)**.

Figure 2. Illustration of Example 3.

While the 2-Facility Cardinality Matching inequality is a non-trivial facet of the polytope defined by the feasible solutions of the un-capacitated version of **($P_{z\text{-}y}$)**, it need not be so for the polytope associated with **($P_{z\text{-}y}$)**. This is due to the presence of knapsack constraints (7). The inequality (31) can be strengthened by using a sequential lifting procedure on the missing z variables in it. We now briefly describe a heuristic procedure. Let $z^c = \{z_{ijk_i} | \, i \in M, j \in N, k_{il} \leq k_i \leq k_{iu}\}$ denote the complete z variable set, while z^r denote the restricted variable set of z variables that appear in (31). Specifically, z^r is obtained by removing from z^c those z variables that correspond to hidden assignments of terminals in H_q. In (31), the coefficients of variables belonging to $\{z^c\text{-}z^r\}$ are currently zero. The effort in the lifting procedure is to increase sequentially their coefficients to a positive value, while still ensuring that the inequality remains valid.

Let U_z denote the index set of all z variables in $\{z^c\text{-}z^r\}$, while $V_z \subseteq U_z$ represents those whose coefficients have already been considered for lifting. At stage l of the sequential lifting procedure, the coefficient of $z_{ijk_i}(l)$ for $l \in \{U_z\text{-}V_z\}$ is attempted to be lifted by solving the optimization problem:

$\pi_{ijk_i}(l) = Min$ $\{1 + \Sigma\pi_{yc}y^c - \Sigma z^r - \Sigma\pi_r(V_z)z_r(V_z)|$ $(5^{\leq})$, (6), (7), (8), (9), $z_{ijk_i}(l)$=1, $z_{ijk_i}(s)$=0, $\forall s\in\{U_z$-$V_z\}\}$. (33)

In (33), $(1+\Sigma\pi_{yc}y^c)$ represents the right-hand-side of (31), Σz^r the left-hand-side of (31), $\pi_r(V_z)$ the lifted coefficients of variables in V_z till iteration l-1, and $(5^{\leq})$ represents (5) in ≤ form. To begin with, note that by using $(5^{\leq})$ instead of (5), a relaxation is solved in (33). Further, if (33) is infeasible, i.e. $\pi_{ijk_i}(l)$ =∞, then it implies that $z_{ijk_i}(l)$=0. If (33) is feasible, then $\pi_{ijk_i}(l) \geq 0$, implying that $\Sigma z^r+\Sigma\pi_r(V_z)z_r(V_z) \leq 1+\Sigma\pi_{yc}y^c$, ensuring that the lifted inequality is still valid. Further, at optimality it is ensured that $\Sigma z^r+\Sigma\pi_r(V_z)z_r(V_z) + \pi_{ijk_i}(l) = 1+\Sigma\pi_{yc}y^c$ and therefore the resulting inequality is a facet of the polytope that now includes $z_{ijk_i}(l)$.

The determination of coefficients $\pi_{ijk_i}(l)$ falls into three broad categories: i) those in which $i\in M$-W and $j\in H_q$, ii) $i = i_1$, and iii) $i = i_2$. Consider first the case of $i\in M$-W and $j_q\in H_q$. Suppose that exists a set $S_i(k_i$-1) $\subset N$-H_q, where $|S_i(k_i$-1)$|$ = k_i-1, which along with j_q can be feasibly assigned to i. Without loss of generality, let $k_{i1l} \leq k_{i2l}$. Now suppose as well that there exists a set $S_{i1}(k_{i1l}) \subseteq \{N$-$H_q$-$S_i(k_i$-1)$\}$ that can be feasibly assigned to i_1, then such an assignment solves (33) with $\pi_{ij_qk_i}(l)$ = 0. If no feasible assignment with $\pi_{ij_qk_i}(l)$ = 0 exists, a j_a(a)$\in\{H_q$-$j_q\}$ is chosen to be assigned to either i or to i_1. That is, either $S_i(k_i$-2) $\subset N$-H_q along with j_q and j_q(a) are all assigned to i while $S_{i1}(k_{i1l}) \subseteq \{N$-$H_q$-$S_i(k_i$-2)$\}$ are assigned to i_1, or that $S_i(k_i$-1) $\subset N$-H_q along with j_q are assigned to i while $S_{i1}(k_{i1l}$-1) $\subseteq \{N$-H_q-$S_i(k_i$-1)$\}$ along with $j_q(a)$ are assigned to i_1. If either of these two assignments are feasible, then the optimal value $\pi_{ij_qk_i}(l)$ = 1. If no feasible solution to (33) exists with the assignment of j_q(a) to either i or i_1 then the possibility of a solution that involves assigning a $\{j_q$(a), j_q(b)$\}\in\{H_q$-$j_q\}$ to either i or i_1, while satisfying $(5^{\leq})$, (7), (8) and (9), is explored. If such a solution is feasible, then $\pi_{ij_qk_i}(l)$ = 2. However, if no such feasible solution exists, then the possibility of assigning all three remaining terminals in $\{H_q$-$j_q\}$ to either i or i_1, while satisfying $(5^{\leq})$, (7), (8) and (9), is explored. If such a feasible solution exists, then $\pi_{ij_qk_i}(l)$ = 3, otherwise $\pi_{ij_qk_i}(l)$ = ∞ (i.e., $z_{ij_qk_i}(l)$ = 0). Clearly, the determination of $\pi_{ij_qk_i}(l)$ involves solving a Bin Packing problem consisting of two 'bins', i and i_1, which is NP-Hard. Therefore, we propose solving a relaxation of (33) wherein the two bins are combined to a single bin with a capacity of $(C_i + C_{i_1})$. The respective cardinality requirements of i and i_1 are still maintained at k_i and k_{i1}, respectively. This relaxed problem is easy to solve. It provides a lower bound value of $\bar{\pi}_{ij_qk_i}(l)$ on $\pi_{ij_qk_i}(l)$.

Consider next the case where $i = i_1$. This consists of two sub-cases: a) $k_{i1l} \leq k_{i1} \leq k_1$-1, and b) $k_1 \leq k_{i1} \leq k_{i1u}$. When $k_{i1l} \leq k_{i1} \leq k_1$-1, the variables whose coefficients to be lifted are $z_{i_1j_{q2}k_{i1}}(l)$ and $z_{i_1j_{q4}k_{i1}}(l)$. Consider first the determination of $\pi_{i_1j_{q2}k_{i1}}(l)$. It is worth observing here that if there exists a set $S_{i1}(k_{i1}$-1) $\subset N$-$\{j_{q2}, j_{q4}\}$ with $|$ $S_{i1}(k_{i1}$-1)$|$ = k_{i1}-1, that along with j_{q2} can be feasibly assigned to i_1, and at the same time a set $S_{i2}(k_{i2l}) \subseteq N$-$S_{i1}(k_{i1}$-1)-$\{j_{q1}, j_{q2}, j_{q4}\}$ can be feasibly assigned to i_2, then $\pi_{i_1j_{q2}k_{i1}}(l) = 0$. If no such feasible solution exists, then $\pi_{i_1j_{q2}k_{i1}}(l) = 1$ if there exists a set $S_{i1}(k_{i1}$-2) $\subset N$-$\{j_{q2}, j_{q4}\}$, which along with j_{q2} and j_{q4} can be feasibly assigned to i_1, while at the same time $S_{i2}(k_{i2l}) \subseteq N$-$S_{i1}(k_{i1}$-1)-$\{j_{q1}, j_{q2}, j_{q4}\}$ can be feasibly assigned to i_2. If even this assignment is not possible then $\pi_{i_1j_{q2}k_{i1}}(l) = \infty$. Here as well, the determination of $\pi_{i_1j_{q2}k_{i1}}(l)$ involves solving a 2-Bin Packing problem with the two bins being i_1 and i_2. Therefore, a relaxation is solved by combining the two bins with a capacity of $(C_{i_1} + C_{i_2})$ and obtaining $\bar{\pi}_{i_1j_{q2}k_{i1}}(l)$. The lower bound $\bar{\pi}_{i_1j_{q4}k_{i1}}(l)$ is obtained the same way. For the case in which $k_1 \leq k_{i1} \leq k_{i1u}$, the assignments of j_{q1} and j_{q3} are missing. Here as well, the lower bound values of $\bar{\pi}_{i_1 j_{q1} k_{i1}}(l)$ and $\bar{\pi}_{i_1 j_{q3} k_{i1}}(l)$ are obtained in the same way, with them taking values of either 0, 1 or ∞. This same approach can be replicated for the case of $i = i_2$ with $\bar{\pi}_{i_2 j k_{i2}}(l)$ taking on possible values of 0, 1 or ∞.

The separation heuristic employed to identify a 2-Facility Cardinality Matching inequality (31) that the current LP solution violates is as follows. First, a pair $W = \{i_1, i_2\}$ with respective cardinalities k_1 and k_2 is identified such that $0 < y_{i_1 k_1} < 1$ and $0 < y_{i_2 k_2} < 1$. In addition, $0 < y_{i_1 k_1 - 1} < 1$ and $0 < y_{i_2 k_2 + 1} < 1$. Given this, for each $j \in N$, the sum $SUM_{q1}(j) = \sum_{k_{i_1}=k_{i_1 l}}^{k_1 - 1} z_{i_1 j k_{i_1}} + \sum_{k_{i_2}=k_2+1}^{k_{i_2 u}} z_{i_2 j k_{i_2}} + \sum_{i \in M-W} \sum_{k=k_{il}}^{k_{iu}} z_{ijk}$ is computed, followed by $MINSUM_{q1} = Min_{j \in N}\{SUM_{q1}(j)\}$. Terminal j_{q1} is the designated terminal that corresponds to $MINSUM_{q1}$. Clearly, $MINSUM_{q1} = 0$ implies that the assignment of j_{q1} is hidden from (i_1, k_{i_1}) for $k_{i_1 l} \leq k_{i_1} \leq k_1 - 1$, (i_2, k_{i_2}) for $k_2 + 1 \leq k_{i_2} \leq k_{i_2 u}$, and (i, k_i) for all $i \in M\text{-}W$ and all cardinalities. Similarly, $SUM_{q2}(j) = \sum_{k_{i_1}=k_1}^{k_{i_1 u}} z_{i_1 j k_{i_1}} + \sum_{k_{i_2}=k_{i_2 l}}^{k_2} z_{i_2 j k_{i_2}} + \sum_{i \in M-W} \sum_{k=k_{il}}^{k_{iu}} z_{ijk}$, followed by $MINSUM_{q2} = Min_{j \in N}\{SUM_{q2}(j)\}$ determines j_{q2}. To determine j_{q3}, the sum used is $SUM_{q3}(j) = \sum_{k_{i_1}=k_{i_1 l}}^{k_1 - 1} z_{i_1 j k_{i_1}} + \sum_{k_{i_2}=k_{i_2 l}}^{k_2} z_{i_2 j k_{i_2}} + \sum_{i \in M-W} \sum_{k=k_{il}}^{k_{iu}} z_{ijk}$, with j_{q3} corresponding to $MINSUM_{q3} = Min_{j \in N}\{SUM_{q3}(j)\}$. Finally, j_{q4} corresponds to $MINSUM_{q4} = Min_{j \in N}\{SUM_{q4}(j)\}$ where $SUM_{q4}(j) = \sum_{k_{i_1}=k_1}^{k_{i_1 u}} z_{i_1 j k_{i_1}} + \sum_{k_{i_2}=k_2+1}^{k_{i_2 u}} z_{i_2 j k_{i_2}} + \sum_{i \in M-W} \sum_{k=k_{il}}^{k_{iu}} z_{ijk}$.

If $MINSUM = MINSUM_{q1} + MINSUM_{q2} + MINSUM_{q3} + MINSUM_{q4} < 1$, then (31) is added as a cut. If MINSUM ≥ 1, then the lifting procedures described earlier can be employed on the hidden variables.

4.0 Computational Results

In this section we present a detailed computational study that compares the performance of our proposed branch-and-cut algorithm that uses the disaggregated formulation **($P_{z\text{-}y}$)** along with the valid inequalities described in this paper to a generic branch-and-cut algorithm applied to the traditional formulation **($P_{x\text{-}y}$)**. The proposed branch & cut method has been developed in CPLEX 12.7, compiled with GNU g++ 4.4.5 (with -O3 optimization option) and ran single-threaded on a machine with 8 processors (4 cores, 2.2 GHz), each with 16 GB of RAM, under a i686 GNU/Linux operating system. The separation algorithms for the valid inequalities along with the disaggregated model have been implemented in CPLEX using the call-backs.

Computationally, a potential disadvantage of using **($P_{z\text{-}y}$)** over **($P_{x\text{-}y}$)** is its large size, even though the increase is polynomial in nature. This can be of significant concern for large problems. In this paper, we addressed this issue 'locally' as follows. First, the LP relaxation of **($P_{x\text{-}y}$)** is solved. Next, we measure $\chi_i = \sum_{j \in N} x_{ij}$, which in some sense, represents the 'number' of terminals assigned to concentrator i, in the LP solution. Therefore, χ_i for each $i \in M$ can be fractional. The y and x variable for each i are now split into a prespecified number of cardinalities or levels. In this paper, we tested our approach with these prespecified levels L_i being 3, 7 and 11. To illustrate, with $L_i = 3$, the three cardinalities used are: i) $k_{i1} = \lfloor \chi_i \rfloor - 1$, ii) $k_{i2} = \lfloor \chi_i \rfloor$ and iii) $k_{i3} = \lfloor \chi_i \rfloor + 1$ for each $i \in M$. Accordingly, the variable sets $\{y_{ik_{i1}}, y_{ik_{i2}}, y_{ik_{i3}}\}$ and $\{z_{ijk_{i1}}, z_{ijk_{i2}}, z_{ijk_{i3}}\}$ are defined for each $i \in M$. Using these variables, constraints (6) and (7) can be written as:

$$z_{ijk_l} \leq y_{ik_l} \qquad \forall\, i \in M, j \in N,\, l = i1, i2, i3 \tag{34}$$

$$\sum_{j \in N} d_j z_{ijk_l} \leq C_i y_{ik_l} \qquad \forall i \in M, l = i1, i2, i3. \tag{35}$$

Finally, the cardinality constraints (8) take the form:

$$\sum_{j \in N} z_{ijk_{i1}} \leq k_{i1} y_{ik_{i1}}, \qquad \forall\, i \in M \tag{36}$$

$$\sum_{j \in N} z_{ijk_{i2}} = k_{i2} y_{ik_{i2}}, \qquad \forall\, i \in M \tag{37}$$

$$\sum_{j \in N} z_{ijk_{i3}} \geq k_{i3} y_{ik_{i3}}. \qquad \forall\, i \in M \tag{38}$$

It is worth observing that the use of constraints (36), (37) and (38), in place of (8), makes the resulting formulation a relaxation of (**$P_{z\text{-}y}$**). However, as our computational results show, the slight loss in the lower bound is more than made up by a dramatic reduction in the size of the formulation. It is also worth mentioning that the inequalities described in this paper apply to all levels except $l = i1$. When L_i = 7, then there are seven levels with (37) defined for l = 2,…, 6. With L_i = 11, there will be eleven levels with (37) defined for l = 2,…,10.

The branch-and-cut approaches described above were tested on 36 medium to large size problem instances. These instances can be found in http://people.brunel.ac.uk/~mastjjb/jeb/info.html. The problem size instances can be found in Table 3. The 1st eight instances have sizes of $|M|$ = 16 and $|N|$ = 50, followed by the next eight instances having sizes of $|M|$ = 25 and $|N|$ = 50. The 3rd set of eight problems comprised of instances with sizes of $|M|$ = 50 and $|N|$ = 50. Finally, we tested a set of 12 problem instances with sizes of $|M|$ = 100 and $|N|$ = 1000.

Table 1. Comparison of bounds obtained at the root node between the Traditional formulation and Disaggregated Formulation with L_i = 3

Problems	*Traditional Formulation*					*Disaggregated Formulation - L_i = 3*					
	before cutting		*after cutting*			*before cutting*		*after cutting*			
# Name	Time (sec)	GAP	Time (sec)	#CPX	GAP	Time (sec)	GAP	Time (sec)	#user	#CPX	GAP
Pr.VI – cap61	0.00	7.7497%	0.01	50	0.0536%	0.01	0.0000%	0.01	0	0	0.0000%
Pr.VI – cap62	0.00	10.6211%	0.01	71	0.0852%	0.01	0.0000%	0.01	0	0	0.0000%
Pr.VI – cap63	0.00	12.4061%	0.01	85	0.0148%	0.01	0.1011%	0.05	1	5	0.0135%
Pr.VI – cap64	0.00	13.3786%	0.03	113	0.0799%	0.02	0.7154%	0.09	31	6	0.1060%
Pr.VII – cap71	0.00	10.3600%	0.00	60	0.0000%	0.01	0.0000%	0.01	0	0	0.0000%
Pr.VII – cap72	0.00	15.0625%	0.01	74	0.0000%	0.01	0.0000%	0.02	0	0	0.0000%
Pr.VII – cap73	0.00	18.2687%	0.01	92	0.0000%	0.01	0.0000%	0.02	0	0	0.0000%
Pr.VII – cap74	0.00	20.1189%	0.01	105	0.0000%	0.01	0.0000%	0.02	0	0	0.0000%

Pr.IX – cap91	0.00	17.0337%	0.01	80	0.0000%	0.01	0.0000%	0.01	0	0	0.0000%
Pr.IX – cap92	0.00	22.6502%	0.01	106	0.0000%	0.02	0.3534%	0.04	2	3	0.0000%
Pr.IX – cap93	0.00	25.3742%	0.02	136	0.2386%	0.02	0.6567%	0.06	2	6	0.2319%
Pr.IX – cap94	0.00	27.4029%	0.03	155	0.1439%	0.02	0.8873%	0.07	4	3	0.1619%
Pr.X – cap101	0.00	20.7767%	0.01	82	0.0000%	0.01	0.0000%	0.02	0	0	0.0000%
Pr.X – cap102	0.00	28.6275%	0.01	101	0.0000%	0.02	0.0000%	0.02	0	0	0.0000%
Pr.X – cap103	0.00	33.5287%	0.01	141	0.0000%	0.01	0.0000%	0.02	0	0	0.0000%
Pr.X – cap104	0.00	37.2816%	0.01	166	0.0000%	0.01	0.0000%	0.02	0	0	0.0000%
Pr.XII – cap121	0.00	21.5199%	0.02	187	0.0000%	0.01	0.0000%	0.02	0	0	0.0000%
Pr.XII – cap122	0.00	27.1854%	0.02	254	0.0000%	0.01	0.3547%	0.06	2	3	0.0000%
Pr.XII – cap123	0.00	29.9184%	0.05	300	0.0531%	0.01	0.4339%	0.14	2	6	0.0224%
Pr.XII – cap124	0.00	32.0601%	0.03	300	0.9703%	0.02	0.9007%	0.12	4	4	0.1927%
Pr.XIII – cap131	0.00	25.6435%	0.02	195	0.0004%	0.01	0.0000%	0.02	0	0	0.0000%
Pr.XIII – cap132	0.00	33.7876%	0.03	259	0.0000%	0.02	0.0000%	0.03	0	0	0.0000%
Pr.XIII – cap133	0.00	39.2374%	0.04	300	0.0389%	0.01	0.0000%	0.02	0	0	0.0000%
Pr.XIII – cap134	0.00	43.1707%	0.03	300	1.3070%	0.02	0.0000%	0.02	0	0	0.0000%
Pr.A – capa1	0.21	39.2222%	85.87	3279	5.3554%	373.50	2.1672%	373.50	0	0	2.1672%
Pr.A – capa2	0.20	57.0841%	273.20	3276	6.0115%	218.02	3.1074%	218.02	0	0	3.1074%
Pr.A – capa3	0.16	71.6843%	506.51	3252	5.7933%	146.79	1.8431%	146.79	0	0	1.8431%
Pr.A – capa4	0.14	83.7912%	478.48	3273	2.1163%	36.98	0.0010%	41.05	0	2	0.0006%
Pr.B – capb1	0.16	35.8444%	27.56	3297	2.5636%	197.75	0.3750%	197.75	0	0	0.3750%
Pr.B – capb2	0.15	49.9976%	83.62	3295	4.8634%	157.77	0.4060%	157.77	0	0	0.4060%
Pr.B – capb3	0.14	63.3218%	161.31	3296	5.0810%	135.45	0.3331%	135.45	0	0	0.3331%
Pr.B – capb4	0.12	75.4272%	187.14	3260	5.1424%	82.49	0.0929%	82.49	0	0	0.0929%
Pr.C – capc1	0.13	45.7482%	35.19	3291	3.6659%	90.35	0.5361%	125.48	0	2	0.5350%
Pr.C – capc2	0.13	57.8666%	60.58	3274	4.1103%	64.03	0.4205%	78.59	0	2	0.4085%
Pr.C – capc3	0.12	68.9805%	100.27	3290	3.5686%	43.05	0.0920%	52.25	0	2	0.0797%
Pr.C – capc4	0.11	79.5210%	136.89	3283	3.3855%	26.49	0.0500%	33.47	0	2	0.0404%

In Table 1, the results present a comparison of the performance of the disaggregated formulation with $L_i = 3$ and that of the traditional formulation. Specifically, the gap between the best lower bound and upper bound in percentage terms, is measured at two points in time – once before any cuts are added to the respective LP relaxations, and once after all possible cuts are added at the root node of the branch-and-bound tree. Also measured are the total CPU time in seconds at both these points in time. In the case of the traditional formulation, CPLEX generates a variety of cuts using its separation heuristics. The number of such cuts added for each instance are reported under the heading #CPX. In the case of the disaggregated formulation, we first look for violated $(n_i, k_i, \hat{r}_{ik_i})$-*cover*, $(1, \hat{p}_k)$-

configuration and 2-Facility Cardinality Matching inequalities. The number of such cuts added to the disaggregated formulation is reported under the heading #user.

A feature that stands out from the results in Table 1 is that the bounds obtained from the initial LP relaxation of the traditional formulation is poor. The percentage gap varies from 7.75% to 79.52%. In contrast, bounds obtained from the initial LP relaxation of the disaggregated formulation with $L_i = 3$ are very tight. In fact, in 16 of the 36 problem instances, the optimal integer solution was obtained from the initial LP relaxation. Even among the remaining 20 instances, in all but 3 of those instances, the percentage gap was found to be less than 1%. Consequently, for the traditional formulation, a large number of cuts was generated by CPLEX. These numbers varied from 50 for the smallest instance to more than 3000 cuts for all the big problem instance. Computationally, this translates to that many re-optimisations of the resulting LPs. In contrast, very few cuts were generated for the disaggregated formulation. With $L_i = 3$, all the cuts generated were of the type, (n_i, k_i, $\hat{r}_{ik_i}$)-*cover*, (1, $\hat{p}_k$)-configuration, besides the cuts generated by CPLEX. Because of the large number of CPLEX cuts generated for the traditional formulation, the overall time taken at the root node for the traditional formulation is of the same order of magnitude as that for the disaggregated formulation. This in spite of the disaggregated formulation being much larger than the traditional formulation. Finally, the percentage gap obtained after all the cuts were added at the root node was found to be smaller for the disaggregated formulation as compared to the traditional formulation in 33 of the 36 instances. More importantly, for the large problem instances, percentage gap was found to be significantly higher for the traditional formulation as compared to the disaggregated formulation.

In Table 2, the same set of measurements that were made for Table 1, are reported for the disaggregated formulation with $L_i = 7$ and $L_i = 11$. Clearly, the disaggregated model sizes are larger with $L_i = 7$ and with $L_i = 11$. As against $L_i = 3$, both for $L_i = 7$ and $L_i = 11$, a few more (n_i, k_i, $\hat{r}_{ik_i}$)-*cover* and (1, $\hat{p}_k$)-configuration as well as a few 2-Facility Cardinality matching inequalities were generated. However, the overall improvement in the lower bound was found to be marginal, given that in most cases, the lower bounds were already very close to the optimal value. Of course, the time taken to solve the LP relaxations for $L_i = 7$ and $L_i = 11$ are much more.

Table 2: Bounds obtained at the root node for the Disaggregated Formulation with L_i = 7 and L_i = 11.

Problems	*Disaggregated Formulation – L_i = 7*						*Disaggregated Formulation – L_i =11*					
	before cutting		*after cutting*				*before cutting*		*after cutting*			
# Name	Time (sec)	GAP	Time (sec)	#user	#CPX	GAP	Time (sec)	GAP	Time (sec)	#user	#CPX	GAP
Pr.VI – cap61	0.04	0.0000%	0.04	0	0	0.0000%	0.02	0.0000%	0.03	0	0	0.0000%
Pr.VI – cap62	0.04	0.0000%	0.05	0	0	0.0000%	0.03	0.0000%	0.03	0	0	0.0000%
Pr.VI – cap63	0.04	0.1011%	0.17	3	8	0.0135%	0.05	0.1011%	0.15	3	9	0.0117%
Pr.VI – cap64	0.05	0.7151%	0.23	67	2	0.1141%	0.06	0.7151%	0.64	40	2	0.1265%

Pr.VII – cap71	0.03	0.0000%	0.04	0	0	0.0000%	0.03	0.0000%	0.03	0	0	0.0000%
Pr.VII – cap72	0.03	0.0000%	0.03	0	0	0.0000%	0.03	0.0000%	0.04	0	0	0.0000%
Pr.VII – cap73	0.04	0.0000%	0.04	0	0	0.0000%	0.03	0.0000%	0.04	0	0	0.0000%
Pr.VII – cap74	0.03	0.0000%	0.04	0	0	0.0000%	0.03	0.0000%	0.04	0	0	0.0000%
Pr.IX – cap91	0.02	0.0000%	0.02	0	0	0.0000%	0.03	0.0000%	0.04	0	0	0.0000%
Pr.IX – cap92	0.02	0.3534%	0.09	18	4	0.0000%	0.03	0.3521%	0.17	6	3	0.0000%
Pr.IX – cap93	0.03	0.6567%	0.11	16	5	0.2395%	0.06	0.6555%	0.22	8	8	0.2386%
Pr.IX – cap94	0.06	0.8873%	0.22	20	3	0.0594%	0.11	0.8840%	0.40	10	5	0.1624%
Pr.X – cap101	0.02	0.0000%	0.03	0	0	0.0000%	0.03	0.0000%	0.04	0	0	0.0000%
Pr.X – cap102	0.02	0.0000%	0.03	0	0	0.0000%	0.03	0.0000%	0.05	0	0	0.0000%
Pr.X – cap103	0.02	0.0000%	0.03	0	0	0.0000%	0.03	0.0000%	0.05	0	0	0.0000%
Pr.X – cap104	0.02	0.0000%	0.04	0	0	0.0000%	0.04	0.0000%	0.06	0	0	0.0000%
Pr.XII – cap121	0.03	0.0000%	0.04	0	0	0.0000%	0.04	0.0000%	0.06	0	0	0.0000%
Pr.XII – cap122	0.04	0.3547%	0.21	18	3	0.0000%	0.06	0.3534%	0.33	6	3	0.0000%
Pr.XII – cap123	0.05	0.4339%	0.31	18	6	0.0224%	0.08	0.4327%	0.52	8	8	0.0224%
Pr.XII – cap124	0.07	0.9007%	0.42	20	2	0.0818%	0.16	0.8947%	0.70	10	5	0.1927%
Pr.XIII – cap131	0.03	0.0000%	0.04	0	0	0.0000%	0.04	0.0000%	0.06	0	0	0.0000%
Pr.XIII – cap132	0.03	0.0000%	0.05	0	0	0.0000%	0.06	0.0000%	0.08	0	0	0.0000%
Pr.XIII – cap133	0.04	0.0000%	0.05	0	0	0.0000%	0.06	0.0000%	0.08	0	0	0.0000%
Pr.XIII – cap134	0.05	0.0000%	0.06	0	0	0.0000%	0.07	0.0000%	0.10	0	0	0.0000%
Pr.A – capa1	1995.86	2.1672%	1988.94	0	0	2.1672%	-----	-----	-----	-----	-----	-----
Pr.A – capa2	1098.61	3.1074%	1148.92	0	0	3.1074%	-----	-----	-----	-----	-----	-----
Pr.A – capa3	753.91	1.8431%	789.24	0	0	1.8431%	1518.21	1.8431%	1518.21	0	0	1.8431%
Pr.A – capa4	205.82	0.0010%	210.67	0	2	0.0006%	158.46	0.0010%	375.30	0	2	0.0006%
Pr.B – capb1	1192.25	0.3750%	1191.88	0	0	0.3750%	2598.40	0.3750%	2598.40	0	0	0.3750%
Pr.B – capb2	798.36	0.4060%	855.19	0	0	0.4060%	2080.06	0.4060%	2080.06	0	0	0.4060%
Pr.B – capb3	575.48	0.3331%	634.25	0	0	0.3331%	1318.68	0.3331%	1318.68	0	0	0.3331%
Pr.B – capb4	432.83	0.0929%	467.15	0	0	0.0929%	973.98	0.0929%	973.98	0	0	0.0929%
Pr.C – capc1	410.37	0.5361%	603.31	0	3	0.5086%	750.57	0.5361%	1558.26	0	3	0.5204%
Pr.C – capc2	337.14	0.4205%	340.28	0	1	0.4138%	829.35	0.4205%	829.35	0	0	0.4205%
Pr.C – capc3	232.28	0.0920%	258.18	0	0	0.0920%	339.66	0.0920%	505.53	0	1	0.0863%
Pr.C – capc4	179.74	0.0500%	186.18	0	0	0.0500%	244.33	0.0500%	434.58	0	1	0.0311%

Table 3 presents results on the same set of 36 problem instances in which the branch-and-cut procedure using the traditional formulation is compared to the branch-and-cut procedure using the disaggregated formulation with $L_i = 3$. In both approaches, the tolerance limit on the gap between the lower bound and the incumbent solution was set at e^{-6}. In addition, both the procedures were aborted, as soon as the time taken reached 1 hour. In the table, the total number of branch-and-bound nodes visited is reported under column labelled *#nodes*. It is clear from the table that the performance of the branch-and-cut procedure using the disaggregated formulation is found to be superior to the branch-and-cut procedure using the traditional formulation, both in terms of time taken and the number of branch-and-bound nodes visited. What is indeed significant is that the dominance of our proposed procedure becomes even more pronounced for large problem instances. For instance, in problem labelled "Pr.A – capa4", our procedure solved the problem in 199.35 seconds using 101 branch-and-bound node visits. In contrast, in the traditional formulation, the total time taken was 2160.63 seconds with 746 branch-and-bound node visits. In the case of problem instance PrC – capc2, our branch-and-cut procedure took 298.24 seconds with 78 branch-and-bound node visits, while the branch-and-cut procedure using the traditional formulation took 3104.63 seconds with 19070 branch-and-bound node visits. In 6 of the 12 large problem instances, the branch-and-cut procedure using the traditional formulation had to be aborted before solving to optimality as time limit of 3600 seconds was reached. In contrast, our proposed procedure had to be aborted in just 2 of the 12 instances. In these two problem instances, the percentage gap obtained from our procedure was found to be smaller than that obtained using the traditional formulation. In summary, the computational results presented confirm the value of using the disaggregated formulation along with its associated valid inequalities for solving large size and difficult problem instances to optimality over the traditional formulation.

Table 3. Comparison in terms the total time taken and the # of branch-and-bound nodes visited for the Traditional formulation versus Disaggregated Formulation with L_i = 3

Problems			*Traditional Formulation*				*Disaggregated Formulation - L_i = 3*				
# Name	*M*	*N*	*Time* (sec)	*#CPX*	*#nodes*	*GAP*	*Time* (sec)	*#user*	*#CPX*	*#nodes*	*GAP*
Pr.VI – cap61	16	50	0.02	42	12	0.0000%	0.01	0	0	1	0.0000%
Pr.VI – cap62	16	50	0.02	48	1	0.0000%	0.01	0	0	1	0.0000%
Pr.VI – cap63	16	50	0.04	74	25	0.0000%	0.07	1	5	3	0.0000%
Pr.VI – cap64	16	50	0.05	110	54	0.0000%	0.32	6	9	6	0.0000%
Pr.VII – cap71	16	50	0.01	68	1	0.0000%	0.01	0	0	1	0.0000%
Pr.VII – cap72	16	50	0.01	75	1	0.0000%	0.01	0	0	1	0.0000%
Pr.VII – cap73	16	50	0.01	98	1	0.0000%	0.02	0	0	1	0.0000%
Pr.VII – cap74	16	50	0.01	98	1	0.0000%	0.01	0	0	1	0.0000%
Pr.IX – cap91	25	50	0.02	83	1	0.0000%	0.02	0	0	1	0.0000%
Pr.IX – cap92	25	50	0.04	73	6	0.0000%	0.04	2	3	1	0.0000%
Pr.IX – cap93	25	50	0.05	77	21	0.0000%	0.50	2	8	52	0.0000%
Pr.IX – cap94	25	50	0.07	149	59	0.0000%	0.49	2	6	40	0.0000%
Pr.X – cap101	25	50	0.01	98	1	0.0000%	0.01	0	0	1	0.0000%
Pr.X – cap102	25	50	0.02	102	1	0.0000%	0.02	0	0	1	0.0000%
Pr.X – cap103	25	50	0.01	154	1	0.0000%	0.02	0	0	1	0.0000%
Pr.X – cap104	25	50	0.02	159	1	0.0000%	0.02	0	0	1	0.0000%

Pr.XII – cap121	50	50	0.08	167	1	0.0000%	0.02	0	0	1	0.0000%
Pr.XII – cap122	50	50	0.08	232	1	0.0000%	0.06	2	3	1	0.0000%
Pr.XII – cap123	50	50	0.11	20	1	0.0000%	0.24	2	6	6	0.0000%
Pr.XII – cap124	50	50	0.78	300	1193	0.0000%	0.86	2	9	67	0.0000%
Pr.XIII – cap131	50	50	0.03	243	1	0.0000%	0.02	0	0	1	0.0000%
Pr.XIII – cap132	50	50	0.05	217	1	0.0000%	0.02	0	0	1	0.0000%
Pr.XIII – cap133	50	50	0.07	272	9	0.0000%	0.02	0	0	1	0.0000%
Pr.XIII – cap134	50	50	0.12	294	210	0.0000%	0.02	0	0	1	0.0000%
Pr.A – capa1	100	1000	3584.89*	5500	17703	2.0647%	3598.88*	0	2	353	1.8325%
Pr.A – capa2	100	1000	3589.94*	5500	6858	4.3752%	3590.82*	0	0	1278	3.0066%
Pr.A – capa3	100	1000	3585.66*	5500	7808	2.6201%	1882.17	0	2	568	0.0000%
Pr.A – capa4	100	1000	2160.63	5160	746	0.0001%	199.35	0	5	101	0.0001%
Pr.B – capb1	100	1000	2827.60	5495	25220	0.0001%	858.43	0	122	1054	0.0001%
Pr.B – capb2	100	1000	3599.00*	5480	14626	0.1236%	1313.42	0	31	587	0.0001%
Pr.B – capb3	100	1000	3585.30*	5500	9871	1.1452%	649.10	0	41	509	0.0001%
Pr.B – capb4	100	1000	3591.17*	5500	10299	1.0689%	261.35	0	89	846	0.0001%
Pr.C – capc1	100	1000	1165.49	5179	7724	0.0001%	485,17	0	100	716	0,0001%
Pr.C – capc2	100	1000	3104.63	5485	19070	0.0000%	298,24	0	0	78	0,0000%
Pr.C – capc3	100	1000	810.56	5497	2888	0.0001%	202,39	0	36	334	0,0001%
Pr.C – capc4	100	1000	707.75	5500	1734	0.0001%	118,08	0	1	14	0,0000%

* Procedure aborted as time limit of 3600 seconds was reached.

5. Concluding Remarks and Future Research Possibilities

In this paper, we present a new extended formulation of CCLP that uses the idea of cardinality associated with each concentrator. Even though the resulting formulation is bigger in terms of the number of variables and constraints, we show theoretically that it is indeed a stronger formulation. That is, its LP relaxation provides a tighter lower bound than the LP relaxation of the traditional formulation. In addition, we show that our proposed disaggregated formulation reveals generalizations of the Cover and (1, k)-Configuration inequalities which collectively are stronger than the original Cover and (1, k)-Configuration inequalities. Finally, we present another class of inequalities called: 2-Facility Cardinality Matching Inequalities, which are specific to the disaggregated formulation. We present results of extensive computational tests on 36 benchmark problems which are medium to large sized. These results confirm the value of using our proposed disaggregated formulation of CCLP. We were able to demonstrate that our approach was able to identify the optimal solution at the root node itself in most of the reasonable sized instances. For the much larger sized test problems, the proposed branch-and-cut procedure using the disaggregated formulation outperforms the branch-and-cut procedure applied to the traditional formulation by a significant order both in terms of CPU and the number of branches required to solve the problem to optimality.

The results in this paper, both theoretical and computational, demonstrate an approach that seems very promising. However, much remains to be done to take the idea of cardinality-based disaggregation to its logical conclusion. From a theoretical standpoint, it is worthwhile extending the idea behind 2-Facility Cardinality Matching inequality to more than two concentrators. It is also worthwhile exploring the idea of generalizing the cardinality constrained Cover and (1, $\hat{p}_k$)-configuration inequalities to more than one facility or concentrator. Finally, the ideas presented in this paper can be applied to several other closely related NP-Hard problems such as the Capacitated Steiner Tree problem and the Capacitated Network Design Problem.

References

1. Ahuja R.K., J.B. Orlin, S. Pallottino, M.P. Scaparra, and M.G. Scutella, "A multi-exchange heuristic for the single source capacitated facility location. *Management Science*, (50), (2004), pp. 1749–760.
2. Barcelo, J., A. Hallefjord, E. Fernandez and K. Jornsten, "Lagrangean Relaxation and Constraint Generation Procedures for the Capacitated Plant Location Problems with Single Sourcing," *OR Spektrum* 12, (1990), pp. 79-88.
3. Boffey, T. B., "Location Problems Arising in Computer Networks," *Journal of Operational Research Society*, 40(4), (1989), pp. 347-354.
4. Cánovas, L, M. Landete and A. Marín, "On the facets of the Simple Plant Location Polytope," *Discrete Applied Mathematics*, 124, (2002), pp 27-53.
5. Celani, M., R. Cerulli, M. Gaudioso, and Ya.D. Sergeyev, "A Multiplier Adjustment Technique for the Capacitated Concentrator Location Problem," *Optimization Methods and Software*, 10(11), (1998), pp. 87-102.
6. Chen, C.H. and CJ. Ting, "Combining Lagrangian Heuristic and Ant Colony System to solve the Single Source Capacitated Facility Location Problem," *Transportation Research part E*, 44, (2008), pp. 1099–1122.
7. Cho, D., E. L. Johnson, M. Padberg, and M. R. Rao, "On the Uncapacitated Facility Location Problem. I: Valid Inequalities and Facets," *Mathematics of Operations Research*, 8, (1983), pp. 579-589.
8. Cho, D., E. L. Johnson, M. Padberg, and M. R. Rao, "On the Uncapacitated Facility Location Problem. II: Facets and Lifting Theorems," *Mathematics of Operations Research*, 8, (1983), pp. 590-612.
9. Cortinhal, M.J. and M. E. Captivo, "Upper and Lower Bounds for the Single Source Capacitated Location Problem," *European Journal of Operational Research* 151, (2003), pp. 333-351.
10. Gouveia, L. and F. Saldanha-da-Gama, "On the Capacitated Concentrator Location Problem: A Reformulation by Discretization". *Computers & Operations Research*, 33, (2006), pp. 1242-1258.
11. Holmberg, K., Rönnqvist, M., and Yaun Di, "An Exact Algorithm for the Capacitated Facility Location Problems with Single Sourcing," *European Journal of Operational Research* 113, (1999), pp. 544-559.
12. Labbè, M. and H. Yaman, "Polyhedral analysis for concentrator location problems", *Computational Optimization and Applications*, 34, (2006) pp. 377–407.
13. Murthy, I., "Facets of the Cardinality Induced Disaggregated Formulation of the Capacitated Concentrator Location Problem," Working Paper (2018), IIM Bangalore, Bangalore, India.
14. Lo, C. and A. Kershenbaum, "A Two-phased Algorithm and Performance Bounds for the Concentrator Location Problem", *IEEE Transactions on Communications*, 37(1989), pp. 1151-1163.

15. Padberg, M., "(1, *k*)-Configurations and Facets of packing Problems," *Mathematical Programming* 18(1) 1980, pp. 94-99.
16. Pirkul, H., "Efficient Algorithms for the Capacitated Concentrator Location Problem", *Computers and Operations Research*, 14 (1987), pp. 197-208.
17. Pirkul, H. and V. Nagarajan, "Locating Concentrators in Centralized Computer Networks", *Annals of Operations Research*, 36(1992) 247-261.
18. Shima, T., "Integer Programming Model and Exact Solution for Concentrator Location Problem," *Journal of Operations Research Society of Japan*, 43, (2000), pp. 291-305.
19. Sridharan, R., "A Lagrangean Heuristic for the Capacitated Plant Problem with Single Source Constraints", *European Journal of Operational Research*, 66, (1993), pp. 305–312.
20. Yang, Z, Feng Chu and Haoxun Chen, "A Cut-and-Solve based Algorithm for the Single-source Capacitated Facility Location Problem," *European Journal of Operational Research*, 221 (2012), pp. 521-532.