

Generating irreducible copositive matrices using the stable set problem

Peter J.C. Dickinson* Reinier de Zeeuw†

January 27, 2019

Abstract

In this paper it is considered how graphs can be used to generate copositive matrices, and necessary and sufficient conditions are given for these generated matrices to then be irreducible with respect to the set of positive semidefinite plus nonnegative matrices. This is done through combining the well known copositive formulation of the stable set problem with recent results on necessary and sufficient conditions for a copositive matrix to be irreducible. By applying this result, tens of thousands of new irreducible copositive matrices of order less than or equal to thirteen were found. These matrices were then tested for being extreme copositive matrices, and it was found that in all but three cases this was indeed the case. It is also demonstrated in this paper that these matrices provide difficult instances to test approximations of the copositive cone against.

MSC: 05C30, 05C50, 15A21, 15B48, 90C25.

Keywords: α -critical, Copositive, Extreme, Irreducible, Stability number.

1 Introduction

A symmetric matrix is defined to be copositive if its quadratic product is nonnegative with all nonnegative vectors, and the set of copositive matrices of order n is referred to as the copositive cone, denoted COP^n . Linear optimisation with the additional constraint of a matrix being copositive is then referred to as copositive optimisation. We direct the reader to interesting surveys into this provided by [6, 15, 18, 26]. These surveys include numerous applications connected to this set, and in this paper we will focus on the well known result in the field of copositive optimisation, that the stability number of a graph can be formulated as a simple copositive optimisation problem [3, 5, 11, 38]. Using this formulation, any graph can be considered to give a unique matrix on the boundary of the copositive cone, as will be discussed in Section 3.

The copositive cone is a proper cone, i.e. closed, convex, pointed and full-dimensional, and in studying this cone it is of interest to study its extreme rays, or in particular the extreme matrices, i.e. the copositive matrices generating extreme rays of the copositive cone [1, 13, 23, 24, 27]. Since the 1960s, all of the extreme matrices of the copositive cone have been known for $n \leq 4$ [23], however it took another 50 years before all the

*University of Twente and Rabobank, The Netherlands. Email: peter.jc.dickinson@gmail.com

†University of Twente, The Netherlands. Email: reinier.dezeeuw@gmail.com.

extreme matrices were found for $n = 5$ [24]. At the time of writing, it is still an open question to provide a constructive characterisation of the extreme matrices for $n \geq 6$.

A necessary condition for a copositive matrix to be extreme is that it is what is referred in the copositive optimisation community as being irreducible with respect to the set of positive semidefinite plus nonnegative matrices. This property will be discussed further in Section 4. Recently necessary and sufficient conditions for a matrix to be extreme or irreducible were found, with these conditions being related to the so-called sets of minimal zeros of the matrices [25]. We will use these conditions to give necessary and sufficient conditions on a graph such that the copositive matrix it generates is irreducible with respect to the set of positive semidefinite plus nonnegative matrices. We refer to graphs with these properties as cop-irreducible graphs, and the main result of this paper, proven in Section 4, will be to show that a graph is cop-irreducible if and only if it is connected, α -critical and has an additional property that we will refer to as being α -covered.

For $n \leq 13$ we find all the graphs with these properties, which total over 57 thousand. We test the matrices resulting from these graphs with necessary and sufficient conditions for being extreme from [17] and find that the vast majority of them are indeed extreme.

Along with considering the geometric properties of the copositive cone, extreme (or irreducible) copositive matrices can be used as difficult instances to test inner approximations of the copositive cone against. In Section 6 we will see that cop-irreducible graphs do indeed provide difficult instances to test approximations against.

The results of this paper are closely related to the Bachelor thesis of the second author [12].

Notation

We let \mathbb{R}^n denote the set of real n vectors, \mathbb{R}_+^n the set of nonnegative real n vectors and \mathcal{S}^n the set of real symmetric $n \times n$ matrices. Matrices are denoted by capital letters, for example the identity matrix I and the all-ones matrix E (with the order of both of these being apparent from the context). Vectors are denoted by lower case letters, for example $\mathbf{1}$ being the vector of all ones and \mathbf{e}_i being the unit vector with i th entry equal to 1 and all other entries equal to zero. For a set $\mathcal{A} \subseteq \{1, \dots, n\}$, we let $\mathbf{e}_{\mathcal{A}} := \sum_{i \in \mathcal{A}} \mathbf{e}_i \in \mathbb{R}^n$ and we let $\overline{\mathcal{A}} := \{1, \dots, n\} \setminus \mathcal{A}$.

2 Simple graphs

In this paper we consider simple graphs (referred to from now on simply as graphs), i.e. undirected graphs with finite numbers of vertices and edges, and no weights nor loops nor multiple edges, see e.g. [7].

For a graph G we let $V(G)$ be its set of vertices and $E(G)$ its set of edges. An edge between distinct vertices i, j is denoted as ij . Note that as we are considering an undirected graph, ij and ji refer to the same edge. We additionally let A_G denote the graph's adjacency matrix, i.e. a binary symmetric matrix indexed by elements in $V(G)$ such that $(A_G)_{ij} = 1$ if and only if $ij \in E(G)$. We let $N_G[i]$ denote the closed neighbourhood of the vertex i in the graph G , i.e. $N_G[i] = \{i\} \cup \{j \in V(G) : ij \in E(G)\}$.

For a graph G , we let \overline{G} be its complement, i.e. the graph with the same vertex set, but for $i \neq j$ we have $ij \in E(\overline{G})$ if and only if $ij \notin E(G)$.

In adding a vertex v to a graph G , we denote this by $G + v$, and for adding an edge ij we denote this by $G + ij$. Similarly for removing a vertex v or an edge ij from a graph G , we denote this by $G - v$ and $G - ij$ respectively.

In this paper we will consider five classes of graph, three taken from the literature, namely connected graphs, α -critical graphs and co-point-determining graphs, and two that we introduce here, namely α -covered graphs and cop-irreducible graphs.

We say that a graph G is connected if for all pairs of distinct vertices $i, j \in V(G)$, there exists a path between i and j , i.e. there exists $m \geq 2$ and $k_1, \dots, k_m \in V(G)$ such that $k_1 = i$, $k_m = j$ and $k_l k_{l+1} \in E(G)$ for all $l \in \{1, \dots, m-1\}$.

We will discuss α -critical and α -covered graphs below, whilst cop-irreducible and co-point-determining graphs will be discussed in Sections 4 and 5 respectively.

Stability number

A stable set in a graph G (also called an independent set or a co-clique) is a subset of its vertices such that no two vertices in this subset are connected by an edge. We say that a stable set is a maximum stable set if there is no stable set of strictly greater cardinality. The stability number of a graph G (also called the independence number), denoted $\alpha(G)$, is the cardinality of its maximum stable sets. For a survey of this problem we direct the reader to [4]. This problem also has strong connections to copositive optimisation, as will be discussed in Section 3.

A graph G is α -critical if removing any edge from G will increase its stability number, and such graphs have previously been investigated numerous papers in (see e.g. [20, 41, 43, 44, 50]).

We define a graph G to be α -covered if for all $ij \in E(\overline{G})$, there exists a maximum stable set \mathcal{A} of G such that $\{i, j\} \subseteq \mathcal{A}$. We trivially have that complete graphs are α -covered.

It is trivial to see that complete graphs have stability number equal to one, and are connected, α -critical and α -covered. Below we will look at the less trivial example of antiwebs (see e.g. [9, 10, 39, 48]). For $n \geq 4$ and $2 \leq t \leq \lfloor n/2 \rfloor$ these are graphs \overline{W}_t^n with $V(\overline{W}_t^n) = \{1, \dots, n\}$ and $E(\overline{W}_t^n) = \{ij \in V(\overline{W}_t^n)^2 : i < j < i+t \text{ or } j < i+t-n\}$, see Fig. 1. Before being able to consider these, we will first need the following lemma, which is not only useful in the proof of Lemma 2, but was also useful in testing whether a graph of α -critical and/or α -covered, which we needed to do for Section 6.

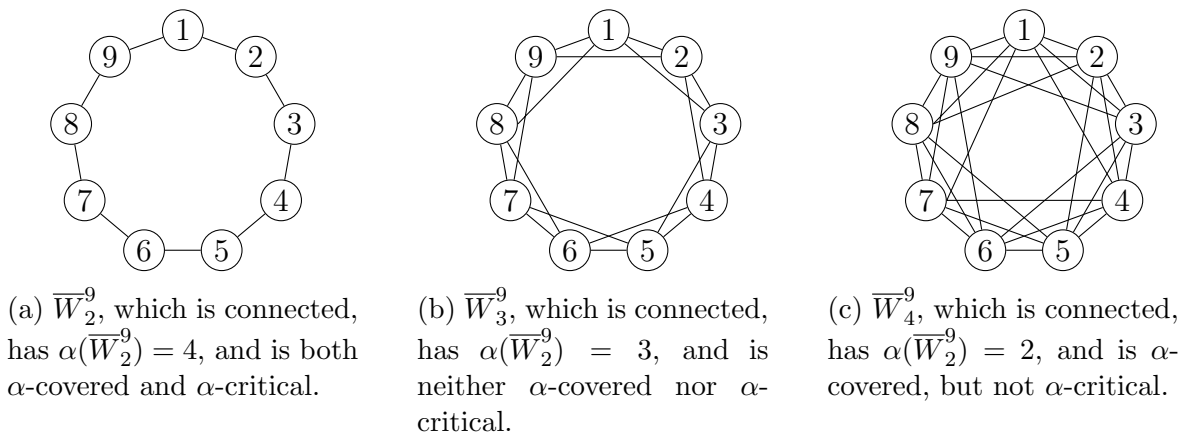


Figure 1: The antiwebs with nine vertices (excluding the edgeless graph).

Lemma 1. For a graph G and $i, j \in V(G)$ with $i \neq j$ let $H = G - N_G[i] - N_G[j]$.

If $ij \in E(\overline{G})$, then the cardinality of the maximum stable set of G which contains both i and j is equal to $2 + \alpha(H)$.

If $ij \in E(G)$, then $\alpha(G - ij) = \max\{\alpha(G), 2 + \alpha(H)\}$.

Proof. This follows trivially from the definitions. \square

Lemma 2. For the antiweb \overline{W}_t^n with $n \geq 4$ and $2 \leq t \leq \lfloor n/2 \rfloor$ we have that:

1. \overline{W}_t^n is a connected graph;
2. $\alpha(\overline{W}_t^n) = \lfloor n/t \rfloor$;
3. \overline{W}_t^n is α -critical if and only if $n = (a + 1)t - 1$ for some $a \in \{1, 2, 3, \dots\}$ (in which case $\alpha(\overline{W}_t^n) = a$);
4. \overline{W}_t^n is α -covered if and only if either $n \leq 3t - 1$ (in which case $\alpha(\overline{W}_t^n) = 2$) or $n = (a + 1)t - 1$ for some $a \in \{1, 2, 3, \dots\}$ (in which case $\alpha(\overline{W}_t^n) = a$).

Proof. Trivially \overline{W}_t^n is connected, and as no two vertices in a stable set can be closer than t , it can be seen that the set $\{t, 2t, \dots, \lfloor n/t \rfloor t\}$, of cardinality $\lfloor n/t \rfloor$, is a maximum stable set of \overline{W}_t^n . This has also previously been shown in [48, Corollary 3.6].

In order to assist with the rest of the proof, for an integer m we define the graph l_m as being the empty graph when $m \leq 0$, and for $m \geq 1$ as having

$$V(l_m) = \{1, \dots, m\}, \quad E(l_m) = \{ij \in V(l_m)^2 : i < j < i + t\}.$$

If $m \geq 1$ then it can be seen that the set $\{1, 1 + t, 1 + 2t, \dots, 1 + \lfloor (m - 1)/t \rfloor t\}$, of cardinality $\lfloor (m - 1)/t \rfloor + 1 = \lfloor (m + t - 1)/t \rfloor$, is a maximum stable set of l_m . Therefore $\alpha(l_m) = \max\{0, \lfloor (m + t - 1)/t \rfloor\}$.

Additionally, for $j \in \{2, \dots, n\}$, let $H_j = \overline{W}_t^n - N_{\overline{W}_t^n}[1] - N_{\overline{W}_t^n}[j]$.

We will now prove the result of when \overline{W}_t^n is α -critical. Due to the symmetry of \overline{W}_t^n , this is equivalent to showing that for all $j \in \{2, \dots, t\}$ we have $\alpha(G - 1j) = \alpha(G) + 1 = \lfloor n/t \rfloor + 1$. For all $j \in \{2, \dots, t\}$ we have that H_j is isomorphic to $l_{n-j-2t+2}$ and thus from Lemma 1 we have

$$\begin{aligned} \alpha(G - 1j) &= \max\{\lfloor n/t \rfloor, 2 + \lfloor (n - j - t + 1)/t \rfloor\} \\ &= \max\{\lfloor n/t \rfloor, 1 + \lfloor (n + 1 - j)/t \rfloor\}. \end{aligned}$$

This is monotonically decreasing in j , and thus \overline{W}_t^n is α -critical if and only if we have $\lfloor (n - t + 1)/t \rfloor \geq \lfloor n/t \rfloor$. This is the case if and only if $n = (a + 1)t - 1$ for some $a \in \{1, 2, 3, \dots\}$, completing the proof of when \overline{W}_t^n is α -critical.

We will now prove the result of when \overline{W}_t^n is α -covered. To do this we will consider three cases:

1. $n \leq 3t - 1$: Then we have that $\alpha(\overline{W}_t^n) \leq 2$ and thus \overline{W}_t^n is trivially α -covered.
2. $3t - 1 < n \neq (a + 1)t - 1$ for all $a \in \{1, 2, 3, \dots\}$: Consider $1j \in E(\overline{W}_t^n)$ with $j = 2t$. We then have that H_j is isomorphic to $l_{n-2j-2t+2}$, and thus the largest stable set

containing the vertices 1 and j has cardinality equal to

$$\begin{aligned} 2 + \alpha(l_{n-4t+2}) &= \begin{cases} 2 & \text{if } n \leq 4t - 2 \\ 2 + \lfloor (n - 3t + 1)/t \rfloor & \text{if } n \geq 4t - 1 \end{cases} \\ &= \begin{cases} 2 & \text{if } n \leq 4t - 2 \\ \lfloor (n - t + 1)/t \rfloor & \text{if } n \geq 4t \end{cases} \\ &= \alpha(\overline{W}_t^n) - 1. \end{aligned}$$

Therefore in this case \overline{W}_t^n is not α -covered.

3. $n = (a + 1)t - 1$ for some $a \in \{3, 4, 5, \dots\}$: Consider an arbitrary $ij \in E(\overline{W}_t^n)$. Due to the symmetry of \overline{W}_t^n , without loss of generality $i = 1$ and $t < j \leq \lfloor n/2 \rfloor$. If $t < j \leq 2t$ then H_j is isomorphic to $l_{n-j-2t+2}$, and thus the largest stable set containing the vertices i and j has cardinality equal to

$$\begin{aligned} 2 + \alpha(l_{n-j-2t+2}) &= 2 + \lfloor (n - j - t + 1)/t \rfloor \\ &= a + \lfloor (2t - j)/t \rfloor \\ &= a = \alpha(\overline{W}_t^n). \end{aligned}$$

If on the other hand $2t < j \leq \lfloor n/2 \rfloor$ then H_j is isomorphic to the disjoint union of $l_{n-j-2t+2}$ and l_{j-2t} , and thus the largest stable set containing the vertices i and j has cardinality equal to

$$\begin{aligned} 2 + \alpha(l_{n-j-2t+2}) + \alpha(l_{j-2t}) &= 2 + \lfloor (n - j - t + 1)/t \rfloor + \lfloor (j - t - 1)/t \rfloor \\ &= 2 + \lfloor (at - j)/t \rfloor + \lfloor (j - 2t)/t \rfloor \\ &= a + \lfloor (2t - j)/t \rfloor - \lfloor (2t - j)/t \rfloor \\ &= a = \alpha(\overline{W}_t^n). \end{aligned}$$

Therefore in this case \overline{W}_t^n is α -covered. □

3 Copositivity

A matrix $\mathbf{A} \in \mathcal{S}^n$ is defined to be copositive if $\mathbf{v}^\top \mathbf{A} \mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{R}_+^n$, and the set of copositive matrices of order n is referred to as the copositive cone, denoted \mathcal{COP}^n . This is a proper cone, i.e. a cone which is closed, convex, pointed and full-dimensional. Letting \mathcal{PSD}^n denote the set of positive semidefinite matrices, and \mathcal{N}^n denote the set of nonnegative symmetric matrices, we have that $\mathcal{PSD}^n + \mathcal{N}^n \subseteq \mathcal{COP}^n$, with equality if and only if $n \leq 4$ [13]. There are numerous applications connected to this set, but in this paper we will focus on the well known result in the field of copositive optimisation, that the stability number of a graph G on n vertices is the minimum $\lambda \in \mathbb{R}$ such that $\lambda(\mathbf{I} + \mathbf{A}_G) - \mathbf{E}$ is a copositive matrix [3, 5, 11, 38]. This is closely related to the Lovász number, which is a semidefinite based approximation of the stability number [22, 32]. As \mathbf{I} is in the interior of the positive semidefinite cone and \mathbf{A}_G is a nonnegative matrix, we have that $\lambda(\mathbf{I} + \mathbf{A}_G) - \mathbf{E}$ is in the interior of the copositive cone for all $\lambda > \alpha(G)$. As the copositive cone is closed, we have that $\alpha(G)(\mathbf{I} + \mathbf{A}_G) - \mathbf{E}$ is on the boundary of the copositive cone, and we will let

$$\mathbf{Z}_G := \alpha(G)(\mathbf{I} + \mathbf{A}_G) - \mathbf{E}.$$

This analysis actually allows us to consider all copositive graphs with only two distinct entries. Consider $\mathbf{X} \in \mathcal{S} \cap \{\beta, \gamma\}^{n \times n}$ for some $\beta, \gamma \in \mathbb{R}$. If $\beta, \gamma \geq 0$ then \mathbf{X} is elementwise nonnegative and thus also copositive. If $\beta, \gamma \leq 0$ then \mathbf{X} is elementwise nonpositive, and it is well known that in this case \mathbf{X} is copositive if and only if it is the all zeros matrix. Now consider the case when \mathbf{X} is neither nonnegative nor nonpositive. Then as multiplication of a matrix by a positive scalar maintains the matrix properties that we are interested in in this paper, without loss of generality we have that $\beta = -1$ and $0 < \gamma$. A necessary condition for copositivity of \mathbf{X} is then that $a_{ii} = \gamma$ for all i , and from now on we assume that this holds. Then letting G be the graph with $ij \in E(G)$ when $a_{ij} = \gamma$, we have that $\mathbf{X} = (1 + \gamma)(\mathbf{I} + \mathbf{A}_G) - \mathbf{E}$.

4 Necessary and sufficient conditions for a graph to be cop-irreducible

We say that a matrix $\mathbf{X} \in \mathcal{COP}^n$ is irreducible with respect to a cone $\mathcal{K} \subseteq \mathcal{COP}^n$ if $\nexists \mathbf{Y} \in \mathcal{K} \setminus \{\mathbf{O}\}$ such that $\mathbf{X} - \mathbf{Y} \in \mathcal{COP}^n$ [2, 13, 16, 17, 23, 25]. For convex cones $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{COP}^n$, it is trivial to see that \mathbf{X} is irreducible with respect to $\mathcal{K}_1 + \mathcal{K}_2$ if and only if it is irreducible with respect to both \mathcal{K}_1 and \mathcal{K}_2 .

The concept of irreducibility is closely related to that of extremality, see Section 8.

Another application of irreducibility is in connection to testing inner approximations of the copositive cone. Let \mathcal{K} be a convex cone such that $(\mathcal{PSD}^n + \mathcal{N}^n) \subseteq \mathcal{K} \subseteq \mathcal{COP}^n$. Then the following are equivalent:

1. $\mathcal{K} = \mathcal{COP}^n$;
2. $\mathbf{X} \in \mathcal{K}$ for all matrices $\mathbf{X} \in \mathcal{COP}^n$;
3. $\mathbf{Y} \in \mathcal{K}$ for all matrices $\mathbf{Y} \in \mathcal{COP}^n$ which are irreducible with respect to $\mathcal{PSD}^n + \mathcal{N}^n$.

We would thus expect copositive matrices which are irreducible with respect to $\mathcal{PSD}^n + \mathcal{N}^n$ to give difficult instances of copositive optimisation problems to test approximations against.

So a natural question is for which graphs G do we have that \mathbf{Z}_G is irreducible with respect to $\mathcal{PSD}^n + \mathcal{N}^n$. There is in fact a simple characterisation for this, which is given in the following theorem. This is the main result of this paper and will be proven later in this section.

Theorem 3. *For a graph G we have that \mathbf{Z}_G is irreducible with respect to $\mathcal{PSD}^n + \mathcal{N}^n$ if and only if G is connected, α -critical and α -covered.*

Due to this result, in the rest of this paper, we will refer to graphs which are connected, α -critical and α -covered as cop-irreducible graphs.

Note from our earlier results, we have that complete graphs are always cop-irreducible (although for these we have that \mathbf{Z}_{K_n} is the all-zeros matrix), and antiweb graphs, \overline{W}_t^n are cop-irreducible if and only if $n = (a + 1)t - 1$ for some $a \in \{1, 2, 3, \dots\}$ (and we then have that the stability number of such a graph is equal to a).

In the rest of this section we will prove a number of results from which Theorem 3 will directly follow.

In the following two lemmas we will see that being α -critical and connected are necessary conditions for \mathbf{Z}_G to be irreducible with respect to \mathcal{N}^n and \mathcal{PSD}^n respectively, and thus are also necessary conditions for being irreducible with respect to $\mathcal{PSD}^n + \mathcal{N}^n$.

Lemma 4. *Let G be a graph which is not α -critical. Then Z_G is not irreducible with respect to \mathcal{N}^n .*

Proof. If G is not α -critical then there exists $ij \in E(G)$ such that for $H = G - ij$ we have $\alpha(G) = \alpha(H)$. We then have $Z_H \in \mathcal{COP}^n$ and $Z_G - Z_H = \alpha(G)(\mathbf{e}_i \mathbf{e}_j^\top + \mathbf{e}_j \mathbf{e}_i^\top) \in \mathcal{N}^n$, completing the proof. \square

Lemma 5. *Let G be a graph which is not connected. Then Z_G is not irreducible with respect to \mathcal{PSD}^n .*

Proof. If G is not a connected graph, then without loss of generality, there exists $m \in \{1, \dots, n-1\}$ such that $ij \notin E(G)$ for all $1 \leq i \leq m < j \leq n$. Let G_1 be the subgraph of G induced by the vertices $\{1, \dots, m\}$ and let G_2 be the subgraph of G induced by the vertices $\{m+1, \dots, n\}$. Furthermore, for $i \in \{1, 2\}$, let $\alpha_i = \alpha(G_i)$, $A_i = A_{G_i}$ and $Z_i = Z_{G_i}$. Then we have

$$\begin{aligned} \alpha(G) &= \alpha_1 + \alpha_2, \\ Z_1 &= \alpha_1 (I + A_1) - E \in \mathcal{COP}^m, \\ Z_2 &= \alpha_2 (I + A_2) - E \in \mathcal{COP}^{n-m}, \\ Z_G &= \alpha(G) \begin{pmatrix} I + A_1 & O \\ O & I + A_2 \end{pmatrix} - \begin{pmatrix} E & E \\ E & E \end{pmatrix} \\ &= \underbrace{\frac{1}{\alpha_1 \alpha_2} \begin{pmatrix} \alpha_2 \mathbf{1} \\ -\alpha_1 \mathbf{1} \end{pmatrix} \begin{pmatrix} \alpha_2 \mathbf{1} \\ -\alpha_1 \mathbf{1} \end{pmatrix}^\top}_{\in \mathcal{PSD}^n \setminus \{O\}} + \underbrace{\frac{\alpha(G)}{\alpha_1} \begin{pmatrix} Z_1 & O \\ O & O \end{pmatrix} + \frac{\alpha(G)}{\alpha_2} \begin{pmatrix} O & O \\ O & Z_2 \end{pmatrix}}_{\in \mathcal{COP}^n}. \end{aligned} \quad \square$$

In order to show that being α -covered is also necessary, and that these conditions combined are sufficient, requires some technical results connected to the so called set of minimal zeros of Z_G . We will not go in to the details of this set in this paper, but instead direct the reader to the paper [25] for the definition of this set and some interesting results connected to it.

Lemma 6. *Consider a graph G with $\alpha(G) \geq 2$ and let $Z = Z_G$. Then we have*

$$\mathcal{V}_{\min}^Z = \{\mu \mathbf{e}_{\mathcal{A}} : \mathcal{A} \text{ is a maximum stable set of } G, \mu \in \mathbb{R}, \mu > 0\}.$$

Furthermore, for a maximum stable set \mathcal{A} of G , a vector $\mathbf{v} = \mathbf{e}_{\mathcal{A}} \in \mathcal{V}_{\min}^Z$ and an index $i \in \{1, \dots, n\}$ the following are equivalent, where for a vector $\mathbf{a} \in \mathbb{R}^n$ we define its support as $\text{supp}(\mathbf{a}) = \{j \in \{1, \dots, n\} : a_j \neq 0\}$:

1. $i \in \overline{\text{supp}(Z\mathbf{v})}$;
2. $\exists \mathbf{w} \in \mathcal{V}_{\min}^Z$ s. t. $i \in \text{supp}(\mathbf{w}) \subseteq \overline{\text{supp}(Z\mathbf{v})}$;
3. Either $i \in \mathcal{A}$ or the subgraph induced by $\{i\} \cup \mathcal{A}$ contains exact one edge;

Proof. For $X \in \mathcal{COP}^n$ we have that

$$\mathcal{V}_{\min}^X := \left\{ \mathbf{v} \in \mathbb{R}_+^n \setminus \{O\} : \begin{array}{l} \mathbf{v}^\top X \mathbf{v} = 0, \\ \mathbf{w}^\top X \mathbf{w} > 0 \forall \mathbf{w} \in \mathbb{R}_+^n \setminus \{O\} \text{ with } \text{supp}(\mathbf{w}) \subsetneq \text{supp}(\mathbf{v}) \end{array} \right\}.$$

We now split this proof into four parts:

1. We will first show that for all $\mathbf{v} \in \mathcal{V}_{\min}^Z$ we have that $\text{supp}(\mathbf{v})$ is a stable set of G . Suppose for the sake of contradiction there exists $\mathbf{v} \in \mathcal{V}_{\min}^Z$ such that $\{i, j\} \subseteq \text{supp}(\mathbf{v})$ for some $ij \in E(G)$. Without loss of generality $(Z\mathbf{v})_i \geq (Z\mathbf{v})_j$, and letting $\mathbf{w} = \mathbf{v} + v_i(\mathbf{e}_j - \mathbf{e}_i)$ we have $\mathbf{w} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ and as $Z \in \mathcal{COP}^n$ we have

$$\begin{aligned} 0 \leq \mathbf{w}^T Z \mathbf{w} &= \mathbf{v}^T Z \mathbf{v} + 2v_i((Z\mathbf{v})_j - (Z\mathbf{v})_i) + v_i^2(z_{ii} - 2z_{ij} + z_{jj}) \\ &\leq 0 + 0 + v_i^2(\alpha - 1)(1 - 2 + 1) = 0. \end{aligned}$$

Therefore $\mathbf{w}^T Z \mathbf{w} = 0$ and $\text{supp}(\mathbf{w}) = \text{supp}(\mathbf{v}) \setminus \{i\} \subsetneq \text{supp}(\mathbf{v})$, which contradicts \mathbf{v} being a minimal zero.

2. Now considering an arbitrary $\mathbf{x} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ such that $\text{supp}(\mathbf{x})$ is a stable set of G , it can be shown that

$$\mathbf{x}^T Z \mathbf{x} = \frac{1}{2} \sum_{\{i,j\} \subseteq \text{supp}(\mathbf{x})} (x_i - x_j)^2 + (\alpha - |\text{supp}(\mathbf{x})|) \|\mathbf{x}\|_2^2,$$

which is greater than or equal to zero, with equality if and only if $\mathbf{x} \in \{\mu \mathbf{e}_{\mathcal{A}} : \mu > 0\}$ for some maximum stable set \mathcal{A} of G . This completes the proof of the characterisation of \mathcal{V}_{\min}^Z .

3. We will now prove that statements 1 and 3 are equivalent.

We have that $v_i \in \{0, 1\}$, with $v_i = 1$ if and only if $|\{j \in \mathcal{A} : ij \in E(G)\}| = 0$. Therefore

$$(Z\mathbf{v})_i = \alpha(G)(v_i + |\{j \in \mathcal{A} : ij \in E(G)\}| - 1) \geq 0,$$

with equality if and only if either $i \in \mathcal{A}$ or $|\{j \in \mathcal{A} : ij \in E(G)\}| = 1$. This completes the proof that $1 \Leftrightarrow 3$.

4. Finally we will prove that statements 1 and 2 are equivalent.

Trivially $2 \Rightarrow 1$.

Now suppose that statement 1 is true, noting that from the previous part of the proof, this immediately implies that statement 3 is also true.

A corollary of statements 1 and 3 being equivalent is that $\text{supp}(\mathbf{e}_{\mathcal{B}}) = \mathcal{B} \subseteq \overline{\text{supp}(Z\mathbf{e}_{\mathcal{B}})}$ for all maximum stable sets. Therefore $\{i\} \cup \mathcal{A} \subseteq \overline{\text{supp}(Z\mathbf{v})}$.

If $i \in \mathcal{A}$ then letting $\mathbf{w} = \mathbf{e}_{\mathcal{A}} \in \mathcal{V}_{\min}^Z$ we have $i \in \text{supp}(\mathbf{w}) \subseteq \overline{\text{supp}(Z\mathbf{v})}$.

If alternatively the subgraph induced by $\{i\} \cup \mathcal{A}$ contains exactly one edge, then there exists $j \in \mathcal{A}$ such that $ij \in E(G)$, and letting $\mathcal{B} = \{i\} \cup \mathcal{A} \setminus \{j\}$, we have that \mathcal{B} is a stable set of G of cardinality $\alpha(G)$, and thus a maximum stable set. Letting $\mathbf{w} = \mathbf{e}_{\mathcal{B}}$, we then have $i \in \text{supp}(\mathbf{w}) = \mathcal{B} \subset \{i\} \cup \mathcal{A} \subseteq \overline{\text{supp}(Z\mathbf{v})}$. \square

We next need the following technical result on cop-irreducible graphs.

Lemma 7. *Let G be a connected, α -critical, α -covered graph. Then for all $\{x, y\} \subseteq V(G)$ with $x \neq y$, there exists a maximum stable set \mathcal{A} of G such that $\{x\} \subseteq \mathcal{A} \subseteq V(G) \setminus \{y\}$.*

Proof. First suppose that $xy \in E(G)$. Then as G is an α -critical graph, there exists a stable set \mathcal{B} of $G - xy$ such that $|\mathcal{B}| \geq \alpha(G) + 1$. We must then have that $\{x, y\} \subseteq \mathcal{B}$, otherwise we get the contradiction that \mathcal{B} is also a stable set of G . Now letting $\mathcal{A} = \mathcal{B} \setminus \{y\}$ we get the required maximum stable set of G .

From now on suppose that $xy \notin E(G)$. As G is connected, there exists $z \in V(G)$ such that $yz \in E(G)$. If $xz \notin E(G)$ then as G is α -covered, there exists a maximum stable set \mathcal{A} of G such that $\{x, z\} \subseteq \mathcal{A}$, and as $yz \in E(G)$, we have $y \notin \mathcal{A}$. If, on the other hand, $xz \in E(G)$, then as G is α -critical, there exists a maximum stable set \mathcal{B} of $G - xz$, and we have $|\mathcal{B}| = \alpha(G) + 1$ and $\{x, z\} \subseteq \mathcal{B}$. As $yz \in E(G - xz)$, we have $y \notin \mathcal{B}$. Then letting $\mathcal{A} = \mathcal{B} \setminus \{z\}$, we are done. \square

We will now give a necessary and sufficient condition for Z_G to be irreducible with respect to \mathcal{N}^n .

Lemma 8. *Consider a graph G on n vertices with $\alpha(G) \geq 2$. Then Z_G is irreducible with respect to \mathcal{N}^n if and only if G is both α -critical and α -covered.*

Proof. From [25], we have that Z_G is irreducible with respect to \mathcal{N}^n if and only if for all $i, j \in \{1, \dots, n\}$, there exists $\mathbf{v} \in \mathcal{V}_{\min}^{Z_G}$ such that $v_i + v_j > 0$ and $(Z_G \mathbf{v})_i = (Z_G \mathbf{v})_j = 0$.

We will first prove the forward implication. We already saw in Lemma 4 that if Z_G is irreducible with respect to \mathcal{N}^n then G is α -critical. We will now suppose that Z_G is irreducible with respect to \mathcal{N}^n and show that G is then also α -covered. We will do this by considering an arbitrary $ij \in E(\overline{G})$ and showing that there exists a maximum stable set of G which contains both i and j . As Z_G is irreducible with respect to \mathcal{N}^n , from Lemma 6, there exists a maximum stable set \mathcal{A} of G such that $|\{i, j\} \cap \mathcal{A}| \geq 1$ and $\{i, j\} \subseteq \text{supp}(Z_G \mathbf{e}_{\mathcal{A}})$. If $\{i, j\} \subseteq \mathcal{A}$ then we are done, and from now on, without loss of generality, we suppose that $j \in \mathcal{A}$ and $i \notin \mathcal{A}$. From Lemma 6, this implies that the subgraph of G induced by $\{i\} \cup \mathcal{A}$ contains exactly one edge. As \mathcal{A} is a stable set and $ij \in E(\overline{G})$, this implies there exists $k \in \mathcal{A} \setminus \{j\}$ such that $ik \in E(G)$, and we then have that $\{i\} \cup \mathcal{A} \setminus \{k\}$ is a stable set of cardinality $\alpha(G)$ containing both i and j .

We will now prove the reverse implication by supposing that G is both α -critical and α -covered and showing that Z_G is then irreducible with respect to \mathcal{N}^n . For arbitrary $i, j \in \{1, \dots, n\}$, consider three cases:

1. $i = j$: From Lemma 7, there exists a maximum stable set \mathcal{A} of G such that $i \in \mathcal{A}$. From Lemma 6 we then have $\mathbf{v} = \mathbf{e}_{\mathcal{A}} \in \mathcal{V}_{\min}^{Z_G}$ with $v_i + v_i = 2$ and $(Z_G \mathbf{v})_i = 0$.
2. $ij \in E(\overline{G})$: As G is α -covered, there exists a maximum stable set \mathcal{A} of G with $\{i, j\} \subseteq \mathcal{A}$. From Lemma 6 we then have $\mathbf{v} = \mathbf{e}_{\mathcal{A}} \in \mathcal{V}_{\min}^{Z_G}$ with $v_i + v_j = 2$ and $(Z_G \mathbf{v})_i = (Z_G \mathbf{v})_j = 0$.
3. $ij \in E(G)$: As G is α -critical, there exists a stable set \mathcal{B} of $G - ij$ such that $|\mathcal{B}| = \alpha(G) + 1$ and we have $\{i, j\} \in \mathcal{B}$. Letting $\mathcal{A} = \mathcal{B} \setminus \{i\}$, from Lemma 6 we have $\mathbf{v} = \mathbf{e}_{\mathcal{A}} \in \mathcal{V}_{\min}^{Z_G}$ with $v_i + v_j = 1$ and $(Z_G \mathbf{v})_i = (Z_G \mathbf{v})_j = 0$. \square

We were unable to find a necessary and sufficient condition for Z_G to be irreducible with respect to \mathcal{PSD}^n , however we have already seen that being connected is a necessary condition, and we will now give a closely related sufficient condition, which together with the rest of the results in this section completes the proof of Theorem 3.

Lemma 9. *For a graph G we have that if G is α -critical and connected then Z_G is irreducible with respect to \mathcal{PSD}^n .*

Proof. Suppose that G is α -critical and connected, and assume for the sake of contradiction that Z_G is not irreducible with \mathcal{PSD}^n . From [25] this implies that there exists $\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\mathbf{a}^\top \mathbf{v} = 0$ for all $\mathbf{v} \in \mathcal{V}_{\min}^{Z_G}$.

Consider an arbitrary $ij \in E(G)$. As G is α -critical, there exists stable set \mathcal{A} of $G - ij$ such that $|\mathcal{A}| = \alpha(G) + 1$, and we have $\{i, j\} \subseteq \mathcal{A}$. Therefore both $\mathcal{A} \setminus \{i\}$ and $\mathcal{A} \setminus \{j\}$ are maximum stable sets of G . Letting $\mathbf{v} = \mathbf{e}_{\mathcal{A} \setminus \{i\}} = \mathbf{e}_{\mathcal{A} \setminus \{i, j\}} + \mathbf{e}_j$ and $\mathbf{w} = \mathbf{e}_{\mathcal{A} \setminus \{j\}} = \mathbf{e}_{\mathcal{A} \setminus \{i, j\}} + \mathbf{e}_i$, we have $\mathbf{v}, \mathbf{w} \in \mathcal{V}_{\min}^{Z_G}$, and thus $0 = \mathbf{a}^\top \mathbf{v}$ and $0 = \mathbf{a}^\top \mathbf{w}$, implying that $0 = \mathbf{a}^\top \mathbf{v} - \mathbf{a}^\top \mathbf{w} = a_j - a_i$, or equivalently $a_i = a_j$.

As G is connected, this implies that $\mathbf{a} = \mu \mathbf{1}$ for some $\mu \in \mathbb{R} \setminus \{0\}$. Considering an arbitrary maximum stable set \mathcal{B} of G , we then have $\mathbf{e}_{\mathcal{B}} \in \mathcal{V}_{\min}^{Z_G}$ and thus $0 = \mathbf{e}_{\mathcal{B}}^\top \mathbf{a} = \mu \alpha(G) \neq 0$, a contradiction. \square

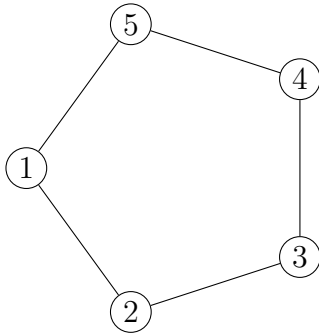
5 Vertex duplication

In Section 6 we will consider all cop-irreducible graphs with at most 13 vertices, however, before doing this, we will first consider a simple operation for taking a cop-irreducible graph and constructing a larger cop-irreducible graph from this. We consider this first as it simplifies the analysis of the graphs in the next section. This operation is known as vertex duplication, which is defined as follows:

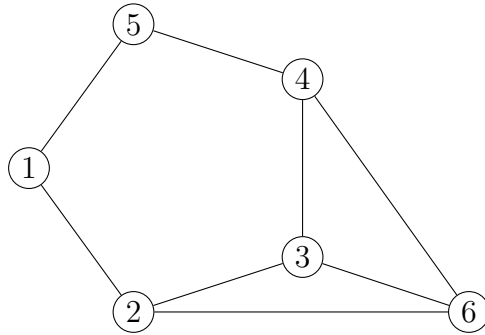
We take a graph G and a vertex $v \in V(G)$. We then add a new vertex w and some edges connecting this vertex to the graph so that we get the new (larger) graph

$$H = G + w + vw + \sum_{x: xv \in E(G)} xw.$$

An example of this operation is given in Fig. 2, where we duplicate vertex 3, adding the new vertex 6.



(a) A graph G .



(b) The graph G after duplicating the vertex 3.

Figure 2: An example of vertex duplication.

It is trivial to see that duplicating vertices maintains the stability number of a graph. We also have the following results on duplicating vertices.

Lemma 10. *Consider graphs G and H such that H is produced from G by duplicating one of its vertices. Then for the following properties, we have that G has this property if and only if H has this property:*

- *connected;*

- α -critical;
- α -covered;
- cop-irreducible.

Proof. It is trivial to see that this holds for being connected, and it was shown in [49] that this also holds for being α -critical (see also [44, Theorem 3.5]). If we can show that this also holds for being α -covered then it will trivially follow that it additionally holds for being cop-irreducible. We are thus left to show that G is α -covered if and only if H is α -covered.

Let $v \in V(G)$ and $H = G + w + vw + \sum_{x: xv \in E(G)} xw$. Trivially $\alpha(H) = \alpha(G)$.

First suppose that G is α -covered and consider an arbitrary $ij \in E(\overline{H})$. If $w \notin \{i, j\}$ then $ij \in E(\overline{G})$, and there exists a maximum stable set \mathcal{A} of G such that $\{i, j\} \subseteq \mathcal{A}$, and this is also a maximum stable set of H . If on the other hand we have $w \in \{i, j\}$ then without loss of generality $i = w$ and we have $j \in V(G) \setminus \{v\}$. There exists a maximum stable set \mathcal{B} of G such that $\{v, j\} \subseteq \mathcal{B}$, and $\mathcal{B} \cup \{w\} \setminus \{v\}$ is then a maximum stable set of H containing $\{w, j\}$. Therefore H is also α -covered.

Now suppose that H is α -covered and consider an arbitrary $ij \in E(\overline{G})$. Then $ij \in E(\overline{H})$, and there exists a maximum stable set \mathcal{C} of H such that $\{i, j\} \subseteq \mathcal{C}$. If $w \notin \mathcal{C}$ then \mathcal{C} is a maximum stable set of G . If $w \in \mathcal{C}$, then $\mathcal{C} \cup \{v\} \setminus \{w\}$ is a maximum stable set of G which contains $\{i, j\}$. Therefore G is also α -covered. \square

We now consider what vertex duplication does to Z_G . The following result follows directly from the definitions, and noting that vertex duplication does not affect the stability number of a graph.

Lemma 11. *For a graph G on n vertices, let*

$$Z_G = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^\top & \gamma \end{pmatrix}$$

for some $\mathbf{A} \in \mathcal{S}^{n-1}$, $\mathbf{b} \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$. Then the graph H is produced by duplicating the vertex n of G if and only if

$$Z_H = \begin{pmatrix} \mathbf{A} & \mathbf{b} & \mathbf{b} \\ \mathbf{b}^\top & \gamma & \gamma \\ \mathbf{b}^\top & \gamma & \gamma \end{pmatrix}.$$

We thus see that vertex duplication is equivalent to repeating a row. When we consider duplicating a row of a matrix, many properties are maintain, including, but not limited to, being:

- copositive [1];
- irreducible with respect to \mathcal{PSD}^n or \mathcal{N}^n or $\mathcal{PSD}^n + \mathcal{N}^n$ (trivially follows from [1]);
- extreme [1];
- exposed [14];
- a member of many approximations of the copositive cone, including the \mathcal{Q}_n^r approximations that will be discussed in Section 6.

For this reason, vertex duplication can be seen as a trivial operation, and from now on we will primarily focus on graphs which can not be produced from a smaller graph by vertex duplication. Such graphs are referred to in the literature as co-point-determining graphs [8, 21, 30, 42, 45].

6 Analysis of upto 13 vertices

In order to further analyse cop-irreducible graphs, we wished to consider all such co-point-determining cop-irreducible graphs for low total number of vertices. In order to do this we used a combination of Geogebra, Matlab, Mathematica, Nauty, Octave, SageMath, SDPT3 and Yalmip [36, 19, 28, 29, 31, 33, 35, 37, 46, 47].

As the properties of graphs considered in this paper are maintained under isomorphisms, and the properties of matrices considered in this paper are maintained under permutations, it was sufficient to only consider unlabelled graphs.

The first way that we attempted to enumerate these graphs was to generate all connected graphs with low total number of vertices using Nauty, and then use Mathematica to check which ones were also α -critical and α -covered. Unfortunately the sheer number of connected graphs, meant that we were only about to consider when the total number of vertices less than or equal to 9.

Fortunately we then came across the work of Ben Small on α -critical graphs with up to 13 vertices [44]. He was generous enough to provide us with a file of all such graphs, which we could then check for being co-point-determining, connected and α -covered.

In Fig. 3 the number of different types of graphs is given. As we can see, the total number of cop-irreducible graphs is tiny in comparison to the total number of connected graphs, but there is still a significant number of them. The exact numbers for each $n \leq 13$ are given in Table 1.

n			≤ 4	5	6	7	8	9	10	11	12	13	≤ 13
Number of graphs which are...	... connected and α -critical	all	4	2	2	5	10	31	113	573	4,470	57,158	62,368
		cpd	1	1	0	1	3	8	27	177	1,678	27,731	29,627
	... cop-irreducible	all	4	2	2	5	10	31	109	549	4,179	52,568	57,459
		cpd	1	1	0	1	3	8	23	166	1,536	25,124	26,863

Table 1: Numbers of all and co-point-determining (cpd) connected α -critical and cop-irreducible graphs with at most 13 vertices.

For low values of n , almost all α -critical graphs are cop-irreducible. In fact the smallest graphs which are α -critical but not cop-irreducible have 10 vertices. The number of graphs which are α -critical but not cop-irreducible increases with n , and for $n = 13$, out of the 27,731 connected co-point-determining α -critical graphs, 25,124 are also cop-irreducible.

Limiting to co-point-determining graphs does not affect the order of magnitude of the numbers of different types of graphs, but can still make a significant difference when considering the absolute number of such graphs. For example, there are 57,459 cop-irreducible graphs with at most 13 vertices, but only 26,863 are also co-point determining.

In Table 2 we display all 37 of the co-point-determining cop-irreducible graphs with at most ten vertices.

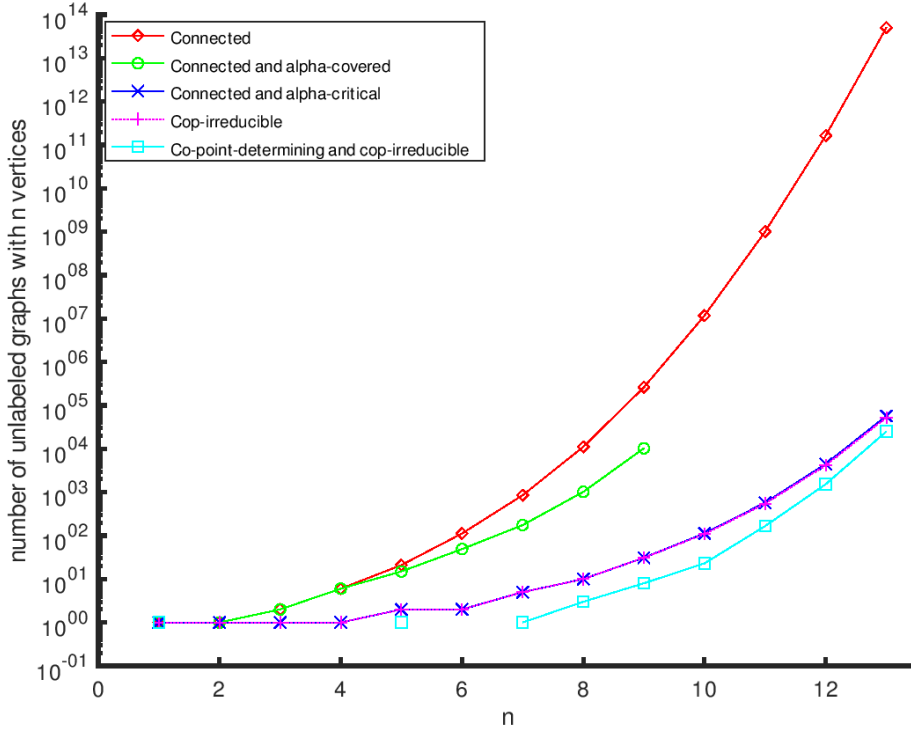


Figure 3: The graph above is a log plot displaying the number of unlabelled graphs on $n \in \{1, \dots, 13\}$ vertices of different types. For $n \geq 10$ the number of α -covered (co-point-determining) graphs is unknown, and thus not shown on the graph. Elsewhere, when no point is given for a value of n , this is due to the number of such graphs being equal to zero. The plots for connected alpha-critical graphs and cop-irreducible graphs are barely distinguishable.

As mentioned earlier, one reason to generate irreducible matrices is to test inner approximations of the copositive cone. We will consider the inner approximations \mathcal{Q}_n^r of the copositive cone, introduced by P  na, Vera and Zuluaga [40]. This is a hierarchy of approximations such that

$$\mathcal{PSD}^n + \mathcal{N}^n = \mathcal{Q}_n^0, \quad \mathcal{Q}_n^r \subseteq \mathcal{Q}_n^{r+1} \text{ for all } r \in \mathbb{N}, \quad \text{int}(\mathcal{COP}^n) \subseteq \bigcup_{r \in \mathbb{N}} \mathcal{Q}_n^r \subseteq \mathcal{COP}^n.$$

As in their paper, for $r \in \mathbb{N}$ we let $\nu^{(r)} = \min_{\lambda} \{ \lambda : \lambda (\mathbf{I} + \mathbf{A}_G) - \mathbf{E} \in \mathcal{Q}_n^r \}$. This then provides an upper bound on $\alpha(G) = \min_{\lambda} \{ \lambda : \lambda (\mathbf{I} + \mathbf{A}_G) - \mathbf{E} \in \mathcal{COP}^n \}$. But how good is this upper bound for different graphs? We have compared cop-irreducible graphs with random graphs, with these comparisons being shown in Figs. 4 and 5.

For $n = 13$ we generated 1,200 random graphs though considering adjacency matrices, by randomly generating binary symmetric n by n matrices with all on-diagonal matrices equal to zero and each entry above the diagonal independently having a probability of 0.324 of being equal to one. This probability was chosen so that the average edge density would match that of cop-irreducible graphs with $(n, \alpha(G)) = (13, 4)$. This was then compared against unlabelled co-point-determining cop-irreducible graphs with $(n, \alpha(G)) = (13, 4)$. However this comparison could be argued to be unfair as over 10% of these random graphs were disconnected and additionally graphs with lots of isomorphisms are more likely to appear.

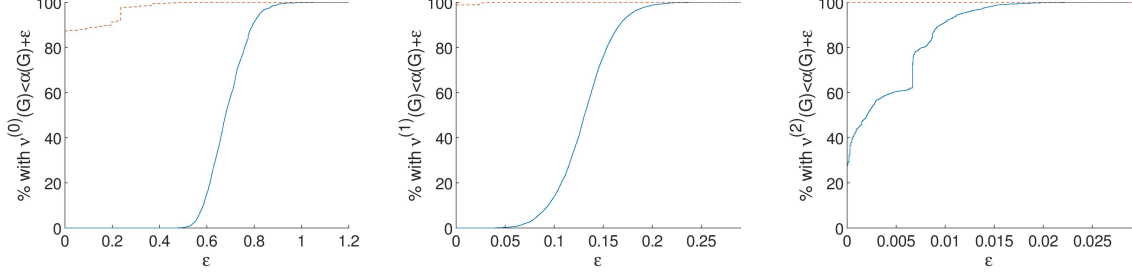


Figure 4: Comparing approximations of graphs with 13 vertices. The blue solid plot is for all 10,454 of the co-point-determining cop-irreducible graphs with 13 vertices and stability number equal to 4. The dashed red plot is for 1,200 randomly generated graphs with 13 vertices.

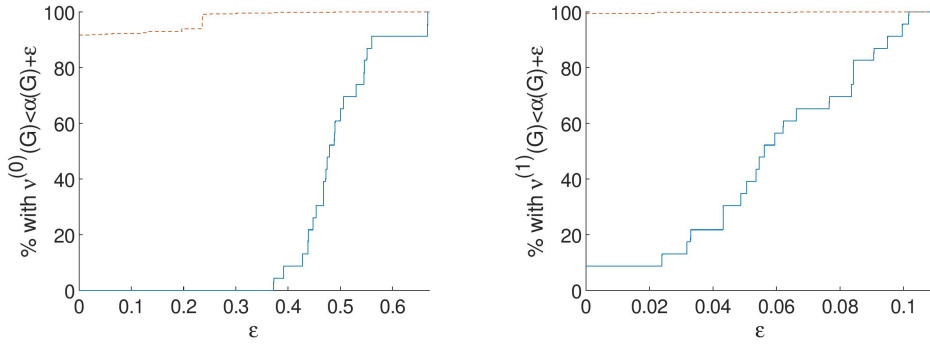


Figure 5: Comparing approximations of graphs with 10 vertices. The blue solid plot is for all 21 of the co-point-determining cop-irreducible graphs with 10 vertices. The dashed red plot is for 1,000 randomly selected connected graphs with 10 vertices. There is no plot for the case of $\nu^{(2)}(G)$, as in this case we always had $\nu^{(2)}(G) = \alpha(G)$.

Thus, as another comparison, for $n = 10$, we looked 1,000 random connected graphs by taking the file of all unlabelled connected graphs from [34] and drawing 1,000 instances of these at random. This was then compared against all unlabelled co-point-determining cop-irreducible graphs with ten vertices.

In both cases we see that the bounds are in general worse for the cop-irreducible graphs than the random graphs. A natural question is which are the hardest cop-irreducible graphs, i.e. the ones with the worse bounds? However in Fig. 6 we see that the graphs with the worse bounds for one approximation do not necessarily have the worse bounds in another approximation, and thus this question does not appear to have a consistent answer.

7 Vertex stretching

In Section 5 we saw a trivial method of generating larger cop-irreducible graphs from smaller ones, allowing us to produce cop-irreducible graphs of an arbitrarily large size. In this section we will consider another less trivial method which can be used to generate large instances to test algorithms against.

This work is inspired by the work on generating larger α -critical graphs from small ones, see e.g. [44, Section 2.4]. Some operations for generating larger cop-irreducible

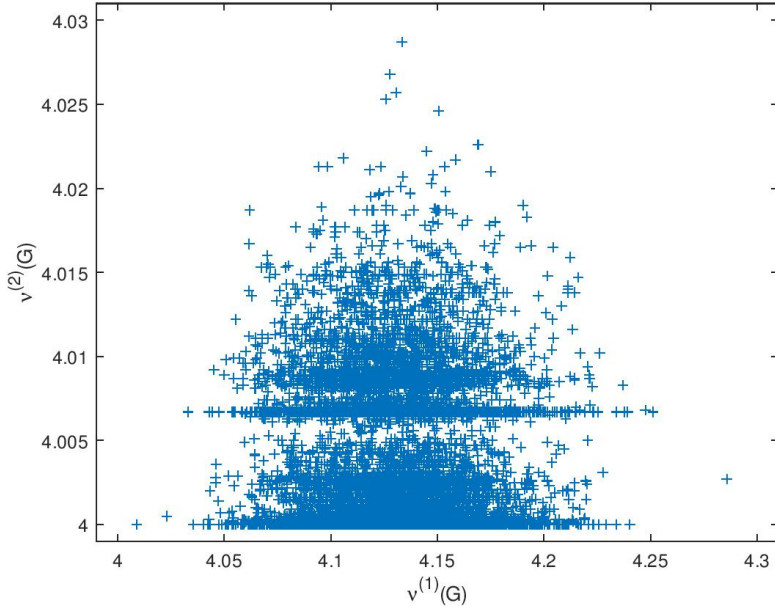


Figure 6: $\nu^{(1)}(G)$ against $\nu^{(2)}(G)$ for cop-irreducible graphs with $(n, \alpha(G)) = (13, 4)$.

graphs are given in the bachelor thesis of the second author [12], and in this section we will consider yet another operation which we will refer to as vertex stretching. This is defined as follows:

We take a graph G , a vertex $v \in V(G)$ and $\{\mathcal{P}, \mathcal{Q}\}$ being a nontrivial partition of the neighbours of v in G , i.e.

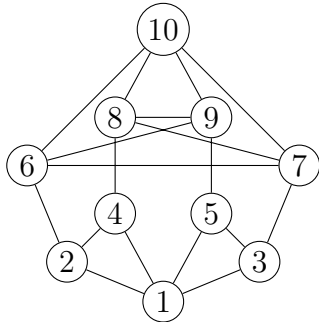
$$\mathcal{P} \cap \mathcal{Q} = \emptyset, \quad \mathcal{P} \neq \emptyset, \quad \mathcal{Q} \neq \emptyset, \quad \mathcal{P} \cup \mathcal{Q} = N_G[v] \setminus \{v\}.$$

We then let $H = G - v + u' + v' + w' + \sum_{x \in \mathcal{P}} xu' + u'v' + v'w' + \sum_{x \in \mathcal{Q}} xw'$.

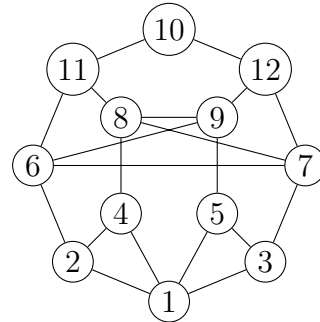
Note that if $\mathcal{P} = \{y\}$ for some $y \in V(G)$, then this operation is equivalent to subdividing the edge vy twice.

By applying this operation to the graph in Fig. 2b, considering $v = 6$, $(u', v', w') = (7, 6, 8)$, $\mathcal{P} = \{4\}$ and $\mathcal{Q} = \{2, 3\}$ we get instance (8,3,2) from Table 2.

Another example of this operation is given in Fig. 7.



(a) A graph which is α -critical but not α -covered.



(b) A cop-irreducible graph.

Figure 7: An example of vertex stretching, where the graph in Fig. 7b is created from Fig. 7a via vertex expansion with $v = 10$, $(u', v', w') = (11, 10, 12)$, $\mathcal{P} = \{6, 8\}$ and $\mathcal{Q} = \{7, 9\}$.

We have the following result on this operation.

Theorem 12. *Consider graphs G, H such that H is produced from G by vertex stretching. Then we have:*

1. $\alpha(H) = \alpha(G) + 1$;
2. G is connected if and only if H is connected;
3. G is α -critical if and only if H is α -critical;
4. If G is cop-irreducible then H is also cop-irreducible;
5. There exist instances of G, H such that H is cop-irreducible but G is not;

Furthermore, if one of the partitions used in the vertex splitting has cardinality one then we have

6. G is cop-irreducible if and only if H is cop-irreducible.

Proof. We will let $G, H, v, u', v', w', \mathcal{P}, \mathcal{Q}$ be as in the definition of vertex stretching above and consider each statement in turn:

1. Let \mathcal{A} be a maximum stable set of G . If $v \in \mathcal{A}$ then $\mathcal{B} = \{u', w'\} \cup \mathcal{A} \setminus \{v\}$ is a stable set of H of cardinality $\alpha(G) + 1$. Alternatively, if $v \notin \mathcal{A}$ then $\mathcal{B} = \mathcal{A} \cup \{v'\}$ is a stable set of H of cardinality $\alpha(G) + 1$. Therefore $\alpha(H) \geq \alpha(G) + 1$.

Now let \mathcal{B} be a stable set of H . If $|\{u', v', w'\} \cap \mathcal{B}| \leq 1$ then $\mathcal{A} = \mathcal{B} \setminus \{u', v', w'\}$ is a stable set of G of cardinality greater than or equal to $\alpha(H) - 1$. If on the other hand $|\{u', v', w'\} \cap \mathcal{B}| \geq 2$ then $\{u', v', w'\} \cap \mathcal{B} = \{u', w'\}$ and $\mathcal{A} = \{v\} \cup \mathcal{B} \setminus \{u', w'\}$ is a stable set of G of cardinality $\alpha(H) - 1$. Therefore $\alpha(G) \geq \alpha(H) - 1$, completing the proof that $\alpha(H) = \alpha(G) + 1$.

2. It is trivial that G is connected if and only if H is connected.
3. This is closely related to [44, Theorem 3.13] and [49, Satz 81], and we include a proof for the sake of completeness.

First suppose that G is α -critical and consider an arbitrary $ij \in E(H)$. We now consider three cases:

- (a) If $|\{i, j\} \cap \{u', v', w'\}| = 0$ then $ij \in E(G)$ and there exists a stable set \mathcal{A} of $G - ij$ of cardinality $\alpha(G) + 1 = \alpha(H)$. If $v \in \mathcal{A}$ then $\mathcal{A} \cup \{u', w'\} \setminus \{v\}$ is a stable set of $H - ij$ of cardinality $\alpha(H) + 1$. If on the other hand $v \notin \mathcal{A}$ then $\mathcal{A} \cup v'$ is a stable set of $H - ij$ of cardinality $\alpha(H) + 1$.
- (b) If $|\{i, j\} \cap \{u', v', w'\}| = 1$ then without loss of generality $i = u'$ and $j \in \mathcal{P}$. We then have $vj \in E(G)$, and thus there exists a stable set \mathcal{A} of $G - vj$ of cardinality $\alpha(G) + 1 = \alpha(H)$ with $\{v, j\} \in \mathcal{A}$. We then have that $\mathcal{A} \cup \{u', w'\} \setminus \{v\}$ is a stable set of $H - ij$ of cardinality $\alpha(H) + 1$.
- (c) If $|\{i, j\} \cap \{u', v', w'\}| = 2$ then without loss of generality $i = u'$ and $j = v'$. Considering an arbitrary $z \in \mathcal{Q}$, we have that $vz \in E(G)$ and thus there exists a stable set \mathcal{A} of $G - vz$ of cardinality $\alpha(G) + 1 = \alpha(H)$. We then have that $\mathcal{A} \cup \{u', v'\} \setminus \{v\}$ is a stable set of $H - ij$ of cardinality $\alpha(H) + 1$.

Therefore G being α -critical implies that H is also α -critical.

We will now suppose that H is α -critical and consider an arbitrary $ij \in E(G)$. We now consider two cases:

- (a) If $v \notin \{i, j\}$ then $ij \in E(H)$ and there exists a stable set \mathcal{B} of $H - ij$ of cardinality $\alpha(H) + 1 = \alpha(G) + 2$. If $|\{u', v', w'\} \cap \mathcal{B}| \leq 1$ then $\mathcal{A} = \mathcal{B} \setminus \{u', v', w'\}$ is a stable set of $G - ij$ of cardinality greater than or equal to $\alpha(G) + 1$. If on the other hand $|\{u', v', w'\} \cap \mathcal{B}| \geq 2$ then $\{u', v', w'\} \cap \mathcal{B} = \{u', w'\}$ and $\mathcal{A} = \{v\} \cup \mathcal{B} \setminus \{u', w'\}$ is a stable set of $G - ij$ of cardinality $\alpha(G) + 1$.
- (b) If $v \in \{i, j\}$ then without loss of generality $i = v$ and $j \in \mathcal{P}$, and there exists a stable set \mathcal{B} of $H - u'j$ of cardinality $\alpha(H) + 1 = \alpha(G) + 2$. We have that $w' \in \mathcal{B}$, otherwise we would get the contradiction that $\mathcal{B} \cup \{v'\} \setminus \{u'\}$ is a stable set of H of cardinality $\alpha(H) + 1$. Therefore $\mathcal{B} \cup \{v\} \setminus \{u'\}$ is a stable set of $G - ij$ of cardinality $\alpha(G) + 1$.

Therefore H being α -critical implies that G is also α -critical.

4. We will now complete the proof of statement 4 by showing that if G is cop-irreducible then H is α -covered. Consider an arbitrary $xy \in E(\overline{H})$. We will show that there exists a maximum stable set \mathcal{A} of H such that $\{x, y\} \subseteq \mathcal{A}$. We split this into 5 cases:

- (a) $\{x, y\} \cap \{u', v', w'\} = \emptyset$: Then $xy \in E(\overline{G})$, and there exists a stable set \mathcal{B} of \overline{G} such that $\{x, y\} \subseteq \mathcal{B}$. If $v \in \mathcal{B}$ then let $\mathcal{A} = \{u', w'\} \cup \mathcal{B} \setminus \{v\}$, otherwise let $\mathcal{A} = \{v'\} \cup \mathcal{B}$. In either case, we have that \mathcal{A} is a maximum stable set of H containing $\{x, y\}$.
- (b) $\{x, y\} \subseteq \{u', v', w'\}$: We then have $\{x, y\} = \{u', w'\}$. From Lemma 7, there exists a maximum stable set \mathcal{B} of G such that $v \in \mathcal{B}$. Letting $\mathcal{A} = \{u', w'\} \cup \mathcal{B} \setminus \{v\}$, we have that \mathcal{A} is a maximum stable set of H containing $\{x, y\}$.
- (c) $\{x, y\} \cap \{u', v', w'\} = \{v'\}$: Without loss of generality $y = v'$. From Lemma 7, there exists a maximum stable set \mathcal{B} of G which contains x but not v . Letting $\mathcal{A} = \{v'\} \cup \mathcal{B}$, we have that \mathcal{A} is a maximum stable set of H containing $\{x, y\}$.
- (d) $\{x, y\} \cap \{u', v', w'\} \subseteq \{u', w'\}$ and $\{x, y\} \cap (\mathcal{P} \cup \mathcal{Q}) \neq \emptyset$: Without loss of generality $x = u'$ and $y \in \mathcal{Q}$. We have $vy \in E(G)$, and thus there exists a stable set \mathcal{B} of $G - vy$ such that $|\mathcal{B}| = \alpha(G) + 1$, and we have $\{v, y\} \subseteq \mathcal{B}$. Letting $\mathcal{A} = \{u'\} \cup \mathcal{B} \setminus \{v\}$, we have that \mathcal{A} is a maximum stable set of H containing $\{x, y\}$.
- (e) $\{x, y\} \cap \{u', v', w'\} \subseteq \{u', w'\}$ and $\{x, y\} \cap (\mathcal{P} \cup \mathcal{Q}) = \emptyset$: Without loss of generality $x \in \{u', w'\}$. We have $vy \in E(\overline{G})$, and thus there exists a maximum stable set \mathcal{B} of G such that $\{v, y\} \subseteq \mathcal{B}$. Letting $\mathcal{A} = \{u', w'\} \cup \mathcal{B} \setminus \{v\}$, we have that \mathcal{A} is a maximum stable set of H containing $\{x, y\}$.

This completes the proof of statement 4.

5. Fig. 7 is an example where H is cop-irreducible but G is not.

6. We will now finish this proof by showing that if $\mathcal{P} = \{z\}$ for some $z \in V(G) \setminus \{v\}$ and H is cop-irreducible then G is α -covered. Consider an arbitrary $xy \in E(\overline{G})$. We will show that there exists a maximum stable set \mathcal{A} of G such that $\{x, y\} \subseteq \mathcal{A}$. We split this into 2 cases:

- (a) $v \notin \{x, y\}$: Then $xy \in E(\overline{H})$, and as H is α -covered, there exists a maximum stable set \mathcal{B} of H such that $\{x, y\} \subseteq \mathcal{B}$. We have that $\{u', v', w'\} \cap \mathcal{B} \neq \emptyset$, otherwise we get the contradiction that $\mathcal{B} \cup \{v'\}$ is a stable set of H of cardinality $\alpha(H) + 1$. If $|\{u', v', w'\} \cap \mathcal{B}| = 1$ then $\mathcal{A} = \mathcal{B} \setminus \{u', v', w'\}$ is a maximum stable set of G containing $\{x, y\}$. If $|\{u', v', w'\} \cap \mathcal{B}| \geq 2$ then $\{u', v', w'\} \cap \mathcal{B} = \{u', w'\}$ and $\mathcal{A} = \{v\} \cup \mathcal{B} \setminus \{u', w'\}$ is a maximum stable set of G containing $\{x, y\}$.
- (b) $v \in \{x, y\}$: Without loss of generality $x = v$, and we have $y \notin (\mathcal{P} \cup \mathcal{Q})$. We have $yw' \in E(\overline{H})$, and thus exists a maximum stable set \mathcal{B} of H such that $\{y, w'\} \subseteq \mathcal{B}$, and as $v'w' \in E(H)$, we have $v' \notin \mathcal{B}$. Without loss of generality, we may assume that $u' \in \mathcal{B}$, as if this is not the case then $\{u'\} \cup \mathcal{B} \setminus \{z\}$ is a stable set of H of cardinality at least $\alpha(H)$. As $\{u', w'\} \subseteq \mathcal{B}$, this implies that $\mathcal{A} = \{v\} \cup \mathcal{B} \setminus \{u', w'\}$ is a stable set of G such that $\{x, y\} \subseteq \mathcal{A}$. \square

By combining the operations of vertex duplication and vertex stretching, and applying them to the known cop-irreducible graphs from Table 2 we can build quite complicated cop-irreducible graphs of an arbitrarily large size. This could be useful generating large instances to test algorithms against. Not only are we able to generate instances, but by keeping track of the operations, we also maintain knowledge of the stability number.

As an example, we could start with a single point, i.e. instance $(1, 1, 1)$ from Table 2. We can then apply vertex duplication to this graph three times to get the complete graph on four vertices. By then subdividing edges (which is a special case of vertex stretching) we can generate instances $(8, 3, 1)$, $(8, 3, 2)$, $(10, 4, 1)$, $(10, 4, 2)$, $(10, 4, 3)$, $(10, 4, 4)$ and $(10, 4, 5)$ from Table 2.

There is already a large amount of work on operations that maintain the property of being α -critical (see [44] of a survey of this), and it would be an interesting source of future research to further consider which of these operations also maintain the property of being cop-irreducible (possibly under extra conditions).

8 Extreme copositive matrices

As mentioned earlier, one of the motivations for looking at irreducible matrices is in the investigation of extreme matrices. A matrix $\mathbf{X} \in \mathcal{S}^n$ is an extreme copositive matrix if $\mathbf{X} \in \mathcal{COP}^n$ and $\mathbf{X} = \mathbf{Y} + \mathbf{Z}$ with $\mathbf{Y}, \mathbf{Z} \in \mathcal{COP}^n$ implies that $\{\mathbf{Y}, \mathbf{Z}\} \subseteq \mathbb{R}\{\mathbf{X}\}$. A subset of these matrices is the set of exposed copositive matrices, however we will not get in to the details of this here and instead direct the interested reader to [14]. In fact the difference between these is inconsequential for considering matrices \mathbf{Z}_G , as we will see in the following lemma.

Lemma 13. *For a graph G we have that \mathbf{Z}_G is an extreme copositive matrix if and only if it is an exposed copositive matrix.*

Proof. This follows directly from combining the equivalence of statements 1 and 2 in Lemma 6 with [17, Theorems 17 and 19]. \square

In the paper [17], it was investigated how to test if a copositive matrix is extreme given its set of minimal zeros. Combining this result with the result on the set of minimal zeros in Lemma 6, it was possible to implement a simple test to check if a matrix Z_G is an extreme copositive matrix. We tested which of the 26,863 co-point-determining cop-irreducible graphs with at most 13 vertices were extreme matrices, expecting this to be the case in only a few instances. Amazingly, we in fact found that out of all these thousands of graphs, only 3 were not extreme. These were the graphs \overline{W}_2^9 , \overline{W}_2^{11} and \overline{W}_2^{13} (with stability numbers equal to 4, 5 and 6 respectively). Inspired by this, we then tested all antiwebs with at most 27 vertices which give cop-irreducible graphs. For these graphs we found that the matrices were extreme if and only if the stability number was less than or equal to 3. This provides a very interesting preliminary result, which we consider to be worthy of further investigation in the future.

9 Conclusion

In this paper we have considered an injective mapping from simple graphs to matrices on the boundary of the copositive cone, with this mapping being closely related to the stability number problem. We have given a necessary and sufficient condition on a graph for the matrix generated by it to be irreducible with respect to the positive semidefinite plus nonnegative cone, and we refer to such graphs as cop-irreducible graphs. There are 57,459 unlabelled graphs on at most 13 vertices with this property, of which 26,863 are co-point-determining. Of these co-point-determining graphs, all but 3 in fact generate extreme copositive matrices.

In this paper we have also seen how cop-irreducible graphs can be used as difficult instances to test inner approximations of the copositive cone against, as well as looking at how we can create larger cop-irreducible graphs out of smaller ones.

Along with what we consider to be some interesting results, this paper has also contained a number of open problems, and we will finish with one more open problem. In our paper, we have considered how we can use simple graphs and the copositive optimisation formulation of the stability number problem to generate copositive matrices, and then studied the properties of these matrices. The copositive formulation of the stability number problem can in fact be extended for finding the maximum weight stable set, where the weights are on the vertices [3, 5, 15]. Using this, we see that there is a similar mapping from weighted graphs to matrices on the boundary of the copositive cone, and it would be of interest to consider how results from our paper could be extended for this.

Acknowledgements. The first author would like to gratefully acknowledge support from the Netherlands Organisation for Scientific Research (NWO) through grant no. 613.009.021.

References


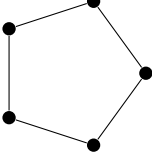
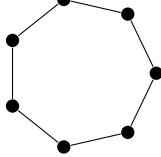
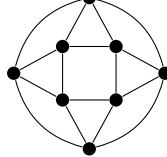
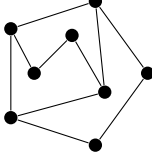
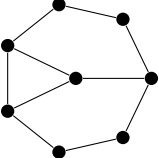
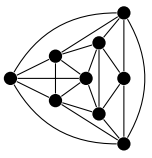
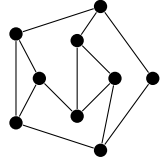
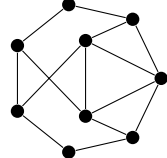
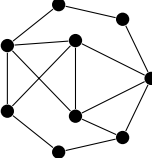
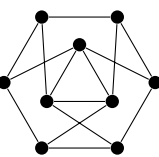
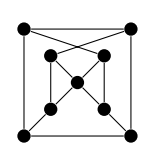
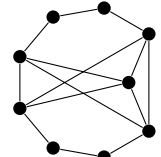
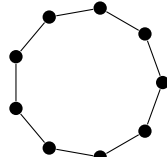
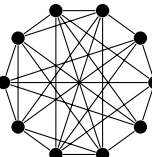
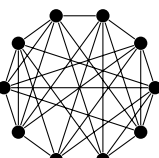
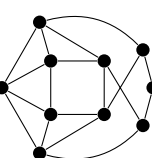
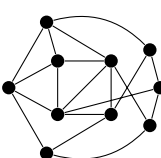
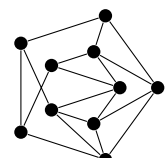
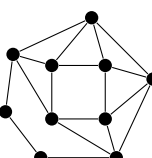
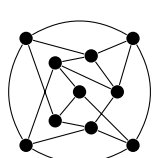
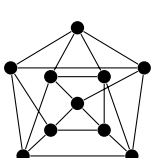
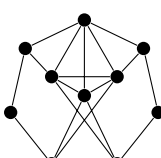
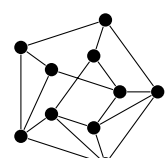
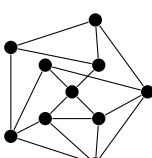
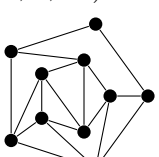
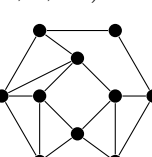
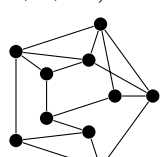
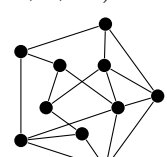
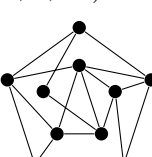
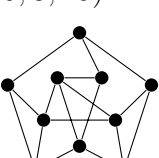
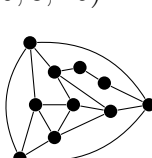
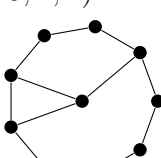
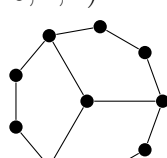
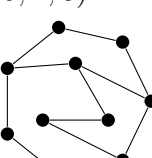
- [1] Leonard D. Baumert. Extreme copositive quadratic forms. *Pacific Journal of Mathematics*, 19:197–204, 1966.

- [2] Leonard D. Baumert. Extreme copositive quadratic forms, ii. *Pacific Journal of Mathematics*, 20:1–20, 1967.
- [3] Immanuel M. Bomze. On standard quadratic optimization problems. *Journal of Global Optimization*, 13(4):369–387, 1998.
- [4] Immanuel M. Bomze, Marco Budinich, Panos M. Pardalos, and Marcello Pelillo. *The Maximum Clique Problem*, pages 1–74. Springer US, 1999.
- [5] Immanuel M. Bomze, Mirjam Dür, Etienne de Klerk, Cornelis Roos, Arie J. Quist, and Tamàs Terlaky. On copositive programming and standard quadratic optimization problems. *Journal of Global Optimization*, 18:301–320, 2000.
- [6] Immanuel M. Bomze, Werner Schachinger, and Gabriele Uchida. Think co(mpletely)positive! matrix properties, examples and a clustered bibliography on copositive optimization. *Journal of Global Optimization*, 52(3):423–445, 2012.
- [7] John Adrian Bondy and U. S. R. Murty. *Graph theory with applications*. The Macmillan Press Ltd., 1976.
- [8] James J. Bull and Craig M. Pease. Combinatorics and variety of mating-type systems. *Evolution*, 43(3):667–671, 1988.
- [9] Manoel Campêlo, Ricardo C. Corrêa, Phablo F. S. Moura, and Marcio C. Santos. On optimal k-fold colorings of webs and antiwebs. *Discrete Applied Mathematics*, 161(1):60–70, 2013.
- [10] Eddie Cheng and Sven de Vries. Antiweb-wheel inequalities and their separation problems over the stable set polytopes. *Mathematical Programming*, 92(1):153–175, 2002.
- [11] Etienne de Klerk and Dmitrii V. Pasechnik. Approximation of the stability number of a graph via copositive programming. *SIAM Journal on Optimization*, 12(4):875–892, 2002.
- [12] Reinier de Zeeuw. Which graphs produce irreducible copositive matrices? Bachelor thesis, University of Twente, 2018.
- [13] Palahenedi H. Diananda. On nonnegative forms in real variables some or all of which are nonnegative. *Mathematical Proceedings of the Cambridge Philosophical Society*, 58:17–25, 1962.
- [14] Peter J.C. Dickinson. Geometry of the copositive and completely positive cones. *Journal of Mathematical Analysis and Applications*, 380:377–395, 2011.
- [15] Peter J.C. Dickinson. *The Copositive Cone, the Completely Positive Cone and their Generalisations*. PhD thesis, University of Groningen, 2013.
- [16] Peter J.C. Dickinson, Mirjam Dür, Luuk Gijben, and Roland Hildebrand. Irreducible elements of the copositive cone. *Linear Algebra and its Applications*, 439(6):1605–1626, 2013.
- [17] Peter J.C. Dickinson and Roland Hildebrand. Considering copositivity locally. *Journal of Mathematical Analysis and Applications*, 437(2):1184–1195, 2016.

- [18] Mirjam Dür. *Copositive Programming - a Survey*, pages 3–20. Springer Berlin Heidelberg, 2010.
- [19] John W. Eaton, David Bateman, Søren Hauberg, and Rik Wehbring. *GNU Octave version 4.2.2 manual: a high-level interactive language for numerical computations*, 2018.
- [20] Paul Erdős and Tibor Gallai. On the minimal number of vertices representing the edges of a graph. *Publ. Math. Inst. Hungar. Acad. Sci.*, 6(18):1–203, 1961.
- [21] Ira M. Gessel and Ji Li. Enumeration of point-determining graphs. *Journal of Combinatorial Theory, Series A*, 118(2):591–612, 2011.
- [22] Martin Grötschel, László Lovász, and Alexander Schrijver. *Geometric algorithms and combinatorial optimization*. Springer-Verlag (Berlin), 1988.
- [23] Marshall Hall Jr and Morris Newman. Copositive and completely positive quadratic forms. *Mathematical Proceedings of the Cambridge Philosophical Society*, 59:329–339, 1963.
- [24] Roland Hildebrand. The extreme rays of the 5×5 copositive cone. *Linear Algebra and its Applications*, 437(7):1538–1547, 2012.
- [25] Roland Hildebrand. Minimal zeros of copositive matrices. *Linear Algebra and its Applications*, 459:154–174, 2014.
- [26] Jean-Baptiste Hiriart-Urruty and Alberto Seeger. A variational approach to copositive matrices. *SIAM Review*, 52(4):593–629, 2010.
- [27] Alan J. Hoffman and Francisco Pereira. On copositive matrices with -1,0,1 entries. *Journal of Combinatorial Theory*, 14(3):302–309, 1973.
- [28] Markus Hohenwarter. GeoGebra: Ein Softwaresystem für dynamische Geometrie und Algebra der Ebene. Master’s thesis, Paris Lodron University, Salzburg, Austria, 2002.
- [29] Wolfram Research, Inc. Mathematica, Version 11.3. Champaign, IL, 2018.
- [30] Goran Kilibarda. Enumeration of unlabelled mating graphs. *Graphs and Combinatorics*, 23(2):183–199, 2007.
- [31] Johan Lofberg. Yalmip: a toolbox for modeling and optimization in matlab. page 284–289. IEEE, 2004.
- [32] László Lovász. On the shannon capacity of a graph. *IEEE Transactions on Information Theory*, 25(1):1–7, 1979.
- [33] MATLAB. *version 9.4 (R2018a)*. The MathWorks Inc., Natick, Massachusetts, 2018.
- [34] Brendan D. McKay. Combinatorial data. <http://users.cecs.anu.edu.au/~bdm/data/graphs.html>.
- [35] Brendan D. McKay. Practical graph isomorphism. *Congressum Numerantium*, 30:45–87, 1981.

- [36] Brendan D. McKay and Adolfo Piperno. Nauty traces. <http://pallini.di.uniroma1.it>.
- [37] Brendan D. McKay and Adolfo Piperno. Practical graph isomorphism, II. *Journal of Symbolic Computation*, 60:94–112, 2014.
- [38] Theodore S. Motzkin and Ernst G. Straus. Maxima for graphs and a new proof of a theorem of turán. *Canadian Journal of Mathematics*, 17:533–540, 1965.
- [39] Gintaras Palubeckis. Facet-inducing web and antiweb inequalities for the graph coloring polytope. *Discrete Applied Mathematics*, 158(18):2075–2080, 2010.
- [40] Javier F. Peña, Juan C. Vera, and Luis F. Zuluaga. Computing the stability number of a graph via linear and semidefinite programming. *SIAM Journal on Optimization*, 18(1):87–105, 2007.
- [41] Michael D. Plummer. On a family of line-critical graphs. *Monatshefte für Mathematik*, 71(1):40–48, 1967.
- [42] Ronald Cedric Read. *The enumeration of mating-type graphs*. Faculty of Mathematics, University of Waterloo, 1989.
- [43] Edward C. Sewell and Leslie E. Trotter. Stability critical graphs and even subdivisions of K_4 . *Journal of Combinatorial Theory, Series B*, 59(1):74–84, 1993.
- [44] Benjamin Luke Small. *On alpha-critical graphs and their construction*. PhD thesis, Washington State University, 2015.
- [45] David P. Sumner. Point determination in graphs. *Discrete Mathematics*, 5(2):179–187, 1973.
- [46] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.2)*, 2018. <http://www.sagemath.org>.
- [47] Kim-Chuan Toh, Michael J. Todd, and Reha H. Tütüncü. *On the Implementation and Usage of SDPT3 – A Matlab Software Package for Semidefinite-Quadratic-Linear Programming, Version 4.0*, pages 715–754. International Series in Operations Research & Management Science. Springer US, 2012.
- [48] Leslie E. Trotter. A class of facet producing graphs for vertex packing polyhedra. *Discrete Mathematics*, 12(4):373–388, 1975.
- [49] Walter Wessel. *On the problem of determining whether a given graph is edge critical or not*, volume 4 of *Colloquia Mathematica Societatis János Bolyai*, pages 1123–1139. 1970.
- [50] Alexander Aleksandrovich Zykov. On some properties of linear complexes. *Matematicheskii sbornik*, 66(2):163–188, 1949.

Table 2: All cop-irreducible graphs with at most 10 vertices which are co-point determining. In brackets above the graph, the first number gives the number of vertices, the second number gives the stability number, and the third number gives the instance number that we have given the graph (in order to distinguish them easier).

(1, 1, 1)	(5, 2, 1)	(7, 3, 1)	(8, 2, 1)	(8, 3, 1)
				
(8, 3, 2)	(9, 2, 1)	(9, 3, 1)	(9, 3, 2)	(9, 3, 3)
				
(9, 3, 4)	(9, 3, 5)	(9, 3, 6)	(9, 4, 1)	(10, 2, 1)
				
(10, 2, 2)	(10, 3, 1)	(10, 3, 2)	(10, 3, 3)	(10, 3, 4)
				
(10, 3, 5)	(10, 3, 6)	(10, 3, 7)	(10, 3, 8)	(10, 3, 9)
				
(10, 3, 10)	(10, 3, 11)	(10, 3, 12)	(10, 3, 13)	(10, 3, 14)
				
(10, 3, 15)	(10, 3, 16)	(10, 4, 1)	(10, 4, 2)	(10, 4, 3)
				
(10, 4, 4)	(10, 4, 5)			
