The Noncooperative Fixed Charge Transportation Problem

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Abstract

We introduce the noncooperative fixed charge transportation problem (NFCTP), which is a game-theoretic extension of the fixed charge transportation problem. In the NFCTP, competing players solve coupled fixed charge transportation problems simultaneously. Three versions of the NFCTP are discussed and compared, which differ in their treatment of shared social costs. This may be used from central authorities in order to find a socially balanced framework which is illustrated in a numerical study. Using techniques from generalized Nash equilibrium problems with mixed-integer variables we show the existence of Nash equilibria for these models and examine their structural properties. Since there is no unique equilibrium for the NFCTP, we also discuss how to solve the Nash selection problem and, finally, propose numerical methods for the computation of Nash equilibria which are based on mixed-integer programming.

Keywords: Noncooperative transportation problem, mixed integer Nash games, linear generalized Nash equilibrium problem

AMS subject classifications: 91A06, 91A10, 90B06, 90C11

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1 Introduction

1.1 Motivation

In the *fixed charge transportation problem* (FCTP), one forwarder transports a given good from sources to sinks while minimizing her transportation costs. In addition to variable costs that are proportional to the amount of transported goods, there exist also fixed costs for establishing a route which distinguishes FCTPs from classical transportation problems.

We extend this setting and introduce several competing forwarders who minimize their transportation costs simultaneously while sharing their supply and demand constraints. This results in a noncooperative game and therefore, in the following, we will refer to this model as *noncooperative fixed charge transportation problem* (NFCTP). The setting is illustrated in figure 1.

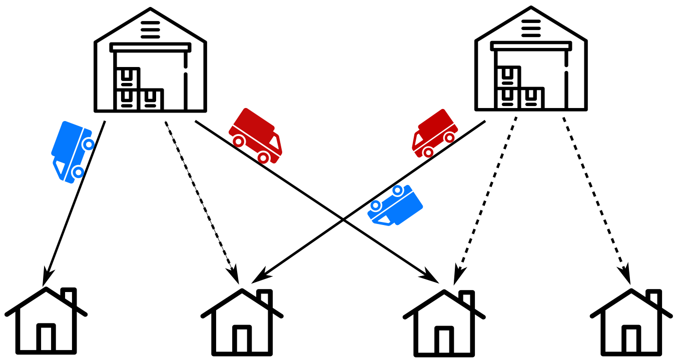

Figure 1: An NFCTP with two supply and four demand nodes is illustrated. Competing players are represented by differently colored trucks. They can activate routes, which turns dotted into solid lines.

The NFCTP can be interpreted from both a *descriptive* and a *normative* point of view. For descriptive purposes, the NFCTP is a realistic extension of the FCTP, which takes into account that in most applications there is not one single forwarder. While the NFCTP does not contain a description of the corresponding time continuous dynamics it gives a transparent model of the competing goals of the several forwarders and enables an observant to draw conclusions regarding an assessment of the situation. From a normative perspective, suppose the existence of a central authority like a state or a big company. The authority may pose rules, e.g. laws or contracts, and is therefore interested in stable situations where no forwarder has an incentive to deviate from its contractual conditions. In other words, the central authority is interested in an equilibrium, which is why we choose noncooperative game theory as modeling tool whose notion of a Nash equilibrium matches these requirements.

1.2 Literature overview

In 1954, Dantzig and Hirsch first introduced the fixed charge transportation problem in a Rand paper on behalf of the Rand Corporation which was then published some years later (cf. [5]). The FCTP extended the classical transportation problem (cf. [20]) in the sense that additional to variable transportation costs the forwarders are charged with fixed costs for establishing a route. The FCTP possesses a rich variety of applications (cf., e.g., [1, 5, 17, 30, 31]). Since it is also a very challenging problem, in the last decades, there have been many publications on numerical solution techniques. See [2, 18, 22] for an overview of recent algorithmic approaches for the FCTP.

Noncooperative game theory was established in 1951, when John Nash investigated the problem of finding equilibria in situations where several competing players optimize their objective functions over strategy sets that are independent of the decisions of the remaining players (cf. [21]). In the remainder of this article, this model is referred to as the (classical) *Nash equilibrium problem* (NEP). In 1954, Kenneth Arrow and Gérard Debreu extended the setting towards a model with coupled strategy sets, that is, strategy sets that may depend on the decisions of the remaining players (cf. [3, 6]). Coupled strategy sets arise in a very natural way if, for instance, players share at least one constraint which could be a common budget or commonly used infrastructure. A Nash equilibrium problem with coupled constraints is called a *generalized Nash equilibrium problem* (GNEP). Despite its early introduction it took over 40 years until GNEPs attracted attention in the operations research community. During this time, that is, until the mid nineties, mainly existence results for (generalized) Nash equilibria were available and the numerical computation of equilibria was less developed (cf. [10]). However, the field of operations research had a deep impact on game theory and provided powerful numerical methods for the computation of Nash equilibria. Excellent overviews of theoretical and numerical results as well as numerous applications of GNEPs are given in [10] and [14]. GNEPs with a linear structure are called *linear generalized Nash equilibrium problems* (LGNEPs) and were examined in [8, 9, 28]. All GNEPs considered so far dealt with the situation where competing players control *continuous* decision variables.

In 2017, GNEPs with mixed-integer variables (MIGNEPs) were first studied systematically in [25, 26]. These recent developments enable the treatment of the NFCTP as a MIGNEP.

There exists some literature which examines the connection between noncooperative game theory and transportation. [13] is an early and well-known contribution to this field of research where the author examines a model of carriers competing for intercity passenger travel as an example of a Nash game. In [32], the authors consider a generalized Nash equilibrium (GNEP) in transportation. [15] also uses a noncooperative game between forwarders as part of a larger model for dynamic pricing. The authors of [16] present a comparative literature review of noncooperative games that describe transport problems in order to show their opportunities and risks. All reviewed articles in this context are categorized into *games against a demon*, *games between travelers*, *games between travelers and authorities* as well as *games between authorities*. The NFCTP is a specific game between authorities which has not been investigated before. A first explicit treatment of the classical transportation problem as GNEP is presented in [29] but due to the mixed-integer structure the techniques and results are not applicable to the NFCTP.

1.3 Statement of contribution

We introduce a game-theoretic extension of the FTCP - the *noncooperative fixed charge transportation problem* (NFCTP). Three versions of the NFCTP are presented and studied extensively. In particular, we prove the existence of Nash equilibria for all models and suggest numerical methods for their computation using recent results and techniques from the framework of generalized Nash equilibrium problems with mixed-integer variables. Finally, we suggest how to solve the Nash selection problem for the NFCTP and show how the three different versions of the NFCTP can be used by central authorities to enforce Nash equilibria with different social properties. This is also illustrated in a numerical study.

Kenneth Arrow was the first to study generalized Nash equilibrium problems in 1954. It is probably a coincidence that fixed charge transportation problems were introduced in the same year. Now, sixty-four years later, theoretical results and numerical tools are available that enable us to treat the NFCTP in an game-theoretical setting. Thus, we aim at building a bridge between a celebrated mixed-integer application and the world of noncooperative game theory.

1.4 Overview

After discussing some basic game-theoretical concepts and results in section 2, we introduce three different versions of the NFCTP in section 3. Each model is illustrated with examples of players acting in a transportation setting where both, private and shared (social) fixed costs occur. In section 4, we show that all NFCTP models are so-called potential games for which Nash equilibria exist that can be computed solving some auxiliary mixed-integer optimization problems. In section 5, we use numerical results to show that the three NFCTP versions represent different ways to distribute social costs which can be used to give players incentives to act in different ways. Finally, we close with a conclusion and some final remarks in section 6.

2 Preliminaries

Before we examine the NFCTP in section 3, we recall some basic notions and results from generalized Nash equilibrium problems which are based on [10, 19, 23, 25, 28].

2.1 Basic notions

Assume that we have N competing players $\nu = 1, \ldots, N$ each controlling a decision variable x^ν in an Euclidean space of dimension n_ν. Complementary, we denote the decisions of the remaining players by $x^{-\nu} := (x^1, \ldots, x^{\nu-1}, x^{\nu+1}, \ldots, x^N)$. The decision variable x^ν governs player ν's objective function $\theta_\nu(\cdot, x^{-\nu}) : \mathbb{R}^{n_\nu} \to \mathbb{R}$. In the constrained case, all feasible decisions of player ν are contained in her strategy set $X_\nu(x^{-\nu}) = \{x^{\nu,1},\ x^{\nu,2}, \ldots\} \subset \mathbb{R}^{n_\nu}$. Altogether, in a *generalized Nash equilibrium problem*(GNEP) all players optimize simultaneously their coupled optimization problems

$$\min_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}) \quad \text{s.t.} \quad x^\nu \in X_\nu(x^{-\nu}).$$

We define $n := \sum_{\nu=1}^N n_\nu$ and the *unfolded common strategy set*

$$M \ := \ \{x \in \mathbb{R}^n |\ x^\nu \in X_\nu(x^{-\nu})\ \forall \nu = 1, \ldots, N\}$$

as well as

$$X(x) \ := \ X_1(x^{-1}) \times \ldots \times X_N(x^{-N}).$$

The notion of a Nash equilibrium is crucial for our analysis.

Definition 2.1 (Nash equilibrium) *Let $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^N) \in \mathbb{R}^n$ be a combination of decisions. Then, $\bar{x}$ is a* Nash equilibrium, *if and only if its components are optimal for all players, that is, for all $\nu = 1, \ldots, N$ we have $\bar{x} \in X_\nu(\bar{x}^{-\nu})$ and $\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}) \;\leq\; \theta_\nu(x^\nu, \bar{x}^{-\nu})$ for all $x^\nu \in X_\nu(\bar{x}^{-\nu})$.*

If player ν's strategy set is independent of $x^{-\nu}$, the GNEP reduces to a *Nash equilibrium problem* (NEP) which is much easier to solve (cf. [10]). We are going to exploit this case in section 4.4. If θ_ν is a linear function in the whole vector $x = (x^1, \ldots, x^N)$, we call the resulting game a *linear generalized Nash equilibrium problem* (LGNEP). If some of the variables in a GNEP are integer, we refer to it as an *GNEP with mixed integer variables* (MIGNEP).

2.2 Rosen's law and existence issues

A GNEP satisfies *Rosen's law* (cf. [23]) if there exists a closed convex set $X \subset \mathbb{R}^n$ and for all $\nu = 1, \ldots, N$ we have

$$X_\nu(x^{-\nu}) \;=\; \{x^\nu \in \mathbb{R}^{n_\nu} |\; (x^\nu, x^{-\nu}) \in X\}. \tag{1}$$

It is important to remark that Rosen's law ensures the existence of Nash equilibria for a given GNEP only if some continuity and convexity assumptions hold. Specifically, if any objective function θ_ν is continuous in the whole vector x and convex in x^ν for each fixed $x^{-\nu}$, and if X is compact and convex, then all the assumptions of the Kakutani fixed-point theorem are satisfied and an equilibrium of the GNEP exists (see e.g. [10, 23]). However, in the context of games with integer variables, the convexity of X cannot be assumed and the existence of equilibria may fail as witnessed by the following example.

Example 2.2 *Consider a game with two players:*

$$\min_{x^1 \in [0,1]} \; \theta_1(x) := \frac{1}{2}(x^1)^2 + x^1 x^2 - x^1, \qquad \min_{x^2 \in \{0,1\}} \; \theta_2(x) := \frac{1}{2}(x^2)^2 - x^1 x^2.$$

Both θ_1 and θ_2 satisfy the continuity and convexity assumptions above and $X = [0,1] \times \{0,1\}$ is compact but not convex. It is easy to see that the game does not have any equilibria. •

Existence results for mixed-integer Nash games are difficult to obtain. To date, existence results are only available for potential games (described in section 2.3) and for 2-groups partitionable games which are generalizations of supermodular games (cf. [24, 25, 26]).

2.3 Potential games

Potential games constitute an important class of Nash equilibrium problems that was first introduced in [19] and immediately attracted great attention because of its nice properties [27]. The fundamental feature of a potential game is the existence of a continuous not necessarily convex function $P : \mathbb{R}^n \to \mathbb{R}$ that is simultaneously relevant for all the players in the game. Specifically, for any $\nu \in \{1, \dots, N\}$ and all $(x^{\nu,1}, x^{-\nu,1}), (x^{\nu,2}, x^{-\nu,2}) \in M$ it holds that

$$\theta_\nu(x^{\nu,1}, x^{-\nu,1}) < \theta_\nu(x^{\nu,2}, x^{-\nu,2}) \iff P(x^{\nu,1}, x^{-\nu,1}) < P(x^{\nu,2}, x^{-\nu,2}).$$

Such P is called an *ordinal potential function* for the game. Roughly speaking, in these games, all the players are unknowingly minimizing the same function P, even if they are acting in a noncooperative manner.

Thanks to its particular structure and assuming a set M satisfying (1), a potential game always admits at least one equilibrium because any minimal point of P over M is also an equilibrium of the game. However, it is well known that equilibria may exist that are not minimal points of P over M. To overcome this, some provably convergent best-response algorithms were recently proposed in [11, 25] to compute any equilibrium of these games .

Among the family of ordinal potential functions an *exact potential function* $\bar{P}$ may exist. In this case we have

$$\theta_\nu(x^{\nu,1}, x^{-\nu,1}) - \theta_\nu(x^{\nu,2}, x^{-\nu,2}) = \bar{P}(x^{\nu,1}, x^{-\nu,1}) - \bar{P}(x^{\nu,2}, x^{-\nu,2}).$$

for any $\nu \in \{1, \dots, N\}$ and all $(x^{\nu,1}, x^{-\nu,1}), (x^{\nu,2}, x^{-\nu,2}) \in M$.

2.4 Pseudo-NEPs

A Pseudo-NEP is a GNEP whose solution set enjoys a nice property. Specifically the set of all its equilibria is equivalent to the union of the sets of the equilibria of all the NEPs with the same objectives θ_ν and feasible sets equal to $X(z)$ for any $z \in \mathbb{R}^n$ such that $z \in X(z)$ (see [4]). Therefore any equilibrium of a Pseudo-NEP can be computed by simply finding a point $z \in X(z)$ and then solving the NEP with a fixed feasible set $X(z)$. This useful property is related to the concept of reproducibility of the coupling constraints. In particular, as shown in [4], all the shared linear equality constraints are reproducible. Thus, any LGNEP with mixed-integer variables whose coupling constraints only consist of shared equality constraints is a Pseudo-NEP.

3 The noncooperative fixed charge transportation models their distribution of social costs

In this section we will present three different versions of the NFCTP as well as some possible applications. The three versions of the NFCTP represent three different ways to distribute so-called shared social costs among a given number of players. In some applications it could be interesting to compare these alternative strategies in order to force the players to act in different ways. This issue will be analyzed in the numerical section 5.

3.1 Notation

Consider N competing players who want to transport one good from R sources to T sinks. At each source $r \in \{1, \ldots, R\}$ a capacity of $S_r \geq 0$ is available and the demand at sink $t \in \{1, \ldots, T\}$ is given by $D_t \geq 0$. We assume $\sum_{r=1}^{R} S_r = \sum_{t=1}^{T} D_t$. The unitary transportation cost and the number of transported units from source r to sink t by player $\nu \in \{1, \ldots, N\}$ are denoted by c_{rt}^{ν} and x_{rt}^{ν}, respectively. The decision variable $y_{rt}^{\nu} \in \{0, 1\}$ indicates whether player ν has activated the edge from source r to sink t or not. Establishing a route causes fixed costs of $f_{rt}^{\nu} \geq 0$. Further, u_{rt}^{ν} is an upper bound on x_{rt}^{ν}. Eventually, we have social fixed costs $\widehat{f}_{rt}$ for establishing the route from r to t which will be specified in the remainder of this section.

3.2 NFCTP(1): Separate activation of edges

In the first version of the NFCTP, which we denote by *NFCTP(1)*, we assume that player ν faces the problem of minimizing the sum of her variable transportation costs and her fixed costs for establishing a new route (r, t)

while respecting the coupled demand and supply constraints.

Formally, NFCTP(1) consists of N players simultaneously solving the optimization problem

$$\begin{array}{lll} \min\limits_{(x^\nu,y^\nu)} & \sum\limits_{r=1}^{R}\sum\limits_{t=1}^{T}\Big(c^\nu_{rt}x^\nu_{rt} + (f^\nu_{rt} + \widehat{f}_{rt})y^\nu_{rt}\Big) & \\ \text{s.t.} & \sum\limits_{\ell=1}^{N}\sum\limits_{t=1}^{T} x^\ell_{rt} \;=\; S_r, \; \sum\limits_{\ell=1}^{N}\sum\limits_{r=1}^{R} x^\ell_{rt} \;=\; D_t, & \\ & x^\nu_{rt} \;\leq\; u^\nu_{rt}y^\nu_{rt}, \; x^\nu_{rt} \geq 0, \; y^\nu_{rt} \in \{0,1\}, & \end{array}$$

where the constraints are posed for all $r \in \{1,\dots,R\}$ and $t \in \{1,\dots,T\}$. We observe that NFCTP(1) describes a linear mixed-integer game.

Example 3.1 *Assume, the players are competing airlines, which are transporting passengers from Frankfurt and Bonn to three holiday islands: Mallorca, Corsica and Rhodes. The number of people leaving from Frankfurt is 500 and that from Bonn is 500, while 500 passengers are expected in Mallorca, 300 in Corsica and 200 in Rhodes. Every edge corresponds to a specific route. Offering services on a route requires valid licenses which is charged with some fixed costs. Airline 1 uses bigger planes than Airline 2 and therefore has lower variable costs, but higher private fixed costs. Those are described using cost matrices* $C^\nu := (c^\nu_{rt})$ *and* $F^\nu := (f^\nu_{rt})$ *and are given by*

$$C^1 = \begin{pmatrix} 2, & 2, & 2 \\ 1, & 1, & 1 \end{pmatrix} \text{ and } C^2 = \begin{pmatrix} 3, & 3, & 3 \\ 2, & 2, & 2 \end{pmatrix},$$

$$F^1 = \begin{pmatrix} 70, & 60, & 90 \\ 50, & 40, & 80 \end{pmatrix} \text{ and } F^2 = \begin{pmatrix} 60, & 50, & 80 \\ 40, & 30, & 70 \end{pmatrix}.$$

The airplanes emit carbon dioxide and therefore cause social costs which are given by

$$\widehat{F} = \begin{pmatrix} 50, & 50, & 60 \\ 50, & 50, & 60 \end{pmatrix}.$$

An upper bound is given by $u^\nu_{rt} = 200$.

This example is illustrated in figure 2.

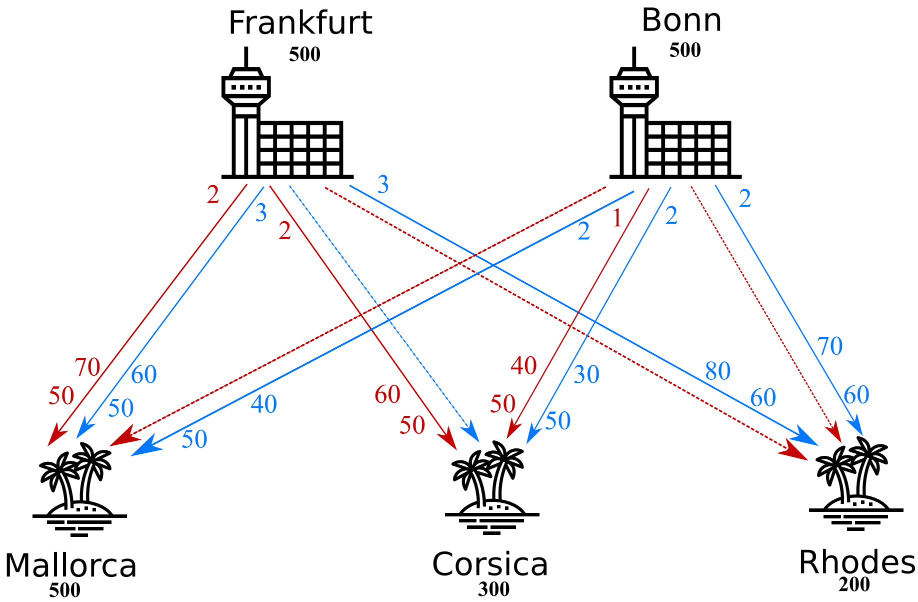

Figure 2: Illustration of Example 3.1, where each airline needs to pay variable costs as well as private and social fixed costs for route usage.

3.3 NFCTP(2): Mixed activation of edges

In contrast to the model in section 3.2, in NFCTP(2), the social fixed costs for player ν depend on the decisions of the remaining players. This happens in example 3.2 where several forwarders decide to establish a new route and the corresponding costs affecting nature and society are taken by each player.

Private fixed costs of f^ν_{rt} occur if y^ν_{rt} is not zero. The social fixed costs $\widehat{f}_{rt}$ are governed by the sum $\sum_{\ell=1}^N z^\ell_{rt}$. Further, let $M_{rt} := \sum_{\ell=1}^N u^\ell_{rt}$.

If social fixed costs occur, they are distributed equally among the players. Player ν faces the optimization problem

$$\begin{aligned}
\min_{(x^\nu, y^\nu, z^\nu)} \quad & \sum_{r=1}^R \sum_{t=1}^T \left(c^\nu_{rt} x^\nu_{rt} + f^\nu_{rt} y^\nu_{rt} + \frac{\widehat{f}_{rt}}{N} \sum_{\ell=1}^N z^\ell_{rt} \right) \\
\text{s.t.} \quad & \sum_{\ell=1}^N \sum_{t=1}^T x^\ell_{rt} = S_r, \ \sum_{\ell=1}^N \sum_{r=1}^R x^\ell_{rt} = D_t, \\
& x^\nu_{rt} \leq u^\nu_{rt} y^\nu_{rt}, \ \textstyle\sum_{\ell=1}^N x^\ell_{rt} \leq M_{rt} \sum_{\ell=1}^N z^\ell_{rt}, \\
& x^\nu_{rt} \geq 0, \ y^\nu_{rt} \in \{0,1\}, \ z^\nu_{rt} \in \{0,1\}
\end{aligned}$$

with $r \in \{1, \ldots, R\}$ and $t \in \{1, \ldots, T\}$. NFCTP(2) is also a linear mixed-integer game.

Example 3.2 *Two competing freight forwarders want to transport thousand units to Munich (300), Hamburg (200), Cologne (400) and Stuttgart (100). Half of the units are produced in Berlin and Bremen, respectively. The competitors have variable costs of*

$$C^1 = \begin{pmatrix} 2, & 1, & 1, & 1 \\ 3, & 0.5, & 1, & 1 \end{pmatrix} \text{ and } C^2 = \begin{pmatrix} 3, & 2, & 2, & 2 \\ 3, & 0.5, & 1, & 2 \end{pmatrix}.$$

Both firms need to pay tolls for using the roads. Those private fixed costs are

$$F^1 = \begin{pmatrix} 70, & 60, & 90, & 100 \\ 80, & 10, & 90, & 100 \end{pmatrix} \text{ and } F^2 = \begin{pmatrix} 70, & 80, & 90, & 100 \\ 80, & 10, & 80, & 100 \end{pmatrix}.$$

When a road is being built, because one of the forwarders is going to use it, social costs of $\widehat{F} = (\widehat{f}_{rt})$ occur:

$$\widehat{F} = \begin{pmatrix} 60, & 50, & 60, & 60 \\ 50, & 30, & 50, & 70 \end{pmatrix}.$$

Those are shared among all competitors.

Figure 3 illustrates this example. In this setting, the routes (Berlin, Munich), (Berlin, Hamburg), (Berlin, Cologne), (Bremen, Cologne) and (Bremen, Stuttgart) are only used by one competitor, but both share the corresponding social fixed costs.

3.4 NFCTP(3): Decreasing fixed costs

Another possible model could be the following. Here, the fixed cost $\widehat{f}_{rt}$ paid by any player using edge (r,t) is shared by all players using the same edge. One possible application is setting up a very expensive infrastructure like a charging network in the context of electric mobility. These costs are usually shared by the future users of this infrastructure.

$$\begin{aligned} \min_{(x^\nu, y^\nu)} \quad & \sum_{r=1}^{R} \sum_{t=1}^{T} \left(c_{rt}^\nu x_{rt}^\nu + f_{rt}^\nu y_{rt}^\nu + \widehat{f}_{rt} \left(1 - \frac{1}{N} \sum_{\nu \neq \ell = 1}^{N} y_{rt}^\ell \right) y_{rt}^\nu \right) \\ \text{s.t.} \quad & \sum_{\ell=1}^{N} \sum_{t=1}^{T} x_{rt}^\ell = S_r, \ \sum_{\ell=1}^{N} \sum_{r=1}^{R} x_{rt}^\ell = D_t, \\ & x_{rt}^\nu \leq u_{rt}^\nu y_{rt}^\nu, \ x_{rt}^\nu \geq 0, \ y_{rt}^\nu \in \{0,1\} \end{aligned}$$

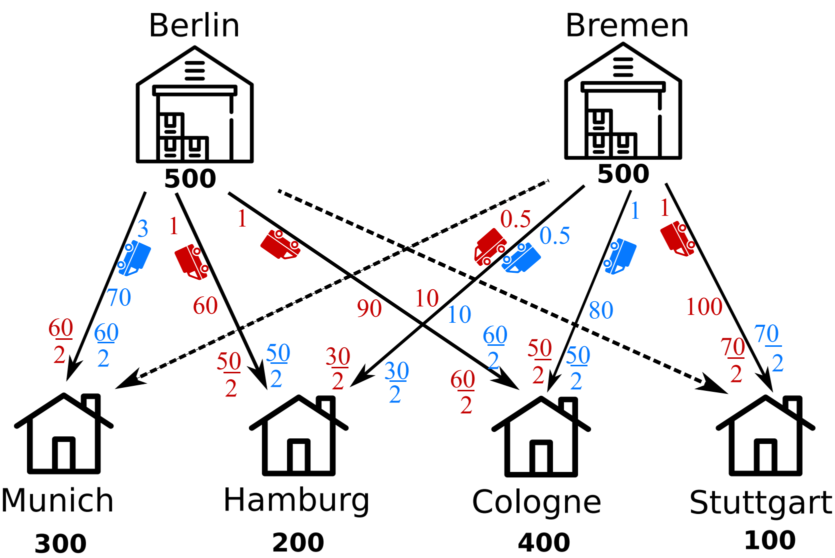

Figure 3: Illustration of Example 3.2. Each competitor pays variable and private fixed costs for using a road. A truck indicates by whom the road is being used.

All constraints are claimed for all $r \in \{1, \dots, R\}$ and $t \in \{1, \dots, T\}$.

In its original formulation, NFCTP(3) is a nonlinear mixed-integer game. By introducing an additional binary variable $b_{rt}^{\ell\nu}$, we can reformulate NFCTP(3) as a linear mixed-integer game.

$$
\begin{aligned}
\min_{(x^\nu, y^\nu, b^\nu)} \quad & \sum_{r=1}^{R} \sum_{t=1}^{T} \left(c_{rt}^\nu x_{rt}^\nu + (f_{rt}^\nu + \widehat{f}_{rt}) y_{rt}^\nu - \frac{1}{N} \widehat{f}_{rt} \sum_{\nu \neq \ell = 1}^{N} b_{rt}^{\ell\nu} \right) \\
\text{s.t.} \quad & \sum_{\ell=1}^{N} \sum_{t=1}^{T} x_{rt}^\ell = S_r, \quad \sum_{\ell=1}^{N} \sum_{r=1}^{R} x_{rt}^\ell = D_t, \\
& x_{rt}^\nu \le u_{rt}^\nu y_{rt}^\nu, \quad b_{rt}^{\ell\nu} \le y_{rt}^\ell, \\
& b_{rt}^{\ell\nu} \le y_{rt}^\nu, \quad b_{rt}^{\ell\nu} \in \{0,1\}, \ \forall \ell \neq \nu, \\
& 0 \le x_{rt}^\nu \le u_{rt}^\nu, \quad y_{rt}^\nu \in \{0,1\}
\end{aligned}
$$

All constraints hold true for all $r \in \{1, \dots, R\}$ and $t \in \{1, \dots, T\}$. However, the drawback here is that we have to introduce $(N-1) \cdot T \cdot R$ additional binary variables for each player which makes this approach numerically intractable in practical applications. Therefore, in the remainder of this article, we will stick to the original nonlinear formulation of NFCTP(3).

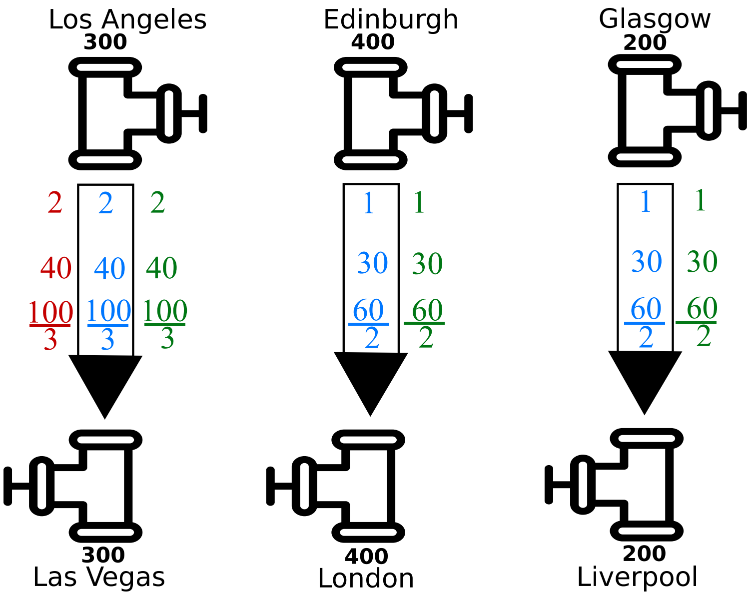

Figure 4: Illustration of Example 3.3. Variable and private fixed costs for a route are paid by everyone who uses a route. The social costs are shared among those who use the road.

Example 3.3 *Three travel agencies offer a new transportation system where people can be sent in a special capsule through a pipeline. The infrastructure consists of three routes: Los Angeles to Las Vegas, Edinburgh to London and Glasgow to Liverpool, whose demand is 300, 400, 200, respectively. For each passenger, they have to pay variable costs*

$$C^1 = C^2 = C^3 = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}.$$

Further, each agency pays private fixed costs as rent for the vehicle when they use a route:

$$F^1 = F^2 = F^3 = \begin{pmatrix} 40 & 30 & 30 \end{pmatrix}.$$

The social costs $\widehat{F} = (\widehat{f}_{rt})$ for building a pipeline between two cities are shared between the agencies that use this specific route. Those costs are:

$$\widehat{F} = \begin{pmatrix} 100 & 60 & 60 \end{pmatrix}.$$

In the scenario in figure 4, two agencies use all routes, whereas one only operates in the United States.

4 Nash equilibria of the NFCTP models

In this section, we prove existence of Nash equilibria for the NFCTP models and investigate their structure which allows us to compute these equilibria numerically.

4.1 Unfolded common strategy sets

In the following sections we need the unfolded common strategy sets M_1, M_2 and M_3 of NFCTP(1), NFCTP(2) and NFCTP(3), respectively, which are given by

$$\begin{aligned} M_1 := M_3 \;\; := \;\; & \Big\{(x,y)\Big| \; \textstyle\sum_{\ell=1}^{N}\sum_{t=1}^{T} x_{rt}^{\ell} = S_r, \; \sum_{\ell=1}^{N}\sum_{r=1}^{R} x_{rt}^{\ell} = D_t, \\ & x_{rt}^{\nu} \le u_{rt}^{\nu} y_{rt}^{\nu}, \; x_{rt}^{\nu} \ge 0, \; y_{rt}^{\nu} \in \{0,1\}, \\ & \forall r \in \{1,\dots,R\}, \; \forall t \in \{1,\dots,T\}, \; \forall \nu \in \{1,\dots,N\}\Big\} \end{aligned}$$

and

$$\begin{aligned} M_2 \;\; := \;\; & \Big\{(x,y,z)\Big| \; \textstyle\sum_{\ell=1}^{N}\sum_{t=1}^{T} x_{rt}^{\ell} \; = \; S_r, \; \sum_{\ell=1}^{N}\sum_{r=1}^{R} x_{rt}^{\ell} \; = \; D_t, \\ & x_{rt}^{\nu} \le u_{rt}^{\nu} y_{rt}^{\nu}, \; \textstyle\sum_{\ell=1}^{N} x_{rt}^{\ell} \le M_{rt} \sum_{\ell=1}^{N} z_{rt}^{\ell}, \\ & x_{rt}^{\nu} \ge 0, \; y_{rt}^{\nu} \in \{0,1\}, \; z_{rt}^{\nu} \in \{0,1\}, \\ & \forall r \in \{1,\dots,R\}, \; \forall t \in \{1,\dots,T\}, \; \forall \nu \in \{1,\dots,N\}\Big\}. \end{aligned}$$

The unfolded common strategy sets contain all feasible points that are possible candidates for Nash equilibria. In section 4.2, we derive some objective functions whose global minimal points over the sets M_1, M_2 and M_3 are exactly the Nash equilibria of the corresponding games.

4.2 Structural properties

Our first main result is that all considered versions of the NFCTP are potential games (cf. section 2.3). This facilitates both, the theoretical investigations as well as their numerical treatment.

Theorem 4.1 *NFCTP(1), NFCTP(2) and NFCTP(3) are potential games, whose exact potential functions are*

$$\begin{aligned}
\bar{P}_1 :=\; & \textstyle\sum_{r=1}^{R} \sum_{t=1}^{T} \left(\sum_{\nu=1}^{N} c_{rt}^{\nu} x_{rt}^{\nu} + \sum_{\nu=1}^{N} (f_{rt}^{\nu} + \widehat{f}_{rt}) y_{rt}^{\nu} \right), \\
\bar{P}_2 :=\; & \textstyle\sum_{r=1}^{R} \sum_{t=1}^{T} \left(\sum_{\nu=1}^{N} c_{rt}^{\nu} x_{rt}^{\nu} + \sum_{\nu=1}^{N} f_{rt}^{\nu} y_{rt}^{\nu} + \frac{\widehat{f}_{rt}}{N} \sum_{\nu=1}^{N} z_{rt}^{\nu} \right) \text{ and} \\
\bar{P}_3 :=\; & \textstyle\sum_{r=1}^{R} \sum_{t=1}^{T} \left(\sum_{\nu=1}^{N} c_{rt}^{\nu} x_{rt}^{\nu} + (f_{rt}^{\nu} + \widehat{f}_{rt}) \sum_{\nu=1}^{N} y_{rt}^{\nu} - \widehat{f}_{rt} \frac{1}{N} \sum_{\nu=1}^{N} \sum_{\nu \neq \ell = 1}^{N} y_{rt}^{\nu} y_{rt}^{\ell} \right),
\end{aligned}$$

respectively.

Proof. It is a direct consequence of the definition of exact potential functions. •

Further, we notice that two versions of the NFCTP are Pseudo-NEPs (cf. section 2.4) which means that we can exploit many structural properties of NEPs when analyzing this GNEP.

Theorem 4.2 *NFCTP(1) and NFCTP(3) are Pseudo-NEPs.*

Proof. By [4, Proposition 2], the mapping

$$\begin{aligned}
K(x) = \Bigg\{ y : & \sum_{t=1}^{T} y_{rt}^{\nu} + \sum_{\nu \neq \ell = 1}^{N} \sum_{t=1}^{T} x_{rt}^{\ell} = S_r,\ \forall \nu \in \{1, \dots, N\},\ \forall r \in \{1, \dots, R\}, \\
& \sum_{r=1}^{R} y_{rt}^{\nu} + \sum_{\nu \neq \ell = 1}^{N} \sum_{r=1}^{R} x_{rt}^{\ell} = D_t,\ \forall \nu \in \{1, \dots, N\},\ \forall t \in \{1, \dots, T\} \Bigg\},
\end{aligned} \tag{2}$$

is reproducible, i.e., for any $x \in K(x)$ and any $z \in K(x)$, it holds that $K(x) = K(z)$. Therefore, by [4, Corollary 2] the games are Pseudo-NEPs. •

We remark that NFCTP(2) is not a Pseudo-NEP due to the presence of the shared inequality constraints that are not reproducible.

4.3 Existence

In order to avoid trivial games we assume that each version of the NFCTP possesses a nonempty unfolded strategy set, that is M_i is nonempty for any $i \in \{1, 2, 3\}$. Then the existence of Nash equilibria for the NFCTP is ensured by the following theorem whose proof is based on the fact that all the NFCTPs are potential games.

Theorem 4.3 *Let M_1, M_2 and M_3 be nonempty. Then there exists at least one Nash equilibrium for NFCTP(1), NFCTP(2) and NFCTP(3).*

Proof. In view of Theorem 4.1, any minimal point of $\bar{P}_i$ over M_i is an equilibrium of NFCTP(i), for $i = 1, 2, 3$. Therefore, the statement holds true by the Weierstrass theorem. •

While we have shown the existence of at least one Nash equilibrium in the NFCTPs, we cannot expect uniqueness. This is illustrated in the following examples and was shown for generic GNEPs in [7].

4.4 Computation

In this section, we show how to compute Nash equilibria for the NFCTP models by solving some auxiliary mixed integer optimization problems.

4.4.1 About NFCTP(1)

The following proposition gives a way to compute any equilibrium of NFCTP(1). It is based on the fact that NFCTP(1) is a Pseudo-NEP (cf. Theorem 4.2).

Proposition 4.4 *Let $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^N)$ be any point such that*

$$\sum_{\ell=1}^{N}\sum_{t=1}^{T} \bar{x}_{rt}^{\ell} \;=\; S_r,\; \sum_{\ell=1}^{N}\sum_{r=1}^{R} \bar{x}_{rt}^{\ell} \;=\; D_t,\; 0 \le \bar{x}_{rt}^{\ell} \le u_{rt}^{\ell} \tag{3}$$

for all $r \in \{1, \ldots, R\}$, $t \in \{1, \ldots, T\}$ and $\ell \in \{1, \ldots, N\}$.

Any point $(\widehat{x}, \widehat{y}) = ((\widehat{x}^1, \widehat{y}^1), \ldots, (\widehat{x}^N, \widehat{y}^N))$ that solves the mixed-integer linear program

$$\begin{aligned}
\min_{x,y} \quad & \sum_{\ell=1}^{N}\sum_{r=1}^{R}\sum_{t=1}^{T} \left(c_{rt}^{\ell} x_{rt}^{\ell} + f_{rt}^{\ell} y_{rt}^{\ell}\right) \\
s.t. \quad & \sum_{t=1}^{T} x_{rt}^{\ell} \;=\; \sum_{t=1}^{T} \bar{x}_{rt}^{\ell},\; \sum_{r=1}^{R} x_{rt}^{\ell} \;=\; \sum_{r=1}^{R} \bar{x}_{rt}^{\ell}, \\
& x_{rt}^{\ell} \le u_{rt}^{\nu} y_{rt}^{\ell},\; x_{rt}^{\ell} \ge 0,\; y_{rt}^{\ell} \in \{0, 1\},
\end{aligned} \tag{4}$$

where the constraints are posed for all $r \in \{1, \dots, R\}$, $t \in \{1, \dots, T\}$ *and* $\ell \in \{1, \dots, N\}$, *is an equilibrium of NFCTP(1).*

On the other hand, any equilibrium $(\bar{x}, \bar{y}) = ((\bar{x}^1, \bar{y}^1), \dots, (\bar{x}^N, \bar{y}^N))$ *of NFCTP(1) is a solution of the mixed-integer linear program* (4).

Proof. The first part of the theorem is a direct consequence of [4, Theorem 4], Theorem 4.2, and Theorem 4.1.

The last part is due to the fact that problem (4) is a mere aggregation of the different independent optimization problems of the players coming from the parametrization with $\bar{x}$:

$$\begin{aligned} \min_{x^\nu, y^\nu} \quad & \sum_{r=1}^{R} \sum_{t=1}^{T} \left(c_{rt}^\ell x_{rt}^\ell + f_{rt}^\ell y_{rt}^\ell \right) \\ \text{s.t.} \quad & \sum_{t=1}^{T} x_{rt}^\ell = \sum_{t=1}^{T} \bar{x}_{rt}^\ell, \ \sum_{r=1}^{R} x_{rt}^\ell = \sum_{r=1}^{R} \bar{x}_{rt}^\ell, \\ & x_{rt}^\ell \leq u_{rt}^\nu y_{rt}^\ell, \ x_{rt}^\ell \geq 0, \ y_{rt}^\ell \in \{0, 1\}. \end{aligned} \tag{5}$$

Therefore, solving problem (4) is equivalent to solving NEP (5). •

Example 4.5 *Let us consider again Example 3.1. We can easily compute equilibria of the game by following the steps given in Proposition 4.4.*

For example the points

$$\bar{x}^1 = \begin{pmatrix} 100, & 100, & 100 \\ 100, & 100, & 100 \end{pmatrix}, \qquad \bar{x}^2 = \begin{pmatrix} 150, & 50, & 0 \\ 150, & 50, & 0 \end{pmatrix}$$

satisfy conditions (3). *Therefore, an equilibrium of the game can be computed by solving problem* (4) *with the point* $\bar{x}$ *as parameter. This equilibrium is:*

$$\widehat{x}^1 = \begin{pmatrix} 0, & 100, & 200 \\ 200, & 100, & 0 \end{pmatrix}, \qquad \widehat{y}^1 = \begin{pmatrix} 0, & 1, & 1 \\ 1, & 1, & 0 \end{pmatrix},$$

$$\widehat{x}^2 = \begin{pmatrix} 200, & 0, & 0 \\ 100, & 100, & 0 \end{pmatrix}, \qquad \widehat{y}^2 = \begin{pmatrix} 1, & 0, & 0 \\ 1, & 1, & 0 \end{pmatrix},$$

which yields a total cost of 1350 *for the first company, and a total cost of* 1280 *for the second company.*

Note that different initial configurations yield different equilibria. For example:

$$\bar{x}^1 = \begin{pmatrix} 100, & 100, & 0 \\ 100, & 150, & 50 \end{pmatrix}, \qquad \bar{x}^2 = \begin{pmatrix} 150, & 50, & 100 \\ 150, & 0, & 50 \end{pmatrix},$$

satisfy conditions (3). *Therefore, an equilibrium of the game can be computed by solving problem* (4) *with the point* $\bar{x}$ *as parameter. This equilibrium is:*

$$\widehat{x}^1 = \begin{pmatrix} 0, & 200, & 0 \\ 200, & 50, & 50 \end{pmatrix}, \qquad \widehat{y}^1 = \begin{pmatrix} 0, & 1, & 0 \\ 1, & 1, & 1 \end{pmatrix},$$

$$\widehat{x}^2 = \begin{pmatrix} 150, & 0, & 150 \\ 150, & 50, & 0 \end{pmatrix}, \qquad \widehat{y}^2 = \begin{pmatrix} 1, & 0, & 1 \\ 1, & 1, & 0 \end{pmatrix},$$

which yields a total cost of 1140 *for the first company, and a total cost of* 1720 *for the second company.*

4.4.2 About NFCTP(2)

The following proposition gives a way to compute equilibria of NFCTP(2). It is based on the fact that NFCTP(2) is a potential game, and that it enjoys some nice properties based on reproducibility.

Proposition 4.6 *Let* $\bar{x} = (\bar{x}^1, \dots, \bar{x}^N)$ *be any point that satisfies* (3). *Any point* $(\widehat{x}, \widehat{y}, \widehat{z}) = ((\widehat{x}^1, \widehat{y}^1, \widehat{z}^1), \dots, (\widehat{x}^N, \widehat{y}^N, \widehat{z}^N))$ *that solves the following mixed-integer linear program*

$$\begin{array}{ll} \min\limits_{x,y,z} & \sum\limits_{\ell=1}^{N} \sum\limits_{r=1}^{R} \sum\limits_{t=1}^{T} \left(c_{rt}^\ell x_{rt}^\ell + f_{rt}^\ell y_{rt}^\ell + \widehat{f}_{rt} z_{rt}^\ell \right) \\ s.t. & \sum\limits_{t=1}^{T} x_{rt}^\ell = \sum\limits_{t=1}^{T} \bar{x}_{rt}^\ell, \; \sum\limits_{r=1}^{R} x_{rt}^\ell = \sum\limits_{r=1}^{R} \bar{x}_{rt}^\ell, \\ & x_{rt}^\ell \leq u_{rt}^\nu y_{rt}^\ell, \; \sum_{\ell=1}^N x_{rt}^\ell \leq M_{rt} \sum_{\ell=1}^N z_{rt}^\ell, \\ & x_{rt}^\ell \geq 0, \; y_{rt}^\ell \in \{0,1\}, \; z_{rt}^\ell \in \{0,1\}, \end{array} \tag{6}$$

for all $r \in \{1, \dots, R\}$, $t \in \{1, \dots, T\}$ *and* $\ell \in \{1, \dots, N\}$ *is an equilibrium of NFCTP(2).*

Proof. The proof holds by the same reasonings of the proof of proposition 4.4, and observing that NFCTP(2) is a potential game. •

The contrary does not hold, i.e., it is not necessarily true that any equilibrium $(\bar{x}, \bar{y}, \bar{z}) = ((\bar{x}^1, \bar{y}^1, \bar{z}^1), \dots, (\bar{x}^N, \bar{y}^N, \bar{z}^N))$ of NFCTP(2) is a solution of the mixed-integer linear program (6). This is due to the fact that the NFCTP(2) is a generalized nonconvex potential game, and therefore the solution set of its potential optimization problem could not include the whole set of equilibria of the game, see e.g. [25, Example 2].

Example 4.7 *Let us consider again Example 3.2. We can easily compute equilibria of the game by following the steps given in Proposition 4.6.*

For example the points

$$\bar{x}^1 = \begin{pmatrix} 100, & 100, & 0, & 0 \\ 0, & 0, & 250, & 50 \end{pmatrix}, \qquad \bar{x}^2 = \begin{pmatrix} 100, & 100, & 100, & 0 \\ 100, & 0, & 50, & 50 \end{pmatrix},$$

satisfy conditions (3)*. Therefore, an equilibrium of the game can be computed by solving problem* (6) *with the point* $\bar{x}$ *as parameter. This equilibrium is:*

$$\widehat{x}^1 = \begin{pmatrix} 100, & 0, & 50, & 50 \\ 0, & 100, & 200, & 0 \end{pmatrix}, \ \widehat{y}^1 = \begin{pmatrix} 1, & 0, & 1, & 1 \\ 0, & 1, & 1, & 0 \end{pmatrix},$$

$$\widehat{x}^2 = \begin{pmatrix} 200, & 0, & 50, & 50 \\ 0, & 100, & 100, & 0 \end{pmatrix}, \ \widehat{y}^2 = \begin{pmatrix} 1, & 0, & 1, & 1 \\ 0, & 1, & 1, & 0 \end{pmatrix},$$

$$\widehat{z}^1 + \widehat{z}^2 = \begin{pmatrix} 1, & 0, & 1, & 1 \\ 0, & 1, & 1, & 0 \end{pmatrix},$$

which yields a total cost of 1040 *for the first company, and a total cost of* 1430 *for the second company. Again, different initial configurations gives different equilibria.*

The following is a result linking NFCTP(1) and NFCTP(2).

Proposition 4.8 *Let* $(\bar{x}, \bar{y})$ *be an equilibrium of NFCTP(1). Assume that a player* ν *exists such that for all* (rt) *with* $\bar{y}^\nu_{rt} = 1$ *another player* $\ell \neq \nu$ *exists such that* $\bar{y}^\ell_{rt} = 1$*. Then* $(\bar{x}, \bar{y}, \bar{z})$*, where* $\bar{z}^\nu = \max_\ell\{\bar{y}^\ell\}$ *and* $\bar{z}^\ell = 0$ *for all* $\ell \neq \nu$*, is an equilibrium of NFCTP(2).*

Proof. The tuple $(\bar{x}, \bar{y}, \bar{z})$ is feasible for NFCTP(2). To show that it is also optimal, we observe that for any player ℓ, all feasible tuples $(x^\ell, y^\ell, z^\ell, \bar{x}^{-\ell}, \bar{y}^{-\ell}, \bar{z}^{-\ell})$

are such that $z^\ell \geq \bar{z}^\ell$. Therefore, for any player ℓ and any feasible tuple $(x^\ell, y^\ell, z^\ell, \bar{x}^{-\ell}, \bar{y}^{-\ell}, \bar{z}^{-\ell})$ we obtain

$$\begin{aligned}\sum_{r=1}^{R}\sum_{t=1}^{T}\left(c^\ell_{rt}\bar{x}^\ell_{rt} + f^\ell_{rt}\bar{y}^\ell_{rt} + \widehat{f}_{rt}\bar{z}^\ell_{rt}\right) &\leq \sum_{r=1}^{R}\sum_{t=1}^{T}\left(c^\ell_{rt}x^\ell_{rt} + f^\ell_{rt}y^\ell_{rt} + \widehat{f}_{rt}\bar{z}^\ell_{rt}\right) \\ &\leq \sum_{r=1}^{R}\sum_{t=1}^{T}\left(c^\ell_{rt}x^\ell_{rt} + f^\ell_{rt}y^\ell_{rt} + \widehat{f}_{rt}z^\ell_{rt}\right),\end{aligned}$$

where the first inequality follows from the fact that $(\bar{x}, \bar{y})$ is an equilibrium of NFCTP(1), and the second inequality from $z^\ell \geq \bar{z}^\ell$ and $\widehat{f} \geq 0$. •

4.4.3 About NFCTP(3)

The following proposition suggests a way to simplify NFCTP(3). Namely it makes available a recasting as a NEP. This is possible because the NFCTP(3) enjoys some nice properties based on reproducibility.

Proposition 4.9 *Let $\bar{x} = (\bar{x}^1, \dots, \bar{x}^N)$ be any point that satisfies* (3)*. Any equilibrium $(\widehat{x}, \widehat{y}) = ((\widehat{x}^1, \widehat{y}^1), \dots, (\widehat{x}^N, \widehat{y}^N))$ of the following NEP*

$$\begin{aligned}\min_{(x^\nu, y^\nu)} \quad & \sum_{r=1}^{R}\sum_{t=1}^{T}\left(c^\nu_{rt}x^\nu_{rt} + f^\nu_{rt}y^\nu_{rt} + \widehat{f}_{rt}\left(1 - \frac{1}{N}\sum_{\nu\neq\ell=1}^{N} y^\ell_{rt}\right)y^\nu_{rt}\right) \\ s.t. \quad & \sum_{t=1}^{T} x^\ell_{rt} = \sum_{t=1}^{T}\bar{x}^\ell_{rt}, \ \sum_{r=1}^{R} x^\ell_{rt} = \sum_{r=1}^{R}\bar{x}^\ell_{rt}, \\ & x^\nu_{rt} \leq u^\nu_{rt}y^\nu_{rt}, \ x^\nu_{rt} \geq 0, \ y^\nu_{rt} \in \{0,1\}\end{aligned} \tag{7}$$

for all $r \in \{1, \dots, R\}$, $t \in \{1, \dots, T\}$ and $\ell \in \{1, \dots, N\}$ is an equilibrium of GNEP (3.4)*.*

On the other hand, any equilibrium $(\bar{x}, \bar{y}) = ((\bar{x}^1, \bar{y}^1), \dots, (\bar{x}^N, \bar{y}^N))$ of GNEP (3.4) *is an equilibrium of NEP* (7)*.*

Proof. It is the same as in the first part of Proposition 4.4. •

Example 4.10 *Consider Example 3.3. By exploiting Proposition 4.9, it is easy to see that any point $\bar{x}$ that satisfies* (3)*, actually gives an equilibrium of the game.*

For example the point:

$$\bar{x}^1 = \begin{pmatrix}100, & 100, & 100\end{pmatrix}, \ \bar{x}^2 = \begin{pmatrix}100, & 100, & 100\end{pmatrix}, \ \bar{x}^3 = \begin{pmatrix}100, & 200, & 0\end{pmatrix},$$

satisfies (3). *The unique feasible point of NEP* (7) *is:*

$$\begin{aligned} \widehat{x}^1 &= \bar{x}^1, \quad y^1 = \begin{pmatrix}1, & 1, & 1\end{pmatrix}, \\ \widehat{x}^2 &= \bar{x}^2, \quad y^2 = \begin{pmatrix}1, & 1, & 1\end{pmatrix}, \\ \widehat{x}^3 &= \bar{x}^3, \quad y^3 = \begin{pmatrix}1, & 1, & 0\end{pmatrix}. \end{aligned}$$

Therefore it is the unique equilibrium of NEP (7)*, and thus an equilibrium of the NFCTP(3).*

Moreover note that NEP (7) is a potential game whose potential function $\bar{P}_3$ is given in Proposition 4.1. Therefore, to compute any equilibrium of NEP (7) one can use Algorithm 1 in [25] with $\varepsilon \to 0$ (see [25, Remark 2]).

4.5 Selection of Nash equilibria

Consider the mixed-integer linear program

$$\begin{aligned} \min_{x,y,z} \quad & z + \varepsilon \sum_{\ell=1}^{N} \sum_{r=1}^{R} \sum_{t=1}^{T} \left(c_{rt}^{\ell} x_{rt}^{\ell} + (f_{rt}^{\ell} + \widehat{f}_{rt}) y_{rt}^{\ell} \right) \\ \text{s.t.} \quad & z \geq \sum_{r=1}^{R} \sum_{t=1}^{T} \left(c_{rt}^{\ell} x_{rt}^{\ell} + (f_{rt}^{\ell} + \widehat{f}_{rt}) y_{rt}^{\ell} \right), \ \sum_{\ell=1}^{N} \sum_{t=1}^{T} x_{rt}^{\ell} \ = \ S_r, \\ & \sum_{\ell=1}^{N} \sum_{r=1}^{R} x_{rt}^{\ell} \ = \ D_t, \ x_{rt}^{\ell} \leq u_{rt}^{\nu} y_{rt}^{\ell}, \\ & x_{rt}^{\ell} \geq 0, \ y_{rt}^{\ell} \in \{0, 1\}, \end{aligned} \tag{8}$$

for all $r \in \{1, \ldots, R\}$, $t \in \{1, \ldots, T\}$ and $\ell \in \{1, \ldots, N\}$.

Following the developments in section 3 of [25], we know that any solution of problem (8) with $\varepsilon > 0$, is an equilibrium of NFCTP(1). If ε is small, then program (8) aims at selecting the equilibria at which the costs of all the players are as similar as possible. A quantified version of this statement is given in Proposition 4.12.

Notice that this method to select equilibria can be done only for potential games whose objective functions of the players are independent from the variables of the other players, see e.g. [25]. This is why we cannot give a similar result for NFTCP(2) and NFTCP(3).

Example 4.11 *Consider again the problem of Example 3.1. An equilibrium of the game that is a solution of problem* (8) *with* $\varepsilon = 1$ *is the following:*

$$\widehat{x}^1 = \begin{pmatrix} 152, & 0, & 0 \\ 200, & 100, & 200 \end{pmatrix}, \qquad \widehat{y}^1 = \begin{pmatrix} 1, & 0, & 0 \\ 1, & 1, & 1 \end{pmatrix},$$

$$\widehat{x}^2 = \begin{pmatrix} 148, & 200, & 0 \\ 0, & 0, & 0 \end{pmatrix}, \qquad \widehat{y}^2 = \begin{pmatrix} 1, & 1, & 0 \\ 0, & 0, & 0 \end{pmatrix},$$

which yields a total cost of 1254 *for both the companies.*

On the other hand, if we set $\varepsilon = 10$*, we get:*

$$\widehat{x}^1 = \begin{pmatrix} 200, & 200, & 0 \\ 200, & 100, & 200 \end{pmatrix}, \qquad \widehat{y}^1 = \begin{pmatrix} 1, & 1, & 0 \\ 1, & 1, & 1 \end{pmatrix},$$

$$\widehat{x}^2 = \begin{pmatrix} 100, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix}, \qquad \widehat{y}^2 = \begin{pmatrix} 1, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix},$$

which yields a total cost of 1860 *for the first company, and of* 410 *for the second one. Therefore the value of* ε *must be sufficiently small in order to have the objectives as similar as possible.*

Proposition 4.12 *Assume that all* c_{rt}^{ℓ} *are positive. Let* $(\bar{x}, \bar{y}, \bar{z})$ *be a solution of* (8) *with* $\varepsilon = 0$*. Then for every* ν *and* μ *we obtain*

$$\left| \sum_{r=1}^{R} \sum_{t=1}^{T} \left(c_{rt}^{\nu} \bar{x}_{rt}^{\nu} + (f_{rt}^{\nu} + \widehat{f}_{rt}) \bar{y}_{rt}^{\nu} \right) - \sum_{r=1}^{R} \sum_{t=1}^{T} \left(c_{rt}^{\mu} \bar{x}_{rt}^{\mu} + (f_{rt}^{\mu} + \widehat{f}_{rt}) \bar{y}_{rt}^{\mu} \right) \right| \leq RT f^{\max},$$

where $f^{\max} = \max_{\ell,r,t}(f_{rt}^{\ell} + \widehat{f}_{rt})$.

Proof. Assume, w.l.o.g., that

$$\bar{z} = \sum_{r=1}^{R} \sum_{t=1}^{T} \left(c_{rt}^{\nu} \bar{x}_{rt}^{\nu} + (f_{rt}^{\nu} + \widehat{f}_{rt}) \bar{y}_{rt}^{\nu} \right) > \sum_{r=1}^{R} \sum_{t=1}^{T} \left(c_{rt}^{\mu} \bar{x}_{rt}^{\mu} + (f_{rt}^{\mu} + \widehat{f}_{rt}) \bar{y}_{rt}^{\mu} \right).$$

For every rt such that $\left(c_{rt}^{\nu} \bar{x}_{rt}^{\nu} + (f_{rt}^{\nu} + \widehat{f}_{rt}) \bar{y}_{rt}^{\nu} \right) > 0$, we have

$$\left(c_{rt}^{\mu} \bar{x}_{rt}^{\mu} + (f_{rt}^{\mu} + \widehat{f}_{rt}) \bar{y}_{rt}^{\mu} \right) = 0,$$

because, otherwise, we would decrease the value of $\bar{x}_{rt}^{\nu}$ and, correspondingly, increase the value of $\bar{x}_{rt}^{\mu}$ to obtain a feasible point that deny the optimality of

$(\bar{x}, \bar{y}, \bar{z})$ w.r.t (8). Moreover, $\left(c^\nu_{rt}\bar{x}^\nu_{rt} + (f^\nu_{rt} + \widehat{f}_{rt})\bar{y}^\nu_{rt}\right)$ cannot be greater than $f^\mu_{rt} + \widehat{f}_{rt}$, because, otherwise, we would activate $\bar{y}^\mu_{rt}$ and deny the optimality of $(\bar{x}, \bar{y}, \bar{z})$ in the same way as above. Therefore,

$$\left(c^\nu_{rt}\bar{x}^\nu_{rt} + (f^\nu_{rt} + \widehat{f}_{rt})\bar{y}^\nu_{rt}\right) - \left(c^\mu_{rt}\bar{x}^\mu_{rt} + (f^\mu_{rt} + \widehat{f}_{rt})\bar{y}^\mu_{rt}\right) \leq f^\mu_{rt} + \widehat{f}_{rt}$$

and this implies the assertion. •

5 Numerical results

In this section we exploit the theoretical results of section 4.4 to compute equilibria of, and, thus, to make a comparison among, the three NFCTP models described in section 3. We consider games with three players ($N = 3$) because Nash games are particularly relevant if the number of the competing agents is small and perfect competition is not a good model (cf. [12]). We set a network in which any of the five sources ($R = 5$) is connected with any of the ten sinks ($T = 10$). The capacities of the sources, the demands of the sinks, the marginal costs, and the fixed costs relative to any link and for any firm were randomly generated using the uniform distribution and then approximated by the ceil operator: $S_r \in [100, 300]$, $D_t \in [50, 200]$, $u^\nu_{rt} \in [10, 50]$, $c^\nu_{rt} \in [1, 3]$, $f^\nu_{rt} \in [10, 100]$. We set $D_T = \sum_{r\in\{1,\dots,R\}} S_r - \sum_{t\in\{1,\dots,T\}} D_t$ to make the instances feasible. In order to make a thorough analysis, we consider two different settings: one in which the level of shared costs $\widehat{f}_{rt}$ is high and one in which it is low. High level shared costs are randomly distributed within the interval: $\widehat{f}_{rt} \in [50, 500]$. While the vector of the low level shared costs is the same but divided by 10. We used exactly the same data for all the three NFCTPs.

All the experiments were carried out on an Intel Core i7-4702MQ CPU @ 2.20GHz x 8 with Ubuntu 14.04 LTS 64-bit using AMPL. As optimization solver we used CPLEX 12.6.0.1 with default options. We never report CPU time consumption since, in all our tests, AMPL returns a solution in less than few seconds.

By exploiting its separable potential structure, see [25, Sect. 3], we computed equilibria of NFCTP(1) by adopting two different strategies: we denote by NFCTP$(1)_{\text{sum}}$ an equilibrium of NFCTP(1) computed by minimizing the exact potential function $\bar{P}_1$ defined in Theorem 4.1; we denote by NFCTP$(1)_{\max}$ an equilibrium of the NFCTP(1) computed by solving (8) with $\varepsilon = 1e - 3$. We recall that in the equilibrium NFCTP$(1)_{\text{sum}}$ the social cost is minimized, while in the equilibrium $\text{NFCTP}(1)_{\max}$ all the players have optimal costs as similar as possible. On the other hand both NFCTP(2) and NFCTP(3) do not possess this separable structure. Thus we computed equilibria of NFCTP(2) and NFCTP(3) only by minimizing the exact potential functions $\bar{P}_2$ and $\bar{P}_3$, described in Theorem 4.1, respectively.

We report the numerical results in tables 1-4. Specifically tables 1 and 2 report the results obtained on two instances, A and B, in which the shared costs are high. While in tables 3 and 4 we consider the same istances but with low shared costs. The rows in the tables have the following meanings: cost_tot is the sum of the total costs of all the players (variable + fixed + shared costs); cost_var_tot is the sum of the variable costs of all the players; cost_fix_tot is the sum of the fixed costs of all the players; cost_sha_tot is the sum of the shared costs of all the players; #routes_3players is the amount of the links that are used by all the three players together; #routes_2players is the amount of the links that are used by only two players; #routes_1player is the amount of the links that are used by only one player; #routes_tot is the amount of the links that are used by at least one player; for any $\nu = 1, 2, 3$, cost_ν, cost_var_ν, cost_fix_ν, and cost_sha_ν are the total, variable, fixed and shared costs of player ν respectively; while quantities_ν is the total amount of units transported by player ν.

Focusing on the column $\text{NFCTP}(1)_{\max}$ in tables 1-4, we observe that Proposition 4.12 gives an upper bound on the maximum deviation between the costs borne by two different players that is confirmed in our experiments. Specifically this deviation is greater than zero only in table 1: cost_1 - cost_3 = 13.25. However it is much smaller that the upper bound given by Proposition 4.12.

The sum of the total costs of all the players turns out to be lower for NFCTP(2) and NFCTP(3) rather than for NFCTP(1) ($\text{NFCTP}(1)_{\text{sum}}$ or $\text{NFCTP}(1)_{\max}$). This is consistent with the fact that in NFCTP(2) and NFCTP(3) the shared costs are really partitioned among the players if they partially cooperate by using, as much as possible, the same links in order to activate as less links as possible. This behaviour is confirmed by observing the row #routes_tot in the tables. In fact, the amount of the links that are used by at least one player for NFCTP(2) and NFCTP(3) is always less than that of NFCTP(1).

Finally, by focusing on the amount of the links that are used by all the three players together (#routes_3players), we note that in the case of NFCTP(3) it is always greater than in all the other cases. This should be justified by observing that in NFCTP(3) the players are stimulated to use the same links in order to obtain the greatest possible discount on the shared costs.

	NFCTP(1)$_{\text{sum}}$	NFCTP(1)$_{\text{max}}$	NFCTP(2)	NFCTP(3)
cost_tot	9833	9893.5	6967	7208
cost_var_tot	2502	2538.5	2467	2538
cost_fix_tot	1403	1298	1459	1780
cost_sha_tot	5928	6057	3041	2890
#routes_3players	0	0	5	9
#routes_2players	7	5	5	3
#routes_1player	13	17	5	2
#routes_tot	20	22	15	14
cost_1	2668	3302.25	2120.67	2318.67
cost_var_1	604	876.25	693	721
cost_fix_1	278	370	414	552
cost_sha_1	1786	2056	1013.67	1045.67
quantities_1	263	338.75	298	288
cost_2	3139	3302.25	2575.67	2633
cost_var_2	823	821.25	969	1006
cost_fix_2	542	542	593	668
cost_sha_2	1774	1939	1013.67	959
quantities_2	317	327.25	370	390
cost_3	4026	3289	2270.67	2256.33
cost_var_3	1075	841	805	811
cost_fix_3	583	386	452	560
cost_sha_3	2368	2062	1013.67	885.33
quantities_3	416	330	328	318

Table 1: Instance (A) - high shared costs $\widehat{f}$

6 Conclusion and final remarks

We introduced the noncooperative fixed charge transportation problem which is a game-theoretic model whose players solve mixed-integer optimization problems. We examined its structure thoroughly and carried out some numerical investigations. In summary, we hope that this contribution will further stimulate the interest of other researchers for this young and promising field of research in the intersection of noncooperative game theory and mixed-integer programming.

	$NFCTP(1)_{sum}$	$NFCTP(1)_{max}$	NFCTP(2)	NFCTP(3)
cost_tot	9789	10007.1	7805	7629.67
cost_var_tot	2692	2651.14	2634	2854
cost_fix_tot	1523	1720	1459	1941
cost_sha_tot	5574	5636	3712	2834.67
#routes_3players	2	2	3	10
#routes_2players	3	7	8	1
#routes_1player	20	12	8	5
#routes_tot	25	21	19	16
cost_1	2934	3335.71	2531.33	2276.67
cost_var_1	717	803.71	772	829
cost_fix_1	592	663	522	601
cost_sha_1	1625	1869	1237.33	846.67
quantities_1	329	340.57	352	365
cost_2	2393	3335.71	2173.33	3110
cost_var_2	732	1005.71	567	1127
cost_fix_2	392	665	369	743
cost_sha_2	1269	1665	1237.33	1240
quantities_2	298	427.86	258	432
cost_3	4462	3335.71	3100.33	2243
cost_var_3	1243	841.71	1295	898
cost_fix_3	539	392	568	597
cost_sha_3	2680	2102	1237.33	748
quantities_3	536	394.57	553	366

Table 2: Instance (B) - high shared costs $\widehat{f}$

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	NFCTP(1)$_{\text{sum}}$	NFCTP(1)$_{\text{max}}$	NFCTP(2)	NFCTP(3)
cost_tot	4170.3	4200.83	4033.3	4062.6
cost_var_tot	2367	2476.43	2343	2407
cost_fix_tot	1029	962	993	1061
cost_sha_tot	774.3	762.4	697.3	594.6
#routes_3players	0	0	0	2
#routes_2players	4	4	7	7
#routes_1player	22	21	17	11
#routes_tot	26	25	24	20
cost_1	1309.3	1400.28	1324.43	992.83
cost_var_1	699	772.87	728	593
cost_fix_1	308	346	364	252
cost_sha_1	302.3	281.4	232.43	147.83
quantities_1	304	319.66	312	242
cost_2	1177	1400.28	1380.43	1834.17
cost_var_2	636	778.27	728	1041
cost_fix_2	329	391	420	527
cost_sha_2	212	231	232.43	266.17
quantities_2	282	328.42	345	437
cost_3	1684	1400.28	1328.43	1235.6
cost_var_3	1032	925.27	887	773
cost_fix_3	392	225	209	282
cost_sha_3	260	250	232.43	180.6
quantities_3	410	347.91	339	317

Table 3: Instance (A) - low shared costs $\widehat{f}$

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	$\text{NFCTP}(1)_{\text{sum}}$	$\text{NFCTP}(1)_{\text{max}}$	NFCTP(2)	NFCTP(3)
cost_tot	4520.2	4576.24	4381.4	4392.13
cost_var_tot	2536	2541.34	2509	2482
cost_fix_tot	1253	1287	1213	1272
cost_sha_tot	731.2	747.9	659.4	638.13
#routes_3players	2	2	2	3
#routes_2players	3	4	6	5
#routes_1player	19	19	15	14
#routes_tot	24	25	23	22
cost_1	1318.7	1525.41	1425.8	1310.43
cost_var_1	699	801.01	825	736
cost_fix_1	412	483	381	376
cost_sha_1	207.7	241.4	219.8	198.43
quantities_1	333	370.34	367	344
cost_2	1108	1525.41	1354.8	1259.9
cost_var_2	570	874.91	733	660
cost_fix_2	339	389	402	411
cost_sha_2	199	261.5	219.8	188.9
quantities_2	285	388.96	352	330
cost_3	2093.5	1525.41	1600.8	1821.8
cost_var_3	1267	865.41	951	1086
cost_fix_3	502	415	430	485
cost_sha_3	324.5	245	219.8	250.8
quantities_3	545	403.71	444	489

Table 4: Instance (B) - low shared costs $\widehat{f}$

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