The convex hull of a quadratic constraint over a polytope

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Abstract

A quadratically constrained quadratic program (QCQP) is an optimization problem in which the objective function is a quadratic function and the feasible region is defined by quadratic constraints. Solving non-convex QCQP to global optimality is a well-known NP-hard problem and a traditional approach is to use convex relaxations and branch-and-bound algorithms. This paper makes a contribution in this direction by showing that the exact convex hull of a general quadratic equation intersected with any bounded polyhedron is second-order cone representable. We present a simple constructive proof to this result and some simple applications of this result.

1 Introduction

A quadratically constrained quadratic program (QCQP) is an optimization problem in which the objective function is a quadratic function and the feasible region is defined by quadratic constraints. A variety of complex systems can be cast as an instance of a QCQP. Combinatorial problems like MAXCUT [27], engineering problems such as signal processing [26, 33], chemical process [31, 43, 4, 21, 29, 61] and power engineering problems such as the optimal power flow [13, 37, 17, 34] are just a few examples.

Solving non-convex QCQP to global optimality is a well-know NP-hard problem and a traditional approach is to use spacial branch-and-bound tree based algorithm. The computational success of any branch-and-bound tree based algorithm depends on the convexification scheme used at each node of the tree. Not surprisingly, there has been a lot of research on deriving strong convex relaxations for general-purpose QCQPs. The most common relaxations found in the literature are based on Linear programming (LP), second order cone programing (SOCP) or semi-definite programming (SDP). Reformulation-linearization technique (RLT) [53, 55] is a LP-based hierarchy, Lasserre hierarchy or the sum-of-square hierarchy [36] is a SDP-based hierarchy which exactly solves QCQPs under some minor technical conditions and, recently, new

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LP and SOCP-based alternatives to sum of squares optimization have also been proposed [2]. While SDP relaxations are know to be strong, they don't always scale very well computationally. SOCP relaxations tend to be more computationally attractive, although they are often derived by further relaxing SDP relaxations [16].

Another direction of research focuses on convexification of specific functions/sets, with the McCormick relaxation [40] being perhaps the most classic example. There have been a lot of work in function convexification (see for instance [3, 54, 5, 51, 38, 11, 41, 6, 8, 7, 44, 20, 52, 50, 42, 61, 62, 39, 14, 18, 1, 30, 56]) and convexification of sets: [59, 46, 47, 58, 28, 35, 49, 19, 37, 45, 15].

A related question when studying convex relaxations is that of representability of the exact convex hull of the feasible set: Is it LP, SOCP or SDP representable? In [23], we prove that the convex hull of the so-called bipartite bilinear constraint (which is a special case of a quadratic constraint) intersected with a box constraint is SOCP representable (SOCr). The proof yields a procedure to compute this convex hull exactly. Encouraging computational results are also reported in [23] in terms of obtaining dual bounds using this construction, which significantly outperform SDP and McCormick relaxations and also bounds produced by commercial solvers.

2   Our result

For an integer $t \geq 1$, we use $[t]$ to describe the set $\{1, \ldots, t\}$. For a set $G \subseteq \mathbb{R}^n$, we use $\operatorname{conv}(G)$, $\operatorname{extr}(G)$ to denote the convex hull of G and the set of extreme points of G, respectively.

In this paper, we generalize one of the main results in [23]. Specifically, we show that the convex hull of a *general* quadratic equation intersected with *any* bounded polyhedron is SOCr. Moreover, the proof is constructive, there by adding to the literature on explicit convexification in the context of QCQPs. The formal result is as following:

Theorem 1. *Let*

$$S := \{x \in \mathbb{R}^n \mid x^\top Q x + \alpha^\top x = g, \ x \in P\}, \tag{1}$$

where $Q \in \mathbb{R}^{n \times n}$ is a symmetric matrix, $\alpha \in \mathbb{R}^n$, $g \in \mathbb{R}$ and $P := \{x \mid Ax \leq b\}$ is a polytope. Then $\operatorname{conv}(S)$ is SOCr.

Notice that we make no assumption regarding the structure or coefficients of the quadratic equation defining S. We require P to be a bounded polyhedron, which is not very restrictive given that in global optimization the variables are often assumed to be bounded to use branch-and-bound algorithms.

The result presented in Theorem 1 is somewhat unexpected since the sum-of-squares approach would build a sequence of SDP relaxations for (1) in order to optimize (exactly) a linear function over S, while even the SDP cone of three-by-three dimensional matrices is not SOCr [25]. Note that optimizing a linear function over S is NP-hard, therefore, while the convex hull is SOCr, the construction involves the introduction of an exponential number of variables (more precisely $\mathcal{O}(\Delta n)$ variables, where Δ is total number of faces of P).

Surprisingly, the proof of Theorem 1 is fairly straightforward. At the heart of our proof of Theorem 1, is the following classification of surfaces defined by one quadratic equation. A quadratic surface $\{x \in \mathbb{R}^n \mid x^\top Q x + \alpha^\top x = g\}$ satisfies exactly one of the following:

1. the surface is that of a convex (such as in an ellipsoid) set; or

2. the surface is that of the union of two convex sets (see Figure 1); or

3. it has the property that, through every point of the surface, there exists a *straight* line that is *entirely* contained in the surface (see Figure 2).
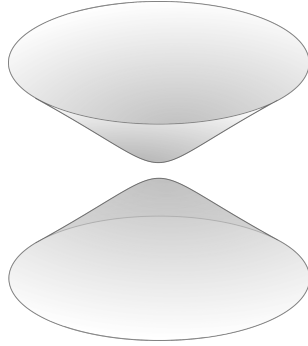

Figure 1: Two-sheets hyperboloid. The surface is the union of two convex peices.
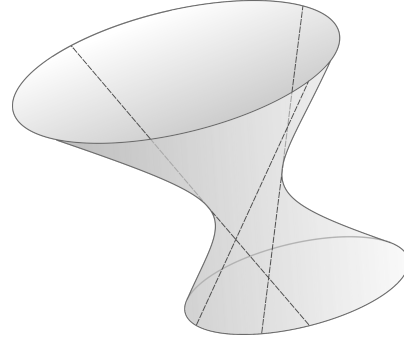
Figure 2: One-sheet hyperboloid. Through every point of the surface, there exists a straight line that is entirely contained in the surface.

To the best of our knowledge, this classification of quadratic surfaces is new and may be of independent interest.

In Case 1 and Case 2, we can easily obtain that the convex hull of S is SOCr as we show in Section 3.5. In Case 3, no point in the interior of the polytope can be an extreme point of S as we show in Section 3.4. Observing that the convex hull of a compact set is also the convex hull of its extreme points, we intersect the surface with each facet of the polytope – the unioin of these intersections will contain all the extreme points of S. Now, each such intersection leads to new sets with the same form as S but in one dimension lower. The argument then goes by recursion. The details of the proof are presented in Section 3.

3   Proof of Theorem 1

The final proof of Theorem 1 is presented in Section 3.6. First, we go through a few rounds of preliminary results and simplifications.

3.1   Convex hulls via disjunctions

In this section, we describe a simple procedure to obtain the convex hull of a compact set S using a disjunctive argument. We use this procedure to prove Theorem 1 in Section 3.6. Let S be a compact set and let $\mathrm{extr}(S)$ be the set of extreme points of S. First, we partition the

extreme points of S. Specifically, suppose there exist $B^1, \ldots, B^k \subseteq S$ such that:

$$S \supseteq \bigcup_{i=1}^{k} B^i \supseteq \text{extr}(S). \tag{2}$$

We observe that (2) implies that

$$\text{conv}(S) \supseteq \text{conv}\left(\bigcup_{i=1}^{k} B^i\right) \supseteq \text{conv}(\text{extr}(S)) = \text{conv}(S), \tag{3}$$

where the last equality holds due to S being compact (this is a consequence of Krein-Milman Theorem, see Theorem B.2.10 in [10]). Finally, we obtain that

$$\text{conv}(S) = \text{conv}\left(\bigcup_{i=1}^{k} B^i\right) = \text{conv}\left(\bigcup_{i=1}^{k} \text{conv}(B^i)\right). \tag{4}$$

Observation 1. *If* $\text{conv}(B^i)$ *is SOCr for all* $i \in [k]$, *then the set*

$$\text{conv}\left(\bigcup_{i=1}^{k} \text{conv}(B^i)\right),$$

is SOCr [10]. Thus, we obtain from (4) that $\text{conv}(S)$ *is SOCr. In addition, we obtain a constructive procedure to compute* $\text{conv}(S)$.

3.2 Dealing with low dimensional polytope

We begin by stating an useful lemma.

Lemma 1. *Let* $G = \{(x, w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid x \in G_0, \ w = C^\top x + h\}$, *where* $G_0 \subseteq \mathbb{R}^{n_1}$ *is bounded, and* $C^\top x + h$ *is an affine function of* x. *Then,*

$$\text{conv}(G) = \{(x, w) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid x \in \text{conv}(G_0), \ w = C^\top x + h\}.$$

Proof. See Lemma 4 in [23]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let S and P be defined as in (1). Next, we show that we may assume without loss of generality that P is full dimensional. In fact, if P is not full dimensional, then P is contained in a non-trivial affine subspace defined by a system of linear equations $Mx = f$. Without loss of generality, we may assume that M has full row-rank k, $1 \leq k < n$. Let $M = \begin{bmatrix} M_B & M_N \end{bmatrix}$ where M_B is invertible. Then, we may write this system as $x_B = -M_B^{-1} M_N x_N + M_B^{-1} f$, where $x_B \in \mathbb{R}^k$, $x_N \in \mathbb{R}^{n-k}$ and, for simplicity, we assume that x_B (resp. x_N) correspond to the first k (resp. last $n-k$) components of x. By defining $C = -M_B^{-1} M_N$ and $h = M_B^{-1} f$ to simplify notation, we obtain

$$x_B = Cx_N + h. \tag{5}$$

4

By partitioning Q in sub-matrices of appropriate sizes, we may explicitly write the quadratic equation defining S in terms of x_B and x_N as follows:

$$\begin{bmatrix} x_B^\top & x_N^\top \end{bmatrix} \begin{bmatrix} Q_{BB} & Q_{BN} \\ Q_{NB} & Q_{NN} \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} + \alpha^\top \begin{bmatrix} x_B \\ x_N \end{bmatrix} = g. \tag{6}$$

Using (5), we replace x_B in (6) to obtain

$$x_N^\top \tilde{Q} x_N + \tilde{\alpha}^\top x_N = \tilde{g},$$

where

$$\tilde{Q} = C^\top Q_{BB} C + C^\top Q_{BN} + Q_{NB} C + Q_{NN},$$
$$\tilde{\alpha} = 2C^\top Q_{BB} h + Q_{BN}^\top h + Q_{NB} h + C^\top \alpha_B + \alpha_N,$$
$$\tilde{g} = g - h^\top Q_{BB} h - \alpha_B^\top h.$$

Therefore, we may write S as

$$S := \{(x_B, x_N) \in \mathbb{R}^n \mid x_N^\top \tilde{Q} x_N + \tilde{\alpha}^\top x_N = \tilde{g}, \ x_N \in \tilde{P}, \ x_B = C x_N + h\}, \tag{7}$$

where $\tilde{P}$ is now a full dimensional polytope. Now by Lemma 1, we may assume from now on that P is full dimensional.

3.3 Reduction to canonical form

In this section, we discuss how to change coordinates to re-write S defined in (1) in a "canonical" form such that all quadratic terms are squared terms. This will allows us to easily classify S into different cases as discussed in Section 2.

The next observation will validate the change of coordinates that we will perform next.

Observation 2. *Let $S \subseteq \mathbb{R}^n$ and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be an affine map. Then*

$$\mathrm{conv}(F(S)) = F(\mathrm{conv}(S)),$$

where $F(S) := \{F(x) \mid x \in S\}$ ([10], Proposition B.1.6). Furthermore if $\mathrm{conv}(S)$ is SOCr, then $\mathrm{conv}(F(S))$ is also SOCr ([10], Section 2.3.2).

Let S be the set defined in (1). Suppose, without loss of generality, that Q is a symmetric matrix. By the spectral theorem $Q = V^\top \Sigma V$, where Σ is a diagonal matrix and the columns of V are a set of orthogonal vectors. Letting $w = Vx$, we have that

$$S := V^{-1} \left(\{w \mid w^\top \Sigma w + \alpha^\top V^{-1} w = d, \ w \in \tilde{P}\} \right),$$

where $\tilde{P} := \{w \mid AV^{-1} w \leq b\}$.

Therefore, by Observation 2, it is sufficient to study the convex hull of a set of the form:

$$S := \left\{ (x, y, z) \in \mathbb{R}^n \mid \sum_{i=1}^{n_q} a_i x_i^2 + \sum_{i=1}^{n_q} \alpha_i x_i + \sum_{j=1}^{n_l} \beta_j y_j = g, \ (x, y, z) \in P \right\},$$

where $n_q + n_l + n_o = n$, $a_i \neq 0$ for $i \in [n_q]$ (i.e., the rank of Q is n_q) and $\beta_j \neq 0$ for $j \in [n_l]$. Thus, $x \in \mathbb{R}^{n_q}$ ($y \in \mathbb{R}^{n_l}$) are the variables present (not present) in quadratic terms, and $z \in \mathbb{R}^{n_o}$ are the variables not present in the quadratic equation (We are using different letters to represent variables, to clarify the various types of different variables). We can further simplify S by completing squares:

$$S := \{ (x, y, z) \in \mathbb{R}^n \mid \sum_{i=1}^{n_q} \sigma(a_i) \left(\sqrt{|a_i|} x_i + \sigma(a_i) \frac{\alpha_i}{2\sqrt{|a_i|}} \right)^2 + \sum_{i=1}^{n_l} \beta_i y_i = g + \sum_{i=1}^{n_q} \frac{\alpha_i^2}{4 a_i}, \ (x, y, z) \in P \},$$

where $\sigma(a)$ denotes the sign of a. Now, since $u_i = \left(\sqrt{|a_i|} x_i + \sigma(a_i) \frac{\alpha_i}{2\sqrt{|a_i|}} \right)$ for $i \in [n_q]$ and $v_i = \beta_i y_i$ for $i \in [n_l]$ define linear bijections, it follows from Observation 2 that it is sufficient to study the convex hull of the following simplified set:

$$S := \{ (w, x, y, z) \in \mathbb{R}^{n_{q+}} \times \mathbb{R}^{n_{q-}} \times \mathbb{R}^{n_l} \times \mathbb{R}^{n_0} \mid \sum_{i=1}^{n_{q+}} w_i^2 - \sum_{j=1}^{n_{q-}} x_j^2 + \sum_{k=1}^{n_l} y_k = g, \ (w, x, y, z) \in P \}, \quad (8)$$

where $w \in \mathbb{R}^{n_{q+}}$ (resp. $x \in \mathbb{R}^{n_{q-}}$) are the variables present in quadratic terms with coefficient $+1$ (resp. -1), $y \in \mathbb{R}^{n_l}$ are the variables present in linear term only, and $z \in \mathbb{R}^{n_o}$ are the variables present in the description of P only. In (8), we assume that $g \geq 0$, since otherwise we may multiply the equation by -1 and apply suitable affine transformations to bring it back to the form of (8).

3.4 Sufficient conditions for points to not be extreme

Next, we prove a sequence of lemmas showing that, depending on the values of n_{q-}, n_{q+}, n_l, and n_0 in (8), S falls in one of three cases discussed in Section 2, where we assume that the polytope P is full dimensional. Consider the set S as defined in (8).

Lemma 2. *Suppose $n_o \geq 1$. If $(w, x, y, z) \in S \cap \text{int}(P)$, then (w, x, y, z) is not an extreme point of S.*

Proof. Since $(w, x, y, z) \in \text{int}(P)$, there exists a vector $\delta \in \mathbb{R}^{n_o} \setminus \{0\}$ such that $(w, x, y, z + \delta), (w, x, y, z - \delta) \in P$. Clearly, these points are in S as well and, therefore, (w, x, y, z) is not an extreme point of S ☐

Lemma 3. *Suppose $n_0 = 0$ and $n_l \geq 2$. If $(w, x, y) \in S \cap \text{int}(P)$, then (w, x, y) is not an extreme point of S.*

6

Proof. Since $n_l \geq 2$, $(w, x, y_1 \pm \lambda, y_2 \mp \lambda, \ldots, y_{n_l})$ are feasible for sufficiently small positive values of λ. Therefore, (w, x, y) is not an extreme point. $\square$

Lemma 4. *Suppose $n_0 = 0$, n_{q+}, $n_{q-} \geq 1$ and $n_l = 1$. If $(w, x, y) \in S \cap \mathrm{int}(P)$, then (w, x, y) is not an extreme point of S.*

Proof. Since $n_{q+}, n_{q-} \geq 1$, and $n_l = 1$, $(w_1 + \lambda, w_2, \ldots, w_{n_{q+}}, x_1 + \lambda, x_2, \ldots, x_{n_{q-}}, y + 2\lambda(-w_1 + x_1))$ are feasible for sufficiently small positive and negative values of λ. Therefore, (w, x, y) is not an extreme point. $\square$

Lemma 5. *Suppose $n_0 = 0$, $n_{q+} \geq 2$, $n_{q-} \geq 1$ and $n_l = 0$. If $(w, x) \in S \cap \mathrm{int}(P)$, then (w, x) is not an extreme point of S.*

Proof. We show that there exists a straight line through (w, x) that is entirely contained in the surface defined by the quadratic equation (a surface that has this property is called as "ruled surface" [24].

More specifically, we prove that there exists a vector $(u, v) \in (\mathbb{R}^{n_{q+}} \times \mathbb{R}^{n_{q-}}) \setminus \{0\}$ such that the line $\{(w, x) + \lambda(u, v) \mid \lambda \in \mathbb{R}\}$ satisfies the quadratic equation and therefore, (w, x) being in the interior of P cannot be an extreme point of S. We consider two cases:

1. $(w, x) \neq \mathbf{0}$: Then observe that $w \neq \mathbf{0}$, since otherwise we would have $w = \mathbf{0}$ and $x = \mathbf{0}$, because $g \geq 0$. Observe that

$$\sum_{i=1}^{n_{q+}} w_i^2 = g + \sum_{j=1}^{n_{q-}} x_j^2 \geq x_1^2 \Leftrightarrow \frac{|x_1|}{\|w\|_2} \leq 1. \tag{9}$$

Next, observe that:

$$
\begin{aligned}
g &= \sum_{i=1}^{n_{q+}} (w_i + \lambda u_i)^2 - \sum_{i=1}^{n_{q-}} (x_i + \lambda v_i)^2 \ \forall \lambda \in \mathbb{R} \\
\Leftrightarrow g &= \left(\sum_{i=1}^{n_{q+}} w_i^2 - \sum_{i=1}^{n_{q-}} x_i^2 \right) + \lambda^2 \left(\sum_{i=1}^{n_{q+}} u_i^2 - \sum_{i=1}^{n_{q-}} v_i^2 \right) + 2\lambda \left(\sum_{i=1}^{n_{q+}} w_i u_i - \sum_{i=1}^{n_{q-}} x_i v_i \right) \ \forall \lambda \in \mathbb{R} \\
\Leftrightarrow 0 &= \lambda \left(\sum_{i=1}^{n_{q+}} u_i^2 - \sum_{i=1}^{n_{q-}} v_i^2 \right) + 2 \left(\sum_{i=1}^{n_{q+}} w_i u_i - \sum_{i=1}^{n_{q-}} x_i v_i \right) \ \forall \lambda \in \mathbb{R} \\
\Leftrightarrow \quad & \sum_{i=1}^{n_{q+}} u_i^2 - \sum_{i=1}^{n_{q-}} v_i^2 = 0, \ \sum_{i=1}^{n_{q+}} w_i u_i - \sum_{i=1}^{n_{q-}} x_i v_i = 0. \tag{10}
\end{aligned}
$$

Suppose we set $v_1 = 1$ and $v_j = 0$ for all $j \in \{2, \ldots, n_{q-}\}$. Then satisfying (10) is equivalent to finding real values of u satisfying:

$$\sum_{i=1}^{n_{q+}} u_i^2 = 1, \ \sum_{i=1}^{n_{q+}} w_i u_i = x_1.$$

7

This is the intersection of a circle of radius 1 in dimension two or higher (since $n_{q+} \geq 2$ in this case) and a hyperplane whose distance from the origin is $\frac{|x_1|}{\|w\|_2}$. Since, by (9), we have that this distance is at most 1, the hyperplane intersects the circle and therefore we know that a real solution exists.

2. $(w, x) = \mathbf{0}$: In this case, observe that $g = 0$ and then $\mathbf{0}$ is a convex combination of

$$(\quad \underbrace{\pm\lambda, 0, \ldots, 0}_{\text{first } n_{q+} \text{ components}} \quad , \quad \underbrace{\pm\lambda, 0, \ldots, 0}_{\text{second } n_{q-} \text{ components}} \quad)$$

for sufficiently small $\lambda > 0$.

$\square$

3.5 Sufficient conditions for convex hull to be SOCr

In this section, we repeatedly use the following result from [57].

Theorem 2. *Let $G \subseteq \mathbb{R}^n$ be a convex set and let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Then*

$$\text{conv}\left(\{G \cap \{x \mid f(x) = 0\}\}\right) = \text{conv}\left(\{G \cap \{x \mid f(x) \leq 0\}\}\right) \cap \text{conv}\left(\{G \cap \{x \mid f(x) \geq 0\}\}\right).$$

For the two lemmas that follows, consider the notation of S defined in (8).

Lemma 6. *Suppose $n_0 = 0$ and $n_l \leq 1$. If $n_{q+} = 0$ or $n_{q-} = 0$, then $\text{conv}(S)$ is SOCr.*

Proof. We prove only the case $n_{q-} = 0$ (case $n_{q+} = 0$ is analogous). Let $(w, y) \in S$. Let $y = y_1$ if $n_l = 1$ and $y = 0$ if $n_l = 0$. In this case, $g - y$ is non-negative for all feasible values of y and we can use the identity $t = \frac{(t+1)^2 - (t-1)^2}{4}$ to write $S = S' \cap S''$, where:

$$S' := \{(w, y) \in P \mid \|2w_1, \ldots, 2w_{n_{q+}}, (g - y - 1)\| \leq (g - y + 1)\},$$
$$S'' := \{(w, y) \in P \mid \|2w_1, \ldots, 2w_{n_{q+}}, (g - y - 1)\| \geq (g - y + 1)\}.$$

Notice that S' is a SOCr convex set. Also notice that S'' is a reverse convex set intersected with a polytope and hence $\text{conv}(S'' \cap P)$ is polyhedral and contained in P (see [32],Theorem 1). Therefore, by Theorem 2, we have that $\text{conv}(S) = \text{conv}(S') \cap \text{conv}(S'')$ is SOCr. $\square$

Lemma 7. *Suppose $n_0 = 0$, $n_{q+} \leq 1$ and $n_l = 0$. Then $\text{conv}(S)$ is SOCr.*

Proof. If $n_{q+} = 0$, then S is empty set or contains a single point, the origin.

Therefore, consider the case where $n_{q+} = 1$, thus $w = w_1$. Notice that $S = S' \cap S''$, where

$$S' := \{(w, x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid w^2 \geq g + \sum_{j=1}^{n_{q-}} x_j^2, \ (w, x) \in P\},$$

$$S'' := \{(w, x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid w^2 \leq g + \sum_{j=1}^{n_{q-}} x_j^2, \ (w, x) \in P\}.$$

8

By Theorem 2, $\mathrm{conv}(S) = \mathrm{conv}(S') \cap \mathrm{conv}(S'')$. Next, we show that both $\mathrm{conv}(S')$ and $\mathrm{conv}(S'')$ are SOCr. Notice that S' is the union of the following two SOCr sets:

$$S'_+ := \left\{ (w,x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid w \geq \left(g + \sum_{j=1}^{n_{q-}} x_j^2 \right)^{\frac{1}{2}} , \ w \geq 0, \ (w,x) \in P \right\}$$

$$= \mathrm{Proj}_{w,x} \left(\left\{ (w,x,t) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \times \mathbb{R} \mid w \geq \left((\sqrt{g}t)^2 + \sum_{j=1}^{n_{q-}} x_j^2 \right)^{\frac{1}{2}} , \ x \geq 0, \ t = 1, \ (w,x) \in P \right\} \right),$$

$$S'_- := \left\{ (w,x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid -w \geq \left(g + \sum_{j=1}^{n_{q-}} x_j^2 \right)^{\frac{1}{2}} , \ w \leq 0, \ (w,x) \in P \right\}$$

$$= \mathrm{Proj}_{w,x} \left(\left\{ (w,x,t) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \times \mathbb{R} \mid -w \geq \left((\sqrt{g}t)^2 + \sum_{j=1}^{n_{q-}} x_j^2 \right)^{\frac{1}{2}} , \ w \leq 0, \ t = 1, \ (w,x) \in P \right\} \right).$$

Thus, $\mathrm{conv}(S') = \mathrm{conv}(S'_+ \cup S'_-)$ is SOCr.

Notice that $S'' = \{(w,x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid |w| \leq (g + \sum_{j=1}^{n_{q-}} x_j^2)^{\frac{1}{2}}, \ (w,x) \in P\}$ and is therefore the union of two sets:

$$S''_+ := \left\{ (w,x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid w \leq \left(g + \sum_{j=1}^{n_{q-}} x_j^2 \right)^{\frac{1}{2}} , \ w \geq 0, \ (w,x) \in P \right\},$$

$$S''_- := \left\{ (w,x) \in \mathbb{R}^1 \times \mathbb{R}^{n_{q-}} \mid -w \leq \left(g + \sum_{j=1}^{n_{q-}} x_j^2 \right)^{\frac{1}{2}} , \ w \leq 0, \ (w,x) \in P \right\},$$

each of them being a reverse convex set intersected with a polyhedron. Therefore, $\mathrm{conv}(S''_+)$ and $\mathrm{conv}(S''_-)$ are polyhedral and therefore $\mathrm{conv}(S'') = \mathrm{conv}(\mathrm{conv}(S''_+) \cup \mathrm{conv}(S''_-))$ is a polyhedral set. $\quad\square$

3.6 Proof of Theorem 1

Finally, we bring the pieces together to prove Theorem 1.

Proof. (of Theorem 1) Let S be defined as in (8). Recall that appropriate transformations have been applied to ensure that P is full-dimensional (see Section 3.2), and subsequent transformations brought S in a "canonical" form shown in (8). Also, notice that S is defined in the n-dimensional space, where $n = n_{q+} + n_{q-} + n_l + n_0$. The proof goes by induction on n. Notice that if $n = 1$, then S is a polytope and hence $\mathrm{conv}(S)$ is SOCr. Suppose S is SOCr

9

in dimension n (induction hypothesis). We now show that S is SOCr in dimension $n + 1$. If $n_0 = 0$, $n_l \leq 1$, and $n_{q+} = 0$ or $n_{q-} = 0$, then the result follows from Lemma 6. Similarly, if n_0, $n_{q+} \leq 1$, and $n_l = 0$, then the result follows from Lemma 7. Otherwise, it follows from Lemma 2, 3, 4 and 5 that no point in the interior of P can be an extreme point of S. Let N be the number of facets of P, each of which given by one equation of a linear system $Fx = f$. Let $B^i = S \cap \{x \in \mathbb{R}^{n+1} \mid F_{i.}x = f_i\}$ be the intersection of S with the ith facet of P. By the discussion in Section 3.1, it is enough to show that the convex hull of each B^i is SOCr. Let $i \in \{1, \ldots, N\}$. Choose j_0 such that $F_{ij_0} \neq 0$. For simplicity, suppose $j_0 = 1$. Then, we may write $B^i = \{x \in \mathbb{R}^{n+1} \mid (x_2, \ldots, x_{n+1}) \in B_0^i, \ x_1 = b_i - \sum_{j=2}^{n+1} F_{ij}x_j\}$, where B_0^i is obtained from B^i by replacing $x_1 = f_i - \sum_{j=2}^{n+1} F_{ij}x_j$ in all the constraints defining S. Now, $\mathrm{conv}(B_0^i) \subseteq \mathbb{R}^n$ is SOCr by induction hypothesis. Therefore, $\mathrm{conv}(B^i)$ is SOCr by Lemma 1. $\square$

4 Applications

As mentioned in the introduction, Theorem1 is a generalization of a convexification result presented in [23]. Encouraging computational results were reported in [23] in terms of obtaining dual bounds using this construction, which significantly outperform SDP and McCormick relaxations and also bounds produced by commercial solvers. In this section, we illustrate how the result of Theorem1 can have other applications.

Computationally useful extended formulations Consider the simple quadratic set defined by a single bilinear: $S = \{(x, y, w) \in \mathbb{R}^3 \parallel w = xy, \ l \leq w \leq u, \ (x, y) \in [0, 1]^2\}$. When $l \leq 0$ and $1 \leq u$, the convex hull of S is a polytope given by the McCormick envelope of the bilinear term. However, if $0 < l$ or $u < 1$, then the convex hull of S is no longer polyhedral [48, 60, 9]. Indeed, [48] shows that the convex hull is very complicated in the orginal space and the resulting inequalities describing the convex hull cannot be used in computation. Theorem 1 shows that the convex hull of S is SOCr and the proof advises an implementable method to compute this convex hull. Specifically, we intersect the bilinear term $w = xy$ with each facet of the box $[0, 1]^2 \times [l, u]$. Each intersection yields a two dimensional conic section over a box (these will form the B^i sets of Section 3.1) whose convex hull can be easily computed. We then obtain the convex hull of S via a disjunctive formulation. We are currently numerically testing this convexification vs McCormick inequalities.

More convexification results In [22], Theorem 1 is used prove that the convex hull of more general quadratic systems of the form

$$\{(x, y, X) \mid \langle A^i, X \rangle \leq 0 \ i \in [m], X = xy^\top\} \tag{11}$$

(where A^i have specific properties, see [22]), are SOCr and for showing how linear functions can be optimized over these sets in polynomial time assuming m is fixed.

Other structural results   To show the power of the techniques used to prove Theorem 1, we use the same techniques to prove the following structural results on box quadratic programs (QP):

Proposition 1. *Consider the box QP:*

$$\begin{aligned} \max \quad & x^\top A x + b^\top x \\ \text{s.t.} \quad & x \in [0,\ 1]^n. \end{aligned} \tag{12}$$

where A is a symmetric matrix. If A has k^+ positive eigenvalues and k^- negative eigenvalues, then (12) has an optimal solution with atleast $\min\{k^+, k^- + 1\} - 1$ components set to either 0 or 1.

Proof. Let x^* be an optimal solution of (12) with l components of x^* being either 0 or 1. If $l \leq \{k^+, k^- + 1\} - 2$, then we show that there exists another optimal solution of (12) with $l + 1$ components being 0 or 1.

Let us fix the l variables which are binary in x^* to their values of 0 or 1 and re-write the objective in terms of $n - l$ variables:

$$x^\top \tilde{A} x + \tilde{b}^\top x,$$

where x is now assumed to be $n - l$ dimensional variable and $\tilde{A}$ is a $n - l \times n - l$ symmetric matrix. By the interlacing theorem applied to A and $\tilde{A}$, we know that $\tilde{A}$ has atleast $k^+ - l$ (≥ 2) positive eigenvalues and $k^- - l$ (≥ 1) negative eigenvalues. Let z^* be the optimal objective function value. Examine the quadratic equation:

$$x^\top \tilde{A} x + \tilde{b}^\top x = z^*. \tag{13}$$

It is clear (after applying bijective affine transformations), that the above quadratic equation satisfies the conditions of one of the following: Lemma 2, Lemma 3, Lemma 4, or Lemma 5. In all these cases there is a line (which lives in the space orthogonal to the fixed variables) that passes through x^* satisfying (13). Thus, we can move along this line until one more variable gets fixed. □

In a recent paper [12], it was shown that Chvátal-Gomory (CG) cuts are effective in solving QPs. While, we do not claim any strong connection, Proposition 1 may indicate why the CG cuts are so effective in solving box QPs.

Acknowledgments

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