Interdiction of a Mixed-Integer Linear System

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A system-interdiction problem can be modeled as a bilevel program in which the upper level models interdiction decisions and the lower level models system operation. This paper studies MILSIP, a mixed-integer linear system interdiction problem, which assumes binary interdiction decisions and models system operations through a mixed-integer linear program. To solve large-scale instances of MILSIP, we apply Benders decomposition to a single-level reformulation of MILSIP. We identify "significant" linear programming subproblems whose feasibility implies the optimality of MILSIP. We propose an algorithm that that solves in each iteration: a relaxed master problem, a lower-level problem, and a significant subproblem. We demonstrate the algorithm's computational capabilities on a natural gas transmission system with 32 interdictable components. Numerical tests show that our algorithm solves certain problem instances an order of magnitude faster than an alternative algorithm from the literature.

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1. Introduction

A system-interdiction problem is a bilevel optimization problem in which a leader and a follower act sequentially. Interdiction, carried out by the leader (the "attacker"), damages or destroys system components to disrupt the system's functionality. We consider a binary model of interdiction, that is, a non-interdicted component functions normally and an interdicted component loses its functionality entirely. The follower (the "defender" or "system operator") then operates the damaged system by minimizing a system disutility function. The leader anticipates the rational response of the follower, and chooses a resource-limited interdiction plan that maximizes the follower's minimum achievable disutility function value. Following the convention in the interdiction literature (Brown et al. 2006), we assume that both the leader and the follower have perfect information. Such a bilevel interdiction model can be viewed as a type of Stackelberg game (von Stackelberg 2011, Chapter 4), specifically, a two-person, two-stage, zero-sum sequential game. Wood (2011) provides a comprehensive review of bilevel interdiction models.

This paper investigates the *mixed-integer linear system interdiction problem* (MILSIP), a system-interdiction problem in which a mixed-integer linear program (MILP) defines the system operator's problem. Integer variables in the lower level are necessitated by discrete operational decisions of the follower, for example, transmission-line switching that changes the network topology of an electric power system (Delgadillo et al. 2010, Zhao and Zeng 2013), or replacement of disabled system components from a spare-parts inventory (Salmeron and Wood 2015). Nonconvexity of the operator's model can also give rise to a lower-level integer program. For example, a nonlinear and nonconvex flow-pressure relationship determines the flow in a natural gas pipeline (Osiadacz 1987, pp. 69–79). A piecewise linear approximation of the nonlinear relationship makes the lower-level problem an MILP (De Wolf and Smeers 2000). In fact, this is the model we study in Section 4.

Simpler versions of our problem with convex operator's problems have been studied extensively. Network interdiction problems in which the system under study can be represented by a network-flow model have received much attention (McMasters and Mustin 1970, Wood 1993, Washburn and Wood 1995). The bilevel interdiction model has also been applied to systems whose operator's model can be represented by a more general linear program (LP). For instance, (a) Salmeron et al. (2004a) analyze interdiction of an electric power system using a linearized power-flow model to represent system operations (see Wood and Wollenberg 2012, pp. 277–278); (b) Lim and Smith (2007) consider a multicommodity flow network; and (c) Brown et al. (2006) discuss a number of applications to infrastructure protection.

While much of the interdiction literature assumes a convex operator's problem, Salmeron et al. (2009) propose a method called "global Benders decomposition" that can solve more general interdiction problems. Salmeron and Wood (2015) apply this method to a power system whose optimal operation is modeled as an MILP. This method involves a problem-specific task: a penalty vector for the upper-level discrete variables must be determined. Although in some practical cases a penalty vector can be found using the structure of the lower-level problem, it is difficult in general to define valid penalty vectors.[1]

We can also view MILSIP as a bilevel program where an embedded lower-level problem appears in the constraints of an upper-level program. Many researchers have investigated bilevel programs; for example, see the reviews of Wen and Hsu (1991), Vicente and Calamai (1994), Dempe (2003), and Colson et al. (2007). If the lower-level problem were convex, one could replace this problem by its Karush-Kuhn-Tucker (KKT) conditions to obtain an equivalent single-level optimization problem under certain regularity conditions (Fortuny-Amat and McCarl 1981, Dempe and Franke 2016). This reformulation technique has been applied successfully to interdiction problems with a convex operator's problem (Salmeron et al. 2004b, Arroyo and Galiana 2005, Arroyo 2010). However, standard reformulation techniques that depend on the convexity of the lower-level problem do not apply to MILSIP because of the lower-level integer variables. MILSIP belongs to a special case of bilevel programs, mixed-integer bilevel program (MIBP).

A few solution methods have been proposed for MIBP. Moore and Bard (1990) and Xu and Wang (2014) consider branch-and-bound algorithms. These algorithms use the high-point problem (Moore and Bard 1990) as the relaxation of MIBP. As Wood (2011) points out, however, relaxations based on the high-point problem are weak for an interdiction problem with its directly conflicting objective functions. Gümüş and Floudas (2005) reformulate the mixed-integer inner problem as continuous via its convex-hull representation (Sherali and Adams 1990). However, finding globally optimal solutions to the resulting nonlinear bilevel programming problem remains challenging. DeNegre (2011) presents a single-level reformulation of MIBP using a lower-level value function defined through a parametrization of the lower-level MILP by the upper-level variables. Exact characterization of this value function of a lower-level MILP is difficult in general, so DeNegre proposes two heuristic methods. Saharidis and Ierapetritou (2008) apply Benders decomposition (Benders 1962) to a type of linear MIBP where all integer variables are controlled by the leader. The subproblem in the proposed method is a nonconvex bilevel linear program, which may result in invalid Benders cuts. Mitsos (2009) investigates nonlinear MIBPs but assumes the lower-level equality constraints to be independent of the upper-level variables. More recently, several branch-and-cut approaches have been proposed for general linear MIBPs. These approaches strengthen the high point relaxation by introducing cuts that eliminate bilevel-infeasible solutions, e.g., intersection cuts (Fischetti et al. 2018, 2017), multiway disjunction cuts (Wang and Xu 2017), and value-function-based disjunction cuts (Lozano and Smith 2017). However, they all assume integer coefficients in the lower-level constraints. In the presence of fractional coefficients, the bilevel-infeasible incumbent solution may lie right on the hyperplane defined by the cuts, resulting in ineffective cuts. These methods are restrictive because most practical infrastructure systems are described by fractional parameters. Zeng and An (2014) propose a column-and-constraint generation (CCG) algorithm that does not assume integral coefficients, which is applied to an interdiction problem in Zhao and Zeng (2013) and two-stage robust optimization problems in Zeng and Zhao (2013). We will investigate CCG as an alternative method for solving MILSIP, but we note one feature that may impose a computational burden: CCG's relaxed master problem not only adds constraints in each major iteration, but also adds variables.

The discussion above indicates that MILSIP remain challenging to solve, especially in the presence of a large-scale follower's problem with fractional coefficients. This paper applies generalized Benders decomposition to a single-level reformulation of MILSIP, which decomposes MILSIP into a master problem and an exponential number of feasibility subproblems. Most importantly, we identify "significant" linear programming subproblems whose feasibility implies the optimality of MILSIP. We propose an algorithm that iteratively solves a relaxed master problem, a lower-level problem, and a single significant subproblem. Our method provides upper and lower bounds on the optimal objective function value, and is finitely convergent.

The remainder of this paper is structured as follows. Section 2 formulates MILSIP, and Section 3 presents the identification of the significant subproblems and our new decomposition algorithm.

Section 4 describes the application of the proposed method to a natural gas transmission system, and Section 5 presents conclusions. Appendix A presents a small numerical example of the decomposition algorithm which the reader may wish to consult after reading Section 3. Appendix B presents the mathematical and algorithmic details that are required to apply the algorithm to the computational example of Section 4.

2. Mixed Integer Linear System Interdiction Problem

This section formulates MILSIP and analyzes its structure.

2.1. Mathematical Formulation

We first define the operator's problem. This problem has decision variables $(\boldsymbol{x}, \boldsymbol{y})$, where $\boldsymbol{x} \in \mathcal{X} \subset \mathbb{Z}^{n_x}$ and $\boldsymbol{y} \in \mathbb{R}_+^{n_y}$. We assume that the cardinality of $\mathcal{X}$, denoted $|\mathcal{X}|$, is finite. We believe this is a mild assumption since integer variables are usually bounded in real-world problems (Geoffrion 1974). For later use, we define $\mathcal{V}$ as the index set for $\mathcal{X}$, implying that $\mathcal{X} = \{\boldsymbol{x}_v : v \in \mathcal{V}\}$ and that $|\mathcal{V}| = |\mathcal{X}|$.

Without loss of generality, we use a minimization formulation, in which a disutility function for system operations defines the objective. When no interdiction takes place, the operator's problem is

$$\min_{\boldsymbol{x} \in \mathcal{X}, \boldsymbol{y} \in \mathbb{R}_+^{n_y}} \{\boldsymbol{c}^\intercal \boldsymbol{x} + \boldsymbol{d}^\intercal \boldsymbol{y} : C\boldsymbol{x} + D\boldsymbol{y} \leq \boldsymbol{g}\}, \tag{1}$$

where $\boldsymbol{c} \in \mathbb{R}^{n_x}$, $\boldsymbol{d} \in \mathbb{R}^{n_y}$, $C \in \mathbb{R}^{n_g \times n_x}$, $D \in \mathbb{R}^{n_g \times n_y}$, and $\boldsymbol{g} \in \mathbb{R}^{n_g}$.

Now, using binary *interdiction variables* $\boldsymbol{z} \in \{0,1\}^{n_z}$, we consider the effect of interdiction. Letting i index interdictable system components, we let $z_i = 1$ if component i is interdicted, and let $z_i = 0$ otherwise. The vector $\boldsymbol{z}$ in the aggregate defines an *interdiction plan.*

Parametrizing C, D, and $\boldsymbol{g}$ by the interdiction plan $\boldsymbol{z}$,

$$\{\boldsymbol{x} \in \mathcal{X}, \boldsymbol{y} \in \mathbb{R}_+^{n_y} : C(\boldsymbol{z})\boldsymbol{x} + D(\boldsymbol{z})\boldsymbol{y} \leq \boldsymbol{g}(\boldsymbol{z})\}, \tag{2}$$

defines the post-interdiction feasible region for the system operator. Linear functions $C(\cdot) : \mathbb{R}^{n_z} \to \mathbb{R}^{n_g \times n_x}$, $D(\cdot) : \mathbb{R}^{n_z} \to \mathbb{R}^{n_g \times n_y}$, and $\boldsymbol{g}(\cdot) : \mathbb{R}^{n_z} \to \mathbb{R}^{n_g}$ define the parametrization. The base-case values are $C(\boldsymbol{0}) = C$, $D(\boldsymbol{0}) = D$, and $\boldsymbol{g}(\boldsymbol{0}) = \boldsymbol{g}$.

A linear matrix-valued function represents the parametrization of each coefficient matrix. Taking $C(\boldsymbol{z})$ as an example, and in the most general case, the i,j-th entry of $C(\boldsymbol{z})$ is a linear function $C_{ij}(\boldsymbol{z}) = \boldsymbol{u}_{ij}^\intercal \boldsymbol{z} + C_{ij}(\boldsymbol{0})$, where $\boldsymbol{u}_{ij} \in \mathbb{R}^{n_z}$.[2] The function $\boldsymbol{g}(\boldsymbol{z}) = \boldsymbol{g} - B\boldsymbol{z}$ represents the parametrization of vector $\boldsymbol{g}$, where $B \in \mathbb{R}^{n_g \times n_z}$; we replace $\boldsymbol{g}(\boldsymbol{z})$ with $\boldsymbol{g} - B\boldsymbol{z}$ from here on.

Given an interdiction plan $\hat{\boldsymbol{z}}$, the operator's problem is

OpMIP($\hat{\boldsymbol{z}}$):

$$\min_{\boldsymbol{x}\in\mathcal{X},\boldsymbol{y}\in\mathbb{R}_+^{n_y}} \{\boldsymbol{c}^\intercal\boldsymbol{x} + \boldsymbol{d}^\intercal\boldsymbol{y} : B\hat{\boldsymbol{z}} + C(\hat{\boldsymbol{z}})\boldsymbol{x} + D(\hat{\boldsymbol{z}})\boldsymbol{y} \le \boldsymbol{g}\}. \tag{3}$$

OpMIP($\hat{\boldsymbol{z}}$) allows bilinear terms in $\boldsymbol{z}$ and $(\boldsymbol{x}, \boldsymbol{y})$. The bilinear terms account for the consequence of interdiction in a compact fashion.[3]

As the final step in developing MILSIP, we model a leader who makes resource-constrained binary interdiction decisions to maximize the system operator's minimum disutility. Let $A\boldsymbol{z} \le \boldsymbol{b}$ denote linear constraints on interdiction plans (e.g., resource limits, logical constraints), where $A \in \mathbb{R}^{n_b \times n_z}$ and $\boldsymbol{b} \in \mathbb{R}^{n_b}$. In the bilevel interdiction model, the upper-level variables $\boldsymbol{z}$ represent the leader's decisions, and the lower-level problem is **OpMIP**($\boldsymbol{z}$). We formulate the bilevel interdiction model as

MILSIP:

$$\max_{\boldsymbol{z}\in\mathcal{Z}} \min_{\boldsymbol{x}\in\mathcal{X},\boldsymbol{y}\in\mathbb{R}_+^{n_y}} \quad \boldsymbol{c}^\intercal\boldsymbol{x} + \boldsymbol{d}^\intercal\boldsymbol{y}, \tag{4}$$

$$\text{s.t.} \quad B\boldsymbol{z} + C(\boldsymbol{z})\boldsymbol{x} + D(\boldsymbol{z})\boldsymbol{y} \le \boldsymbol{g}, \tag{5}$$

where

$$\mathcal{Z} = \{\boldsymbol{z} \in \{0,1\}^{n_z} : A\boldsymbol{z} \le \boldsymbol{b}\} \tag{6}$$

Note that constraints (5) apply only to the inner minimization.

2.2. Properties of MILSIP

MILSIP can be viewed as a special case of a mixed-integer bilevel program (MIBP). Solving a general MIBP can be difficult, because of issues like the inner problem being unbounded or infeasible for certain $\boldsymbol{z} \in \mathcal{Z}$. In addition, the leader may not achieve the optimum (Moore and Bard 1990). These difficulties do not concern us, however, because of the following assumption:

Assumption 1. *For any* $\boldsymbol{z} \in \mathcal{Z}$, $\mathbf{OpMIP}(\boldsymbol{z})$ *has a finite optimal solution.*

This assumption is reasonable for our gas transmission system model considered in Section 4 as well as most practical operator's problems because (a) the operator seeks to minimize the sum of operating costs plus the sum of economic penalties for unserved demand, so a feasible solution always has a non-negative cost, and (b) even the most severe attack results in a feasible response by the system operator, because the operator always has the option of curtailing all demand.

Another notable implication of the structure of **MILSIP** is that when we fix an interdiction plan $\boldsymbol{z}$, the problem becomes a single-level MILP. This suggests that we might view $\boldsymbol{z}$ as complicating variables and try to apply Benders decomposition. However, the optimal objective function value of $\mathbf{OpMIP}(\boldsymbol{z})$ is neither convex nor concave in $\boldsymbol{z}$ (Blair and Jeroslow 1977), so Benders decomposition does not apply directly.

3. A Decomposition Algorithm

This section proposes a decomposition algorithm to solve large-scale instances of **MILSIP**. We begin by adopting a single-level reformulation from the literature. Then, we apply Benders decomposition to the reformulated problem, which decomposes **MILSIP** into a master problem and an exponential number of convex subproblems. Among the exponentially many subproblems, we identify "significant" subproblems whose feasibility implies the optimality of MILSIP. Finally, we develop an algorithm that solves in each iteration a relaxed master problem, a lower-level problem, and only one "significant" subproblem.

3.1. Single-Level Reformulation

As in standard Benders decomposition, we introduce a new variable $\eta \in \mathbb{R}$ and reformulate **MILSIP** into this equivalent form:

EF1:

$$\max_{\boldsymbol{z}\in\mathcal{Z},\eta\in\mathbb{R}} \quad \eta, \tag{7}$$

$$\text{s.t.} \quad \eta \le \min_{\boldsymbol{x}\in\mathcal{X},\boldsymbol{y}\in\mathbb{R}_+^{n_y}} \{\boldsymbol{c}^\intercal\boldsymbol{x} + \boldsymbol{d}^\intercal\boldsymbol{y} : B\boldsymbol{z} + C(\boldsymbol{z})\boldsymbol{x} + D(\boldsymbol{z})\boldsymbol{y} \le \boldsymbol{g}\}. \tag{8}$$

We then enumerate the integer vectors $\boldsymbol{x} \in \mathcal{X}$. Using the index set $\mathcal{V}$ defined for $\mathcal{X}$, **EF1** is equivalent to the following (see Alderson et al. 2011 and Zeng and An 2014):

EF2:

$$\max_{\boldsymbol{z}\in\mathcal{Z},\eta\in\mathbb{R}} \quad \eta, \tag{9}$$

$$\text{s.t.} \qquad \eta \le \boldsymbol{c}^\intercal \boldsymbol{x}_v + \min_{\boldsymbol{y}_v \in \mathbb{R}_+^{n_y}} \{\boldsymbol{d}^\intercal \boldsymbol{y}_v : B\boldsymbol{z} + C(\boldsymbol{z})\boldsymbol{x}_v + D(\boldsymbol{z})\boldsymbol{y}_v \le \boldsymbol{g}\}, \quad \forall v \in \mathcal{V}. \tag{10}$$

For a given $v \in \mathcal{V}$, the minimization problem on the right hand side of (10) becomes an LP after fixing the interdiction plan to $\hat{\boldsymbol{z}}$:

$$\min_{\boldsymbol{y}_v \in \mathbb{R}_+^{n_y}} \{\boldsymbol{d}^\intercal \boldsymbol{y}_v : B\hat{\boldsymbol{z}} + C(\hat{\boldsymbol{z}})\boldsymbol{x}_v + D(\hat{\boldsymbol{z}})\boldsymbol{y}_v \le \boldsymbol{g}\}. \tag{11}$$

Assumption 1 implies that LP (11) is never unbounded below for any $v \in \mathcal{V}$. Given a specific $\hat{\boldsymbol{z}}$, LP (11) may have a finite optimal solution or it may be infeasible. We consider these two cases.

First, if LP (11) is feasible, we can replace it by its dual:

$$\max_{\boldsymbol{\mu}_v \in \mathbb{R}_+^{n_g}} \{\boldsymbol{\mu}_v^\intercal (B\hat{\boldsymbol{z}} + C(\hat{\boldsymbol{z}})\boldsymbol{x}_v - \boldsymbol{g}) : D(\hat{\boldsymbol{z}})^\intercal \boldsymbol{\mu}_v + \boldsymbol{d} \ge \boldsymbol{0}\}, \tag{12}$$

where $\boldsymbol{\mu}_v$ denotes the dual vector associated with the constraints in (11). Applying this replacement to constraints (10) yields the following set of constraints:

$$\eta \le \boldsymbol{c}^\intercal \boldsymbol{x}_v + \max_{\boldsymbol{\mu}_v \in \mathbb{R}_+^{n_g}} \{\boldsymbol{\mu}_v^\intercal (B\boldsymbol{z} + C(\boldsymbol{z})\boldsymbol{x}_v - \boldsymbol{g}) : D(\boldsymbol{z})^\intercal \boldsymbol{\mu}_v + \boldsymbol{d} \ge \boldsymbol{0}\}, \quad \forall v \in \mathcal{V}. \tag{13}$$

Second, when LP (11) is infeasible for some $\hat{\boldsymbol{z}} \in \mathcal{Z}$ and $\hat{v} \in \mathcal{V}$, the dual LP (12) can be either unbounded or infeasible. In the unbounded case, we can still replace LP (11) by its dual. This is because we can view the right-hand-side value of constraint (10) and that of constraint (10) as $+\infty$. In this case, this constraint is redundant for interdiction plan $\hat{\boldsymbol{z}}$, because $\boldsymbol{x}_{\hat{v}}$ cannot correspond to an optimal decision for the system operator. When the dual LP (12) is infeasible, we can no longer make the replacement.

We prove that the dual LP (12) can only be unbounded under Assumption 1. Therefore, we can obtain an equivalent formulation of **EF2** by replacing constraints (10) with (13).

Theorem 1. *Under Assumption 1, if the primal LP* (11) *is infeasible for a given* $\hat{\boldsymbol{z}} \in \mathcal{Z}$ *and* $\hat{v} \in \mathcal{V}$*, then the corresponding dual LP* (12) *can only be unbounded.*

Proof. For the given $\hat{\boldsymbol{z}}$, Assumption 1 implies that there exists a $v' \in \mathcal{V}$ for which the primal LP (11) has an optimal solution. By strong duality, the dual LP (12) is also feasible with a finite maximum when for the same v'. This implies that the linear system $\{\boldsymbol{\mu} : D(\hat{\boldsymbol{z}})^\intercal \boldsymbol{\mu} + \boldsymbol{d} \geq \boldsymbol{0}\}$ is feasible, because the feasible region is independent from the choice of v. Therefore, the dual LP (12) for the given $\hat{\boldsymbol{z}}$ and $\hat{v}$ can only be unbounded above. □

Consider the following single-level problem,

EF3:

$$\max_{\boldsymbol{z}\in\mathcal{Z},\, \eta\in\mathbb{R},\, \boldsymbol{\mu}_v\in\mathbb{R}_+^{n_g}\,\forall v\in\mathcal{V}} \quad \eta, \tag{14}$$

$$\text{s.t.} \quad \eta \leq \boldsymbol{c}^\intercal \boldsymbol{x}_v + \boldsymbol{\mu}_v^\intercal (B\boldsymbol{z} + C(\boldsymbol{z})\boldsymbol{x}_v - \boldsymbol{g}), \quad \forall v \in \mathcal{V}, \tag{15}$$

$$D(\boldsymbol{z})^\intercal \boldsymbol{\mu}_v + \boldsymbol{d} \geq \boldsymbol{0}, \qquad \forall v \in \mathcal{V}. \tag{16}$$

Theorem 2. *An interdiction plan* $\boldsymbol{z}^*$ *is optimal for* **MILSIP** *if and only if there exist* $\eta^* \in \mathbb{R}$ *and* $\boldsymbol{\mu}_v^* \in \mathbb{R}_+^{n_g}$ *for all* $v \in \mathcal{V}$*, such that* $(\eta^*, \boldsymbol{z}^*, \boldsymbol{\mu}_1^*, \ldots, \boldsymbol{\mu}_{|\mathcal{V}|}^*)$ *is optimal for* **EF3**.

Proof. We first show the equivalence of **EF2** and **EF3**.

Given $v \in \mathcal{V}$, $(\eta, \boldsymbol{z})$ satisfies constraint (13) if and only if there exists $\boldsymbol{\mu}_v \in \mathbb{R}_+^{n_g}$, such that $\eta \leq \boldsymbol{c}^\intercal \boldsymbol{x}_v + \boldsymbol{\mu}_v^\intercal (B\boldsymbol{z} + C(\boldsymbol{z})\boldsymbol{x}_v - \boldsymbol{g})$ and $D(\boldsymbol{z})^\intercal \boldsymbol{\mu}_v + \boldsymbol{d} \geq \boldsymbol{0}$. Therefore, for the same v, the set of values for $(\eta, \boldsymbol{z})$ that satisfies (13) is identical to the set of values for $(\eta, \boldsymbol{z})$ that satisfies constraints (15) and (16). Following the strong duality between (11) and (12), the projection of the feasible region defined by constraints (15) and (16) onto the $(\eta, \boldsymbol{z})$-space is identical to the set of values for $(\eta, \boldsymbol{z})$ that satisfies (10).

Since **EF2** and **EF3** share the same objective function, and since they define the same feasible region for $(\eta, \boldsymbol{z})$, $(\eta^*, \boldsymbol{z}^*)$ is an optimal solution to **EF2** if and only if there exists $\boldsymbol{\mu}_v^* \in \mathbb{R}_+^{n_g}$ for all $v \in \mathcal{V}$, such that $(\eta^*, \boldsymbol{z}^*, \boldsymbol{\mu}_1^*, \ldots, \boldsymbol{\mu}_{|\mathcal{V}|}^*)$ is optimal for **EF3**. Because of the equivalence of **MILSIP** and **EF2**, the theorem follows. □

3.2. Benders Decomposition and Significant Subproblems

Both the number of decision variables and the number of constraints in **EF3** are exponential in n_x. The column-and-constraint generation (CCG) algorithm developed by Zeng and An (2014), when applied to **MILSIP**, would solve a relaxed version of **EF3** that includes only a subset of constraints (15) and (16). Both the number of variables and the number of constraints in the relaxed problem grow linearly with the iteration count.

Unlike the general-purpose approach of Zeng and An (2014), we exploit the structure of **MILSIP**, and propose a decomposition algorithm that iteratively solves three optimization problems with a fixed number of variables. Note that **EF3** lends itself to generalized Benders decomposition; the existence of bilinear terms in $\boldsymbol{z}$ and $\boldsymbol{\mu}$ requires the "generalized" approach. See Geoffrion (1972) and Floudas (1995) for a detailed description of generalized Benders decomposition.

If we fix $\hat{\eta}$ and the interdiction plan $\hat{\boldsymbol{z}}$, **EF3** separates into $|\mathcal{V}|$ feasibility subproblems, each differing in the value of $\boldsymbol{x}_v$. The feasibility problem for each $v \in \mathcal{V}$ is the following LP

$\mathbf{FP}_v(\hat{\eta}, \hat{\boldsymbol{z}})$:

$$\text{find} \quad \boldsymbol{\mu}_v \in \mathbb{R}_+^{n_g}, \tag{17}$$

$$\text{s.t.} \quad \hat{\eta} \le \boldsymbol{c}^\intercal \boldsymbol{x}_v + \boldsymbol{\mu}_v^\intercal (B\hat{\boldsymbol{z}} + C(\hat{\boldsymbol{z}})\boldsymbol{x}_v - \boldsymbol{g}), \quad : \phi_v^{\text{Eta}} \tag{18}$$

$$D(\hat{\boldsymbol{z}})^\intercal \boldsymbol{\mu}_v + \boldsymbol{d} \ge \boldsymbol{0}, \quad : \boldsymbol{\phi}_v^{\text{D}} \tag{19}$$

where dual variables $\phi_v^{\text{Eta}} \in \mathbb{R}_+$ and $\boldsymbol{\phi}_v^{\text{D}} \in \mathbb{R}_+^{n_y}$ are displayed to the right of their corresponding constraints.

A naive implementation of Benders involve solving the exponentially many subproblems. However, the exponentially many subproblems are not of equal importance. Given $\hat{\boldsymbol{z}} \in \mathcal{Z}$, by solving $\mathbf{OpMIP}(\hat{\boldsymbol{z}})$ to ϵ-optimality, we can identify a subproblem whose feasibility implies the optimality of MILSIP.

Suppose we obtain an ϵ-optimal solution to $\mathbf{OpMIP}(\hat{\boldsymbol{z}})$, where $\epsilon \ge 0$. We identify a "ϵ-significant" feasibility subproblem based on this solution.

Definition 1. Given $\hat{\boldsymbol{z}} \in \mathcal{Z}$, $\hat{\eta} \geq \eta^*$, and $\epsilon \geq 0$, $\mathbf{FP}_{v*}(\hat{\eta}, \hat{\boldsymbol{z}})$ is *ϵ-significant* if there exists $\boldsymbol{y}^* \in \mathbb{R}_+^{n_y}$ such that $(\boldsymbol{x}_{v^*}, \boldsymbol{y}^*)$ is an ϵ-optimal solution to $\mathbf{OpMIP}(\hat{\boldsymbol{z}})$.

An ϵ-significant feasibility problem has the following property:

Theorem 3. *Given $\hat{\boldsymbol{z}} \in \mathcal{Z}$, $\hat{\eta} \geq \eta^*$, and $\epsilon \geq 0$, if the ϵ-significant feasibility problem $\mathbf{FP}_{v^*}(\hat{\eta}, \hat{\boldsymbol{z}})$ is feasible, then $\hat{\boldsymbol{z}}$ is an ϵ-optimal solution to* **MILSIP**.

Proof. Because $\mathbf{FP}_{v^*}(\hat{\eta}, \hat{\boldsymbol{z}})$ is ϵ-significant, there exists $\boldsymbol{y}^* \in \mathbb{R}_+^{n_y}$ such that $(\boldsymbol{x}_{v^*}, \boldsymbol{y}^*)$ is an ϵ-optimal solution to $\mathbf{OpMIP}(\hat{\boldsymbol{z}})$. Let η^ϵ be the objective function value of $\mathbf{OpMIP}(\hat{\boldsymbol{z}})$ evaluated at $(\boldsymbol{x}_{v^*}, \boldsymbol{y}^*)$. Since $\hat{\boldsymbol{z}}$ is a feasible interdiction plan, $\eta' \leq \eta^* \leq \hat{\eta}$, where η' is the optimal objective value of $\mathbf{OpMIP}(\hat{\boldsymbol{z}})$. By ϵ-optimality of $(\boldsymbol{x}_{v^*}, \boldsymbol{y}^*)$, we have $\eta^\epsilon \leq \eta' + \epsilon$.

The optimal objective value of the primal LP (11) with $v = v^*$ is also η^ϵ. By strong duality, the optimal objective value of the dual LP (12) with $v = v^*$ is also η^ϵ. Therefore, $\mathbf{FP}_{v^*}(\hat{\eta}, \hat{\boldsymbol{z}})$ is feasible only if $\hat{\eta} \leq \eta^\epsilon$. It follows that

$$\eta' \leq \eta^* \leq \hat{\eta} \leq \eta^\epsilon \leq \eta' + \epsilon,$$

and that $\hat{\boldsymbol{z}}$ is an ϵ-optimal solution to **MILSIP**. □

Theorem 3 has an important algorithmic implication: a Benders feasibility cut generated from an ϵ-significant feasibility problem always cuts off a candidate solution $(\hat{\eta}, \hat{\boldsymbol{z}})$ if $\hat{\boldsymbol{z}}$ is not an ϵ-optimal interdiction plan. Therefore, compared to an arbitrarily chosen feasibility problem, an ϵ-significant feasibility problem can be of more help for convergence. In practice, obtaining an optimal ($\epsilon = 0$) solution might take an exorbitant amount of time because of the NP-hardness of $\mathbf{OpMIP}(\hat{\boldsymbol{z}})$. We usually terminate upon finding an ϵ-optimal solution with $\epsilon > 0$.

Instead of solving all subproblems, we propose an algorithm in which we only solve an operator's problem to identify an ϵ-significant feasibility problem, and in which we solve this one feasibility problem at each iteration and generate a Benders cut. However, the bilinear terms in $\hat{\boldsymbol{z}}$ and $\boldsymbol{\mu}_v$ in $\mathbf{FP}_v(\hat{\eta}, \hat{\boldsymbol{z}})$ make it difficult to define a Benders cut explicitly.

3.3. Benders Feasibility Cuts

Once an ϵ-significant subproblem $\mathbf{FP}_v(\hat{\eta}, \hat{\boldsymbol{z}})$ is solved, we can generate a Benders feasibility cut using dual information from that solution. Let $\boldsymbol{\phi}_v = (\phi_v^{\text{Eta}}, \boldsymbol{\phi}_v^{\text{D}}) \in \Phi_v$ denote the dual vector for $\mathbf{FP}_v(\hat{\eta}, \hat{\boldsymbol{z}})$. The Lagrangian function for $\mathbf{FP}_v(\hat{\eta}, \hat{\boldsymbol{z}})$ is

$$L_v(\hat{\eta}, \hat{\boldsymbol{z}}, \boldsymbol{\mu}_v, \boldsymbol{\phi}_v) = \phi_v^{\text{Eta}}[\hat{\eta} - \boldsymbol{c}^\intercal \boldsymbol{x}_v - \boldsymbol{\mu}_v^\intercal (B\hat{\boldsymbol{z}} + C(\hat{\boldsymbol{z}})\boldsymbol{x}_v - \boldsymbol{g})] - (\boldsymbol{\phi}_v^{\text{D}})^\intercal [D(\hat{\boldsymbol{z}})^\intercal \boldsymbol{\mu}_v + \boldsymbol{d}]. \tag{20}$$

Suppose we solve $\mathbf{FP}_v(\hat{\eta}, \hat{\boldsymbol{z}})$ and obtain an optimal dual vector $\boldsymbol{\phi}_v^*$. A Benders feasibility cut can be formulated as follows (Geoffrion 1972):

$$0 \geq \min_{\boldsymbol{\mu}_v \in \mathbb{R}_+^{n_g}} L_v(\eta, \boldsymbol{z}, \boldsymbol{\mu}_v, \boldsymbol{\phi}_v^*). \tag{21}$$

Constraint (21) has an inner optimization problem with respect to $\boldsymbol{\mu}_v$ that is parametric in $(\eta, \boldsymbol{z})$. The Lagrangian we are minimizing has bilinear terms in $\boldsymbol{z}$ and $\boldsymbol{\mu}_v$. In the presence of such bilinear terms, explicit optimal objective values cannot be determined in general (Floudas 1995).

If we linearize the bilinear terms with the big-M method, the feasible region for $\boldsymbol{\mu}_v$ will remain unchanged. Moreover, $(\eta, \boldsymbol{z})$ and $\boldsymbol{\mu}_v$ are now linearly separable. In this case, a Benders cut can be found explicitly as an inequality constraint linear in η and $\boldsymbol{z}$ (Geoffrion 1972, Floudas 1995), and the master problem is an MILP. It is vital for the tightness and computational precision of the master problem to define the smallest valid big-M values possible. The choice of such big-M values is problem-specific (see Appendix B for an example).

Let $\tilde{L}_v(\eta, \boldsymbol{z}, \boldsymbol{\mu}_v, \boldsymbol{\phi}_v)$ denote the Lagrangian of the linearized feasibility problem (**LFP**). With a slight abuse of notation, we also use $\boldsymbol{\mu}_v$ and $\boldsymbol{\phi}_v$ to denote the primal and dual variables, respectively, for the linearized problem. Suppose we solve $\mathbf{LFP}_v(\hat{\eta}, \hat{\boldsymbol{z}})$ and obtain an optimal primal vector $\boldsymbol{\mu}_v^*$ and an optimal dual vector $\boldsymbol{\phi}_v^*$. The Benders feasibility cut can be expressed as

$$0 \geq \tilde{L}_v(\eta, \boldsymbol{z}, \boldsymbol{\mu}_v^*, \boldsymbol{\phi}_v^*). \tag{22}$$

3.4. Algorithm

We propose an algorithm in which three problems are solved iteratively: (i) a relaxed master problem (**RMP**), (ii) an operator's problem, and (iii) a ϵ-significant linearized feasibility problem.

A Bilevel Benders Decomposition Algorithm for MILSIP (BLBD).

Input: An instance of **MILSIP**, absolute convergence tolerance for **OpMIP** $\epsilon \geq 0$, absolute convergence tolerance for **MILSIP** $\tilde{\epsilon} \geq \epsilon$, an upper bound on the objective function value (e.g., largest possible disutility of the system) $\overline{\overline{\eta}}$.

Output: An $\tilde{\epsilon}$-optimal interdiction plan and associated objective value.

Step 0: $\overline{\eta}^0 \leftarrow \overline{\overline{\eta}}$; $\underline{\eta}^{\text{best}} \leftarrow -\infty$; $\boldsymbol{z}^{\text{best}} \leftarrow \mathbf{0}$; $k \leftarrow 0$.

Step 1: $k \leftarrow k+1$; Solve the current **RMP**:

$$\begin{aligned} \max_{\boldsymbol{z}\in\mathcal{Z},\eta\in\mathbb{R}} \quad & \eta, \\ \text{s.t.} \quad & \eta \leq \overline{\eta}^{k-1}, \\ & \tilde{L}_{v^j}(\eta, \boldsymbol{z}, \boldsymbol{\mu}^j_{v^j}, \boldsymbol{\phi}^j_{v^j}) \leq 0, \quad j = 1, \ldots, k-1 \end{aligned}$$

for $(\overline{\eta}^k, \boldsymbol{z}^k)$.

Step 2: Solve **OpMIP**$(\boldsymbol{z}^k)$ for ϵ-optimal solution $(\boldsymbol{x}_{v^k}, \boldsymbol{y}^k)$ with objective value $\underline{\eta}^k$;

If $\underline{\eta}^k > \underline{\eta}^{\text{best}}$, then $\underline{\eta}^{\text{best}} \leftarrow \underline{\eta}^k$ and $\boldsymbol{z}^{\text{best}} \leftarrow \boldsymbol{z}^k$;

If $\overline{\eta}^k - \underline{\eta}^{\text{best}} \leq \tilde{\epsilon}$, then go to Step 4.

Step 3: Solve $\textbf{LFP}_{v^k}(\overline{\eta}^k, \boldsymbol{z}^k)$ for $(\boldsymbol{\mu}^k_{v^k}, \boldsymbol{\phi}^k_{v^k})$;

Add a Benders feasibility cut $\tilde{L}_{v^k}(\eta, \boldsymbol{z}, \boldsymbol{\mu}^k_{v^k}, \boldsymbol{\phi}^k_{v^k}) \leq 0$ to **RMP**; Go to Step 1.

Step 4: Print "The solution is" $\boldsymbol{z}^{\text{best}}$, "with objective value" $\underline{\eta}^{\text{best}}$;

Print "Provable optimality gap is" $\overline{\eta}^k - \underline{\eta}^{\text{best}}$;

Stop.

Logical constraints can be added optionally to **RMP**. A type of logical constraint, a solution-elimination constraint (SEC),

$$\sum_{i:z_i^k=1} z_i - \sum_{i:z_i^k=0} z_i - \left(\sum_i z_i^k - 1\right) \leq 0 \tag{23}$$

excludes a single interdiction plan $\boldsymbol{z}^k$ from the leader's feasibility set to prevent repeating suboptimal interdiction plans (Brown et al. 2009, Alderson et al. 2011). Since $\tilde{\epsilon} \geq \epsilon$, Theorem 3 implies that a Benders cut in BLBD always cuts off the current solution to **RMP**, but this Benders cut does not necessarily dominate an SEC.

BLBD converges in a finite number of iterations when SECs are present, since the set $\mathcal{Z}_I$ is finite. Without SECs, given the convexity of the subproblems, Assumption 1, and the finiteness of set $\mathcal{X}$, Theorem 2.5 in Geoffrion (1972) implies finite convergence for any $\tilde{\epsilon} > 0$.

4. Application to a Natural Gas Transmission System Interdiction Problem

To demonstrate the capabilities of BLBD, this section formulates and solves an interdiction model for a natural gas transmission system.

4.1. Operator's Model

We consider an operator of a natural gas transmission system who wishes to satisfy demand at a set of locations, each with a guaranteed pressure. The components of the system are natural gas sources, pipelines, and compressor stations. The performance of the system is measured by total supply cost plus economic losses resulting from unmet demand.

We model the gas transmission system as a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{L} \cup \mathcal{C})$ where nodes $\mathcal{N}$ represent junctions, supply points and demand points, arcs $\mathcal{L}$ represent pipelines, and arcs $\mathcal{C}$ represent compressor stations. For any arc $\ell \in \mathcal{L} \cup \mathcal{C}$, let $\mathrm{o}(\ell)$ and $\mathrm{d}(\ell)$ denote that arc's origin node and destination node, respectively.

Our formulation is based on the steady-state gas transmission system model described by De Wolf and Smeers (2000), with modifications to incorporate the effects of interdiction.

Indices and sets:

$n \in \mathcal{N}$ nodes of the gas transmission system;

$l \in \mathcal{L}$ natural gas pipelines;

$c \in \mathcal{C}$ compressor stations;

$m \in \mathcal{M}$ gas sources;

Parameters:

h_m cost of gas supply at source m [\$/10^6 scm (standard cubic meter)];

q_n value of unserved demand at node n [\$/10^6 scm];

γ_l pipeline constant;

$\overline{r}_c$ maximum compression ratio for station c;

A^{Pipe} entry A_{nl}^{Pipe} equals 1 if $\text{o}(l) = n$, -1 if $\text{d}(l) = n$, and 0 otherwise;

A^{Comp} entry A_{nl}^{Comp} equals 1 if $\text{o}(l) = n$, -1 if $\text{d}(l) = n$, and 0 otherwise;

K entry K_{nm} equals 1 if gas source m is located at node n, and 0 otherwise;

$\overline{s}_m$ maximum gas supply at source m [10^6 scm/day];

$\overline{p}_n$ maximum pressure at node n [bar];

$\underline{p}_n$ minimum pressure at node n [bar];

$\overline{d}_n$ gas demand at node n [10^6 scm/day].

Decision variables:

f_l^{Pipe} gas flow through pipeline l [10^6 scm/day];

f_c^{Comp} gas flow through compressor station c [10^6 scm/day];

d_n served demand at node n [10^6 scm/day];

s_m gas output from source m [10^6 scm/day];

p_n pressure at node n [bar].

Formulation:

$$\min_{\boldsymbol{f}^{\textbf{Pipe}}, \boldsymbol{f}^{\textbf{Comp}}, \boldsymbol{d}, \boldsymbol{s}, \boldsymbol{p}} \boldsymbol{h}^\intercal \boldsymbol{s} + \boldsymbol{q}^\intercal (\overline{\boldsymbol{d}} - \boldsymbol{d}) \tag{24}$$

$$\text{s.t.} \quad |f_l^{\text{Pipe}}| f_l^{\text{Pipe}} = \gamma_l (p_{\text{o}(l)}^2 - p_{\text{d}(l)}^2), \qquad \forall\, l \in \mathcal{L}, \tag{25}$$

$$\overline{r}_c p_{\text{o}(c)} \geq p_{\text{d}(c)} \geq p_{\text{o}(c)}, \qquad \forall c \in \mathcal{C}, \tag{26}$$

$$\begin{bmatrix} A^{\text{Pipe}} & A^{\text{Comp}} \end{bmatrix} \begin{bmatrix} \boldsymbol{f}^{\text{Pipe}} \\ \boldsymbol{f}^{\text{Comp}} \end{bmatrix} = K\boldsymbol{s} - \boldsymbol{d}, \tag{27}$$

$$\boldsymbol{d} \leq \overline{\boldsymbol{d}}, \tag{28}$$

$$\underline{\boldsymbol{p}} \leq \boldsymbol{p} \leq \overline{\boldsymbol{p}}, \tag{29}$$

$$\boldsymbol{s} \leq \overline{\boldsymbol{s}}, \tag{30}$$

$$\boldsymbol{f}^{\text{Pipe}} \in \mathbb{R}^{|\mathcal{L}|}, \boldsymbol{f}^{\text{Comp}} \in \mathbb{R}^{|\mathcal{C}|}, \boldsymbol{d} \in \mathbb{R}_+^{|\mathcal{N}|}, \tag{31}$$

$$\boldsymbol{s} \in \mathbb{R}_+^{|\mathcal{M}|}, \boldsymbol{p} \in \mathbb{R}^{|\mathcal{N}|}. \tag{32}$$

The objective function (24) sums supply costs and economic losses resulting from unserved demand over a day. For simplicity, this formulation ignores the gas consumed by gas-powered compressor stations. Typically, this consumption amounts to less than 2% of the total gas supply entering the system (Martin et al. 2005), and we will see that this is negligible in the context of interdiction. Constraints (25) characterize flows in natural gas pipelines. Constant γ_l is defined through a function of the length, the diameter and the roughness of the pipeline, along with the composition of the gas being transported; see Osiadacz (1987, pp. 69–79) for more details. Constraints (26) limit the pressure increase that each compressor station can provide. Constraints (27) represent conservation of flow. Constraints (28) ensure that no served load exceeds what is demanded. Constraints (29) limit pressure at each node. Constraints (30) limit output from each gas source.

Brown et al. (2006) present a bilevel model to analyze the interdiction of a crude-oil pipeline network, but their (pure) network-flow operator's model cannot account for the range of gas pressures guaranteed to power plants, industrial users, and local distribution companies. Indeed, without flow-pressure relationship (25) and allied constructs, a network-flow model does not completely describe the physics of a gas pipeline network and, as we shall see in computational tests, a suboptimal interdiction plan may result. We can also more accurately model attacks on compressor stations, which Steinhäusler et al. (2008) identify as security risks.

4.2. Piecewise Linearization

Constraints (25) make the operator's problem nonconvex. To tackle this nonconvexity, we construct a piecewise linear approximation for each nonlinear flow-pressure relationship. The resulting operator's problem is an MILP.

We first define $\pi_n = p_n^2$ for all $n \in \mathcal{N}$. Accordingly, $\overline{\pi}_n = \overline{p}_n^2$ and $\underline{\pi}_n = \underline{p}_n^2$. We also define parameter $\tau_l = \max(\overline{\pi}_{\mathrm{o}(l)} - \underline{\pi}_{\mathrm{d}(l)}, \overline{\pi}_{\mathrm{d}(l)} - \underline{\pi}_{\mathrm{o}(l)})$, which denotes the maximum possible deviation of squared pressure across pipeline l. Then, we discretize the functions $|f_l^{\mathrm{Pipe}}|f_l^{\mathrm{Pipe}}$ using breakpoints $(\hat{f}_{li}, \hat{f}_{li}|\hat{f}_{li}|)$, indexed by $i \in \mathcal{I}_l$. Finally, we cast each piecewise linear approximation into the form of a SOS2 constraint (a constraint that incorporate special ordered sets of Type 2; see Beale and Forrest 1976 for details). Denoting the SOS2 variable at breakpoint i for pipeline l by α_{li}, the following linear constraints approximate constraints (25):

$$\mathbf{1}^\intercal \boldsymbol{\alpha}_l = 1, \qquad \forall\, l \in \mathcal{L}, \tag{33}$$

$$\mathrm{SOS2}\{\alpha_{l,1}, \ldots, \alpha_{l,|\mathcal{I}_l|}\}, \qquad \forall\, l \in \mathcal{L}, \tag{34}$$

$$f_l^{\mathrm{Pipe}} = \boldsymbol{\alpha}_l^\intercal \hat{\boldsymbol{f}}_l, \qquad \forall\, l \in \mathcal{L}, \tag{35}$$

$$\boldsymbol{\alpha}_l^\intercal (\hat{\boldsymbol{f}}_l \circ |\hat{\boldsymbol{f}}_l|) = \gamma_l (\pi_{\mathrm{o}(l)} - \pi_{\mathrm{d}(l)}), \quad \forall\, l \in \mathcal{L}, \tag{36}$$

where (a) the operator $\circ$ denotes the entrywise product, and where (b) SOS2 constraints (34) allow at most two of the variables α_{li} to be nonzero for a given l and, if two are nonzero, those nonzeros must be consecutive in the ordering $i = 1, \ldots, |\mathcal{I}_l|$.

4.3. Interdiction Model

Our model concerns itself with interdiction on gas sources, pipelines, and compressor stations, and thus $\boldsymbol{z} = (\boldsymbol{z}^{\mathrm{Pipe}}, \boldsymbol{z}^{\mathrm{Src}}, \boldsymbol{z}^{\mathrm{Comp}})$. We assume that the leader is subject to a cardinality constraint $\sum_{l\in\mathcal{L}} z_l^{\mathrm{Pipe}} + \sum_{c\in\mathcal{C}} z_c^{\mathrm{Comp}} + \sum_{m\in\mathcal{M}} z_m^{\mathrm{Src}} \le \Delta$ for some $\Delta \in \mathbb{Z}_+$. For modeling purposes, we assume that no supply is available from an interdicted gas source, and no flow is possible on an interdicted pipeline. The modeling assumptions regarding compressor stations are more complicated, however.

In most gas transmission systems, a compressor station is supplemented with a bypass pipeline (Hiller et al. 2016). Depending on whether the bypass pipeline is destroyed, two types of compressor-station interdictions might occur. A *minor* interdiction would destroy only the compressors, but not the bypass pipeline. In this case, we would have $\pi_{\mathrm{o}(c)} = \pi_{\mathrm{d}(c)}$. On the other hand, a *major* interdiction would destroy both the compressor station and the bypass pipeline, completely breaking the connection between o(c) and d(c). For simplicity, we model only a minor interdiction of each compressor station because (a) our real-world test problem (see Fig. 1 below) only has two compressor stations, (b) a major interdiction of compressor station (21, 9) can be accomplished, in effect and with the same resource consumption, by interdicting two adjacent pipelines, and (c) a rational attacker would never interdict compressor station (22, 18), because interdicting one of its adjacent pipelines achieves the effect of a major interdiction and only requires one unit of resource.

The formulation of **MILSIP** for a gas transmission system is shown below:

$$\max_{\boldsymbol{z}\in\mathcal{Z}} \min_{\boldsymbol{f}^{\mathbf{Pipe}},\, \boldsymbol{f}^{\mathbf{Comp}},\, \boldsymbol{d},\, \boldsymbol{s},\, \boldsymbol{\pi},\, \boldsymbol{\alpha}} \quad \boldsymbol{h}^{\intercal}\boldsymbol{s} + \boldsymbol{q}^{\intercal}(\overline{\boldsymbol{d}} - \boldsymbol{d}) \tag{37}$$

$$\text{s.t.} \quad (27), (28), (33), (34),$$

$$f_l^{\mathrm{Pipe}} = \boldsymbol{\alpha}_l^{\intercal}\hat{\boldsymbol{f}}_l(1 - z_l^{\mathrm{Pipe}}), \quad \forall\, l \in \mathcal{L}, \tag{38}$$

$$\boldsymbol{\alpha}_l^{\intercal}(\hat{\boldsymbol{f}}_l \circ |\hat{\boldsymbol{f}}_l|) \geq \gamma_l(\pi_{\mathrm{o}(l)} - \pi_{\mathrm{d}(l)}) - \gamma_l\tau_l z_l^{\mathrm{Pipe}}, \quad \forall\, l \in \mathcal{L}, \tag{39}$$

$$\boldsymbol{\alpha}_l^{\intercal}(\hat{\boldsymbol{f}}_l \circ |\hat{\boldsymbol{f}}_l|) \leq \gamma_l(\pi_{\mathrm{o}(l)} - \pi_{\mathrm{d}(l)}) + \gamma_l\tau_l z_l^{\mathrm{Pipe}}, \quad \forall\, l \in \mathcal{L}, \tag{40}$$

$$\pi_{\mathrm{o}(c)} \leq \pi_{\mathrm{d}(c)}, \quad \forall\, c \in \mathcal{C}, \tag{41}$$

$$\pi_{\mathrm{d}(c)} \leq \pi_{\mathrm{o}(c)} + (\overline{r}_c^2 - 1)\pi_{\mathrm{o}(c)}(1 - z_c^{\mathrm{Comp}}), \quad \forall\, c \in \mathcal{C}, \tag{42}$$

$$\underline{\boldsymbol{\pi}} \leq \boldsymbol{\pi} \leq \overline{\boldsymbol{\pi}}, \tag{43}$$

$$\boldsymbol{s} \leq \overline{\boldsymbol{s}} \circ (\mathbf{1} - \boldsymbol{z}^{\mathrm{Src}}), \tag{44}$$

$$\boldsymbol{z} \in \{0,1\}^{|\mathcal{L}|+|\mathcal{C}|+|\mathcal{M}|}, \boldsymbol{f}^{\mathrm{Pipe}} \in \mathbb{R}^{|\mathcal{L}|}, \boldsymbol{f}^{\mathrm{Comp}} \in \mathbb{R}^{|\mathcal{C}|}, \boldsymbol{d} \in \mathbb{R}_+^{|\mathcal{N}|}, \tag{45}$$

$$\boldsymbol{s} \in \mathbb{R}_+^{|\mathcal{M}|}, \boldsymbol{\pi} \in \mathbb{R}^{|\mathcal{N}|}, \boldsymbol{\alpha}_l \in \mathbb{R}_+^{|\mathcal{I}_l|}, \forall\, l \in \mathcal{L}, \tag{46}$$

where

$$\mathcal{Z} = \{\boldsymbol{z} \in \{0,1\}^{|\mathcal{L}|+|\mathcal{C}|+|\mathcal{M}|} : \mathbf{1}^{\intercal}\boldsymbol{z} \leq \Delta\}. \tag{47}$$

We illustrate the new constraints. Constraints (38) build on constraints (35) and force flow to zero on interdicted pipelines. Constraints (39) and (40) build on constraints (36) and remove the flow-pressure requirement for any interdicted pipeline. Constraints (41) and (42) build on constraints (26). They force $\pi_{\mathrm{o}(c)} = \pi_{\mathrm{d}(c)}$ when a minor compressor-station interdiction takes place. After the transformation $\pi_n = p_n^2$, constraints (29) become constraints (43). Constraints (44) build on constraints (30) and disable supply from interdicted sources.

Our interdiction model evaluates the steady-state worst-case disutility of the system after the survival of initial transients. We assume that (a) demand can be curtailed in a continuous fashion, and that (b) the operator's problem remains feasible in any configuration created by interdiction solely by reducing supply in a continuous fashion at appropriate demand nodes. Extensive testing has shown no violations of the second assumption. An artifice of our model is that a connected part of the system with no gas supply and thus no gas flow can exhibit arbitrary pressures within the limits of constraints (43). This artifice is caused by a lack of pressure reference, and does not affect the interdictor's preference since the disutility function does not involve pressure.

4.4. Results

We consider the Belgian gas network described by De Wolf and Smeers (2000), modeled in a steady state. The gas transmission system comprises 22 nodes, 24 pipelines, 6 sources (a storage facility is modeled as a natural gas source with specified maximum daily output), and 2 compressor stations, as shown in Figure 1. Since no relevant data can be found for estimating the value of unserved gas demand in the Belgian network, we set the value of unserved demand at $1/scm, a number roughly 11 times higher than the highest supply price.

In each piecewise linearization of a gas flow equation, we set the number of breakpoints at 31 for each pipeline. The breakpoints are distributed uniformly between the negative and positive flow limits. (See Martin et al. 2005 for a discussion on choosing the number of breakpoints.) As a result, the upper level of the interdiction model has 32 binary variables and the lower level has 720 binary variables.

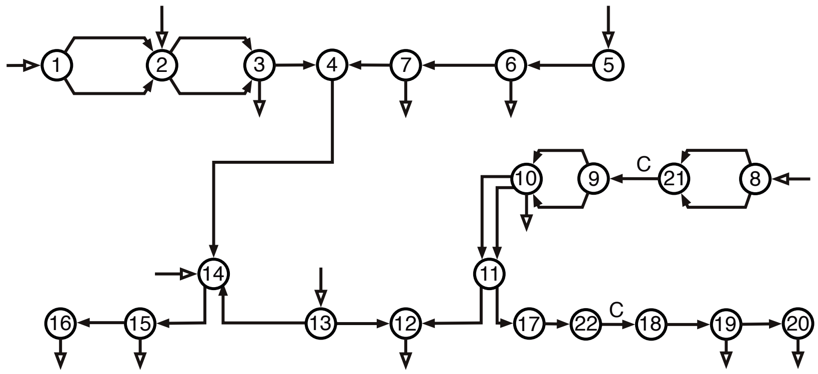

Figure 1 **A Belgian gas transmission system. This figure is based on data and a schematic diagram in De Wolf and Smeers (2000). An arrow connecting two nodes denotes a pipeline or a compressor station; the annotation "C" then identifies each compressor station. A hollow-headed inward-pointing arrow denotes a source and hollow-headed outward-pointing arrow indicates a demand. Note also that each directed pipeline and compressor arc indicates the direction of positive flow, but flow in the opposite direction is allowed.**

We note that as Δ increases, the number of feasible interdiction plans grows exponentially. It is impractical to solve this problem by total enumeration. Therefore, we solve this interdiction model by BLBD. An appendix describes reformulation and decomposition of the interdiction model, Benders feasibility cuts, and the specification of big-M coefficients.

We carry out tests on a personal computer with a 2.4 GHz dual-core CPU and 8 GB of RAM. We implement the algorithms in MATLAB, and solve the optimization problems with GUROBI 6.0 (Gurobi Optimization Inc. 2015). We set $\epsilon = \tilde{\epsilon} = 10$ for detailed testing, but later do provide summary results for a coarser convergence tolerance. We optionally add SECs (23) to BLBD. (The simple form of the interdiction-resource constraint (47) enables the use of potentially stronger SECs as in Brown et al. (2009). We find no computational benefit in using the potentially stronger SECs, however, and therefore use SECs (23) or no SECs at all.) For comparison, we solve the same model with the CCG method (we use the strong-duality-based reformulation, see Zeng and An 2014), with the same choice of big-M coefficients. We set an absolute convergence tolerance of 10 for both the operator's problem and the relaxed master problem in CCG. Table 1 presents computational results.

Table 1 Results with varying cardinality limits Δ on interdiction

Δ	Disutility ($×10^7/day)	Optimal Interdiction Plan		Number of Iterations			CPU Time (sec.)		
		Pipelines	Sources	BLBD		CCG	BLBD		CCG
				w/o SEC	w/ SEC		w/o SEC	w/ SEC	
1	2.392	(14,15)	-	3	3	2	1.75	1.44	0.99
2	3.628	(3,4)	8	7	7	6	3.02	2.77	2.87
3	4.068	(3,4)	5, 8	23	23	12	5.85	6.22	6.75
4	4.427	(3,4)	1, 2, 8	27	26	13	6.33	5.94	6.81
5	4.539	(5,6)	1, 2, 8, 13	53	51	24	9.98	9.41	17.55
6	4.630	(5,6)	1, 2, 8, 13, 14	46	42	18	9.33	9.18	8.12

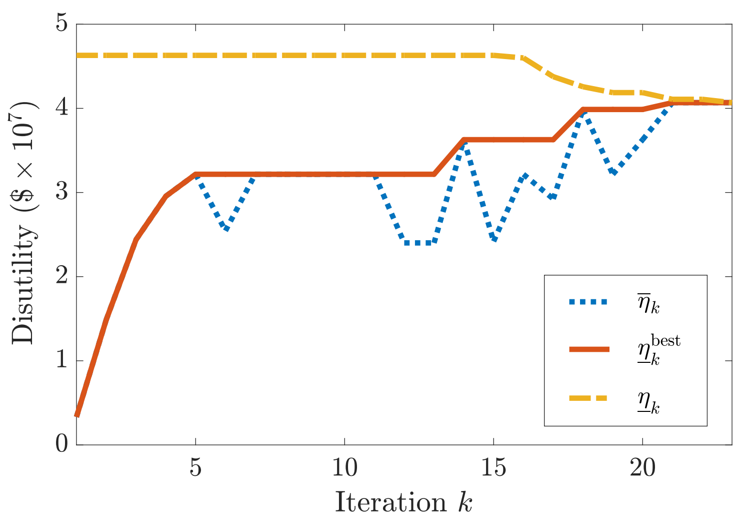

Figure 2 **Upper and lower bounds on optimal objective function value versus iteration using BLBD with $\Delta = 3$, $\epsilon = \tilde{\epsilon} = 10$, and no SECs.**

We illustrate a few of the optimal interdiction plans. When $\Delta = 1$, pipeline (14,15) is optimally interdicted, because significant demand at nodes 15 and 16 is cut off from supply. As Δ increases, the optimal plan interdicts more sources. All demand is interrupted for $\Delta \geq 6$. An alternative optimal solution for $\Delta = 6$, not shown in Table 1, simply interdicts all six sources. We also note that under a resource constraint on cardinality, it is disadvantageous for the leader to interdict a compressor station in this particular system. In fact, when both compressor stations lose their compressors to interdiction and are bypassed, and given that no other components are interdicted, the operator's model shows a demand curtailment of 6.09×10^4 scm/day, only 0.13% of total demand.

MILSIP solves to near optimality quickly under both BLBD and CCG. Figure 2 shows the convergence of upper and lower bounds using BLBD when $\Delta = 3$. Table 1 shows that, compared to BLBD, CCG converges in fewer iterations in all cases. However, when the number of iterations is large, BLBD requires less CPU time to converge, as can be seen from the cases with $\Delta = 5$ and $\Delta = 6$.

To further demonstrate the computational performance of BLBD, we consider a restricted interdiction model that allows only attacks on pipelines.

Table 2 Results with varying cardinality limits Δ on pipeline-only interdiction

Δ	Disutility	Optimal Interdiction Plan	Number of Iterations		CPU Time (sec.)	
	(\$×10⁷/day)	Pipelines	BLBD	CCG	BLBD	CCG
1	2.392	(14,15)	2	2	1.7	1.1
2	2.956	(3,4), (9,10)	19	15	7.5	7.2
3	3.627	(3,4), (9,10), (9,10)	44	31	33.1	114.1
4	4.068	(3,4), (5,6), (8,21), (8,21)	77	38	52.5	244.1
5	4.427	(2,3), (2,3), (5,6), (9,10), (9,10)	81	45	77.4	571.9
6	4.427	(2,3), (2,3), (5,6), (9,10), (9,10), (14,15)	144	62	131.6	1551.4
7	4.539	(2,3), (2,3), (5,6), (9,10), (9,10), (12,13), (13,14)	191	87	201.8	4482.2
8	4.630	(2,3), (2,3), (5,6), (9,10), (9,10), (4,14), (12,13), (15,14)	121	39	102.8	551.3

Table 2 shows an optimal interdiction plan for each value of Δ from one up to eight, at which point all demand goes unmet; we illustrate a few of these cases. When $\Delta = 2$, it is optimal to attack both pipeline $(3,4)$ and one of the parallel pipelines $(9,10)$, specifically, the one with the larger diameter. We note that for this value of Δ, an interdiction model that employs a (pure) network-flow model for the gas pipeline system interdicts pipelines $(3,4)$ and $(11,12)$, which is suboptimal by 4.5%. (If gas pressure constraints are ignored, the interdictor would appear to gain nothing by interdicting only one of the parallel pipelines $(9,10)$, as seen in the optimal solution.) When $\Delta \geq 3$, multiple optimal solutions exist because interdicting both pipelines between nodes 9 and 10 has the same effect as interdicting both pipelines between nodes 8 and 21.

We note that the restricted model is computationally more difficult for both CCG and BLBD. SECs are omitted because they provide no computational benefit for this problem. Although BLBD requires more iterations to converge, BLBD is an order of magnitude faster than CCG when the total iteration count becomes large.

Both the number of variables and the number of constraints in CCG's relaxed master problem grows with the iteration count, and this makes its solution a bottleneck for CCG. For example, when $\Delta = 7$, CCG's relaxed master problem has 31 348 constraints and 17 926 variables in its last iteration, and that problem requires 381.4 seconds to solve. By contrast, only the number of constraints increases by iteration in BLBD's **RMP**. **RMP** has 193 constraints and 33 variables in its final iteration and this master problem requires only 0.59 seconds to solve. When compared with CCG, the computational time per iteration for BLBD grows more slowly with iteration count.

The detailed results above use tight absolute convergence tolerances ($\epsilon = \tilde{\epsilon} = 10$) in order to demonstrate BLBD's capabilities. In practice, coarser convergence tolerances might be appropriate for developing interdiction plans, so we describe aggregate results obtained using $\epsilon = \tilde{\epsilon} = 10^4$; for simplicity, we do not consider SECs. The results summarize as follows: (a) all interdiction-plan solutions are identical to those shown in Tables 1 and 2; (b) solution times for the test cases from Table 1 do not change much for either methods; and (c) BLBD's average solution time for the test cases from Table 2 reduces from 76.0 seconds to 30.5 seconds, and from 940.4 seconds to 237.6 seconds for CCG. BLBD maintains a speed advantage over CCG, sometimes a substantial advantage, even when using coarser optimality tolerances.

5. Conclusions

This paper has investigated the mixed-integer linear system interdiction problem (MILSIP), a system-interdiction problem in which the operator's problem is represented by a mixed-integer linear program. We develop and demonstrate the BLBD (Bilevel Benders Decomposition) algorithm that solves large-scale instances of MILSIP with a pre-specified optimality tolerance. At each iteration, BLBD solves three optimization problems, each with a fixed number of decision variables: a relaxed master problem that generates an interdiction plan, a post-interdiction operator's problem, and a subproblem whose feasibility implies optimality of MILSIP. Numerical tests show that BLBD solves instances of an interdiction problem on a real-world natural gas transmission system an order of magnitude faster than an alternative algorithm from the literature.

The proposed algorithmic framework can be further improved. Adding multiple Benders feasibility cuts during one iteration may be beneficial. Also, special-purpose cuts and fast heuristics that utilize the structure of the system (for example, the ones proposed in Salmeron and Wood 2015) might be integrated to solve the master problem and the operator's problem more efficiently. We are also exploring the application of the proposed algorithm to a wider range of problems, including the defender-attacker-defender model of Alderson et al. (2011).

Appendix A: An Illustrative Example of BLBD

Consider the following linear MIBP:

Example 1.

$$\begin{aligned}\max_{\boldsymbol{z}\in\{0,1\}^2}\ \min_{x\in\{0,1\},y\in\mathbb{R}_+}\quad & -x-y,\\ \text{s.t.}\quad & 4x+y-2z_1\le 4,\\ & -1.6x+y+z_2\le 1.2.\end{aligned} \tag{48}$$

All the constraints in this example are linear. (Section 4 considers an interdiction problem having bilinear terms in both upper- and lower-level variables.) The set of possible values for lower-level integer variables is $\mathcal{X}=\{\hat{x}_1,\hat{x}_2\}$, where $\hat{x}_1=0$ and $\hat{x}_2=1$. **EF3** for this example is

$$\begin{aligned}\max_{\eta\in\mathbb{R},\boldsymbol{z}\in\{0,1\}^2,\boldsymbol{\mu}_1,\boldsymbol{\mu}_2\in\mathbb{R}^2_+}\quad & \eta,\\ \text{s.t.}\quad & \eta\le(4\mu_{11}-1.6\mu_{12}-1)\hat{x}_1-2\mu_{11}z_1-4\mu_{11}+\mu_{12}z_2-1.2\mu_{12},\\ & 1-\mu_{11}-\mu_{12}\le 0,\\ & \eta\le(4\mu_{21}-1.6\mu_{22}-1)\hat{x}_2-2\mu_{21}z_1-4\mu_{21}+\mu_{22}z_2-1.2\mu_{22},\\ & 1-\mu_{21}-\mu_{22}\le 0.\end{aligned} \tag{49}$$

EF3 decomposes into $|\mathcal{X}|=2$ feasibility problems: for each $v\in\{1,2\}$,

$$\begin{aligned}\text{find}\quad & \boldsymbol{\mu}_v\in\mathbb{R}^2_+,\\ \text{s.t.}\quad & \eta\le(4\mu_{v1}-1.6\mu_{v2}-1)\hat{x}_v-2\mu_{v1}z_1-4\mu_{v1}+\mu_{v2}z_2-1.2\mu_{v2},\\ & 1-\mu_{v1}-\mu_{v2}\le 0.\end{aligned} \tag{50}$$

To linearize the bilinear terms in $\boldsymbol{z}$ and $\boldsymbol{\mu}_v$, we define $\xi_{v1}=\mu_{v1}z_1$ and $\xi_{v2}=\mu_{v2}z_2$. Adding a single slack variable ρ_v^{Eta} for the first constraint ensures feasibility; we minimize this slack variable. Using a sufficiently large $M>0$, $\mathbf{LFP}_v(\hat{\eta},\hat{\boldsymbol{z}})$ is

$$
\begin{aligned}
\min \quad & \rho_v^{\text{Eta}}, \\
\text{over} \quad & \boldsymbol{\mu}_v \in \mathbb{R}_+^2, \boldsymbol{\xi}_v \in \mathbb{R}_+^2, \rho_v^{\text{Eta}} \in \mathbb{R}_+, \\
\text{s.t.} \quad & \hat{\eta} \le (4\mu_{v1} - 1.6\mu_{v2} - 1)\hat{x}_v - 2\xi_{v1} - 4\mu_{v1} + \xi_{v2} - 1.2\mu_{v2} + \rho_v^{\text{Eta}}, \quad : \phi_v^{\text{Eta}} \\
& 1 - \mu_{v1} - \mu_{v2} \le 0, \\
& \xi_{vi} \le M\hat{z}_i, \quad i = 1,2, \qquad : \phi_{vi}^{\text{D1}} \\
& \xi_{vi} \ge \mu_{vi} - M(1-\hat{z}_i), \quad i = 1,2, \qquad : \phi_{vi}^{\text{D2}} \\
& \xi_{vi} \le \mu_{vi}, \quad i = 1,2,
\end{aligned}
\tag{51}
$$

for $v \in \{1,2\}$. The constant M bounds both μ_{v1} and μ_{v2} from above for $v \in \{1,2\}$.

Solving (51) yields the optimal objective value $\tilde{L}_v^*$ and optimal dual variables $\boldsymbol{\phi}^* = (\phi_v^{\text{Eta},*}, \boldsymbol{\phi}_v^{\text{D1},*}, \boldsymbol{\phi}_v^{\text{D2},*})$. The resulting Benders feasibility cut is

$$
0 \ge \tilde{L}_v^* + \phi_v^{\text{Eta},*}(\eta - \hat{\eta}) + M\sum_{i=1}^{2}\left[(\phi_{vi}^{\text{D2},*} - \phi_{vi}^{\text{D1},*})(z_i - \hat{z}_i)\right]. \tag{52}
$$

We apply BLBD, without using SECs, to Example 1, letting $\epsilon = 0$, $\overline{\overline{\eta}} = 0$, and $M = 100$.

The constant M is chosen conservatively here for simplicity. Admittedly, big-M constants weaken the LP relaxation of **RMP**, since they appear in the coefficients of Benders feasibility cuts. However, upper bounds on decision variables appear in most algorithms for interdiction problems. Also, in many real-world interdiction problems, good upper bounds on $\boldsymbol{\mu}_v$ can be determined. (See Section 4 for an example, and note that for Example 1, we can determine by inspection that $M = 1$ is sufficiently large.)

At the first iteration, we solve **RMP**

$$
\begin{aligned}
\max_{z \in \{0,1\}^2, \eta \in \mathbb{R}} \quad & \eta, \\
& \eta \le 0.
\end{aligned}
$$

The maximizer $\boldsymbol{z}$ can be any value in $\{0,1\}^2$, so, along with $\overline{\eta}^1 = 0$, suppose that the solver returns $\boldsymbol{z}^1 = (1,1)$. Solving $\mathbf{OpMIP}((1,1))$ yields $x^1 = 1$ and $\underline{\eta}^1 = \underline{\eta}^{\text{best}} = -2.8$. We then solve the significant feasibility problem $\mathbf{FP}_1(0,(1,1))$, which generates this Benders cut:

$$
0 \ge 2.8 + \eta + 180(z_1 - 1).
$$

Since $\overline{\eta}^1 - \underline{\eta}^{\text{best}} = 2.8$, we set $k = 2$, and solve the augmented **RMP**

$$\begin{aligned}
\max_{\boldsymbol{z}\in\{0,1\}^2,\eta\in\mathbb{R}} \quad & \eta, \\
\text{s.t.} \quad & 0 \geq 2.8 + \eta + 180(z_1 - 1), \\
& \eta \leq 0,
\end{aligned}$$

which again has multiple optimal solutions. Along with $\overline{\eta}^2 = 0$, suppose that the solver returns $\boldsymbol{z}^2 = (0, 1)$. We solve **OpMIP**$((0,1))$ to obtain $x^2 = 1$ and $\underline{\eta}^2 = \underline{\eta}^{\text{best}} = -1$, and then solve $\mathbf{FP}_1(0,(0,1))$ to generate the following Benders cut:

$$0 \geq 1 + \eta.$$

Since $\overline{\eta}^2 - \underline{\eta}^{\text{best}} = 1$, we set $k = 3$, and solve **RMP**

$$\begin{aligned}
\max_{\boldsymbol{z}\in\{0,1\}^2,\eta\in\mathbb{R}} \quad & \eta, \\
\text{s.t.} \quad & 0 \geq 2.8 + \eta + 180(z_1 - 1), \\
& 0 \geq 1 + \eta, \\
& \eta \geq 0,
\end{aligned}$$

which yields $\overline{\eta}^3 = -1$. We have $\overline{\eta}^3 = \underline{\eta}^{\text{best}} = -1$, so $\boldsymbol{z}^{\text{best}} = (0, 1)$ is an optimal solution.

Appendix B: Reformulation and Decomposition Procedures for A Gas Transmission System

This appendix describes detailed reformulation and decomposition procedures for the gas transmission system interdiction problem (37)–(46). We first formulate the lower-level operator's problem **OpMIP**$(\hat{\boldsymbol{z}})$. In creating the single-level reformulation for the bilevel interdiction model, one key step is to derive the dual of LP (11), the operator's problem with all its integer variables fixed. We then formulate this dual LP and the feasibility problems resulting from the decomposition. Finally, we specify a Benders feasibility cut as a linear inequality. See Section 4 for notation used here.

B.1. The Operator's Problem

For a given interdiction plan $\hat{\boldsymbol{z}}$, **OpMIP**$(\hat{\boldsymbol{z}})$ is:

$$\min_{\boldsymbol{f}^{\mathbf{Pipe}},\boldsymbol{f}^{\mathbf{Comp}},\boldsymbol{d},\boldsymbol{s},\boldsymbol{\pi},\boldsymbol{\alpha}} \quad \boldsymbol{h}^{\intercal}\boldsymbol{s} + \boldsymbol{q}^{\intercal}(\overline{\boldsymbol{d}} - \boldsymbol{d}) \tag{53}$$

$$\text{s.t.} \quad \mathbf{1}^{\intercal}\boldsymbol{\alpha}_l = 1, \qquad \forall\, l \in \mathcal{L} \qquad : \lambda_l^{\text{P1}} \tag{54}$$

$$\text{SOS2}\{\alpha_{l,1}, \ldots, \alpha_{l,|\mathcal{I}_l|}\}, \qquad \forall\, l \in \mathcal{L} \tag{55}$$

$$f_l^{\text{Pipe}} = \boldsymbol{\alpha}_l^\intercal \hat{\boldsymbol{f}}_l (1 - \hat{z}_l^{\text{Pipe}}), \quad \forall l \in \mathcal{L} \quad : \lambda_l^{\text{P2}} \tag{56}$$

$$\boldsymbol{\alpha}_l^\intercal (\hat{\boldsymbol{f}}_l \circ |\hat{\boldsymbol{f}}_l|) \geq \gamma_l (\pi_{\text{o}(l)} - \pi_{\text{d}(l)}) - \gamma_l \tau_l z_l^{\text{Pipe}}, \quad \forall l \in \mathcal{L} \quad : \mu_l^{\text{P3}} \tag{57}$$

$$\boldsymbol{\alpha}_l^\intercal (\hat{\boldsymbol{f}}_l \circ |\hat{\boldsymbol{f}}_l|) \leq \gamma_l (\pi_{\text{o}(l)} - \pi_{\text{d}(l)}) + \gamma_l \tau_l z_l^{\text{Pipe}}, \quad \forall l \in \mathcal{L} \quad : \mu_l^{\text{P4}} \tag{58}$$

$$\pi_{\text{o}(c)} \leq \pi_{\text{d}(c)}, \quad \forall c \in \mathcal{C} \quad : \mu_c^{\text{C1}} \tag{59}$$

$$\pi_{\text{d}(c)} \leq \pi_{\text{o}(c)} + (\overline{r}_c^2 - 1) \pi_{\text{o}(c)} (1 - \hat{z}_c^{\text{Comp}}), \quad \forall c \in \mathcal{C} \quad : \mu_c^{\text{C2}} \tag{60}$$

$$\begin{bmatrix} A^{\text{Pipe}} & A^{\text{Comp}} \end{bmatrix} \begin{bmatrix} f^{\text{Pipe}} \\ f^{\text{Comp}} \end{bmatrix} = K\boldsymbol{s} - \boldsymbol{d}, \quad : \boldsymbol{\lambda}^{\text{F}} \tag{61}$$

$$\boldsymbol{d} \leq \overline{\boldsymbol{d}}, \quad : \boldsymbol{\mu}^{\text{D}} \tag{62}$$

$$\underline{\boldsymbol{\pi}} \leq \boldsymbol{\pi} \leq \overline{\boldsymbol{\pi}}, \quad : \boldsymbol{\mu}^{\text{L1}}, \boldsymbol{\mu}^{\text{L2}} \tag{63}$$

$$\boldsymbol{s} \leq \overline{\boldsymbol{s}} \circ (\mathbf{1} - \hat{\boldsymbol{z}}^{\text{Src}}), \quad : \boldsymbol{\mu}^{\text{S}} \tag{64}$$

$$\boldsymbol{f}^{\text{Pipe}} \in \mathbb{R}^{|\mathcal{L}|}, \boldsymbol{f}^{\text{Comp}} \in \mathbb{R}^{|\mathcal{C}|}, \boldsymbol{d} \in \mathbb{R}_+^{|\mathcal{N}|}, \tag{65}$$

$$\boldsymbol{s} \in \mathbb{R}_+^{|\mathcal{M}|}, \boldsymbol{\pi} \in \mathbb{R}^{|\mathcal{N}|}, \boldsymbol{\alpha}_l \in \mathbb{R}_+^{|\mathcal{I}_l|}, \forall l \in \mathcal{L}. \tag{66}$$

The dual variables are introduced on the right side of their associated constraints.

B.2. Dual Problem of the Operator's Restricted LP

We obtain the operator's restricted LP by fixing the lower-level integer variables in **OpMIP**$(\hat{\boldsymbol{z}})$. For the interdiction model considered in Section 4, the lower-level integer variables are the binary variables implicit in the SOS2 constraints (34). These constraints require that, for each pipeline l, at most two α_{li} can be nonzero, and that if two are nonzero they must be consecutive in the ordering $i = 1, \ldots, |\mathcal{I}_l|$. The implicit binary variables determine which variables α_{li} are nonzero.

Recall that in the piecewise linearization, the breakpoint i for pipeline l is denoted by $(\hat{f}_{li}, \hat{f}_{li}|\hat{f}_{li}|)$. Without loss of generality, a feasible f_l^{Pipe} in **OpMIP**$(\hat{\boldsymbol{z}})$ is a convex combination of the $\hat{f}_{li}$ values at two "active" breakpoints. The implicit binary variables choose which two consecutive breakpoints are active in the piecewise linearization. Therefore, once we fix the implicit binary variables, we fix the active breakpoints.

We refer to LP of the form of (11) as the Operator's Restricted LP. When deriving its dual, for ease of exposition, we collect the $\hat{f}_{li}$ values of the first active breakpoint for each pipeline into a vector $\hat{\boldsymbol{f}}_{\text{I}}$, and collect the $\hat{f}_{li}$ values of the second active breakpoint for each pipeline into vector $\hat{\boldsymbol{f}}_{\text{II}}$. Let $\text{n}(m)$ denote the node where source m is located, and let matrix $\overline{A}^{\text{Comp}}$ be identical to A^{Comp}, except that every -1 entry in A^{Comp}

is replaced by 0. With implicit binary variables fixed, the operator's restricted LP has the form of (11). The dual variables for this LP are defined in (54)–(66). Then, the dual of the Operator's Restricted LP, which has the form of (12), is:

$$\max \quad -\sum_{l\in\mathcal{L}}\lambda_l^{\mathrm{P1}} - \sum_m \left[\mu_m^{\mathrm{S}}\overline{s}_m(1-\hat{z}_m^{\mathrm{Src}})\right] - \sum_l(\mu_l^{\mathrm{P3}}+\mu_l^{\mathrm{P4}})\gamma_l\tau_l z_l^{\mathrm{Pipe}} \tag{67}$$

$$+\sum_n(-\mu_n^{\mathrm{D}}\overline{d}_n + \mu_n^{\mathrm{L1}}\underline{\pi}_n - \mu_n^{\mathrm{L2}}\overline{\pi}_n) + \boldsymbol{q}^\intercal\overline{\boldsymbol{d}}, \tag{68}$$

$$\text{over} \quad \boldsymbol{\lambda}^{\mathrm{P1}}, \boldsymbol{\lambda}^{\mathrm{P2}} \in \mathbb{R}^{|\mathcal{L}|}, \boldsymbol{\mu}^{\mathrm{P3}}, \boldsymbol{\mu}^{\mathrm{P4}} \in \mathbb{R}_+^{|\mathcal{L}|}, \boldsymbol{\mu}^{\mathrm{C1}}, \boldsymbol{\mu}^{\mathrm{C2}} \in \mathbb{R}_+^{|\mathcal{C}|}, \tag{69}$$

$$\boldsymbol{\lambda}^{\mathrm{F}} \in \mathbb{R}^{|\mathcal{N}|}, \boldsymbol{\mu}^{\mathrm{D}}, \boldsymbol{\mu}^{\mathrm{L1}}, \boldsymbol{\mu}^{\mathrm{L2}} \in \mathbb{R}_+^{|\mathcal{N}|}, \boldsymbol{\mu}^{\mathrm{S}} \in \mathbb{R}_+^{|\mathcal{M}|} \tag{70}$$

$$\text{s.t.} \quad \boldsymbol{\lambda}^{\mathrm{P1}} + \boldsymbol{\lambda}^{\mathrm{P2}} \circ (\hat{\boldsymbol{z}}^{\mathrm{Pipe}} - \mathbf{1}) \circ \hat{\boldsymbol{f}}_{\mathrm{I}} + (\boldsymbol{\mu}^{\mathrm{P4}} - \boldsymbol{\mu}^{\mathrm{P3}}) \circ \hat{\boldsymbol{f}}_{\mathrm{I}} \circ |\hat{\boldsymbol{f}}_{\mathrm{I}}| \geq \mathbf{0}, \tag{71}$$

$$\boldsymbol{\lambda}^{\mathrm{P1}} + \boldsymbol{\lambda}^{\mathrm{P2}} \circ (\hat{\boldsymbol{z}}^{\mathrm{Pipe}} - \mathbf{1}) \circ \hat{\boldsymbol{f}}_{\mathrm{II}} + (\boldsymbol{\mu}^{\mathrm{P4}} - \boldsymbol{\mu}^{\mathrm{P3}}) \circ \hat{\boldsymbol{f}}_{\mathrm{II}} \circ |\hat{\boldsymbol{f}}_{\mathrm{II}}| \geq \mathbf{0}, \tag{72}$$

$$\boldsymbol{\mu}^{\mathrm{L2}} - \boldsymbol{\mu}^{\mathrm{L1}} + A^{\mathrm{Pipe}}((\boldsymbol{\mu}^{\mathrm{P3}} - \boldsymbol{\mu}^{\mathrm{P4}}) \circ \boldsymbol{\gamma}) - A^{\mathrm{Comp}}(\boldsymbol{\mu}^{\mathrm{C2}} - \boldsymbol{\mu}^{\mathrm{C1}}) \tag{73}$$

$$-\overline{A}^{\mathrm{Comp}}\left[\boldsymbol{\mu}^{\mathrm{C2}} \circ (\overline{\boldsymbol{r}} \circ \overline{\boldsymbol{r}} - \mathbf{1}) \circ (\mathbf{1} - \hat{\boldsymbol{z}}^{\mathrm{Comp}})\right] = \mathbf{0}, \tag{74}$$

$$(A^{\mathrm{Pipe}})^\intercal \boldsymbol{\lambda}^{\mathrm{F}} + \boldsymbol{\lambda}^{\mathrm{P2}} = \mathbf{0}, \tag{75}$$

$$(A^{\mathrm{Comp}})^\intercal \boldsymbol{\lambda}^{\mathrm{F}} = \mathbf{0}, \tag{76}$$

$$-\lambda_{\mathrm{n}(m)}^{\mathrm{F}} + \mu_m^{\mathrm{S}} + h_m \geq 0, \qquad \forall m \in \mathcal{M} \tag{77}$$

$$-\boldsymbol{q} + \boldsymbol{\lambda}^{\mathrm{F}} + \boldsymbol{\mu}^{D} \geq \mathbf{0}. \tag{78}$$

B.3. Linearized Feasibility Problem

At each iteration, we solve an ϵ-significant **LFP** (see Section 3.2) to generate a Benders feasibility cut of the form of (22). Recall that we linearize bilinear terms in interdiction variables and dual variables. To this end, we define

$$\boldsymbol{\xi}^{\mathrm{P2}} \in \mathbb{R}^{|\mathcal{L}|}, \quad \xi_l^{\mathrm{Pipe}} = \lambda_l^{\mathrm{P2}} z_l^{\mathrm{Pipe}}, \forall l \in \mathcal{L}, \tag{79}$$

and similarly

$$\boldsymbol{\xi}^{\mathrm{P3}} \in \mathbb{R}_+^{|\mathcal{L}|}, \quad \xi_l^{\mathrm{P3}} = \mu_l^{\mathrm{P3}} z_l^{\mathrm{Pipe}}, \forall l \in \mathcal{L}, \tag{80}$$

$$\boldsymbol{\xi}^{\mathrm{P4}} \in \mathbb{R}_+^{|\mathcal{L}|}, \quad \xi_l^{\mathrm{P4}} = \mu_l^{\mathrm{P4}} z_l^{\mathrm{Pipe}}, \forall l \in \mathcal{L}, \tag{81}$$

$$\boldsymbol{\xi}^{\mathrm{Comp}} \in \mathbb{R}_+^{|\mathcal{C}|}, \quad \xi_c^{\mathrm{Comp}} = \mu_c^{\mathrm{C2}} z_c^{\mathrm{Comp}}, \forall c \in \mathcal{C}, \tag{82}$$

$$\boldsymbol{\xi}^{\mathrm{Src}} \in \mathbb{R}_+^{|\mathcal{M}|}, \quad \xi_m^{\mathrm{Src}} = \mu_m^{\mathrm{S}} z_m^{\mathrm{Src}}, \forall m \in \mathcal{M}. \tag{83}$$

In addition, we define variable ρ^{Eta}, which is the only slack variable needed for the feasibility problem.

The largest value of unserved demand, $\max\{q_1,\ldots,q_{|\mathcal{N}|}\}$, bounds the components of $\boldsymbol{\mu}^{\text{S}}$ from above. The big-M coefficients M^{Src} can take this value. We also use the largest value of unserved demand as a heuristic upper bound M^{Pipe} for components of $\boldsymbol{\lambda}^{\text{P2}}$, $\boldsymbol{\mu}^{\text{P3}}$, and $\boldsymbol{\mu}^{\text{P4}}$. We solve an operator's problem with all compressor stations shut down, and use the optimal dual variables associated with the nodal minimal pressure constraint as heuristic upper bounds for $\boldsymbol{\mu}^{\text{C2}}$.

Given $(\hat{\eta},\hat{\boldsymbol{z}})$ and active breakpoints $\hat{\boldsymbol{f}}_{\text{I}}$ and $\hat{\boldsymbol{f}}_{\text{II}}$, **LFP** may be expressed as follows:

$$\min \quad \rho^{\text{Eta}}, \tag{84}$$

$$\text{over} \quad \boldsymbol{\lambda}^{\text{P1}},\boldsymbol{\lambda}^{\text{P2}}\in\mathbb{R}^{|\mathcal{L}|},\boldsymbol{\mu}^{\text{P3}},\boldsymbol{\mu}^{\text{P4}}\in\mathbb{R}_+^{|\mathcal{L}|},\boldsymbol{\mu}^{\text{C1}},\boldsymbol{\mu}^{\text{C2}}\in\mathbb{R}_+^{|\mathcal{C}|}, \tag{85}$$

$$\boldsymbol{\lambda}^{\text{F}}\in\mathbb{R}^{|\mathcal{N}|},\boldsymbol{\mu}^{\text{D}},\boldsymbol{\mu}^{\text{L1}},\boldsymbol{\mu}^{\text{L2}}\in\mathbb{R}_+^{|\mathcal{N}|},\boldsymbol{\mu}^{\text{S}}\in\mathbb{R}_+^{|\mathcal{M}|} \tag{86}$$

$$\boldsymbol{\xi}^{\text{P2}},\boldsymbol{\xi}^{\text{P3}},\boldsymbol{\xi}^{\text{P4}}\in\mathbb{R}^{|\mathcal{L}|},\boldsymbol{\xi}^{\text{Comp}}\in\mathbb{R}_+^{|\mathcal{C}|},\boldsymbol{\xi}^{\text{Src}}\in\mathbb{R}_+^{|\mathcal{M}|},\rho^{\text{Eta}}\in\mathbb{R}_+, \tag{87}$$

$$\text{s.t.} \quad \hat{\eta}\le-\sum_l\lambda_l^{\text{P1}}+\sum_m\overline{s}_m(\xi_m^{\text{Src}}-\mu_m^{\text{S}})-\sum_l(\xi_l^{\text{P3}}+\xi_l^{\text{P4}})\gamma_l\tau_l \tag{88}$$

$$+\sum_n(-\mu_n^{\text{D}}\overline{d}_n+\mu_n^{\text{L1}}\underline{\pi}_n-\mu_n^{\text{L2}}\overline{\pi}_n)+\boldsymbol{q}^{\intercal}\overline{\boldsymbol{d}}+\rho^{\text{Eta}}, \qquad :\boldsymbol{\phi}^{\text{Eta}} \tag{89}$$

$$\boldsymbol{\lambda}^{\text{P1}}+(\boldsymbol{\xi}^{\text{P2}}-\boldsymbol{\lambda}^{\text{P2}})\circ\hat{\boldsymbol{f}}_{\text{I}}+(\boldsymbol{\mu}^{\text{P4}}-\boldsymbol{\mu}^{\text{P3}})\circ\hat{\boldsymbol{f}}_{\text{I}}\circ|\hat{\boldsymbol{f}}_{\text{I}}|\ge\boldsymbol{0}, \tag{90}$$

$$\boldsymbol{\lambda}^{\text{P1}}+(\boldsymbol{\xi}^{\text{P2}}-\boldsymbol{\lambda}^{\text{P2}})\circ\hat{\boldsymbol{f}}_{\text{II}}+(\boldsymbol{\mu}^{\text{P4}}-\boldsymbol{\mu}^{\text{P3}})\circ\hat{\boldsymbol{f}}_{\text{II}}\circ|\hat{\boldsymbol{f}}_{\text{II}}|\ge\boldsymbol{0}, \tag{91}$$

$$\boldsymbol{\mu}^{\text{L2}}-\boldsymbol{\mu}^{\text{L1}}-A^{\text{Pipe}}((\boldsymbol{\mu}^{\text{P4}}-\boldsymbol{\mu}^{\text{P3}})\circ\boldsymbol{\gamma})-A^{\text{Comp}}(\boldsymbol{\mu}^{\text{C2}}-\boldsymbol{\mu}^{\text{C1}}) \tag{92}$$

$$-\overline{A}^{\text{Comp}}\left[(\overline{\boldsymbol{r}}\circ\overline{\boldsymbol{r}}-\boldsymbol{1})\circ(\boldsymbol{\mu}^{\text{C2}}-\boldsymbol{\xi}^{\text{Comp}})\right]=\boldsymbol{0}, \tag{93}$$

$$(A^{\text{Pipe}})^{\intercal}\boldsymbol{\lambda}^{\text{F}}+\boldsymbol{\lambda}^{\text{P2}}=\boldsymbol{0}, \tag{94}$$

$$(A^{\text{Comp}})^{\intercal}\boldsymbol{\lambda}^{\text{F}}=\boldsymbol{0}, \tag{95}$$

$$-\lambda_{\text{n}(m)}^{\text{F}}+\mu_m^{\text{S}}+h_m\ge0, \qquad \forall m\in\mathcal{M} \tag{96}$$

$$-\boldsymbol{q}+\boldsymbol{\lambda}^{\text{F}}+\boldsymbol{\mu}^{D}\ge\boldsymbol{0}, \tag{97}$$

$$-M^{\text{Pipe}}\hat{\boldsymbol{z}}^{\text{Pipe}}\le\boldsymbol{\xi}^{\text{P2}}, \qquad :\boldsymbol{\phi}_1^{\text{P2}} \tag{98}$$

$$M^{\text{Pipe}}\hat{\boldsymbol{z}}^{\text{Pipe}}\ge\boldsymbol{\xi}^{\text{P2}}, \qquad :\boldsymbol{\phi}_2^{\text{P2}} \tag{99}$$

$$\boldsymbol{\xi}^{\text{P2}}\le\boldsymbol{\lambda}^{\text{P2}}+M^{\text{Pipe}}(\boldsymbol{1}-\hat{\boldsymbol{z}}^{\text{Pipe}}), \qquad :\boldsymbol{\phi}_3^{\text{P2}} \tag{100}$$

$$\boldsymbol{\xi}^{\text{P2}}\ge\boldsymbol{\lambda}^{\text{P2}}-M^{\text{Pipe}}(\boldsymbol{1}-\hat{\boldsymbol{z}}^{\text{Pipe}}), \qquad :\boldsymbol{\phi}_4^{\text{P2}} \tag{101}$$

$$\boldsymbol{\xi}^{\text{P3}} \le M^{\text{Pipe}} \hat{\boldsymbol{z}}^{\text{Pipe}}, \qquad : \boldsymbol{\phi}_1^{\text{P3}} \tag{102}$$

$$\boldsymbol{\xi}^{\text{P3}} \le \boldsymbol{\mu}^{\text{P3}}, \tag{103}$$

$$\boldsymbol{\xi}^{\text{P3}} \ge \boldsymbol{\mu}^{\text{P3}} - M^{\text{Pipe}}(\mathbf{1} - \hat{\boldsymbol{z}}^{\text{Pipe}}), \qquad : \boldsymbol{\phi}_2^{\text{P3}} \tag{104}$$

$$\boldsymbol{\xi}^{\text{P4}} \le M^{\text{Pipe}} \hat{\boldsymbol{z}}^{\text{Pipe}}, \qquad : \boldsymbol{\phi}_1^{\text{P3}} \tag{105}$$

$$\boldsymbol{\xi}^{\text{P4}} \le \boldsymbol{\mu}^{\text{P4}}, \tag{106}$$

$$\boldsymbol{\xi}^{\text{P4}} \ge \boldsymbol{\mu}^{\text{P4}} - M^{\text{Pipe}}(\mathbf{1} - \hat{\boldsymbol{z}}^{\text{Pipe}}), \qquad : \boldsymbol{\phi}_2^{\text{P4}} \tag{107}$$

$$\boldsymbol{\xi}^{\text{Comp}} \le M^{\text{Comp}} \hat{\boldsymbol{z}}^{\text{Comp}}, \qquad : \boldsymbol{\phi}_1^{\text{Comp}} \tag{108}$$

$$\boldsymbol{\xi}^{\text{Comp}} \le \boldsymbol{\mu}^{\text{C2}}, \tag{109}$$

$$\boldsymbol{\xi}^{\text{Comp}} \ge \boldsymbol{\mu}^{\text{C2}} - M^{\text{Comp}}(\mathbf{1} - \hat{\boldsymbol{z}}^{\text{Comp}}), \qquad : \boldsymbol{\phi}_2^{\text{Comp}} \tag{110}$$

$$\boldsymbol{\xi}^{\text{Src}} \le M^{\text{Src}} \hat{\boldsymbol{z}}^{\text{Src}}, \qquad : \boldsymbol{\phi}_1^{\text{Src}} \tag{111}$$

$$\boldsymbol{\xi}^{\text{Src}} \le \boldsymbol{\mu}^{\text{S}}, \tag{112}$$

$$\boldsymbol{\xi}^{\text{Src}} \ge \boldsymbol{\mu}^{\text{S}} - M^{\text{Src}}(\mathbf{1} - \hat{\boldsymbol{z}}^{\text{Src}}). \qquad : \boldsymbol{\phi}_2^{\text{Src}} \tag{113}$$

B.4. Benders Feasibility Cut

A Benders feasibility cut generated from the solution of **LFP** has the following form:

$$\begin{aligned}
0 \ge\ & \tilde{L}^*_{\text{FP}} + \phi^{\text{Eta}}(\eta - \hat{\eta}) \\
& + M^{\text{Pipe}} \langle (-\boldsymbol{\phi}_1^{\text{P2},*} - \boldsymbol{\phi}_2^{\text{P2},*} + \boldsymbol{\phi}_3^{\text{P2},*} + \boldsymbol{\phi}_4^{\text{P2},*} - \boldsymbol{\phi}_1^{\text{P3},*} + \boldsymbol{\phi}_2^{\text{P3},*} - \boldsymbol{\phi}_1^{\text{P4},*} + \boldsymbol{\phi}_2^{\text{P4},*}), (\boldsymbol{z}^{\text{Pipe}} - \hat{\boldsymbol{z}}^{\text{Pipe}}) \rangle \\
& + M^{\text{Comp}} \langle (-\boldsymbol{\phi}_1^{\text{Comp},*} + \boldsymbol{\phi}_2^{\text{Comp},*}), (\boldsymbol{z}^{\text{Comp}} - \hat{\boldsymbol{z}}^{\text{Comp}}) \rangle \\
& + M^{\text{Src}} \langle (-\boldsymbol{\phi}_1^{\text{Src},*} + \boldsymbol{\phi}_2^{\text{Src},*}), (\boldsymbol{z}^{\text{Src}} - \hat{\boldsymbol{z}}^{\text{Src}}) \rangle,
\end{aligned} \tag{114}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors, $\tilde{L}^*_{\text{FP}}$ is the minimum of **LFP**, and the variables $\boldsymbol{\phi}^*$ are the optimal dual variables for **LFP**.

Endnotes

1. Indeed, Salmeron et al. (2009) present a case where sensible assumptions lead to an invalid penalty vector.

2. Typically, interdicting a component of the system changes only a few entries of C and D, and the vectors $\boldsymbol{u}_{ij}$ are sparse.

3. For example, suppose y_1 represents flow on component 1, an arc. Further, suppose that y_1 is determined by a linear constraint $y_1 = \tilde{c}x_1 + \tilde{d}y_2$ and that y_1 must be zero if the arc is interdicted. A natural way of modeling this effect of interdiction is a constraint with bilinear terms: $y_1 = (\tilde{c}x_1 + \tilde{d}y_2)(1 - z_1)$. Section 4.3 provides another example that uses such bilinear terms. Wood (1993) and Israeli (1999) model the consequence of interdiction using constraints that involve upper bounds on the lower-level variables. Using that paradigm, and following the previous example, the constraint $y_1 \le \overline{y}_1(1 - z_1)$ would be introduced, where constant $\overline{y}_1$ is an upper bound on y_1. In the case where bounding y_1 from above is unnecessary, our bilinear formulation is more compact.

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