LOCAL MINIMIZERS OF SEMI-ALGEBRAIC FUNCTIONS

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Abstract. Consider a semi-algebraic function $f\colon \mathbb{R}^n \to \mathbb{R}$, which is continuous around a point $\bar{x} \in \mathbb{R}^n$. Using the so–called *tangency variety* of f at $\bar{x}$, we first provide necessary and sufficient conditions for $\bar{x}$ to be a local minimizer of f, and then in the case where $\bar{x}$ is an isolated local minimizer of f, we define a "tangency exponent" $\alpha_* > 0$ so that for any $\alpha \in \mathbb{R}$ the following four conditions are always equivalent:

(i) the inequality $\alpha \geq \alpha_*$ holds;
(ii) the point $\bar{x}$ is an α-order sharp local minimizer of f;
(iii) the limiting subdifferential ∂f of f is $(\alpha - 1)$-order strongly metrically subregular at $\bar{x}$ for 0; and
(iv) the function f satisfies the Łojaseiwcz gradient inequality at $\bar{x}$ with the exponent $1 - \frac{1}{\alpha}$.

Besides, we also present a counterexample to a conjecture posed by Drusvyatskiy and Ioffe in [Math. Program. Ser. A, 153(2):635–653, 2015].

1. Introduction

Optimality conditions form the foundations of mathematical programming both theoretically and computationally (see, for example, [6, 12, 13, 27, 29, 34]).

To motivate the discussion, consider a function $f\colon \mathbb{R}^n \to \mathbb{R}$, which is continuous around a point $\bar{x} \in \mathbb{R}^n$. It is well known that if $\bar{x}$ is a local minimizer of f then 0 belongs to the limiting subdifferential $\partial f(\bar{x})$ of f at $\bar{x}$ (see the next section for definitions and notations). The converse is known to be true for convex functions, but it is false in the general case. On the other hand, when f is a polynomial function, Barone-Netto defined in [5] a finite family of smooth one-variable functions that can be used to test if $\bar{x}$ is a local minimizer of f. Inspired by this result, under the assumption that f is a semi-algebraic function, we construct a finite sequence of real numbers, say $\{a_1, \ldots, a_p\}$, so that the following statements hold:

- the point $\bar{x}$ is a local minimizer of f if, and only if, $a_k \geq 0$ for all $k = 1, \ldots, p$;
- the point $\bar{x}$ is an isolated local minimizer of f if, and only if, $a_k > 0$ for all $k = 1, \ldots, p$.

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It is essential to mention that there is no gap between these necessary and sufficient conditions. Furthermore, the sequence $\{a_1, \ldots, a_p\}$ does not invoke any second-order subdifferential of f. In fact, as we can see in Sections 3 and 4, this sequence constructed based on the so-called *tangency variety* of f at $\bar{x}$ which is defined purely in subdifferential terms. Moreover, in the case where $\bar{x}$ is an isolated local minimizer of f, we determine a "tangency exponent" $\alpha_* > 0$ such that for all $\alpha \in \mathbb{R}$ the following two statements are equivalent:

- the inequality $\alpha \geq \alpha_*$ is valid;
- the point $\bar{x}$ is an *α-order sharp local minimizer* of f.

The latter means that there exist constants $c > 0$ and $\epsilon > 0$ such that

$$f(x) \quad \geq \quad f(\bar{x}) + c\|x - \bar{x}\|^{\alpha} \quad \text{ for all } \quad x \in \mathbb{B}_{\epsilon}(\bar{x}).$$

It is well-known that second-order growth conditions (i.e., the case of $\alpha = 2$) play an important role in nonlinear optimization, both for convergence analysis of algorithms and for perturbation theory (see, for example, [12, 31, 34]). Under the assumptions that f is convex and $\bar{x}$ is a (necessarily isolated) local minimizer of f, Aragón Artacho and Geoffroy [2] first proved that $\bar{x}$ is a 2-order sharp local minimizer of f if and only if the limiting subdifferential ∂f is *strongly metrically subregular* at $\bar{x}$ for 0 in the sense that there exist constants $c > 0$ and $\epsilon > 0$ such that

$$\mathfrak{m}_f(x) \quad \geq \quad c\,\|x - \bar{x}\| \quad \text{ for all } \quad x \in \mathbb{B}_{\epsilon}(\bar{x}), \tag{1}$$

where $\mathfrak{m}_f(x)$ denotes the minimal norm of subgradients $v \in \partial f(x)$. Afterward, relaxing the convexity of f to the assumption that f is semi-algebraic, Drusvyatskiy and Ioffe [15] proved that the corresponding equivalence still holds. Furthermore, the authors conjecture that a second-order growth condition at a not necessarily isolated minimizer entails (not necessarily strong) subregularity of the limiting subdifferential. We provide a counterexample to this conjecture, see Example 4.2.

Replacing $\|x - \bar{x}\|$ in (1) by $\|x - \bar{x}\|^{\beta}$ with some constant $\beta \in \mathbb{R}$, one can consider the following *β-order strong metric subregularity* of ∂f at $\bar{x}$ for 0: there exist constants $c > 0$ and $\epsilon > 0$ such that

$$\mathfrak{m}_f(x) \quad \geq \quad c\,\|x - \bar{x}\|^{\beta} \quad \text{ for all } \quad x \in \mathbb{B}_{\epsilon}(\bar{x}) \setminus \{\bar{x}\}.$$

(Note that we exclude $\bar{x}$ here because β may be negative; for example, the limiting subdifferential of the continuous function $\mathbb{R} \to \mathbb{R}, x \mapsto \sqrt{|x|}$, is strongly metrically subregular of order $\beta = -\frac{1}{2}$ at $\bar{x} = 0$ for 0). Metric regularity and (strong) metric subregularity are becoming an important and active area of research in variational analysis and optimization theory. For more details, we refer the reader to the books [14, 25, 29] and the survey [23, 24] with references therein. Recently, under the assumptions that f is convex, $\bar{x}$ is a local minimizer of f, and that $\alpha > 1$, Zheng and Ng [36], and independently, Mordukhovich and Ouyang [30] showed that $\bar{x}$ is an α-order sharp local minimizer of f if and only if the limiting subdifferential ∂f is $(\alpha - 1)$-order strong metric subregularity at $\bar{x}$ for 0.

In other lines of development, Bolte, Daniilidis, and Lewis [9] showed that if f is subanalytic and $\bar{x}$ is a critical point of f (i.e., $\mathfrak{m}_f(\bar{x}) = 0$), then f satisfies the *Łojaseiwcz gradient inequality* at $\bar{x}$ with an exponent $\theta \in [0, 1)$, which means that there exist constants $c > 0$ and $\epsilon > 0$ such that

$$\mathfrak{m}_f(x) \quad \geq \quad c\,|f(x) - f(\bar{x})|^\theta \qquad \text{for all} \qquad x \in \mathbb{B}_\epsilon(\bar{x}) \setminus \{\bar{x}\}.$$

It is worth emphasizing that the convergence behavior of many first-order methods can be understood using the Łojasiewicz gradient inequality and its associated exponent; see, for example, [1, 4, 11, 17, 28].

Motivated by the aforementioned works, we show that if f is semi-algebraic and $\bar{x}$ is an isolated local minimizer of f, then for any $\alpha \geq \alpha_*$, the following statements are equivalent:

- the point $\bar{x}$ is an α-order sharp minimizer of f.
- the limiting subdifferential ∂f is $(\alpha - 1)$-order strongly metrically subregular at $\bar{x}$ for 0.
- the function f satisfies the Łojaseiwcz gradient inequality at $\bar{x}$ with the exponent $1 - \frac{1}{\alpha}$.

Note that, for a special value of α, these three equivalences were proved by Gwoździewicz [19] (with f being an analytic function) and by the author [32] (with f being a continuous subanalytic function).

To be concrete, we study only semi-algebraic functions. Analogous results, with essentially identical proofs, also hold for functions definable in a polynomially bounded o-minimal structure (see [35] for more on the subject). However, to lighten the exposition, we do not pursue this extension here.

The rest of this paper is organized as follows. Section 2 contains some preliminaries from variational analysis and semi-algebraic geometry widely used in the proofs of the main results given below. The tangency variety, which plays an important role in this study, is presented in Section 3. The main results are given in Section 4. Finally, several examples are provided in Section 5.

2. Preliminaries

Throughout this work we shall consider the Euclidean vector space $\mathbb{R}^n$ endowed with its canonical scalar product $\langle \cdot, \cdot \rangle$ and we shall denote its associated norm $\| \cdot \|$. The closed ball (resp., the sphere) centered at $\bar{x} \in \mathbb{R}^n$ of radius ϵ will be denoted by $\mathbb{B}_\epsilon(\bar{x})$ (resp., $\mathbb{S}_\epsilon(\bar{x})$). When $\bar{x}$ is the origin of $\mathbb{R}^n$ we write $\mathbb{B}_\epsilon$ instead of $\mathbb{B}_\epsilon(\bar{x})$.

For a function $f\colon \mathbb{R}^n \to \mathbb{R}$, we define the *epigraph* of f to be

$$\text{epi}f \quad := \quad \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y \geq f(x)\}.$$

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be *lower semi-continuous* (or lsc for short) at x if the inequality $\liminf_{x'\to x} f(x') \geq f(x)$ holds.

2.1. **Normals and subdifferentials.** Here we recall the notions of the normal cones to sets and the subdifferentials of real-valued functions used in this paper. The reader is referred to [29, 33] for more details.

Definition 2.1. Consider a set $\Omega \subset \mathbb{R}^n$ and a point $x \in \Omega$.

(i) The *regular normal cone* (known also as the *prenormal* or *Fréchet normal cone*) $\widehat{N}(x;\Omega)$ to Ω at x consists of all vectors $v \in \mathbb{R}^n$ satisfying

$$\langle v, x'-x\rangle \quad \leq \quad o(\|x'-x\|) \quad \text{as} \quad x' \to x \quad \text{with} \quad x' \in \Omega.$$

(ii) The *limiting normal cone* (known also as the *basic* or *Mordukhovich normal cone*) $N(x;\Omega)$ to Ω at x consists of all vectors $v \in \mathbb{R}^n$ such that there are sequences $x^k \to x$ with $x^k \in \Omega$ and $v^k \to v$ with $v^k \in \widehat{N}(x^k;\Omega)$.

If Ω is a manifold of class C^1, then for every point $x \in \Omega$, the normal cones $\widehat{N}(x;\Omega)$ and $N(x;\Omega)$ are equal to the normal space to Ω at x in the sense of differential geometry; see [33, Example 6.8].

Definition 2.2. Consider a function $f\colon \mathbb{R}^n \to \mathbb{R}$ and a point $x \in \mathbb{R}^n$.

(i) The *limiting* and *horizon subdifferentials* of f at x are defined respectively by

$$\begin{aligned} \partial f(x) \quad &:= \quad \big\{v \in \mathbb{R}^n \;\big|\; (v,-1) \in N\big((x,f(x)); \operatorname{epi} f\big)\big\},\\ \partial^\infty f(x) \quad &:= \quad \big\{v \in \mathbb{R}^n \;\big|\; (v,0) \in N\big((x,f(x)); \operatorname{epi} f\big)\big\}. \end{aligned}$$

(ii) The *nonsmooth slope* of f at x is defined by

$$\mathfrak{m}_f(x) \quad := \quad \inf\{\|v\| \mid v \in \partial f(x)\}.$$

By definition, $\mathfrak{m}_f(x) = +\infty$ whenever $\partial f(x) = \emptyset$.

In [29, 33] the reader can find equivalent analytic descriptions of the limiting subdifferential $\partial f(x)$ and comprehensive studies of it and related constructions. For convex f, this subdifferential coincides with the convex subdifferential. Furthermore, if the function f is of class C^1, then $\partial f(x) = \{\nabla f(x)\}$ and so $\mathfrak{m}_f(x) = \|\nabla f(x)\|$. The horizon subdifferential $\partial^\infty f(x)$ plays an entirely different role–it detects horizontal "normal" to the epigraph–and it plays a decisive role in subdifferential calculus; see [33, Corollary 10.9] for more details.

Theorem 2.1 (Fermat rule)**.** *Consider an lsc function $f\colon \mathbb{R}^n \to \mathbb{R}$ and a closed set $\Omega \subset \mathbb{R}^n$. If $\bar{x} \in \Omega$ is a local minimizer of f on Ω and the qualification condition*

$$\partial^\infty f(\bar{x}) \cap N(\bar{x};\Omega) \quad = \quad \{0\}$$

is valid, then the inclusion $0 \in \partial f(\bar{x}) + N(\bar{x};\Omega)$ holds.

2.2. **Semi-algebraic geometry.** Now, we recall some notions and results of semi-algebraic geometry, which can be found in [7, 35].

Definition 2.3. A subset S of $\mathbb{R}^n$ is called *semi-algebraic*, if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f_i(x) = 0,\ i = 1, \ldots, k; f_i(x) > 0,\ i = k+1, \ldots, p\},$$

where all f_i are polynomials. In other words, S is a union of finitely many sets, each defined by finitely many polynomial equalities and inequalities. A function $f\colon S \to \mathbb{R}$ is said to be *semi-algebraic*, if its graph

$$\{(x, y) \in S \times \mathbb{R} \mid y = f(x)\}$$

is a semi-algebraic set.

A major fact concerning the class of semi-algebraic sets is its stability under linear projections (see, for example, [7]).

Theorem 2.2 (Tarski–Seidenberg Theorem). *The image of any semi-algebraic set $S \subset \mathbb{R}^n$ under a projection to any linear subspace of $\mathbb{R}^n$ is a semi-algebraic set.*

Remark 2.1. As an immediate consequence of the Tarski–Seidenberg Theorem, we get semi-algebraicity of any set $\{x \in A : \exists y \in B, (x, y) \in C\}$, provided that A, B, and C are semi-algebraic sets in the corresponding spaces. Also, $\{x \in A : \forall y \in B, (x, y) \in C\}$ is a semi-algebraic set as its complement is the union of the complement of A and the set $\{x \in A : \exists y \in B, (x, y) \notin C\}$. Thus, if we have a finite collection of semi-algebraic sets, then any set obtained from them with the help of a finite chain of quantifiers is also semi-algebraic. In particular, for a semi-algebraic function $f\colon \mathbb{R}^n \to \mathbb{R}$, it is easy to see that the nonsmooth slope $\mathfrak{m}_f\colon \mathbb{R}^n \to \mathbb{R}$ is a semi-algebraic function.

The following three well-known lemmas will be of great for us; see, for example, [21, Theorem 1.8, Theorem 1.11, and Lemma 1.7].

Lemma 2.1 (Monotonicity Lemma). *Let $f\colon (a, b) \to \mathbb{R}$ be a semi-algebraic function. Then there are finitely many points $a = t_0 < t_1 < \cdots < t_k = b$ such that the restriction of f to each interval (t_i, t_{i+1}) are analytic, and either constant, or strictly increasing or strictly decreasing.*

Lemma 2.2 (Curve Selection Lemma). *Consider a semi-algebraic set $S \subset \mathbb{R}^n$ and a point $\bar{x} \in \mathbb{R}^n$ that is a cluster point of S. Then there exists an analytic semi-algebraic curve $\phi\colon (0, \epsilon) \to \mathbb{R}^n$ with $\lim_{t\to 0^+} \phi(t) = \bar{x}$ and with $\phi(t) \in S$ for $t \in (0, \epsilon)$.*

Lemma 2.3 (Growth Dichotomy Lemma). *Let $f\colon (0, \epsilon) \to \mathbb{R}$ be a semi-algebraic function with $f(t) \neq 0$ for all $t \in (0, \epsilon)$. Then there exist constants $a \neq 0$ and $\alpha \in \mathbb{Q}$ such that $f(t) = at^\alpha + o(t^\alpha)$ as $t \to 0^+$.*

In the sequel we will make fundamental use of Hardt's semi-algebraic triviality. We present a particular case–adapted to our needs–of a more general result: see [7, 22, 35] for the statement in its full generality.

Theorem 2.3 (Hardt's semi-algebraic triviality). *Let S be a semi-algebraic set in $\mathbb{R}^n$ and $f\colon S \to \mathbb{R}$ a continuous semi-algebraic map. Then there are finitely many points $-\infty = t_0 < t_1 < \cdots < t_k = +\infty$ such that f is semi-algebraically trivial over each the interval (t_i, t_{i+1}), that is, there are a semi-algebraic set $F_i \subset \mathbb{R}^n$ and a semi-algebraic homeomorphism $h_i\colon f^{-1}(t_i, t_{i+1}) \to (t_i, t_{i+1}) \times F_i$ such that the composition h_i with the projection $(t_i, t_{i+1}) \times F_i \to (t_i, t_{i+1}), (t, x) \mapsto t$, is equal to the restriction of f to $f^{-1}(t_i, t_{i+1})$.*

We will also need the following lemma, whose proof follows immediately from [15, Lemma 2.10] (see also [10, Proposition 4]).

Lemma 2.4. *Consider an lsc semi-algebraic function $f\colon \mathbb{R}^n \to \mathbb{R}$ and a semi-algebraic curve $\phi\colon [a, b] \to \mathbb{R}^n$. Then for all but finitely many $t \in [a, b]$, the mappings ϕ and $f \circ \phi$ are analytic at t and satisfy*

$$\begin{aligned} v \in \partial f(\phi(t)) \quad &\Longrightarrow \quad \langle v, \dot{\phi}(t)\rangle \;=\; (f \circ \phi)'(t), \\ v \in \partial^\infty f(\phi(t)) \quad &\Longrightarrow \quad \langle v, \dot{\phi}(t)\rangle \;=\; 0. \end{aligned}$$

Proof. Thanks to the Monotonicity Lemma 2.1, both the mappings ϕ and $f\circ\phi$ are analytic on $[a, b]$ except at finitely many $t \in [a, b]$.

Let c be the supremum of real numbers $T \in [a, b]$ such that for all but finitely many $t \in [a, T]$, we have

$$\begin{aligned} (f \circ \phi)'(t) \;&=\; \langle v, \dot{\phi}(t)\rangle \quad \text{for all} \quad v \in \partial f(\phi(t)), \\ \langle v, \dot{\phi}(t)\rangle \;&=\; 0 \quad \text{for all} \quad v \in \partial^\infty f(\phi(t)). \end{aligned}$$

An application of [15, Lemma 2.10] yields $c > a$. We must prove that $c = b$. Suppose that this is not the case. Replacing the interval $[a, b]$ by the interval $[c, b]$ and applying [15, Lemma 2.10] again, we find a small real number $\epsilon > 0$ such that for any $t \in (c, c + \epsilon)$, the following relations hold:

$$\begin{aligned} (f \circ \phi)'(t) \;&=\; \langle v, \dot{\phi}(t)\rangle \quad \text{for all} \quad v \in \partial f(\phi(t)), \\ \langle v, \dot{\phi}(t)\rangle \;&=\; 0 \quad \text{for all} \quad v \in \partial^\infty f(\phi(t)), \end{aligned}$$

thus contradicting the definition of c. The proof is complete. □

3. Tangencies

From now on, let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a non-constant semi-algebraic function, which is continuous around a point $\bar{x} \in \mathbb{R}^n$. Using the so–called tangency variety of f at $\bar{x}$ we define finite sets of real numbers that can be used to test if f has a local minimizer at

$\bar{x}$ and if f has an α-order sharp local minimizer at $\bar{x}$. Let us begin with the following definition (see also [21]).

Definition 3.1. The *tangency variety of f at $\bar{x}$* is defined as follows:

$$\Gamma(f) \ := \ \{x \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R} \text{ such that } \lambda(x - \bar{x}) \in \partial f(x)\}.$$

By the Tarski–Seidenberg Theorem 2.2, $\Gamma(f)$ is a semi-algebraic set. Moreover, thanks to the Fermat rule (Theorem 2.1), we can see that for any $t > 0$, the tangency variety $\Gamma(f)$ contains the set of minimizers of the optimization problem $\min_{x \in \mathbb{S}_t(\bar{x})} f(x)$; in particular, $\bar{x}$ is a cluster point of $\Gamma(f)$.

Applying Hardt's triviality Theorem 2.3 for the continuous semi-algebraic function

$$\Gamma(f) \to \mathbb{R}, \quad x \mapsto \|x - \bar{x}\|,$$

we find a constant $\epsilon > 0$ such that the restriction of this function on $\Gamma(f) \cap \mathbb{B}_\epsilon(\bar{x}) \setminus \{\bar{x}\}$ is a topological trivial fibration. Let p be the number of connected components of a fiber of this restriction. Then $\Gamma(f) \cap \mathbb{B}_\epsilon(\bar{x}) \setminus \{\bar{x}\}$ has exactly p connected components, say $\Gamma_1, \ldots, \Gamma_p$, and each such component is a semi-algebraic set. Moreover, for all $t \in (0, \epsilon)$ and all $k = 1, \ldots, p$, the sets $\Gamma_k \cap \mathbb{S}_t(\bar{x})$ are connected. Corresponding to each Γ_k, let

$$f_k \colon (0, \epsilon) \to \mathbb{R}, \quad t \mapsto f_k(t),$$

be the function defined by $f_k(t) := f(x)$, where $x \in \Gamma_k \cap \mathbb{S}_t(\bar{x})$.

Lemma 3.1. *For all $\epsilon > 0$ small enough, the following statements hold:*

(i) *All the functions f_k are well-defined and semi-algebraic.*
(ii) *Each the function f_k is either constant or strictly monotone.*

Proof. (i) Fix $k \in \{1, \ldots, p\}$ and take any $t \in (0, \epsilon)$. We will show that the restriction of f on $\Gamma_k \cap \mathbb{S}_t(\bar{x})$ is constant. To see this, let $\phi \colon [0, 1] \to \mathbb{R}^n$ be a smooth semi-algebraic curve such that $\phi(\tau) \in \Gamma_k \cap \mathbb{S}_t(\bar{x})$ for all $\tau \in [0, 1]$. By definition, we have

$$\|\phi(\tau) - \bar{x}\| = t \qquad \text{and} \qquad \lambda(\tau)(\phi(\tau) - \bar{x}) \in \partial f(\phi(\tau))$$

for some $\lambda(\tau) \in \mathbb{R}$. Moreover, in view of Lemma 2.4, for all but finitely many $\tau \in [a, b]$, the mappings ϕ and $f \circ \phi$ are analytic at τ and satisfy

$$v \in \partial f(\phi(\tau)) \implies \langle v, \dot{\phi}(\tau) \rangle \ = \ (f \circ \phi)'(\tau).$$

Therefore

$$\begin{aligned} (f \circ \phi)'(\tau) &= \langle \lambda(\tau)\big(\phi(\tau) - \bar{x}\big), \dot{\phi}(\tau) \rangle \\ &= \frac{\lambda(\tau)}{2} \frac{d\|\phi(\tau) - \bar{x}\|^2}{d\tau} \\ &= 0. \end{aligned}$$

So f is constant on the curve ϕ.

On the other hand, since the set $\Gamma_k \cap \mathbb{S}_t(\bar{x})$ is connected semi-algebraic, it is path connected. Hence, any two points in $\Gamma_k \cap \mathbb{S}_t(\bar{x})$ can be joined by a piecewise smooth semi-algebraic curve (see [21, Theorem 1.13]). It follows that the restriction of f on $\Gamma_k \cap \mathbb{S}_t(\bar{x})$ is constant and so the function f_k is well-defined. Finally, by the Tarski–Seidenberg Theorem 2.2, f_k is semi-algebraic.

(ii) By decreasing ϵ (if necessary) and applying the Monotonicity Lemma 2.1, it is not hard to get this item. □

For each $t \in (0, \epsilon)$, the sphere $\mathbb{S}_t(\bar{x})$ is a nonempty compact semi-algebraic set. Hence, the function

$$\psi\colon (0,\epsilon) \to \mathbb{R}, \quad t \mapsto \psi(t) := \min_{x \in \mathbb{S}_t(\bar{x})} f(x),$$

is well-defined, and moreover, it is semi-algebraic because of the Tarski–Seidenberg Theorem 2.2 (see the discussion in [21, Section 1.6]). The following lemma is simple but useful.

Lemma 3.2. *For $\epsilon > 0$ small enough, the following statements hold:*

(i) *The functions ψ and $f_1, \ldots, f_p$ are either coincide or disjoint.*

(ii) $\psi(t) = \min_{k=1,\ldots,p} f_k(t)$ *for all* $t \in (0, \epsilon)$.

(iii) $\psi \equiv f_k$ *for some* $k \in \{1, \ldots, p\}$.

Proof. (i) This is an immediate consequence of the Monotonicity Lemma 2.1.

(ii) Without loss of generality, assume $\bar{x} = 0$ and $f(\bar{x}) = 0$. Applying the Curve Selection Lemma 2.2 and shrinking ϵ (if necessary), we find an analytic semi-algebraic curve $\phi\colon (0,\epsilon) \to \mathbb{R}^n$ such that $\|\phi(t)\| = t$ and $(f \circ \phi)(t) = \psi(t)$ for all t. By Lemma 2.4, then we have for any $t \in (0, \epsilon)$,

$$v \in \partial^\infty f(\phi(t)) \implies \langle v, \dot{\phi}(t)\rangle \ = \ 0.$$

Observe

$$\langle \phi(t), \dot{\phi}(t)\rangle \ = \ \frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2,$$

and hence the qualification condition

$$\partial^\infty f(\phi(t)) \cap N\big(\phi(t); \mathbb{S}_t(\bar{x})\big) = \{0\}$$

holds for all $t \in (0, \epsilon)$. Consequently, since $\phi(t)$ minimizes f subject to $\|x\| = t$, applying the Fermat rule (Theorem 2.1), we deduce that $\phi(t)$ belongs to $\Gamma(f)$. Therefore,

$$\psi(t) \ = \ \min_{x \in \mathbb{S}_t(\bar{x})} f(x) \ = \ \min_{x \in \Gamma(f) \cap \mathbb{S}_t(\bar{x})} f(x) \ = \ \min_{k=1,\ldots,p} \min_{x \in \Gamma_k \cap \mathbb{S}_t(\bar{x})} f(x) \ = \ \min_{k=1,\ldots,p} f_k(t),$$

where the third equality follows from the fact that

$$\Gamma(f) \cap \mathbb{S}_t(\bar{x}) \ = \ \bigcup_{k=1}^{p} \Gamma_k \cap \mathbb{S}_t(\bar{x}).$$

(iii) This follows from Items (i) and (ii). □

4. Main results

Recall that $f\colon \mathbb{R}^n \to \mathbb{R}$ is a non-constant semi-algebraic function, which is continuous around a point $\bar{x} \in \mathbb{R}^n$. As in the previous section, we associate to the function f a finite number of functions $f_1, \ldots, f_p$ of a single variable. Let

$$K := \{k \mid f_k \text{ is not constant}\}.$$

Note that $f_k \equiv f(\bar{x})$ for all $k \notin K$. By the Growth Dichotomy Lemma 2.3, we can write for each $k \in K$,

$$f_k(t) \quad = \quad f(\bar{x}) + a_k t^{\alpha_k} + o(t^{\alpha_k}) \quad \text{ as } \quad t \to 0^+,$$

where $a_k \in \mathbb{R}, a_k \neq 0$, and $\alpha_k \in \mathbb{Q}, \alpha_k > 0$. It is convenient to define $a_k = 0$ for $k \notin K$. As we can see the "tangency coefficients" a_k and the "tangency exponents" α_k play important roles in Theorems 4.1 and 4.2 below.

We now arrive to the first main result of this section. This result provides necessary and sufficient conditions for optimality of nonsmooth semi-algebraic functions.

Theorem 4.1 (Necessary and sufficient conditions for optimality)**.** *With the above notations, the following statements hold:*

(i) *The point $\bar{x}$ is a local minimizer of f if, and only if, $a_k \geq 0$ for all $k = 1, \ldots, p$.*
(ii) *The point $\bar{x}$ is an isolated local minimizer of f if, and only if, $a_k > 0$ for all $k = 1, \ldots, p$.*

Proof. Recall that

$$\psi(t) \quad := \quad \min_{x \in \mathbb{S}_t(\bar{x})} f(x) \quad \text{ for } \quad t \geq 0.$$

By definition, it is easy to see that $\bar{x}$ is a local minimizer (resp., an isolated local minimizer) of f if, and only if, for all $t > 0$ small enough, we have $\psi(t) \geq f(\bar{x})$ (resp., $\psi(t) > f(\bar{x})$). This observation, together with Lemma 3.2, implies easily the desired conclusion. □

Remark 4.1. Very recently, using tangency varieties, Guo and Phạm [18] proposed a computational and symbolic method to determine the type (local minimizer, maximizer or saddle point) of a given isolated critical point, which is degenerate, of a multivariate polynomial function.

We know from Łojasiewicz's inequality [21, Theorem 1.14] that $\bar{x}$ is an isolated local minimizer of f if, and only, if there exists a real number $\alpha > 0$ such that $\bar{x}$ is an α-order sharp local minimizer of f. A characteristic of this number α in terms of the "tangency exponents" of f is given in Theorem 4.2 below. To this end, let

$$\alpha_* \quad := \quad \max_{k \in K} \alpha_k \ > \ 0.$$

The second main result of this section reads as follows.

Theorem 4.2 (Isolated local minimizers)**.** *With the above notations, assume that $\bar{x} \in \mathbb{R}^n$ is an isolated local minimizer of f. Then for any $\alpha \in \mathbb{R}$, the following statements are equivalent:*

(i) *The inequality $\alpha \geq \alpha_*$ holds.*

(ii) *The point $\bar{x}$ is an α-order sharp local minimizer of f, i.e., there exist constants $c > 0$ and $\epsilon > 0$ such that*

$$f(x) \;\; \geq \;\; f(\bar{x}) + c\,\|x - \bar{x}\|^{\alpha} \quad \text{for all} \quad x \in \mathbb{B}_{\epsilon}(\bar{x}).$$

(iii) *The limiting subdifferential ∂f of f is $(\alpha - 1)$-order strongly metrically subregular at $\bar{x}$ for 0, i.e., there exist constants $c > 0$ and $\epsilon > 0$ such that*

$$\mathfrak{m}_f(x) \;\; \geq \;\; c\,\|x - \bar{x}\|^{\alpha - 1} \quad \text{for all} \quad x \in \mathbb{B}_{\epsilon}(\bar{x}) \setminus \{\bar{x}\}.$$

(iv) *The function f satisfies the Łojaseiwcz gradient inequality at $\bar{x}$ with the exponent $1 - \frac{1}{\alpha}$, i.e., there exist constants $c > 0$ and $\epsilon > 0$ such that*

$$\mathfrak{m}_f(x) \;\; \geq \;\; c\,|f(x) - f(\bar{x})|^{1 - \frac{1}{\alpha}} \quad \text{for all} \quad x \in \mathbb{B}_{\epsilon}(\bar{x}) \setminus \{\bar{x}\}.$$

In order to prove Theorem 4.2 below, we need the following result which can be seen as a nonsmooth version of the Bochnack–Łojasiewicz inequality [8].

Lemma 4.1. *There exist constants $c > 0$ and $\epsilon > 0$ such that*

$$\mathfrak{m}_f(x)\|x - \bar{x}\| \;\; \geq \;\; c|f(x) - f(\bar{x})| \quad \text{for all} \quad x \in \mathbb{B}_{\epsilon}(\bar{x}).$$

Proof. Without loss of generality, we may assume that $\bar{x} = 0$ and $f(\bar{x}) = 0$.

Arguing by contradiction, suppose that the lemma is false, that is

$$\liminf_{x \to \bar{x}} \frac{\mathfrak{m}_f(x)\|x\|}{|f(x)|} \;\; = \;\; 0.$$

In light of the Curve Selection Lemma 2.2, we find a non-constant analytic semi-algebraic curve $\phi\colon (0, \epsilon) \to \mathbb{R}^n$ with $\lim_{t \to 0^+} \phi(t) = 0$ such that $(f \circ \phi)(t) \neq 0$ and

$$\lim_{t \to 0^+} \frac{\mathfrak{m}_f(\phi(t))\|\phi(t)\|}{|(f \circ \phi)(t)|} \;\; = \;\; 0.$$

Since f is continuous at $\bar{x}$, it holds that

$$\lim_{t \to 0^+} (f \circ \phi)(t) \;\; = \;\; 0.$$

By the Growth Dichotomy Lemma 2.3, we can write

$$\phi(t) \;\; = \;\; at^{\alpha} + o(t^{\alpha}) \quad \text{and} \quad (f \circ \phi)(t) \;\; = \;\; bt^{\beta} + o(t^{\beta}) \quad \text{as } t \to 0^+,$$

for some $a \in \mathbb{R}^n, a \neq 0, \alpha \in \mathbb{Q}, \alpha > 0, b \in \mathbb{R}, b \neq 0$, and $\beta \in \mathbb{Q}, \beta > 0$. Then a direct calculation shows that for all sufficiently small $t > 0$,

$$\|\phi(t)\| \;\; \simeq \;\; t\|\dot{\phi}(t)\| \quad \text{and} \quad |(f \circ \phi)(t)| \;\; \simeq \;\; |t(f \circ \phi)'(t)|,$$

where $A \simeq B$ means that A/B lies between two positive constants. On the other hand, we deduce easily from Lemma 2.4 that

$$|(f \circ \phi)'(t)| \ \leq \ \mathfrak{m}_f(\phi(t))\|\dot{\phi}(t)\|.$$

Therefore,

$$|(f \circ \phi)(t)| \ \simeq \ |t(f \circ \phi)'(t)| \quad \leq \ t\mathfrak{m}_f(\phi(t))\|\dot{\phi}(t)\| \ \simeq \ \mathfrak{m}_f(\phi(t))\|\phi(t)\|.$$

Consequently, there exists a constant $c > 0$ such that

$$c \ \leq \ \frac{\mathfrak{m}_f(\phi(t))\|\phi(t)\|}{|(f \circ \phi)(t)|}$$

for all sufficiently small $t > 0$. Letting t tend to zero in this inequality, we arrive at a contradiction. □

Proof of Theorem 4.2. Without loss of generality, assume $\bar{x} = 0$ and $f(\bar{x}) = 0$.

By Theorem 4.1, $K = \{1, \ldots, p\}$ and $a_k > 0$ for all $k \in K$. Recall that

$$\psi(t) \ := \ \min_{x \in \mathbb{S}_t(\bar{x})} f(x).$$

In light of Lemma 3.2, we can write

$$\psi(t) \ = \ a_* t^{\alpha_*} + o(t^{\alpha_*}) \quad \text{as} \quad t \to 0^+, \tag{2}$$

where $a_* := \min\{a_k \mid k \in K \text{ and } \alpha_k = \alpha_*\}$. In particular, for any real number $c \in (0, a_*)$ there exists $\epsilon \in (0, 1)$ such that

$$\psi(t) \ \geq \ c\, t^{\alpha_*} \quad \text{for all} \quad t \in [0, \epsilon]. \tag{3}$$

(i) ⇔ (ii): Assume that $\alpha \geq \alpha_*$. From (3) we have for all $x \in \mathbb{B}_\epsilon(\bar{x})$,

$$f(x) \ \geq \ \psi(\|x\|) \ \geq \ c\,\|x\|^{\alpha_*} \ \geq \ c\,\|x\|^{\alpha},$$

which proves (ii).

Conversely, assume that there exist constants $c' > 0$ and $\epsilon' > 0$ such that

$$f(x) \ \geq \ c'\|x\|^{\alpha} \quad \text{for all} \quad x \in \mathbb{B}_{\epsilon'}(\bar{x}).$$

Then for all $t \in [0, \epsilon]$ we have

$$\psi(t) \ = \ \min_{x \in \mathbb{S}_t(\bar{x})} f(x) \ \geq \ c' t^{\alpha}.$$

Combining this with (2) we get $\alpha \geq \alpha_*$.

(iv) ⇒ (iii) ⇒ (ii): Clearly, the condition (iii) holds provided that both the conditions (ii) and (iv) hold. So it suffices to show the implications (iii) ⇒ (ii) and (iv) ⇒ (ii).

Note that the minimum in the definition of ψ is attained. In view of the Curve Selection Lemma 2.2, there is an analytic semi-algebraic curve $\phi\colon (0, \epsilon) \to \mathbb{R}^n$ such that $\|\phi(t)\| = t$

and $(f \circ \phi)(t) = \psi(t)$ for all t. Applying Lemma 2.4 and shrinking ϵ (if necessary), we have for any $t \in (0, \epsilon)$,

$$\begin{aligned} v \in \partial f(\phi(t)) &\implies \langle v, \dot{\phi}(t)\rangle = \psi'(t), \\ v \in \partial^\infty f(\phi(t)) &\implies \langle v, \dot{\phi}(t)\rangle = 0. \end{aligned} \tag{4}$$

In particular, as in the proof of Lemma 3.2, we have $\phi(t) \in \Gamma(f)$, i.e., there is a real number $\lambda(t)$ satisfying

$$\lambda(t)\phi(t) \in \partial f(\phi(t)). \tag{5}$$

By definition, then

$$\|\lambda(t)\phi(t)\| \geq \mathfrak{m}_f(\phi(t)).$$

Furthermore, it follows from (4) and (5) that

$$\psi'(t) = \lambda(t)\langle \phi(t), \dot{\phi}(t)\rangle = \lambda(t)\frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2 = \lambda(t)t.$$

Consequently,

$$|\psi'(t)| = |\lambda(t)t| = \|\lambda(t)\phi(t)\| \geq \mathfrak{m}_f(\phi(t)).$$

Therefore, if the condition (iii) holds, then $|\psi'(t)| \geq c\,t^{\alpha-1}$, while if the condition (iv) holds, then $|\psi'(t)| \geq c\,(\psi(t))^{1-\frac{1}{\alpha}}$; in both the cases, we get $\alpha \geq \alpha_*$ and so $\psi(t) \geq c' t^\alpha$ for some constant $c' > 0$. Therefore the condition (ii) holds.

(ii) $\Rightarrow$ (iv): Our assumption implies the existence of constants $c > 0$ and $\epsilon > 0$ such that

$$f(x) \geq c\,\|x\|^\alpha \quad \text{for all} \quad x \in \mathbb{B}_\epsilon(\bar{x}).$$

On the other hand, applying Lemma 4.1, we deduce that there exist constants $c' > 0$ and $\epsilon' > 0$ such that

$$\|x\|\mathfrak{m}_f(x) \geq c'|f(x)| \quad \text{for all} \quad x \in \mathbb{B}_{\epsilon'}(\bar{x}).$$

Therefore, the inequality

$$\left(\frac{1}{c}\,f(x)\right)^{\frac{1}{\alpha}} \mathfrak{m}_f(x) \geq c'|f(x)|$$

holds for all x near $\bar{x}$, from which the desired conclusion follows. □

From [15, Example 3.2] we know that the implication (ii) $\Rightarrow$ (iii), and hence the implication (ii) $\Rightarrow$ (iv), of Theorem 4.2 may easily fail in absence of continuity. The following example shows that the implication (iii) $\Rightarrow$ (iv) of Theorem 4.2 also may fail in absence of continuity.

Example 4.1. Consider the lsc, semi-algebraic function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 + x^2 & \text{if } x < 0, \\ x^2 & \text{otherwise.} \end{cases}$$

Observe that f is not continuous at $\bar{x} = 0$ and that 0 is a 2-order sharp local minimizer of f. A simple computation shows that

$$\mathfrak{m}_f(x) \quad = \quad 2|x| \quad \text{for all} \quad x \in \mathbb{R},$$

and so the condition (iii) of Theorem 4.2 holds with $\alpha = 2$. However, it is easy to check that f does not satisfy the condition (iv) of Theorem 4.2.

Remark 4.2. Consider an lsc function $f\colon \mathbb{R}^n \to \mathbb{R}$, which has a (not necessarily isolated) local minimum at $\bar{x} \in \mathbb{R}^n$. It is well-known (see [2, 3, 16, 15, 30, 36]) that the existence of constants $c > 0$ and $\epsilon > 0$ such that

$$\mathfrak{m}_f(x) \quad \geq \quad c\,\mathrm{dist}\big(x, (\partial f)^{-1}(0)\big) \quad \text{for all} \quad x \in \mathbb{B}_\epsilon(\bar{x})$$

implies the existence of constants $c' > 0$ and $\epsilon' > 0$ satisfying

$$f(x) \quad \geq \quad f(\bar{x}) + c'\,\mathrm{dist}\big(x, (\partial f)^{-1}(0)\big)^2 \quad \text{for all} \quad x \in \mathbb{B}_{\epsilon'}(\bar{x}),$$

where $\mathrm{dist}\big(x, (\partial f)^{-1}(0)\big)$ stands for the Euclidean distance from x to $(\partial f)^{-1}(0)$. In [16, Remark 3.4], Drusvyatskiy and Ioffe conjectured that the converse is also true for semi-algebraic functions. The next example shows that this conjecture does not hold in general.

Example 4.2. Let $f\colon \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto f(x, y)$, be the continuous semi-algebraic function defined by $f(x, y) := |x^2 - y^4|$. A direct calculation shows that

$$\partial f(x, y) = \begin{cases} \{(2x, -4y^3)\} & \text{if} \quad x^2 - y^4 > 0, \\ \{(-2x, 4y^3)\} & \text{if} \quad x^2 - y^4 < 0, \\ \{(2(2t-1)x, -4(2t-1)y^3) \mid t \in [0, 1]\} & \text{otherwise.} \end{cases}$$

In particular, we have

$$f^{-1}(0) \quad = \quad (\partial f)^{-1}(0) \quad = \quad \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^4 = 0\}.$$

Furthermore, according to Kuo's work [26] (see also [20]), we find a constant $c > 0$ such that

$$f(x, y) \quad \geq \quad c\,\mathrm{dist}((x, y), f^{-1}(0))^2 \quad = \quad c\,\mathrm{dist}\big((x, y), (\partial f)^{-1}(0)\big)^2$$

for all (x, y) near $(0, 0) \in \mathbb{R}^2$. On the other hand, it is not hard to check that

$$\mathfrak{m}_f(0, t) \quad = \quad 4t^3 \quad \text{and} \quad \mathrm{dist}\big((0, t), (\partial f)^{-1}(0)\big) \quad \simeq \quad t^2 \quad \text{as} \quad t \to 0,$$

and so

$$\lim_{t \to 0} \frac{\mathfrak{m}_f(0, t)}{\mathrm{dist}\big((0, t), (\partial f)^{-1}(0)\big)} \quad = \quad 0.$$

Therefore, there is no constant $c' > 0$ such that the inequality

$$\mathfrak{m}_f(x, y) \quad \geq \quad c' \operatorname{dist}\big((x, y), (\partial f)^{-1}(0)\big)$$

holds for all (x, y) near $(0, 0)$.

We finish this section with the following corollary.

Corollary 4.1. *Under the assumptions of Theorem 4.2, suppose that $\alpha \geq \alpha_*$. Then for any constant $c \in (0, a_*)$ there exists $\epsilon > 0$ such that*

$$f(x) \quad \geq \quad f(\bar{x}) + c\|x - \bar{x}\|^{\alpha} \quad \text{for all} \quad x \in \mathbb{B}_{\epsilon}(\bar{x}),$$

where $a_* := \min\{a_k \mid k \in K \text{ and } \alpha_k = \alpha_*\}$.

Proof. This follows immediately from the argument given at the beginning of the proof of Theorem 4.2. □

5. Examples

In this section we provide examples to illustrate our main results. For simplicity we consider the case where f is a polynomial function in two variables $(x, y) \in \mathbb{R}^2$. By definition, then

$$\Gamma(f) \quad := \quad \left\{(x, y) \in \mathbb{R}^2 \mid y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y} = 0\right\}.$$

Example 5.1. Let $f(x, y) := 3x^2 + 2y^3$. The tangency variety $\Gamma(f)$ is given by the equation:

$$xy(6 - 6y) \quad = \quad 0.$$

Hence, for any $\epsilon \in (0, 1)$, the set $\Gamma(f) \cap \mathbb{B}_{\epsilon} \setminus \{(0, 0)\}$ has four connected components:

$$\begin{aligned} \Gamma_{\pm 1} \quad &:= \quad \{(0, \pm t) \mid t \in (0, \epsilon]\}, \\ \Gamma_{\pm 2} \quad &:= \quad \{(\pm t, 0) \mid t \in (0, \epsilon]\}. \end{aligned}$$

Consequently,

$$\begin{aligned} f_{\pm 1} \quad &:= \quad f|_{\Gamma_{\pm 1}} \quad = \quad \pm 2t^3, \\ f_{\pm 2} \quad &:= \quad f|_{\Gamma_{\pm 2}} \quad = \quad 3t^2. \end{aligned}$$

It follows that $K = \{\pm 1, \pm 2\}$ and

$$a_{\pm 1} \quad = \quad \pm 2 \quad \text{and} \quad a_{\pm 2} \quad = \quad 3.$$

Therefore, by Theorem 4.1, the origin is not a local minimizer of f.

Example 5.2. Let $f(x,y) := x^2$. The tangency variety $\Gamma(f)$ is given by the equation:

$$2xy = 0.$$

Hence, for any $\epsilon > 0$, the set $\Gamma(f) \cap \mathbb{B}_\epsilon \setminus \{(0,0)\}$ has four connected components:

$$\begin{aligned}\Gamma_{\pm 1} &:= \{(0, \pm t) \mid t \in (0, \epsilon]\}, \\ \Gamma_{\pm 2} &:= \{(\pm t, 0) \mid t \in (0, \epsilon]\}.\end{aligned}$$

Consequently,

$$\begin{aligned}f_{\pm 1} &:= f|_{\Gamma_{\pm 1}} = 0, \\ f_{\pm 2} &:= f|_{\Gamma_{\pm 2}} = t^2.\end{aligned}$$

It follows that $K = \{\pm 2\}$ and

$$a_{\pm 1} = 0 \quad \text{and} \quad a_{\pm 2} = 1.$$

Therefore, by Theorem 4.1, the origin is a non-isolated local minimizer of f.

Example 5.3. Let $f(x,y) := 2x^2 + y^4$. The tangency variety $\Gamma(f)$ is given by the equation:

$$xy(4 - 4y^2) = 0.$$

Hence, for any $\epsilon \in (0,1)$, the set $\Gamma(f) \cap \mathbb{B}_\epsilon \setminus \{(0,0)\}$ has four connected components:

$$\begin{aligned}\Gamma_{\pm 1} &:= \{(0, \pm t) \mid t \in (0, \epsilon]\}, \\ \Gamma_{\pm 2} &:= \{(\pm t, 0) \mid t \in (0, \epsilon]\}.\end{aligned}$$

Consequently,

$$\begin{aligned}f_{\pm 1} &:= f|_{\Gamma_{\pm 1}} = t^4, \\ f_{\pm 2} &:= f|_{\Gamma_{\pm 2}} = 2t^2.\end{aligned}$$

It follows that $K = \{\pm 1, \pm 2\}$ and

$$\begin{aligned}a_{\pm 1} &= 1 \quad \text{and} \quad a_{\pm 2} = 2, \\ \alpha_{\pm 1} &= 4 \quad \text{and} \quad \alpha_{\pm 2} = 2.\end{aligned}$$

By Theorems 4.1 and 4.2, the origin is an α-order sharp local minimizer of f for all $\alpha \geq \max_{k=\pm 1, \pm 2} \alpha_k = 4$.

6. Conclusions

This paper considers local minimizers of semi-algebraic functions. In terms of the tangency variety, we have presented necessary and sufficient conditions for optimality. We have also showed relationships between generalized notions of sharp minima, strong metric subregularity and the Łojasiewicz gradient inequality; these relations may easily fail when the minimizer in question is not isolated. The constrained case will be studied in future research.

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