An extragradient method for solving variational inequalities without monotonicity *

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Abstract

A new extragradient projection method is devised in this paper, which does not obviously require generalized monotonicity and assumes only that the so-called dual variational inequality has a solution in order to ensure its global convergence. In particular, it applies to quasimonotone variational inequality having a nontrivial solution.

Key words. Variational inequality, projection method, quasimonotone, extragradient method, global convergence.

1. Introduction

In the classical variational inequality problem, we are given a closed convex set C in $\mathbb{R}^n$ and a continuous mapping F from C into $\mathbb{R}^n$, and we wish to find an element $x \in C$ such that

$$\langle F(x), y - x\rangle \geq 0, \quad \forall\, y \in C. \tag{1.1}$$

Throughout this paper, we denote the variational inequality problem by $\mathrm{VI}(F, C)$.

Projection methods for solving variational inequalities include extragradient methods, double projection methods, and proximal point methods, etc. We refer the readers to [7-17] for more details. Extragradient methods are a class of effective projection methods and developed by many researchers. Basically, different extragradient methods require different monotonicity assumption on the mapping, some of them even require Lipschitz continuity. The original extragradient method assumes that the mapping F is monotone and Lipschitz continuous. Many subsequent papers aimed at improving the extragradient algorithm so that it could apply to a wider class of problems, especially relax the monotonicity assumption of the mapping. The method suggested in [5] applies to monotone variational inequalities, and that in [4] applies to pseudomonotone variational inequalities. So far, the weakest conditions for the global convergence of improved extragradient methods is pseudomonotonicity and continuity of the mapping; see [11] and [1] for more references.

Recently, [14] put forward a double projection method for solving variational inequalities without assuming any kind of generalized monotonicity, which improves the projection method in [9] by relaxing the monotonicity condition. In particular, this method applies to quasimonotone variational inequalities having a nontrivial solution. However, to obtain the next iteration point, the method in [14] needs to calculate a projection onto the intersection of a finite number of halfspaces and the closed convex set C, and the number of halspaces is equal to the number of the current iteration, so that one needs to add more and more

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halfspaces as iteration number is increasing. This obviously brings difficulty for calculating the projection. To address this difficulty, a new extragradient-type algorithm is proposed in this paper. No halfspace is added when calculates the projection, and the global convergence is proved under the same assumption with that in [14]. In other words, we propose a new extragradient method for solving variational inequalities without assuming any kind of the usual generalized monotonicity, while known extragradient methods assume pseudomonotonicity at least.

2. Preliminaries

Definition 2.1. $x \in C$ is called a solution of the dual variational inequality if

$$\langle F(y), y - x\rangle \geq 0, \quad \forall\, y \in C.$$

Definition 2.2. The mapping F is said to be pseudomonotone if for each pair $x, y \in C$,

$$\langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), y - x\rangle \geq 0.$$

Definition 2.3. The mapping F is said to be quasimonotone if for each pair $x, y \in C$,

$$\langle F(x), y - x\rangle > 0 \Rightarrow \langle F(y), y - x\rangle \geq 0.$$

It can be seen that pseudomonotone mappings are quasimonotone, but not vice versa. For example, if $F(x) := x^2$ and $C := [-1, 1]$, then F is quasidomonotone on C. However, F is not pseudomonotone.

Definition 2.4. $x \in C$ is called a trivial solution if $\langle F(x), y - x\rangle = 0$ for all $y \in C$. If a solution of $\mathrm{VI}(F, C)$ is not a trivial solution, it is called a nontrivial solution.

Proposition 2.1. If the mapping F is pseudomonotone and $\mathrm{VI}(F, C)$ has a solution, then the dual variational inequality has a solution.

Proof. Let x be a solution set of $\mathrm{VI}(F, C)$. Then

$$\langle F(x), y - x\rangle \geq 0, \quad \forall y \in C.$$

Since F is pseudomonotone, it follows that $\langle F(y), y - x\rangle \geq 0$ for all $y \in C$. Thus x is a solution of the dual variational inequality. □

Proposition 2.2. If the mapping F is quasimonotone and $\mathrm{VI}(F, C)$ has a nontrivial solution, then the dual variational inequality has a solution.

Proof. Let $x^* \in C$ be a nontrivial solution of the variational inequality. Fix any $y \in C$. One has $\langle F(x^*), y - x^*\rangle \geq 0$. By Lemma 3.1 in [2], one of the following must hold:

$$\langle F(y), y - x^*\rangle \geq 0, \quad \text{or } \langle F(x^*), x - x^*\rangle \leq 0 \text{ for all } x \in C.$$

Since x^* is a solution of $\mathrm{VI}(F, C)$, the second inequality implies that x^* is a trivial solution, which contradicts that x^* is a nontrivial solution. Thus the first inequality must hold, and hence x^* is a solution of the dual variational inequality. □

Note that the mapping can be not quasimonotone even if the dual variational inequality has a solution. This can be seen from the following example.

Example 2.1. Let

$$K := [-1, 1] \times [-1, 1]$$

and

$$F(x_1, x_2) = (x_1^2, x_2^2), \quad \forall (x_1, x_2) \in K.$$

If we take $x = (-1, 0)$ and $y = (-\frac{1}{2}, -1)$, then $\langle F(x), y - x \rangle = \frac{1}{2} > 0$, but $\langle F(y), y - x \rangle = -\frac{7}{8} < 0$. Therefore F is not quasimonotone on K. However, $(-1, -1)$ solves the dual variational inequality:

$$\langle F(y), y - x \rangle \geq 0, \quad \forall y \in K.$$

To design projection methods for the variational inequality problem, one usually uses the so-called natural residual

$$r(x) := x - \Pi_C(x - F(x)), \tag{2.1}$$

where Π_C denotes the metric projection onto C.

Lemma 2.1. For any $x, d \in R^n$ and $\alpha \geq 0$, define
$x(\alpha) = \Pi_C(x - \alpha d)$.
Then, $\langle d, x - x(\alpha) \rangle$ is nondecreasing for $\alpha > 0$.

Proof. By the property of the projection, one has

$$\langle \Pi_C(x - td) - \Pi_C(x - \alpha d), (\alpha - t)d \rangle \geq \|\Pi_C(x - td) - \Pi_C(x - \alpha d)\|^2.$$

It follows that $\alpha \mapsto \langle \Pi_C(x - \alpha d), d \rangle$ is decreasing. □

One can refer to [1, 10] for the following results.

Lemma 2.2. For any $x, d \in R^n$ and $\alpha \geq 0$, define
$\Psi(\alpha) = \min\{\|y - x + \alpha d\|^2 \mid y \in C\}$.
Then Ψ is differentiable and $\Psi'(\alpha) = 2 \langle d, x(\alpha) - x + \alpha d \rangle$.

Lemma 2.3. Let $y \in C$ and $x \in R^n$. Then $\langle y - \Pi_C(x), x - \Pi_C(x) \rangle \leq 0$.

Lemma 2.4. $\langle F(x), r(x) \rangle \geq \|r(x)\|^2$ for all $x \in C$.

Lemma 2.5. $r(x) = 0 \Leftrightarrow x$ is a solution of $\mathrm{VI}(F, C)$.

3. Main Results

Now we present a new extragradient method and establish its global convergence.

Algorithm 1. (New extragradient method)
Step 0. Select any $\sigma, \gamma \in (0, 1), x^0 \in C, k = 0$.
Step 1. For $x^k \in C$, compute $r(x^k)$. If $r(x^k) = 0$, stop; else compute

$$y^k = x^k - \eta_k r(x^k),$$

where $\eta_k = \gamma^{m_k}$ with m_k being the smallest nonnegative integer m satisfying

$$\left\langle F(x^k) - F(x^k - \gamma^m r(x^k)), r(x^k) \right\rangle \leq \sigma \|r(x^k)\|^2.$$

Let

$$i_k := \arg\max_{0\le j\le k} \frac{\langle F(y^j), x^k - y^j\rangle}{\|F(y^j)\|}, \tag{3.1}$$

that is, $i_k \in \{0, 1, \dots, k\}$ and

$$\frac{\langle F(y^{i_k}), x^k - y^{i_k}\rangle}{\|F(y^{i_k})\|} \ge \frac{\langle F(y^j), x^k - y^j\rangle}{\|F(y^j)\|}, \quad \forall\, j \in \{0, 1, \dots, k\}.$$

Set $z^k := y^{i_k}$.

Step 2. Compute $x^{k+1} = \Pi_C(x^k - \alpha_k F(z^k))$, where the stepsize α_k satisfies the following two conditions

$$\alpha_k \ge \alpha_k^1 = \frac{\langle F(z^k), x^k - z^k\rangle}{\|F(z^k)\|^2}, \tag{3.2}$$

$$\langle F(z^k), \Pi_C(x^k - \alpha_k F(z^k)) - z^k\rangle \ge 0. \tag{3.3}$$

Let $k = k + 1$ and return to Step 1.

Remark 3.1. Since F is a continuous mapping from C into R^n, we have

$$\langle F(x^k) - F(x^k - \gamma^m r(x^k)), r(x^k)\rangle \le \sigma \|r(x^k)\|^2,$$

for sufficiently large m. Therefore the linesearch in Step 1 is well-defined. If $r(x^k) = 0$, then procedure stops. If $r(x^k) \ne 0$ and $\langle F(x^k) - F(y^k), r(x^k)\rangle \le \sigma\|r(x^k)\|^2$, by Lemma 2.4, then $F(y^k) \ne 0$. Therefore (3.1) is well-defined.

In what follows, we show that Step 2 is well-defined.

Lemma 3.1. Let $\psi(\alpha) := \alpha^2\|F(y)\|^2 - 2\alpha\langle F(y), x - y\rangle - d_C^2(x - \alpha F(y))$. Assume that x and y satisfy the following three conditions:

(C1) $x, y \in C$.

(C2) y is not a solution of $\mathrm{VI}(F, C)$.

(C3) $\langle F(y), x - y\rangle > 0$.

Then there exists $\alpha \in (0, +\infty)$ such that the derivative $\psi'(\alpha) = 0$.

Proof. Note that

$$d_C^2(x - \alpha F(y)) = \inf_{z\in C} \|z - x + \alpha F(y)\|^2 = \inf_{z\in C}\{\|z - x\|^2 + 2\alpha\langle F(y), z - x\rangle\} + \alpha^2\|F(y)\|^2.$$

It follows that

$$\psi(\alpha) = \sup_{z\in C}\{-2\alpha\langle F(y), z - y\rangle - \|z - x\|^2\} = 2\sup_{z\in D}\left\{\langle x - \alpha F(y) - y, z\rangle - \frac{1}{2}\|z\|^2\right\} - \|x - y\|^2, \tag{3.4}$$

where $D := C - y$. Define $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ by

$$g(\alpha) := \sup_{z\in D}\{\langle x - \alpha F(y) - y, z\rangle - \frac{1}{2}\|z\|^2\} \text{ and } h(u) := \sup_{z\in D}\{\langle u, z\rangle - \frac{1}{2}\|z\|^2\}.$$

Obviously, h is a convex function. Since $h(u) \equiv \frac{1}{2}\|u\|^2 - \frac{1}{2}d_D^2(u)$, h is a real-valued convex function on $\mathbb{R}^n$. It follows that g is a real-valued convex function on $\mathbb{R}$, as $g(\alpha) = h(x - \alpha F(y) - y)$.

By (3.4), $\psi(\alpha) = 2g(\alpha) - \|x-y\|^2$. To prove the conclusion, it suffices to prove that the minimization problem $\min_{\alpha\geq 0} g(\alpha)$ has a global optimal solution.

By the definition of conjugate function,

$$g^*(\beta) = \sup_{\alpha\in\mathbb{R}}[\alpha\beta - g(\alpha)] = \sup_{\alpha\in\mathbb{R}}[\alpha\beta - h(x-\alpha F(y)-y)]$$

Let $L := \{x - \alpha F(y) - y : \alpha \in \mathbb{R}\}$. It is easy to check that for $z \in \mathbb{R}^n$,

$$z = x - \alpha F(y) - y \text{ if and only if } \alpha = \frac{\langle F(y), x-y-z\rangle}{\|F(y)\|^2} \text{ and } z \in L,$$

where the denominator $\|F(y)\|^2$ is nonzero, as y is not a solution of VI(F,C); see (C2).

Therefore,

$$\begin{aligned} g^*(\beta) &= \sup\left\{\frac{\langle F(y), x-y-z\rangle\,\beta}{\|F(y)\|^2} - h(z) : z \in L\right\} \\ &= \frac{\langle F(y), x-y\rangle\,\beta}{\|F(y)\|^2} + \sup\left\{\frac{-\beta}{\|F(y)\|^2}\langle F(y), z\rangle - h(z) : z\in L\right\} \end{aligned}$$

Let $\xi := -\frac{\beta F(y)}{\|F(y)\|^2}$. Recall that δ_L is the indicator function of L. It follows that

$$g^*(\beta) = (h+\delta_L)^*(\xi) - \langle \xi, x-y\rangle\,. \tag{3.5}$$

Since h is a real-valued convex function on $\mathbb{R}^n$, Theorem 16.4 in [6] shows that

$$(h+\delta_L)^*(\xi) = \inf_{v\in\mathbb{R}^n}\{h^*(\xi - v) + \delta_L^*(v)\}\,. \tag{3.6}$$

Since $h(u) = (\frac{1}{2}\|\cdot\|^2 + \delta_D)^*(u)$ and $\frac{1}{2}\|\cdot\|^2 + \delta_D(\cdot)$ is a lower semicontinuous convex function, we have $h^*(w) = \frac{1}{2}\|w\|^2 + \delta_D(w)$ for every $w \in \mathbb{R}^n$. Note that

$$\delta_L^*(v) = \sup_{z\in L}\langle z, v\rangle = \sup_{\alpha\in\mathbb{R}}\langle x - \alpha F(y) - y, v\rangle = \langle x-y, v\rangle + \delta_E(v),$$

where $E := \{z \in \mathbb{R}^n : \langle F(y), z\rangle = 0\}$. It follows from (3.6) that

$$\begin{aligned} (h+\delta_L)^*(\xi) &= \inf\{h^*(\xi-v) + \langle v, x-y\rangle + \delta_E(v) : v \in \mathbb{R}^n\} \\ &= \inf\{\frac{1}{2}\|v-\xi\|^2 + \delta_D(\xi - v) + \langle v, x-y\rangle + \delta_E(v) : v\in\mathbb{R}^n\} \\ &= \inf\{\frac{1}{2}\|v-\xi\|^2 + \langle v, x-y\rangle : v \in E\cap(\xi - D)\} \\ &= \frac{1}{2}\|\xi\|^2 - \frac{1}{2}\|x-\xi-y\|^2 + \frac{1}{2}d^2_{E\cap(\xi-D)}(\xi+y-x). \end{aligned}$$

Since infimum over empty set is $+\infty$, we have

$$\xi \in \operatorname{dom}(h+\delta_L)^* \iff E\cap(\xi - D) \neq \emptyset \iff \beta \in \{\langle F(y), y-z\rangle : z \in C\}, \tag{3.7}$$

where the second equivalence is due to the fact that $D = C - y$ and $\xi := -\frac{\beta F(y)}{\|F(y)\|^2}$.

By virtue of (3.5), $\beta \in \operatorname{dom}(g^*)$ if and only if $\xi \in \operatorname{dom}(h+\delta_L)^*$. By (3.7), it follows that

$$\operatorname{dom}(g^*) = \{\langle F(y), y-z\rangle : z\in C\}. \tag{3.8}$$

Since y is not a solution of VI(F,C), there exists $u_0 \in C$ such that $\beta_1 := \langle F(y), y-u_0\rangle > 0$. On the other hand, the condition (C3) implies $\beta_2 := \langle F(y), y-x\rangle < 0$. Since $u_0, x \in C$, (3.8) implies that $\{\beta_1,\beta_2\} \subset$

$\text{dom}(g^*)$. The convexity of $\text{dom}(g^*)$ yields $(\beta_2, \beta_1) \subset \text{dom}(g^*)$, which implies $0 \in \text{int}(\text{dom}(g^*))$. It follows from Corollary 13.3.4(c) and Theorem 27.3 in [6] that g attains its minimum over $[0, +\infty)$, that is, the minimization problem $\min_{\alpha \geq 0} g(\alpha)$ has a global optimal solution α_0.

By (C1), we have $\psi(0) = 0$. As verified in [3, Example IV.4.1.6], the derivative $(d_C^2)'(z) = 0$ for each $z \in C$. So $\psi'(0) = -2\langle F(y), x - y\rangle$. By (C3), we have $\psi'(0) < 0$, which implies $\inf_{\alpha \geq 0} \psi(\alpha) < \psi(0)$. It follows that $\alpha_0 > 0$, and hence $\psi'(\alpha_0) = 0$. □

For $\alpha \geq 0$, we define

$$\begin{aligned} x^k(\alpha) &:= \Pi_C(x^k - \alpha F(z^k)), \\ \Phi_k(\alpha) &:= 2\alpha \left\langle F(z^k), x^k - z^k \right\rangle - \alpha^2 \|F(z^k)\|^2 + \text{dist}(x^k - \alpha F(z^k), C)^2. \end{aligned}$$

This implies that

$$\Phi_k(0) = 0. \tag{3.9}$$

By Lemma 2.2, we have

$$\begin{aligned} \Phi_k'(\alpha) &= 2\left\langle F(z^k), x^k - z^k \right\rangle - 2\alpha \|F(z^k)\|^2 - 2\left\langle F(z^k), x^k - \alpha F(z^k) - x^k(\alpha) \right\rangle \\ &= 2\left\langle F(z^k), x^k - z^k - \alpha F(z^k) - x^k + \alpha F(z^k) + x^k(\alpha) \right\rangle \\ &= 2\left\langle F(z^k), x^k(\alpha) - z^k \right\rangle. \end{aligned} \tag{3.10}$$

Moreover,

$$\begin{aligned} \Phi_k'(\alpha) &= 2\left\langle F(z^k), x^k(\alpha) - x^k + x^k - z^k \right\rangle \\ &= 2\left\langle F(z^k), x^k(\alpha) - x^k \right\rangle + 2\left\langle F(z^k), x^k - z^k \right\rangle. \end{aligned}$$

By Lemma 2.1, we know that $\Phi_k'(\alpha)$ is decreasing with respect to $\alpha \geq 0$.

By the construction of z^k, one has

$$\frac{\left\langle F(z^k), x^k - z^k \right\rangle}{\|F(z^k)\|} \geq \frac{\left\langle F(y^k), x^k - y^k \right\rangle}{\|F(y^k)\|}.$$

Since

$$\left\langle F(y^k), x^k - y^k \right\rangle \geq \eta_k (1 - \sigma) \|r(x^k)\|^2 > 0, \tag{3.11}$$

it follows that

$$\Phi_k'(0) = 2\left\langle F(z^k), x^k - z^k \right\rangle > 0. \tag{3.12}$$

We claim that z^k is not a solution of $\text{VI}(F, C)$. Indeed, since x^{i_k} is not a solution, $r(x^{i_k}) \neq 0$, so

$$\begin{aligned} \left\langle F(z^k), \Pi_C(x^{i_k} - F(x^{i_k})) - z^k \right\rangle &= \left\langle F(y^{i_k}), \Pi_C(x^{i_k} - F(x^{i_k})) - x^{i_k} + \eta_{i_k} r(x^{i_k}) \right\rangle \\ &= (\eta_{i_k} - 1) \left\langle F(y^{i_k}), r(x^{i_k}) \right\rangle \\ &\leq (\eta_{i_k} - 1)(1 - \sigma) \|r(x^{i_k})\|^2 \\ &< 0. \end{aligned} \tag{3.13}$$

This verifies the claim.

By (3.12) and (3.13), it follows from Lemma 3.1 that there exists an $\alpha' > 0$ such that $\left\langle F(z^k), x^k(\alpha') - z^k \right\rangle = 0$. Moreover, since

$$\begin{aligned} 0 &< \langle F(z^k), x^k - z^k \rangle \\ &= \langle F(z^k), x^k - x^k(\alpha') \rangle \\ &= \langle F(z^k), \Pi_C(x^k) - \Pi_C(x^k - \alpha' F(z^k)) \rangle \\ &\leq \alpha' \|F(z^k)\|^2, \end{aligned}$$

it follows that

$$0 < \alpha_k^1 = \frac{\langle F(z^k), x^k - z^k \rangle}{\|F(z^k)\|^2} \le \alpha'. \tag{3.14}$$

This, together with the fact that $\Phi_k'(\alpha)$ is a continuous and decreasing function of α on $[0, +\infty)$, implies that $\{\alpha : \Phi_k'(\alpha) = 0, \alpha \in [0, +\infty)\}$ is a closed interval $[\alpha_{kl}, \alpha_{ku}]$ and $\alpha_{kl} \ge \alpha_k^1 > 0$. Let $\alpha_k^2 = \alpha_{kl}$. We have

$$0 < \Phi_k'(\alpha), \forall \alpha \in [0, \alpha_k^2). \tag{3.15}$$

Conditions (3.15) and (3.14) together imply that

$$0 \le \Phi_k'(\alpha_k) = 2 \left\langle F(z^k), x^k(\alpha_k) - z^k \right\rangle, \forall \alpha_k \in [\alpha_k^1, \alpha_k^2].$$

Therefore, if $\alpha_k \in [\alpha_k^1, \alpha_k^2]$, then α_k satisfies (3.2) and (3.3).

This verifies that Step 2 is well-defined.

Now we present a global convergence result under the assumption that the dual variational inequality has a solution, without assuming usual generalized monotonicity. However, this result applies to quasimonotone variational inequalities having a nontrivial solution, as shown by Proposition 2.2 that the dual variational inequality has a solution in this case.

Theorem 3.1. If there exists $z_0 \in C$ such that

$$\langle F(y), y - z_0 \rangle \ge 0, \quad \forall\, y \in C, \tag{3.16}$$

then either the algorithm terminates in a finite number of iterations or generates a sequence $\{x^k\}$ converging to a solution of the variational inequality problem.

Proof. Fix any $x \in C$.

$$\begin{aligned} \|x^{k+1} - x\|^2 &= \|\Pi_C(x^k - \alpha_k F(z^k)) - x\|^2 \\ &\le \|x^k - \alpha_k F(z^k) - x\|^2 - \|x^k - \alpha_k F(z^k) - x^{k+1}\|^2 \\ &= \|x^k - x\|^2 - 2\alpha_k \left\langle F(z^k), x^k - x \right\rangle + (\alpha_k)^2 \|F(z^k)\|^2 - \|x^k - \alpha_k F(z^k) - x^{k+1}\|^2. \end{aligned} \tag{3.17}$$

It follows that for any $z_0 \in C$ satisfying (3.16),

$$\|x^{k+1} - z_0\|^2 \le \|x^k - z_0\|^2 - 2\alpha_k \left\langle F(z^k), x^k - z^k \right\rangle + (\alpha_k)^2 \|F(z^k)\|^2 - \|x^k - \alpha_k F(z^k) - x^{k+1}\|^2. \tag{3.18}$$

Conditions (3.9) and (3.15) together imply that

$$0 < \Phi_k(\alpha_k^1) \le \Phi_k(\alpha_k) \le \Phi_k(\alpha_k^2), \forall \alpha_k \in [\alpha_k^1, \alpha_k^2].$$

Consequently, by (3.18), it follows that

$$\begin{aligned} \|x^{k+1} - z_0\|^2 &\le \|x^k - z_0\|^2 - \Phi_k(\alpha_k) \\ &\le \|x^k - z_0\|^2 - \Phi_k(\alpha_k^1) \\ &= \|x^k - z_0\|^2 - \frac{\left\langle F(z^k), x^k - z^k \right\rangle^2}{\|F(z^k)\|^2} - \|x^k - \alpha_k^1 F(z^k) - x^k(\alpha_k^1)\|^2 \\ &\le \|x^k - z_0\|^2 - \frac{\left\langle F(z^k), x^k - z^k \right\rangle^2}{\|F(z^k)\|^2}, \forall \alpha_k \in [\alpha_k^1, \alpha_k^2]. \end{aligned} \tag{3.19}$$

Thus (3.19) implies that

$$\lim_{k \to +\infty} \frac{\left\langle F(z^k), x^k - z^k \right\rangle}{\|F(z^k)\|} = 0. \tag{3.20}$$

This, (3.1), and (3.11) together imply that

$$\lim_{k\to+\infty} \frac{\langle F(y^k), x^k - y^k \rangle}{\|F(y^k)\|} = 0. \tag{3.21}$$

By (3.19), $\{\|x^k - z_0\|\}$ is a decreasing sequence and hence $\{x^k\}$ is bounded. Moreover, $\{F(y^k)\}$ is bounded. Therefore an immediate consequence of (3.21) and (3.11) is that $\eta_k(1-\sigma)\|r(x^k)\|^2 \to 0$ as $k \to +\infty$. Since the sequence $\{x^k\}$ being bounded, there exists subsequence $\{x^{k_l}\}$ such that $x^{k_l} \to x'$.

Case 1. If $\limsup \eta_{k_l} > 0$, then $r(x') = 0$.

Using (3.1), we know

$$\frac{\langle F(y^{i_{k_l}}), x^{k_l} - y^{i_{k_l}} \rangle}{\|F(y^{i_{k_l}})\|} \geq \frac{\langle F(y^j), x^{k_l} - y^j \rangle}{\|F(y^j)\|}, 0 \leq j \leq k_l.$$

Therefore, let $k_l \to \infty$, we get

$$\frac{\langle F(y^j), x' - y^j \rangle}{\|F(y^j)\|} \leq 0 \Leftrightarrow \langle F(y^j), x' - y^j \rangle \leq 0, j \in \{0, 1, \ldots\}.$$

If we replace z_0 by x' in the above process, then we obtain the decreasing sequence $\{\|x^k - x'\|\}$. Moreover, since x' is an accumulation point of $\{x^k\}$, we have $x^k \to x'$ as $k \to +\infty$.

Case 2. If $\limsup \eta_{k_l} = 0$, then

$$\langle F(x^{k_l}) - F(x^{k_l} - \gamma^{-1}\eta_{k_l} r(x^{k_l})), r(x^{k_l}) \rangle > \sigma\|r(x^{k_l})\|^2.$$

Moreover, we have

$$\sigma\|r(x')\|^2 \leq 0, \sigma \in (0, 1).$$

Therefore, $r(x') = 0$, by repeating the process in the Case 1, we have $x^k \to x'$ as $k \to +\infty$. □

4. Numerical experiments

Algorithm 1 is tested through following examples under MATLAB version R2007b. The termination criterion is $\|r(x)\| \leq 10^{-4}$. We choose $\sigma = 0.99, \gamma = 0.4$ in Algorithm 1. Let nf denote the total number of times that F is evaluated. We denote by x^* the solution of VI(F, C).

Example 4.1. Let $C = [-1, 1]^4$ and $F(x) = (x_1^2, x_2^2, x_3^2, x_4^2), x \in C$. Then $(-1, -1, -1, -1)$ is a solution of dual variational inequality. The performance of Algorithm 1 with different initial points is listed in Table 1.

Table 1: result for Example 4.1

x^0	iter(nf)	time	x^*
$(0.5, 0.5, 0.5, 0.5)$	135(406)	0.0936006	(0.00705625,0.00705625,0.00705625,0.00705625)
$(0.5, 0.5, 0.5, -0.5)$	126(379)	2.55842	(0.00759151,0.00759151,0.00759151,-1)
$(0.5, -0.5, 0.5, 0.5)$	126(379)	2.49602	(0.00759151,-1,0.00759151,0.00759151)
$(0.5, -0.5, 0.5, -0.5)$	114(343)	2.29321	(0.00835991,-1,0.00835991,-1)
$(0.5, -0.5, -0.5, -0.5)$	95(286)	1.96561	(0.00995731,-1,-1,-1)
$(-0.5, -0.5, -0.5, -0.5)$	2(7)	0.0156001	(-1,-1,-1,-1)

Example 4.2. Let $C = [0,1]^2$. We define

$$F(x_1, x_2) = \begin{cases} (\frac{-t}{1+t}, \frac{-1}{1+t}) & \text{if } (x_1, x_2) \neq (0,0) \\ (0, -1) & \text{if } (x_1, x_2) = (0,0), \end{cases} \tag{4.1}$$

where $t = \frac{x_1 + \sqrt{x_1^2 + 4x_2}}{2}$. Then $(1,1)$ is a solution of the dual variational inequality. The performance of Algorithm 1 with different initial points is listed in Table 2.

Table 2: result for Example 4.2

x^0	iter(nf)	time	x^*
$(0,1)$	2(7)	1.06081	$(1,1)$
$(0,0)$	2(7)	0.889206	$(1,1)$
$(1,0)$	3(10)	2.43362	$(1,1)$
$(0.5,0.5)$	1(4)	0.811205	$(1,1)$

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