High-Order Evaluation Complexity for Convexly-Constrained Optimization with Non-Lipschitzian Group Sparsity Terms

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27 February, 2019

Abstract

This paper studies high-order evaluation complexity for partially separable convexly-constrained optimization involving non-Lipschitzian group sparsity terms in a nonconvex objective function. We propose a partially separable adaptive regularization algorithm using a p-th order Taylor model and show that the algorithm can produce an (ϵ, δ)-approximate q-th-order stationary point at most $O(\epsilon^{-(p+1)/(p-q+1)})$ evaluations of the objective function and its first p derivatives (whenever they exist). Our model uses the underlying rotational symmetry of the Euclidean norm function to build a Lipschitzian approximation for the non-Lipschitzian group sparsity terms, which are defined by the group ℓ_2-ℓ_a norm with $a \in (0, 1)$. The new result shows that the partially-separable structure and non-Lipschitzian group sparsity terms in the objective function may not affect the worst-case evaluation complexity order.

Keywords: complexity theory, nonlinear optimization, non-Lipschitz functions, partially-separable problems, group sparsity, isotropic model.

AMS subject classifications:, 90C30, 90C46, 65K05

1 Introduction

Both applied mathematicians and computer scientists have, in recent years, made significant contributions to the fast-growing field of worst-case complexity analysis for nonconvex optimization (see [12] for a partial yet substantial bibliography). The purpose of this paper is to extend the available general theory in two distinct directions. The first is to cover the case where the problem involving non-Lipschitzian group sparsity terms. The second is to show that the ubiquitous partially-separable structure (of which standard sparsity is a special case) can be exploited without affecting the complexity bounds.

We consider the partially-separable convexly constrained nonlinear optimization problem:

$$\min_{x \in \mathcal{F}} f(x) = \sum_{i \in \mathcal{N}} f_i(U_i x) + \sum_{i \in \mathcal{H}} \|U_i x - b_i\|^a \stackrel{\rm def}{=} \sum_{i \in \mathcal{N} \cup \mathcal{H}} f_i(U_i x) \tag{1.1}$$

where $\mathcal{N} \cup \mathcal{H} \stackrel{\rm def}{=} \mathcal{M}$, $\mathcal{N} \cap \mathcal{H} = \emptyset$, f_i is a continuously p times differentiable function from $\mathbb{R}^{n_i}$ into $\mathbb{R}$ for $i \in \mathcal{N}$, and $f_i(x) = \|U_i x - b_i\|^a$ for $i \in \mathcal{H}$, $a \in (0, 1)$, $\|\cdot\|$ is the Euclidean norm,

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$U_i \in \mathbb{R}^{n_i \times n}$ with $n_i \leq n$, and $b_i \in \mathbb{R}^{n_i}$. Without loss of generality, we assume that, for each $i \in \mathcal{M}$, U_i has full row rank and $\|U_i\| = 1$, and that the ranges of the U_i^T for $i \in \mathcal{N}$ span $\mathbb{R}^n$ so that the intersection of the nullspaces of the U_i is reduced to the origin. We also assume that the ranges of the U_i^T (for $i \in \mathcal{H}$) are orthogonal, that is

$$U_i U_j^T = 0 \ \text{ for } \ i \neq j,\, i, j \in \mathcal{H}. \tag{1.2}$$

Without loss of generality, we furthermore assume that the rows of U_i are orthonormal for $i \in \mathcal{H}$. Our final assumption, as in [17], is that the feasible set $\mathcal{F} \subseteq \mathbb{R}^n$ is non-empty closed and convex, and that it is "kernel-centered" in that is, if $P_{\mathcal{X}}[\cdot]$ is the orthogonal projection onto the convex set $\mathcal{X}$ and † denotes the Moore-Penrose generalized inverse, then

$$U_i^\dagger b_i + P_{\ker(U_i)}[\mathcal{F}] \subseteq \mathcal{F} \ \text{ whenever } \ b_i \in U_i\mathcal{F}, \quad i \in \mathcal{H}. \tag{1.3}$$

These assumptions do not restrict our study for applications. For example, consider the row sparse problem in multivariate regression [29, 30, 38]

$$\min_{X \in R^{\nu \times \gamma}} \ \|HX - B\|_F^2 + \lambda \|X\|_{\ell_a/\ell_2}, \tag{1.4}$$

where $H \in \mathbb{R}^{\kappa \times \nu}, B \in \mathbb{R}^{\kappa \times \gamma}, \|\cdot\|_F$ is the Frobenius norm of a matrix,

$$\|HX - B\|_F^2 = \sum_{j=1}^{\gamma} \sum_{i=1}^{\kappa} (\sum_{\ell=1}^{\nu} H_{i\ell} X_{\ell,j} - B_{ij})^2 \quad \text{and} \quad \|X\|_{\ell_a/\ell_2} = \sum_{i=1}^{\nu} \big(\sum_{j=1}^{\gamma} X_{ij}^2 \big)^{\frac{a}{2}}.$$

Let $n = \nu\gamma$, $\mathcal{F} = \mathbb{R}^n, b_i = 0$, $x = (x_{11}, x_{12}, \ldots, x_{\nu\gamma})^T \in \mathbb{R}^n$ and set $U_i \in \mathbb{R}^{\nu \times n}$ for $i \in \mathcal{N} = \{1, \ldots, \gamma\}$ be the projection whose entries are 0, or 1 such that $U_i x$ be the ith column of X and $U_i \in \mathbb{R}^{\gamma \times n}$ for $i \in \mathcal{H} = \{1, \ldots, \nu\}$ be the projection whose entries are 0, or 1 such that $U_i x$ be the ith row of X. Then problem (1.4) can be written in the form of (1.1). It is easy to see that the $\{U_i^T\}_{i \in \mathcal{N}}$ span $\mathbb{R}^n$. Hence, all assumptions mentioned above hold for problem (1.4).

Problem (1.1) encompasses the non-overlapping group sparse optimization problems. Let $G_1, \ldots, G_m$ be subsets of $\{1, \ldots, n\}$ representing known groupings of the decision variable with size $n_1, \ldots, n_m$ and $G_i \cap G_j = \emptyset, i \neq j$. In this case, problem (1.1) reduces to

$$\min_{x \in \mathcal{F}} f_1(x) + \lambda \sum_{i=1}^{m} \|U_i x\|^a, \tag{1.5}$$

where $f_1 : \mathbb{R}^n \to \mathbb{R}_+$ is a smooth loss function, $\lambda > 0$ is a positive number and $U_i \in \mathbb{R}^{n_i \times n}$ is defined in the following way

$$(U_i)_{kj} = \begin{cases} 1 & \text{if } j \in G_i \\ 0 & \text{otherwise} \end{cases} \qquad \text{for} \quad k = 1, \ldots, n_i.$$

Thus $U_i x = x_{G_i}$ is the ith group variable vector in $\mathbb{R}^{n_i}$ with components $x_j, j \in G_i$. If $\mathcal{F} = \{x \,|\, \alpha_i \leq x_i \leq \beta_i, i = 1, \ldots, n\}$ with $\alpha_i < 0 < \beta_i$, then all assumptions mentioned above with $U_1 = I \in \mathbb{R}^{n \times n}$, for $1 \in \mathcal{N}$ hold for problem (1.5).

In problem (1.1), the decision variables have a group structure so that components in the same group tend to vanish simultaneously. Group sparse optimization problems have been

extensively studied in recent years due to numerous applications. In machine learning and statistics, when the explanatory variables have high correlative nature or can be naturally grouped, it is important to study variable selection at the group sparsity setting [8, 29, 30, 34, 36, 39]. In compressed sensing, group sparsity is refereed to as block sparsity and has been efficiently used to recovery signals with special block structures [1, 22, 32, 33, 35]. In spherical harmonic representations of random fields on the sphere, group Lasso penalty grouped the coefficients of homogeneous harmonic polynomials of the same degree is rotationally invariant while Lasso penalty ($G_i = \{i\}$) is not [31].

Problem (1.1) with $a \in (0, 1)$ and $n_i = 1, i \in \mathcal{H}$ has been studied in [5, 6, 14, 15, 16, 18]. Chen, Toint and Wang [17] show that an adaptive regularization algorithm using a p-th order Taylor model for p *odd* needs in general at most $O(\epsilon^{-(p+1)/p})$ evaluations of the objective function and its derivatives (at points where they are defined) to produce an ϵ-approximate first order critical point. Since this complexity bound is identical in order to that already known for convexly constrained Lipschitzian minimization, the result in [17] shows that introducing non-Lipschitzian singularities in the objective function may not affect the worst-case evaluation complexity order.

The unconstrained optimization of smooth partially-separable was first considered in Griewank and Toint [28], studied by many researchers [25, 24, 13, 37, 19, 21] and extensively used in the popular CUTEst testing environment [26] as well as in the AMPL [23], LANCELOT [20] and FILTRANE [27] packages.

In problem (1.1), all these "element functions" f_i depend on $U_i x \in \mathbb{R}^{n_i}$ rather than on x, which is most useful when $n_i \ll n$. Letting

$$x_i = U_i x \in \mathbb{R}^{n_i} \text{ for } i \in \mathcal{M} \quad \text{and} \quad f_{\mathcal{I}}(x) = \sum_{i \in \mathcal{I}} f_i(x) \text{ for any } \mathcal{I} \subseteq \mathcal{M},$$

we define

$$f_{\mathcal{N}}(x) \stackrel{\rm def}{=} \sum_{i \in \mathcal{N}} f_i(U_i x) = \sum_{i \in \mathcal{N}} f_i(x_i) \quad \text{and} \quad f_{\mathcal{H}}(x) \stackrel{\rm def}{=} \sum_{i \in \mathcal{H}} f_i(U_i x) = \sum_{i \in \mathcal{H}} f_i(x_i).$$

The p-th degree Taylor series

$$T_{f_{\mathcal{N}},p}(x, s) = f_{\mathcal{N}}(x) + \sum_{j=1}^{p} \frac{1}{j!} \nabla_x^j f_{\mathcal{N}}(x)[s]^j, \text{ where } \nabla_x^j f_{\mathcal{N}}(x)[s]^j = \sum_{i \in \mathcal{N}} \nabla_{x_i}^j f_i(x_i)[U_i s]^j, \quad (1.6)$$

indicates that, for each j, only the $|\mathcal{N}|$ tensors $\{\nabla_{x_i}^j f_i(x_i)\}_{i \in \mathcal{N}}$ of dimension n_i^j needs to be computed and stored. Exploiting derivative tensors of order larger than 2 — and thus using the high-order Taylor series (1.6) as a local model of $f_{\mathcal{N}}(x + s)$ in the neighbourhood of x — may therefore be practically feasible in our setting since n_i^j is typically orders of magnitude smaller than n. The same comment applies to $f_{\mathcal{H}}(x)$ whenever $\|U_i x - b_i\| \neq 0$.

The main contribution of this paper is twofold.

- We propose a partially separable adaptive regularization algorithm with a p-th order Taylor model which uses the underlying rotational symmetry of the Euclidean norm function for $f_{\mathcal{H}}$ and the first p derivatives (whenever they exist) of the "element functions" f_i, for $i \in \mathcal{M}$.

- We show that the algorithm can produce an (ϵ, δ)-approximate q-th-order critical point of problem (1.1) at most $O(\epsilon^{-(p+1)/(p-q+1)})$ evaluations of the objective function and its first p derivatives for any $q \in \{1, \ldots, p\}$.

Our results extend worst-case evaluation complexity bounds for smooth nonconvex optimization in [11, 12] which do not use the structure of partially separable functions and do not consider the Lipschitzian singularity. Moreover, our results subsume the results for non-Lipschitz nonconvex optimization in [17] which only consider the complexity with $q = 1$ and $n_i = 1$ for $i \in \mathcal{H}$.

This paper is organized as follows. In Section 2, we define an (ϵ, δ) q-order necessary optimality conditions for local minimizers of problem (1.1). A Lipschitz continuous model to approximate f is proposed in Section 3. We then propose the partially separable adaptive regularization algorithm using the p-th order Taylor model in Section 4. In Section 5, we show that the algorithm produces an (ϵ, δ)-approximate q-th-order critical point at most $O(\epsilon^{-(p+1)/(p-q+1)})$ evaluations of f and its first p derivatives.

We end this section by introducing notations used in the next four sections.

Notations. For a symmetric tensor S of order p, $S[v]^p$ is the result of applying S to p copies of the vector v and

$$\|S\|_{[p]} \stackrel{\rm def}{=} \max_{\|v\|=1} |S[v]^p| \tag{1.7}$$

is the associated induced norm for such tensors. If S_1 and S_2 are tensors, $S_1 \otimes S_2$ is their tensor product and $S_1^{k\otimes}$ is the product of S_1 k times with itself. For any set $\mathcal{X}$, $|\mathcal{X}|$ denotes its cardinality.

Because the notion of partial separability hinges on geometric interpretation of the problem, it is useful to introduce the various subspaces of interest for our analysis. We will extensively use the following definitions. As will become clear in Section 2, we will need to identify

$$\mathcal{C}(x,\epsilon) \stackrel{\rm def}{=} \{i \in \mathcal{H} \mid \|U_i x - b_i\| \leq \epsilon\} \;\; \text{and} \;\; \mathcal{A}(x,\epsilon) \stackrel{\rm def}{=} \mathcal{H} \setminus \mathcal{C}(x,\epsilon), \tag{1.8}$$

the collection of hard elements which are close to singularity for a given x and its complement (the "active" elements), and

$$\mathcal{R}(x,\epsilon) \stackrel{\rm def}{=} \bigcap_{i\in\mathcal{C}(x,\epsilon)} \ker(U_i) = \left[\operatorname*{span}_{i\in\mathcal{C}(x,\epsilon)} (U_i^T)\right]^\perp \tag{1.9}$$

the subspace in which those nearly singular elements are invariant. (When $\mathcal{C}(x,\epsilon) = \emptyset$, we set $\mathcal{R}(x,\epsilon) = \mathbb{R}^n$.) For convenience, if $\epsilon = 0$, we denote $\mathcal{C}(x) \stackrel{\rm def}{=} \mathcal{C}(x,0)$, $\mathcal{A}(x) \stackrel{\rm def}{=} \mathcal{A}(x,0)$, $\mathcal{R}(x) \stackrel{\rm def}{=} \mathcal{R}(x,0)$ and $\mathcal{W}(x) \stackrel{\rm def}{=} \mathcal{W}(x,0)$. From these definitions, we have

$$U_i d = 0, \;\; \text{for} \;\; i \in \mathcal{C}(x), \;\; d \in \mathcal{R}(x). \tag{1.10}$$

Also denote by

$$\mathcal{R}_{\{i\}} \stackrel{\rm def}{=} \operatorname{span}(U_i^T) \tag{1.11}$$

and observe that (1.2) implies that the $\mathcal{R}_{\{i\}}$ are orthogonal for $i \in \mathcal{H}$. Hence $\mathcal{R}_{\{i\}}$ is also the subspace in which all singular elements are invariant but the i-th. We also denote the "working" collection of elements not close to singularity by

$$\mathcal{W}(x,\epsilon) \stackrel{\rm def}{=} \mathcal{N} \cup \mathcal{A}(x,\epsilon). \tag{1.12}$$

If $\{x_k\}$ is a sequence of iterates in $\mathbb{R}^n$, we also use the shorthands

$$\mathcal{C}_k = \mathcal{C}(x_k,\epsilon), \quad \mathcal{A}_k = \mathcal{A}(x_k,\epsilon), \quad \mathcal{R}_k = \mathcal{R}(x_k,\epsilon) \;\; \text{and} \;\; \mathcal{W}_k = \mathcal{W}(x_k,\epsilon). \tag{1.13}$$

We will make frequent use of

$$f_{\mathcal{W}_k}(x) \stackrel{\rm def}{=} \sum_{i \in \mathcal{W}_k} f_i(x), \tag{1.14}$$

which is objective function "reduced" to the elements "away from singularity" at x_k.

For some $x, s \in \mathbb{R}^n$, we often use the notations $r_i = U_i x - b_i$ and $s_i = Us$.

2 Necessary optimality conditions

At variance with the theory developed in [17], which solely covers convergence to ϵ-approximate first-order stationary points, we now consider arbitrary orders of optimality. To this aim, we follow [11] and define, for a sufficiently smooth function $h : \mathbb{R}^n \to \mathbb{R}$ and a convex set $\mathcal{F} \subseteq \mathbb{R}^n$, the vector x to be an (ϵ, δ)-approximate q-th-order stationary point ($\epsilon > 0, \delta > 0$, $q \in \{1, \ldots, p\}$) of $\min_{x \in \mathcal{F}} h(x)$ if, for some $\delta \in (0, 1]$

$$\phi^\delta_{h,q}(x) \leq \epsilon \chi_q(\delta) \tag{2.1}$$

where

$$\phi^\delta_{h,q}(x) \stackrel{\rm def}{=} h(x) - \min_{\substack{x+d \in \mathcal{F} \\ \|d\| \leq \delta}} T_{h,q}(x, d), \tag{2.2}$$

and

$$\chi_q(\delta) \stackrel{\rm def}{=} \sum_{\ell=1}^{q} \frac{\delta^\ell}{\ell!}. \tag{2.3}$$

In other words, we declare x to be an (ϵ, δ)-approximate q-th-order stationary point if the scaled maximal decrease that can be obtained on the q-th order Taylor series for h in a neighbourhood of x of radius δ is at most ϵ. We refer the reader to [11] for a detailed motivation and discussion of this measure. For our present purpose, it is enough to observe that $\phi^\delta_{h,q}(x)$ is a continuous function of x and δ for any q. Moreover, for $q = 1$ and $q = 2$, δ can be chosen equal to one and $\phi^1_{h,1}(x)$ and $\phi^1_{h,2}(x)$ are easy to compute. In the unconstrained case,

$$\phi^1_{h,1}(x) = \|\nabla^1_x h(x)\|$$

and computing $\phi^1_{h,2}$ reduces to solving the standard trust-region problem

$$\phi^1_{h,2}(x) = \left| \min_{\|d\| \leq 1} \nabla^1_x h(x)[d] + \tfrac{1}{2} \nabla^2_x h(x)[d]^2 \right|.$$

In the constrained case,

$$\phi^1_{h,1}(x) = \left| \min_{\substack{x+d \in \mathcal{F} \\ \|d\| \leq 1}} \nabla^1_x h(x)[d] \right|,$$

which is the optimality measure used in [9] or [17] among others. However, given the potential difficulty of solving the global optimization problem in (2.2) for $q > 2$, our approach remains, for now, conceptual for such high optimality orders.

We now claim that we can extend the definition (2.1) to cover problem (1.1) as well. The key observation is that, by the definition of $\mathcal{W}(x, \epsilon)$ and $\mathcal{R}(x, \epsilon)$,

$$f_{\mathcal{W}(x,\epsilon)}(x) = f_{\mathcal{W}(x,\epsilon)}(x + d) \leq f(x + d) \leq f_{\mathcal{W}(x,\epsilon)}(x + d) + \epsilon^a |\mathcal{H}| \ \text{ for all } \ d \in \mathcal{R}(x, \epsilon). \tag{2.4}$$

Note now that $f_{\mathcal{W}(x,\epsilon)}$ is smooth around x because it only contains elements which are away from singularity, and hence that $T_{f_{\mathcal{W}(x,\epsilon)},p}(x,s)$ is well-defined. We may therefore define x to be an (ϵ,δ)-approximate q-th-order stationary point for (1.1) if, for some $\delta \in (0,1]$

$$\psi^{\epsilon,\delta}_{f,q}(x) \leq \epsilon\chi_q(\delta), \tag{2.5}$$

where we define

$$\psi^{\epsilon,\delta}_{f,q}(x) \stackrel{\rm def}{=} f(x) - \min_{\substack{x+d\in\mathcal{F}\\ \|d\|\leq\delta,\, d\in\mathcal{R}(x,\epsilon)}} T_{f_{\mathcal{W}(x,\epsilon)},q}(x,d). \tag{2.6}$$

By the definition of $\mathcal{W}(x,\epsilon)$, we have $f_{\mathcal{W}(x,\epsilon)}(x) \leq f(x)$ and thus

$$f_{\mathcal{W}(x,\epsilon)}(x) - \min_{\substack{x+d\in\mathcal{F}\\ \|d\|\leq\delta,\, d\in\mathcal{R}(x,\epsilon)}} T_{f_{\mathcal{W}(x,\epsilon)},q}(x,d) \leq \psi^{\epsilon,\delta}_{f,q}(x).$$

Taking $\epsilon = 0$, x is q-th-order stationary point if $\psi^{0,\delta}_{f_{\mathcal{W}(x)},q} = 0$, as we now prove.

Theorem 2.1 *If x_* is a local minimizer of (1.1), then there is $\delta \in (0,1]$ such that*

$$\psi^{0,\delta}_{f,q}(x_*) = 0. \tag{2.7}$$

Proof. Suppose first that $\mathcal{R}(x_*) = \{0\}$ (which happens if there exists $x_* \in \mathcal{F}$ such that $f_{\mathcal{H}}(x_*) = 0$ and $\mathrm{span}_{i\in\mathcal{H}}\{U_i^T\} = \mathbb{R}^n$). Then (2.7) holds vacuously with any $\delta \in (0,1]$. Now suppose that $\mathcal{R}(x_*) \neq \{0\}$. Let

$$\delta_1 = \min\left[1, \min_{i\in\mathcal{A}(x_*)} \|U_i x_* - b_i\|\right] \in (0,1].$$

Since x_* is a local minimizer of (1.1), there exists $\delta_2 > 0$ such that

$$\begin{aligned}
f(x_*) &= \min_{\substack{x_*+d\in\mathcal{F}\\ \|d\|\leq\delta_2}} f_{\mathcal{N}}(x_*+d) + \sum_{i\in\mathcal{H}} \|U_i(x_*+d) - b_i\|^a \\
&\leq \min_{\substack{x_*+d\in\mathcal{F}\\ \|d\|\leq\delta_2,\, d\in\mathcal{R}(x_*)}} f_{\mathcal{N}}(x_*+d) + \sum_{i\in\mathcal{H}} \|U_i(x_*+d) - b_i\|^a \\
&= \min_{\substack{x_*+d\in\mathcal{F}\\ \|d\|\leq\delta_2,\, d\in\mathcal{R}(x_*)}} f_{\mathcal{N}}(x_*+d) + \sum_{i\in\mathcal{A}(x_*)} \|U_i(x_*+d) - b_i\|^a \\
&= \min_{\substack{x_*+d\in\mathcal{F}\\ \|d\|\leq\delta_2,\, d\in\mathcal{R}(x_*)}} f_{\mathcal{W}(x_*)}(x_*+d),
\end{aligned}$$

where we used (1.10) and (1.12) to derive the last two equalities, respectively.

Now we consider the reduced problem

$$\min_{\substack{x_*+d\in\mathcal{F}\\ \|d\|\leq\delta_2,\, d\in\mathcal{R}(x_*)}} f_{\mathcal{W}(x_*)}(x_*+d). \tag{2.8}$$

Since we have that

$$f_{\mathcal{W}(x_*)}(x_*) = f_{\mathcal{N}}(x_*) + \sum_{i\in\mathcal{A}(x_*)} \|U_i x_* - b_i\|^a = f_{\mathcal{N}}(x_*) + \sum_{i\in\mathcal{H}} \|U_i x_* - b_i\|^a = f(x_*),$$

we obtain that

$$f_{\mathcal{W}(x_*)}(x_*) \leq \min_{\substack{x_*+d\in\mathcal{F}\\ \|d\|\leq\delta_2,\, d\in\mathcal{R}(x_*)}} f_{\mathcal{W}(x_*)}(x_*+d)$$

and x_* is a local minimizer of problem (2.8).

Note that for any x_*+d in the ball $B(x^*,\delta_3)$ with $\delta_3<\delta_1$, we have

$$\|U_i(x_*+d)-b_i\| \geq \|U_ix_*-b_i\| - \|U_id\| \geq \delta_1 - \|U_i\|\|d\| = \delta_1-\delta_3>0, \quad i\in\mathcal{A}(x_*).$$

Hence $f_{\mathcal{W}(x_*)}(x_*+d)$ is q-times continuously differentiable, and has Lipschitz continuous derivatives of orders 1 to q in $B(x^*,\delta_3)$. By Theorem 3.1 in [10], there is a $\delta\in\big(0,\min[\delta_2,\delta_3]\big]$, such that

$$\psi^{0,\delta}_{f_{\mathcal{W}(x_*)},q}(x_*) = f_{\mathcal{W}(x_*)}(x_*) - \min_{\substack{x_*+d\in\mathcal{F}\\ \|d\|\leq\delta,\ d\in\mathcal{R}(x_*)}} T_{f_{\mathcal{W}(x_*)},q}(x_*,d) = 0$$

This, together with $f(x_*)=f_{\mathcal{W}(x_*)}(x_*)$, gives the desired result (2.7). □

We call x_* is a q-th-order stationary point of (1.1) if there is $\delta\in(0,1]$ such that (2.7) holds.

Theorem 2.2 *For each k, let x_k be an (ϵ_k,δ_k)-approximate q-th-order stationary point of (1.1) with $1\geq\delta_k\geq\bar\delta>0$ and $\epsilon_k\to0$. Then any cluster point of $\{x_k\}$ is a q-th-order stationary point of (1.1).*

Proof. Let x_* be a cluster point of $\{x_k\}$. Without loss of generality, we assume that $x_*=\lim_{k\to\infty}x_k$. From $0<\chi_q(\delta)\leq2$ and $\psi^{\epsilon,\delta}_{f,q}(x)\geq0$, we have from (2.5) that $\lim_{k\to\infty}\psi^{\epsilon_k,\delta_k}_{f,q}(x_k)=0$. We now need to prove that $\psi^{0,\bar\delta}_{f,q}(x_*)=0$.

If $\mathcal{R}(x_*)=\{0\}$, (2.7) holds vacuously with any $\delta>0$, and hence x_* is a qth-order-necessary minimizer of (1.1). Suppose now that $\mathcal{R}(x_*)\neq\{0\}$. We first claim that there exists a $k_*\geq0$ such that

$$\mathcal{C}(x_k,\epsilon_k)\subseteq\mathcal{C}(x_*) \quad \text{for} \quad k\geq k_*. \tag{2.9}$$

To prove this inclusion, we choose k_* sufficiently large to ensure that

$$\|x_k-x_*\|+\epsilon_k < \min_{j\in\mathcal{A}(x_*)}\|U_jx_*-b_j\|, \quad \text{for} \quad k\geq k_*. \tag{2.10}$$

Such a k_* must exist, since the right-hand side of this inequality is strictly positive by definition of $\mathcal{A}(x_*)$. For an arbitrary $k\geq k_*$ and an index $i\in\mathcal{C}(x_k,\epsilon_k)$, using the definition of $\mathcal{C}(x,\epsilon)$, the identity $\|U_i\|=1$ and (2.10), we obtain that

$$\|U_ix_*-b_i\| \leq \|U_i(x_*-x_k)\| + \|U_ix_k-b_i\| \leq \|x_*-x_k\|+\epsilon_k < \min_{j\in\mathcal{A}(x_*)}\|U_jx_*-b_i\|.$$

This implies that $\|U_ix_*-b_i\|=0$ and $i\in\mathcal{C}(x_*)$. Hence (2.9) holds. By the definition of $\mathcal{R}(x,\epsilon)$ and $\mathcal{W}(x,\epsilon)$, (2.9) implies that, for all k,

$$\mathcal{R}(x_*)\subseteq\mathcal{R}(x_k,\epsilon_k) \quad \text{and} \quad \mathcal{W}(x_*)\subseteq\mathcal{W}(x_k,\epsilon_k). \tag{2.11}$$

For any fixed $k \geq k_*$, consider now the following three minimization problems:

$$(A,k) \quad \left\{ \begin{array}{ll} \min_d & T_{f_{\mathcal{W}(x_k,\epsilon_k)},q}(x_k,d) \\ \text{s.t.} & x_k + d \in \mathcal{F},\ d \in \mathcal{R}(x_k,\epsilon_k),\ \|d\| \leq \delta_k, \end{array} \right. \tag{2.12}$$

$$(B,k) \quad \left\{ \begin{array}{ll} \min_d & T_{f_{\mathcal{W}(x_k,\epsilon_k)},q}(x_k,d) \\ \text{s.t.} & x_k + d \in \mathcal{F},\ d \in \mathcal{R}(x_*),\ \|d\| \leq \delta_k, \end{array} \right. \tag{2.13}$$

and

$$(C,k) \quad \left\{ \begin{array}{ll} \min_d & T_{f_{\mathcal{W}(x_*)},q}(x_k,d) \\ \text{s.t.} & x_k + d \in \mathcal{F},\ d \in \mathcal{R}(x_*),\ \|d\| \leq \delta_k. \end{array} \right. \tag{2.14}$$

Since $d = 0$ is a feasible point of these three problems, their minimum values, which we respectively denote by $\vartheta_{A,k}$, $\vartheta_{B,k}$ and $\vartheta_{C,k}$, are all smaller than $f(x_k)$. Moreover, it follows from the first part of (2.11) that, for each k,

$$\vartheta_{B,k} \geq \vartheta_{A,k}. \tag{2.15}$$

It also follows from (2.9) and (1.2) that

$$T_{f_{\mathcal{W}(x_k,\epsilon_k)},q}(x_k,d) = T_{f_{\mathcal{W}(x_*)},q}(x_k,d) - f_{\mathcal{W}(x_*)}(x_k) + f_{\mathcal{W}(x_k,\epsilon_k)}(x_k) \leq T_{f_{\mathcal{W}(x_*)},q}(x_k,d) + |\mathcal{H}|\epsilon_k^a$$

for all $d \in \mathcal{R}(x_*)$, and thus (2.15) becomes

$$\vartheta_{A,k} \leq \vartheta_{B,k} \leq \vartheta_{C,k} + |\mathcal{H}|\epsilon_k^a \quad \text{for all} \quad k \geq k_*. \tag{2.16}$$

The assumption that x_k is an (ϵ_k, δ_k)-approximate qth-order necessary minimizer of (1.1) implies that

$$0 \leq f(x_k) - \vartheta_{C,k} - |\mathcal{H}|\epsilon_k^a \leq f(x_k) - \vartheta_{A,k} \leq \epsilon_k \chi_q(\delta_k), \quad \text{for all} \quad k \geq k_*. \tag{2.17}$$

Now (2.11) implies that $T_{f_{\mathcal{W}(x_*)},q}(x_k,d) \leq T_{f_{\mathcal{W}(x_k)},q}(x_k,d) \leq T_{f,q}(x_k,d)$. Hence

$$f(x_k) - \vartheta_{C,k} = f(x_k) - \min_{\substack{x_k+d\in\mathcal{F} \\ \|d\|\leq\delta_k,\ d\in\mathcal{R}(x_*)}} T_{f_{\mathcal{W}(x_*)},q}(x_k,d) \geq \phi_{f,q}^{0,\delta_k}(x_k).$$

As a consequence, (2.17) implies that

$$\phi_{f,q}^{0,\delta_k}(x_k) \leq \epsilon_k \chi_q(\delta_k) + |\mathcal{H}|\epsilon_k^a. \tag{2.18}$$

In addition, the feasible sets of the three problems (2.12)-(2.14) are convex, and the objectives functions are polynomials with degree q. By the perturbation theory for optimization problems [21, Theorem 3.2.8], we can claim that

$$\lim_{k\to\infty} \vartheta_{C,k} = \min_{\substack{x_*+d\in\mathcal{F} \\ \|d\|\leq\delta_*,\ d\in\mathcal{R}(x_*)}} T_{f_{\mathcal{W}(x_*)},q}(x_*,d), \tag{2.19}$$

where $\delta_* = \liminf_{k\to\infty} \delta_k \geq \bar{\delta}$. This implies that letting $k \to \infty$ in (2.18) gives

$$\psi_{f,q}^{0,\bar{\delta}}(x_*) = f(x_*) - \min_{\substack{x_*+d\in\mathcal{F} \\ \|d\|\leq\bar{\delta},\, d\in\mathcal{R}(x_*)}} T_{f_{\mathcal{W}(x_*)},q}(x_*,d) = 0.$$

□

We conclude this section by an important observation. The optimality measure (2.5) may give the impression (in particular in its use of $\mathcal{R}(x,\epsilon)$) that the "singular" and "smooth" parts of the problem are merely separated, and that one could possibly apply the existing theory for smooth problems to the latter. Unfortunately, this is not true, because the "separation" implied by (2.5) does depend on ϵ, and one therefore need to show that the complexity of minimizing the "non-singular" part does not explode (in particular with the unbounded growth of the Lispchitz constant) when ϵ tends to zero. Designing an suitable algorithm and proving an associated complexity result comparable to what is known for smooth problems is the main challenge in what follows.

3 A Lipschitz continuous model of $f_{\mathcal{W}_k}(x+s)$

Our minimization algorithm, described in the next section, involves the approximate minimization of a *model* $m(x_k,s)$ of $f_{\mathcal{W}_k}$ in the intersection of a neighbourhood of x_k with $\mathcal{R}_k$. This model, depending on function and derivatives values computed at x_k, should be able to predict values and derivatives of f at some neighbouring point x_k+s reasonably accurately. This is potentially difficult if the current point happens to be near a singularity.

Before describing our proposal, we need to state a useful technical result.

Lemma 3.1 *Let a be a positive number and $r\neq 0$. Define, for a positive integer j,*

$$\pi(a-j) \stackrel{\rm def}{=} a\prod_{i=1}^{j-1}(a-i). \tag{3.1}$$

Then, if $\nabla_r^j \|r\|^a$ is the value of the j-th derivative tensor of the function $\|\cdot\|^a$ with respect to its argument, evaluated at r, we have that,

$$\nabla_r^j \|r\|^a = \sum_{i=1}^{j} \phi_{i,j}\|r\|^{a-2i}\, r^{(2i-j)\otimes} \otimes I^{(j-i)\otimes} \tag{3.2}$$

for some scalars $\{\phi_{i,j}\}_{i=1}^j$ such that $\sum_{i=1}^j \phi_{i,j} = \pi(a-j)$, and that

$$\left\| \nabla_r^j \|r\|^a \right\|_{[j]} = |\pi(a-j)|\, \|r\|^{a-j}. \tag{3.3}$$

Moreover, if β_1,β_2 are positive reals and $\|r\|=1$, then

$$\left\| \nabla_r^j \|\beta_1 r\|^a - \nabla_r^j \|\beta_2 r\|^a \right\|_{[j]} = |\pi(a-j)| \left|\beta_1^{a-j} - \beta_2^{a-j}\right|. \tag{3.4}$$

Proof. See appendix. □

Consider now the elements f_i for $i\in\mathcal{N}$. Each such element is p times continuously differentiable and, if we assume that its p-th derivative tensor $\nabla_x^p f_i$ is globally Lipschitz continuous with constant $L_i\geq 0$ in the sense that, for all $x_i,y_i\in\mathbb{R}^{n_i}$

$$\|\nabla_{x_i}^p f_i(x_i) - \nabla_{x_i}^p f_i(y_i)\|_{[p]} \leq L_i\|x_i-y_i\|, \tag{3.5}$$

then it can be shown (see [11, Lemma 2.1]) that

$$f_i(x_i+s_i) = T_{f_i,p}(x_i,s_i) + \frac{1}{(p+1)!}\tau_i L_i \|s_i\|^{p+1} \quad \text{with} \quad |\tau_i|\leq 1. \tag{3.6}$$

Because $\tau_i L_i$ in (3.6) is usually unknown in practice, it may not be possible to use (3.6) directly to model f_i in a neighbourhood of x. However, we may replace this term with an adaptive parameter σ_i, which yields the following $(p+1)$-rst order model for the i-th "nice" element

$$m_i(x_i, s_i) = T_{f_i,p}(x_i, s_i) + \frac{1}{(p+1)!}\, \sigma_i \|s_i\|^{p+1}, \quad (i \in \mathcal{N}). \tag{3.7}$$

Using the associated Taylor's expansion would indeed ignore the non-Lipschitzian singularity occurring for $r_i = 0$ and this would restrict the validity of the model to a possibly very small neighbourhood of x_k whenever r_i is small for some $i \in \mathcal{A}(x_k, \epsilon)$. Our proposal is to use the underlying rotational symmetry of the Euclidean norm function to build a better Lipschtzian model. Suppose that $r_i \neq 0 \neq r_i + s_i$ and let

$$u_i = \frac{r_i}{\|r_i\|}, \quad r_i^+ = r_i(x+s) = r_i + s_i \quad \text{and} \quad u_i^+ = \frac{r_i^+}{\|r_i^+\|}. \tag{3.8}$$

Moreover, let R_i be the rotation in the (u_i, u_i^+) plane[(1)] such that

$$R_i u_i^+ = u_i. \tag{3.9}$$

We observe that, given the isotropic nature of the Euclidean norm, the value $\|r_i\|^a$ and of its derivatives with respect to s_i can be deduced from those $\| \|r_i\|u_i^+ \|^a$. More precisely, for any $d \in \mathbb{R}^{n_i}$,

$$\big\| \|r_i\|u_i^+ \big\|^a = \|r_i\|^a \ \text{ and } \ \nabla_{\cdot}^{\ell} \big\| \|r_i\|u_i^+ \big\|^a[d]^\ell = \nabla_{\cdot}^{\ell} \|r_i\|^a [R_i d]^\ell. \tag{3.10}$$

For example, when $j = 1$, one verifies that

$$\begin{aligned}
\nabla_{\cdot}^1 \|r_i\|^a[R_i d] &= a\|r_i\|^{a-2} r_i^T R_i d \\
&= a\|r_i\|^{a-2}\|r_i\|(R_i^T u_i)^T d \\
&= a\big\| \|r_i\|u_i^+ \big\|^{a-2} (\|r_i\|u_i^+)^T d \\
&= \nabla_{\cdot}^1 \big\| \|r_i\|u_i^+ \big\|^a [d].
\end{aligned}$$

We may then choose to compute the Taylor's expansion for the function $\|\cdot\|^a$ around $\|r_i\|u_i^+$, that is

$$\begin{aligned}
\|r_i^+\|^a &= \big\| \|r_i^+\|u_i^+ \big\|^a \\
&= \big\| \|r_i\|u_i^+ \big\|^a + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} \nabla_{\cdot}^{\ell} \big\| \|r_i\|u_i^+ \big\|^a \big[(\|r_i^+\| - \|r_i\|)u_i^+\big]^\ell \\
&= \|r_i\|^a + \sum_{\ell=1}^{\infty} \frac{(\|r_i^+\| - \|r_i\|)^\ell}{\ell!} \nabla_{\cdot}^{\ell} \big\|r_i\big\|^a \big[R_i u_i^+\big]^\ell \\
&= \|r_i\|^a + \sum_{\ell=1}^{\infty} \frac{(\|r_i^+\| - \|r_i\|)^\ell}{\ell!} \nabla_{\cdot}^{\ell} \big\|r_i\big\|^a [u_i]^\ell.
\end{aligned}$$

Using the expression (3.2) applied to ℓ copies of the unit vector u_i and the fact that $r_i^{j\otimes}[u_i]^j = (r_i^T u_i)^j = \|r_i\|^j$ for all $j \in \mathbb{N}$, we now deduce that, for $\zeta_i = \|r_i + s_i\| - \|r_i\| \geq -\|r_i\|$,

$$\|r_i + s_i\|^a = \|r_i\|^a + \sum_{\ell=1}^{\infty} \frac{\pi(a-\ell)}{\ell!} \zeta_i^\ell \|r_i\|^{a-\ell}, \tag{3.11}$$

(1) If $u_i = u_i^+$, $R_i = I$. If $n_i = 1$ and $r_i r_i^+ < 0$, this rotation is just the mapping from $\mathbb{R}_+$ to $\mathbb{R}_-$, defined by a simple sign change, as in the two-sided model of [17].

which is nothing else than the Taylor's expansion of $\|r_i+\zeta_i u_i\|^a$ (or, equivalently, of $\left\|\|r_i\|u_i^+ + \zeta_i u_i^+\right\|^a$) expressed as a function of the scalar variable $\zeta_i \geq -\|r_i\|$. As can be expected from the isotropic nature of the Euclidean norm, the value of $\|r_i^+\|^a$ (and of its derivatives after a suitable rotation) only depend(s) on the distance of r_i^+ to the singularity at zero. Thus, limiting the development (3.11) to degree p (as in (3.7)), it is natural to define

$$\mu(\|r_i\|,\zeta_i) \stackrel{\rm def}{=} \|r_i\|^a + \sum_{\ell=1}^{p} \frac{\pi(a-\ell)}{\ell!}\zeta_i^\ell \|r_i\|^{a-\ell}, \quad (i \in \mathcal{A}(x,\epsilon)), \tag{3.12}$$

which is a unidimensional model of $\|r_i + \zeta_i u_i\|^a$ based of the residual value $\|r_i\|$. Note that $\mu(\|r_i\|,\zeta_i)$ is Lipschitz continuous as a function of ζ_i as long as $\|r_i\|$ remains uniformly bounded away from zero, with a Lipschitz constant depending on the lower bound. We then define the isotropic model

$$m_i(x_i,s_i) \stackrel{\rm def}{=} \mu(\|r_i\|,\zeta_i) = \mu_i(\|r_i\|, \|r_i+s_i\| - \|r_i\|) \quad \text{for} \quad i \in \mathcal{A}(x,\epsilon), \tag{3.13}$$

so that, abusing notation slightly,

$$m_i(x_i,s_i) = T_{m_i,p}(x_i,s_i) \stackrel{\rm def}{=} \|r_i\|^a + \sum_{\ell=1}^{p} \frac{\pi(a-\ell)}{\ell!}\zeta_i^\ell \|r_i\|^{a-\ell}, \quad (i \in \mathcal{A}(x,\epsilon)).$$

We now state some useful properties of this model.

Lemma 3.2 *Suppose that p is odd and that $\mathcal{A}(x) \neq \emptyset$ for some $x \in \mathcal{F}$. Then, for $i \in \mathcal{A}(x)$,*

$$m_i(x_i,s_i) \geq \|r_i+s_i\|^a, \tag{3.14}$$

and, whenever $\|r_i+s_i\| \leq \|r_i(x)\|$,

$$\nabla_\zeta^\ell \mu(\|r_i\|,\zeta_i)) \geq \nabla_\zeta^\ell \mu(\|r_i\|,0) = \pi(a-\ell)\|r_i\|^{a-\ell} \quad (\ell \text{ odd}), \tag{3.15}$$

and

$$\nabla_\zeta^\ell \mu(\|r_i|,\zeta_i) \leq \nabla_\zeta^\ell \mu(\|r_i\|,0) = \pi(a-\ell)\|r_i\|^{a-\ell} \quad (\ell \text{ even}). \tag{3.16}$$

As a consequence, $m_i(x_i,\zeta_i)$ is a concave function of ζ_i on the interval $[-\|r_i\|,0]$.

Proof. Let $i \in \mathcal{A}(x)$. From the mean-value theorem and (3.11), we have that, for some $\nu \in (0,1)$,

$$\|r_i+s_i\|^a = \|r_i\|^a + \sum_{\ell=1}^{p} \frac{\pi(a-\ell)}{\ell!}\zeta_i^\ell \|r_i\|^{a-\ell} + \frac{\pi(a-p-1)}{(p+1)!}\zeta_i^{p+1}\|r_i+\nu\zeta_i u_i\|^{a-p-1}. \tag{3.17}$$

Since p is odd, we obtain that $\pi(a-p-1) < 0$ and $\zeta_i^{p+1} \geq 0$. Thus (3.14) directly results from (3.17), (3.12) and (3.13). Now (3.12) and (3.13) together imply that

$$\nabla_\zeta^\ell \mu(\|r_i\|,\zeta_i) = \nabla_\zeta^\ell \mu(\|r_i\|,0) + \sum_{j=\ell+1}^{p} \frac{\pi(a-j)}{(j-\ell)!}\zeta_i^{j-\ell}\|r_i\|^{a-j} \tag{3.18}$$

for $\zeta_i = \|r_i(x) + s_i\| - \|r_i(x)\| \leq 0$. Suppose first that ℓ is odd. Then we have that $\pi(a-j)$ is negative for even j, that is exactly when $\zeta_i^{j-\ell}$ is non-positive. Hence every term in the sum of the right-hand side of (3.18) is non-negative and (3.15) follows. Suppose now that ℓ is even. Then $\pi(a-j)$ is negative for odd j, which is exactly when $\zeta_i^{j-\ell}$ is non-negative. Hence every term in the sum of the right-hand side of (3.18) is non-positive and (3.16) follows. The last conclusion of the lemma is then deduced by considering $\ell = 2$ in (3.16) and observing that $\pi(a-\ell)\|r_i(x)\|^{a-\ell} = a(a-1)\|r_i(x)\|^{a-2} < 0$. □

Thus the isotropic model $m_i(x_i, s_i)$ overestimates the true function $\|r_i(x)+s_i\|^a$ and correctly reflects its concavity in the direction of its singularity. But $m_i(x_i, s_i) = \mu(\|r_i\|, \zeta_i)$ is now Lipschitz continuous as a function of s_i, while $\|r_i(x) + s_i\|^a$ is not.

Combining (3.7) and (3.13) now allows us to define a model for the complete f on $\mathcal{R}(x, \epsilon)$ as

$$m(x, s) \stackrel{\rm def}{=} \sum_{i \in \mathcal{W}(x,\epsilon)} m_i(x_i, s_i). \tag{3.19}$$

Since $r_i(x) \neq 0$ for $i \in \mathcal{A}(x, \epsilon)$, this model is in turn well defined.

We conclude this section by observing that writing the problem in the partially-separable form (1.1) is the key to expose the singular parts of the objective function, which then allows exploiting their rotational symmetry.

4 The adaptive regularization algorithm

Having defined a model of f, we may then use this model within a regularization minimization method inspired from the ARp algorithm in [11]. In such an algorithm, the step from an iterate x_k is obtained by attempting to (approximately) minimize the model ((3.19) in our case). If an (ϵ, δ)-approximate q-th-order-necessary minimizer is sought, this minimization is terminated as soon as the step s_k is long enough, in that

$$\|s_k\| \geq \varpi \epsilon^{\frac{1}{p-q+1}} \tag{4.1}$$

for some constant $\varpi \in (0, 1]$, or as soon as the trial point $x_k + s_k$ is an approximate q-th-order-necessary minimizer *of the model*, in the sense that

$$\psi_{m,q}^{\epsilon,\delta_k}(x_k, s_k) \leq \min\left[\frac{\theta\|s_k\|^{p-q+1}}{(p-q+1)!}, a \min_{i \in \mathcal{A}(x_k+s_k,\epsilon)} \|r_i(x_k + s_k)\|\right] \chi_q(\delta_k) \tag{4.2}$$

for some $\theta, \delta_k \in (0, 1]$, where $\psi_{m,q}^{\epsilon,\delta_k}(x_k + s_k)$ is the optimality measure (2.6) computed for the model $m(x, s)$, that is

$$\psi_{m,q}^{\epsilon,\delta}(x, s) \stackrel{\rm def}{=} m(x, s) - \min_{\substack{x+s+d \in \mathcal{F} \\ \|d\| \leq \delta, d \in \mathcal{R}(x,\epsilon)}} T_{m,q}(x, s + d). \tag{4.3}$$

In view of the optimality condition (2.5), we also require that, if $\|r_i(x + s)\| \leq \epsilon$ occurs for some $i \in \mathcal{H}$ in the course of the model minimization, the value of $r_i(x + s)$ is fixed, implying that the remaining minimization is carried out on $\mathcal{R}(x_k + s, \epsilon)$. As a consequence, the dimension of $\mathcal{R}(x_k + s, \epsilon)$ (and thus of $\mathcal{R}_k$) is monotonically non-increasing during the step computation and across all iterations. It was shown in [11, Lemma 2.5] that, unless x_k

is an $(\epsilon, 1)$-approximate p-th-order-necessary minimizer (which is obviously enough for the whole algorithm to terminate), a step satisfying (4.2) can always be found. The fact that this condition must hold on a subspace of potentially diminishing dimension clearly does not affect the result, and indicates that (4.2) is well-defined. This model minimization is in principle simpler than the original problem because the general nonlinear f_i have been replaced by locally accurate polynomial approximations and also because the model is now Lipschitz continuous, albeit still non-smooth. Importantly, the model minimization *does not involve any evaluation of the objective function* or its derivatives, and model evaluations within this calculation therefore do not affect the overall evaluation complexity of the algorithm.

We now introduce some useful notation for describing our algorithm. Define

$$x_{i,k} \stackrel{\rm def}{=} U_i x_k, \quad r_{i,k} \stackrel{\rm def}{=} U_i x_k - b_i, \quad s_{i,k} \stackrel{\rm def}{=} U_i s_k, \quad u_{i,k} \stackrel{\rm def}{=} \frac{r_{i,k}}{\|r_{i,k}\|}$$

and

$$\mathcal{A}_k^+ \stackrel{\rm def}{=} \mathcal{A}(x_k + s_k, \epsilon), \quad \mathcal{R}_k^+ \stackrel{\rm def}{=} \mathcal{R}(x_k + s_k, \epsilon) \ \text{ and } \ \mathcal{W}_k^+ \stackrel{\rm def}{=} \mathcal{W}(x_k + s_k, \epsilon).$$

Also let

$$\Delta f_{i,k} \stackrel{\rm def}{=} f_i(x_{i,k}) - f_i(x_{i,k} + s_{i,k}), \quad \Delta f_k \stackrel{\rm def}{=} f_{\mathcal{W}_k^+}(x_k) - f_{\mathcal{W}_k^+}(x_k + s_k) = \sum_{i \in \mathcal{W}_k^+} \Delta f_{i,k},$$

$$\Delta m_{i,k} \stackrel{\rm def}{=} m_i(x_{i,k}, 0) - m_i(x_{i,k}, s_{i,k}), \quad \Delta m_k = \sum_{i \in \mathcal{W}_k^+} \Delta m_{i,k},$$

and

$$\begin{aligned} \Delta T_k &\stackrel{\rm def}{=} T_{f_{\mathcal{W}_k^+},p}(x_k, 0) - T_{f_{\mathcal{W}_k^+},p}(x_k, s_k) \\ &= [T_{f_{\mathcal{N}},p}(x_k, 0) - T_{f_{\mathcal{N}},p}(x_k, s_k)] + \left[m_{\mathcal{A}_k^+}(x_k, 0) - m_{\mathcal{A}_k^+}(x_k, s_k)\right] \\ &= \Delta m_k + \frac{1}{(p+1)!} \sum_{i \in \mathcal{N}} \sigma_{i,k} \|s_{i,k}\|^{p+1}. \end{aligned} \tag{4.4}$$

Our partially-separable adaptive regularization degree-p algorithm PSARp is then given by Algorithm 4.1 on the following page.

Note that an $x_0 \in \mathcal{F}$ can always be computed by projecting an infeasible starting point onto $\mathcal{F}$. The motivation for the second and third parts of (4.9) and (4.10) is to identify cases where the isotropic model m_i overestimates the element function f_i to an excessive extent, leaving some room for reducing the regularization and hence allowing longer steps. The requirement that $\rho_k \geq \eta$ in both (4.9) and (4.10) is intended to prevent a situation where a particular regularization parameter is increased and another decreased at a given unsuccessful iteration, followed by the opposite situation at the next iteration, potentially leading to cycling.

It is worthwhile to note the differences between the PSARp algorithm and the algorithm discussed in [17]. The first and most important is that the new algorithm is intended to find an (ϵ, δ)-approximate q-th-order necessary minimizer for problem (1.1), rather than a first-order critical point. This is made possible by using the q-th-order termination criterion (4.5) instead of a criterion only involving the first-order model decrease, and by simultaneously

Algorithm 4.1: Partially-Separable Adaptive Regularization (PSARp)

Step 0: Initialization: $x_0 \in \mathcal{F}$ and $\{\sigma_{i,0}\}_{i\in\mathcal{N}} > 0$ are given as well as the accuracy $\epsilon \in (0,1]$ and constants $0 < \gamma_0 < 1 < \gamma_1 \leq \gamma_2$, $\eta \in (0,1)$, $\theta \geq 0$, $\delta_{-1} = 1$, $\sigma_{\min} \in (0, \min_{i\in\mathcal{N}} \sigma_{i,0}]$ and $\kappa_{\rm big} > 1$. Set $k = 0$.

Step 1: Termination: Evaluate $f(x_k)$ and $\{\nabla_x^j f_{\mathcal{W}_k}(x_k)\}_{j=1}^q$. If

$$\psi_{f,q}^{\epsilon,\delta_{k-1}}(x_k) \leq \epsilon\chi_q(\delta_{k-1}) \tag{4.5}$$

return $x_\epsilon = x_k$ and terminate. Otherwise evaluate $\{\nabla_x^j f_{\mathcal{W}_k}(x_k)\}_{j=q+1}^p$.

Step 2: Step computation: Attempt to compute a step $s_k \in \mathcal{R}_k$ such that $x_k + s_k \in \mathcal{F}$, $m(x_k, s_k) < m(x_k, 0)$ and either (4.1) holds or (4.2) holds for some $\delta_k \in (0,1]$. If no such step exists, return $x_\epsilon = x_k$ and terminate.

Step 3: Step acceptance: Compute

$$\rho_k = \frac{\Delta f_k}{\Delta T_k} \tag{4.6}$$

and set $x_{k+1} = x_k$ if $\rho_k < \eta$, or $x_{k+1} = x_k + s_k$ if $\rho_k \geq \eta$.

Step 4: Update the "nice" regularization parameters: For $i \in \mathcal{N}$, if

$$f_i(x_{i,k} + s_{i,k}) > m_i(x_{i,k}, s_{i,k}) \tag{4.7}$$

set

$$\sigma_{i,k+1} \in [\gamma_1\sigma_{i,k}, \gamma_2\sigma_{i,k}]. \tag{4.8}$$

Otherwise, if either

$$\rho_k \geq \eta \quad \text{and} \quad \Delta f_{i,k} \leq 0 \quad \text{and} \quad \Delta f_{i,k} < \Delta m_{i,k} - \kappa_{\rm big}|\Delta f_k| \tag{4.9}$$

or

$$\rho_k \geq \eta \quad \text{and} \quad \Delta f_{i,k} > 0 \quad \text{and} \quad \Delta f_{i,k} > \Delta m_{i,k} + \kappa_{\rm big}|\Delta f_k| \tag{4.10}$$

then set

$$\sigma_{i,k+1} \in [\max[\sigma_{\min}, \gamma_0\sigma_{i,k}], \sigma_{i,k}], \tag{4.11}$$

else set

$$\sigma_{i,k+1} = \sigma_{i,k}. \tag{4.12}$$

Increment k by one and go to Step 1.

using the step termination criteria (4.1) and (4.2) which again replace a simpler version again based solely on first-order information. The second is that the PSARp algorithm applies to the more general problem (1.1), in particular using the isotropic model (3.12) to allow $n_i > 1$ for $i \in \mathcal{H}$.

As alluded to above and discussed in [12] and [4], the potential termination of the algorithm in Step 2 can only happen whenever $q > 2$ and $x_k = x_\epsilon$ is an $(\epsilon, 1)$-approximate p-th-order-necessary minimizer within $\mathcal{R}_k$, which, together with (2.5), implies that the same property holds for problem (1.1). This is a significantly stronger optimality condition than what is required by (4.5). Also note that the potentially costly calculation of (4.2) may be avoided if (4.1) holds.

Let the index set of the "successful" and "unsuccessful" iterations be given by

$$\mathcal{S} \stackrel{\rm def}{=} \{k \geq 0 \mid \rho_k \geq \eta\} \ \text{ and } \ \mathcal{U} \stackrel{\rm def}{=} \{k \geq 0 \mid \rho_k < \eta\}.$$

Also define

$$\mathcal{S}_k \stackrel{\rm def}{=} \mathcal{S} \cap \{0, \ldots, k\} \quad \text{and} \quad \mathcal{U}_k \stackrel{\rm def}{=} \{0, \ldots, k\} \setminus \mathcal{S}_k.$$

We then state a bound on $|\mathcal{U}_k|$ as a function of $|\mathcal{S}_k|$. This is a standard result for non-partially-separable problems (see [7, Theorem 2.4] for instance), but needs careful handling of the model's overestimation properties to apply to our present context.

Lemma 4.1 *Suppose that AS.2 and AS.3 hold and that $\sigma_{i,k} \leq \sigma_{\max}$ for all $i \in \mathcal{M}$ and all $k \geq 0$. Then, for all $k \geq 0$,*

$$k \leq \kappa^a |\mathcal{S}_k| + \kappa^b,$$

where

$$\kappa^a \stackrel{\rm def}{=} 1 + \frac{|\mathcal{N}|\,|\log \gamma_0|}{\log \gamma_1} \quad \textit{and} \quad \kappa^b \stackrel{\rm def}{=} \frac{|\mathcal{N}|}{\log \gamma_1} \log\left(\frac{\sigma_{\max}}{\sigma_{\min}}\right).$$

Proof. See [17, Lemma 4.11]. The proof hinges on (3.14). □

5 Evaluation complexity analysis

We are now ready for a formal analysis of the evaluation complexity of the PSARp algorithm for problem (1.1), under the following assumptions.

AS.1 The feasible set $\mathcal{F}$ is closed, convex, non-empty and kernel-centered (in the sense of (1.3)).

AS.2 Each element function f_i ($i \in \mathcal{N}$) is p times continuously differentiable in an open set containing $\mathcal{F}$, where p is odd whenever $\mathcal{H} \neq \emptyset$.

AS.3 The p-th derivative of each f_i ($i \in \mathcal{N}$) is Lipschitz continuous on $\mathcal{F}$ with associated Lipschitz constant L_i (in the sense of (3.5)).

AS.4 There exists a constant $f_{\rm low}$ such that $f_{\mathcal{N}}(x) \geq f_{\rm low}$ for all $x \in \mathcal{F}$.

AS.5 If $\mathcal{H} \neq \emptyset$, there exists a constant $\kappa_{\mathcal{N}} \geq 0$ such that $\|\nabla_x^j f_i(U_i x)\| \leq \kappa_{\mathcal{N}}$ for all $x \in \mathcal{V}$, $i \in \mathcal{N}$ and $j \in \{1, \ldots, p\}$, where

$$\mathcal{V} \stackrel{\rm def}{=} \Big\{ x \in \mathcal{F} \mid \text{ there exists } i \in \mathcal{H} \text{ with } \|r_i(x)\| \leq \frac{a}{16} \Big\}. \tag{5.1}$$

Note that AS.4 is necessary for problem (1.1) to be well-defined. Also note that, because of AS.2, AS.5 automatically holds if $\mathcal{F}$ is bounded or if the iterates $\{x_k\}$ remain in a bounded set. It is possible to weaken AS.2 and AS.3 by replacing $\mathcal{F}$ with the level set $\mathcal{L} = \{x \in \mathcal{F} \mid f(x) \leq f(x_0)\}$ without affecting the results below. Finally observe that $\mathcal{V}$ need not to be bounded, in particular if $\mathrm{span}_{i\in\mathcal{H}}(U_i)$ is a proper subspace of $\mathbb{R}^n$. AS.5 is, of course, unnecessary if $\mathcal{F}$ or $\mathcal{V}$ are bounded or the iterates remain in a bounded set. The motivation for the particular choice of $\frac{1}{16}a$ in (5.1) will become clear in Lemma 5.5 below.

We first recall a result providing useful bounds.

Lemma 5.1 *There exist a constant $\varsigma > 0$ such that, for all $s \in \mathbb{R}^m$ and all $v \geq 1$,*

$$\varsigma^v \|s\|^v \leq \sum_{i\in\mathcal{N}} \|s_i\|^v \leq |\mathcal{N}|\, \|s\|^v. \tag{5.2}$$

Proof. See [17, Lemma 4.1]. □

This lemma states that $\sum_{i\in\mathcal{N}} \|\cdot\|$ is a norm on $\mathbb{R}^n$ whose equivalence constants with respect to the Euclidean one are ς and $|\mathcal{N}|$.

Our next step is to specify under which conditions the standard ϵ-independent overestimation and derivative accuracy bounds typical of the Lipschitz case (see [11, Lemma 2.1] for instance) can be obtained for the elements functions of (1.1). We define, for a given $k \geq 0$ and a given constant $\phi > 0$ independent of ϵ,

$$\mathcal{O}_{k,\phi} \stackrel{\rm def}{=} \{i \in \mathcal{A}_k^+ \mid \min[\, \|r_{i,k}\|,\, \|r_{i,k} + s_{i,k}\|\,] \geq \phi\}. \tag{5.3}$$

Observe that if, for some $i \in \mathcal{H}$ and $b_i \notin U_i\mathcal{F}$, then both $\|r_{i,k}\|$ and $\|r_{i,k} + s_{i,k}\|$ are bounded away from zero, so $i \in \mathcal{O}_{k,\phi}$ for all k and all ϕ such that $\phi \leq \min_{x\in\mathcal{F}} \|U_i x - b_i\|$. Thus we assume, without loss of generality, that

$$b_i \in U_i\mathcal{F} \quad \text{for all} \quad i \in \mathcal{H}. \tag{5.4}$$

We then obtain the following crucial error bounds.

Lemma 5.2 *Suppose that AS.2 and AS.3 hold. Then, for $k \geq 0$ and $L_{\max} \stackrel{\rm def}{=} \max_{i\in\mathcal{N}} L_i$,*

$$f_i(x_{i,k} + s_{i,k}) = m_i(x_{i,k}, s_{i,k}) + \frac{1}{(p+1)!}\Big[\tau_{i,k} L_{\max} - \sigma_{i,k}\Big]\|s_{i,k}\|^{p+1} \quad \textit{with} \quad |\tau_{i,k}| \leq 1, \tag{5.5}$$

for all $i \in \mathcal{N}$. If, in addition, $\phi > 0$ is given and independent of ϵ, then there exists a constant $L(\phi)$ independent of ϵ such that, for $\ell \in \{1, \ldots, p\}$,

$$\|\nabla_x^\ell f_{\mathcal{N}\cup\mathcal{O}_{k,\phi}}(x_k + s_k) - \nabla_s^\ell T_{f_{\mathcal{N}\cup\mathcal{O}_{k,\phi}},p}(x_k, s_k)\| \leq \frac{L(\phi)}{(p-\ell+1)!}\|s_k\|^{p-\ell+1}. \tag{5.6}$$

Proof. First note that, if f_i has a Lipschitz continuous p-th derivative as a function of $x_i = U_i x$, then (1.6) shows that it also has a Lipschitz continuous p-th derivative as a function of x. It is therefore enough to consider the element functions as functions of x_i.

Observe now that, for each k and $i \in \mathcal{N}$, AS.2 and AS.3 ensure that (5.5) and the inequality

$$\|\nabla^\ell_{x_i} f_i(x_{i,k}+s_{i,k}) - \nabla^\ell_{s_i} T_{f_i,p}(x_{i,k},s_{i,k})\| \le \frac{L_i}{(p-\ell+1)!}\|s_{i,k}\|^{p-\ell+1} \tag{5.7}$$

immediately follow from the known bounds for p times continuously differentiable functions with Lipschitz continuous p-th derivative (see [11, Lemma 2.1]). Consider now $i \in \mathcal{O}_{k,\phi}$ for some k and some fixed $\phi > 0$, implying that $\min[\|r_{i,k}\|, \|r_{i,k}+s_{i,k}\|] \ge \phi > 0$. Then

$$\nabla^\ell_{\cdot}\|r_{i,k}+s_{i,k}\|^a[d]^\ell = \nabla^\ell_{\cdot}\big\|\|r_{i,k}+s_{i,k}\|u^+_{i,k}\big\|^a[d]^\ell = \nabla^\ell_{\cdot}\big\|\|r_{i,k}+s_{i,k}\|u_{i,k}\big\|^a[R_{i,k}d]^\ell \tag{5.8}$$

where $R_{i,k}$ is the rotation such that $R_{i,k}u^+_{i,k} = u_{i,k}$. We also have from (3.10) with x replaced by $x_k + s_k$ that

$$\nabla^\ell_{s_i} T_{m_i,p}(x_{i,k},s_{i,k})[d]^\ell = \nabla^\ell_{\cdot}\big\|\|r_{i,k}\|u_{i,k}\big\|^a[R_{i,k}d]^\ell. \tag{5.9}$$

Taking the difference between (5.8) and (5.9), we obtain, successively using the definition of the tensor norm, the fact that $R_{i,k}$ is orthonormal and (3.4) in Lemma 3.1, that

$$\begin{aligned}
&\big\|\nabla^\ell_{\cdot}\|r_{i,k}+s_{i,k}\|^a - \nabla^\ell_{s_i} T_{m_i,p}(x_{i,k},s_{i,k})\big\|_{[\ell]} \\
&\qquad = \max_{\|d\|=1} \Big|\nabla^\ell_{\cdot}\|r_{i,k}+s_{i,k}\|^a[d]^\ell - \nabla^\ell_{s_i} T_{m_i,p}(x_{i,k},s_{i,k})[d]^\ell\Big| \\
&\qquad = \max_{\|d\|=1} \Big|\nabla^\ell_{\cdot}\big\|\|r_{i,k}+s_{i,k}\|u_{i,k}\big\|^a[R_{i,k}d]^\ell - \nabla^\ell_{\cdot}\big\|\|r_{i,k}\|u_{i,k}\big\|^a[R_{i,k}d]^\ell\Big| \\
&\qquad = \big\|\ \nabla^\ell_{\cdot}\big\|\|r_{i,k}+s_{i,k}\|u_{i,k}\big\|^a - \nabla^\ell_{\cdot}\big\|\|r_{i,k}\|u_{i,k}\big\|^a\ \big\|_{[\ell]} \\
&\qquad = |\pi(a-\ell)|\ \big|\|r_{i,k}+s_{i,k}\|^{a-\ell} - \|r_{i,k}\|^{a-\ell}\big|\,.
\end{aligned}$$

Now the univariate function $\nu(t) \stackrel{\rm def}{=} t^a$ is (more than) $p+1$ times continuously differentiable with bounded $(p+1)$-rst derivative on the interval $[t_1,t_2]$ and thus, from Lemma 5.2, we have that

$$\pi(a-\ell)\left|t_1^{a-\ell} - t_2^{a-\ell}\right| = \left|\frac{d^\ell\nu}{dt^\ell}(t_1) - \frac{d^\ell\nu}{dt^\ell}(t_2)\right| \le \frac{L_\nu}{(p-\ell+1)!}|t_1-t_2|^{p-\ell+1},$$

where L_ν is the upper bound on the $(p+1)$-rst derivative of $\nu(t)$ on interval $[t_1,t_2]$, that is $L_\nu = |\pi(a-p-1)|\min[t_1,t_2]^{a-p-1}$. As a consequence, we obtain that

$$\big\|\nabla^\ell_{\cdot}\|r_{i,k}+s_{i,k}\|^a - \nabla^\ell_{s_i} T_{m_i,p}(x_{i,k},s_{i,k})\big\|_{[\ell]} \le \frac{L(\phi)}{(p-\ell+1)!}\big|\|r_{i,k}+s_{i,k}\| - \|r_{i,k}\|\big|^{p-\ell+1},$$

where we use the fact that $\min[\|r_{i,k}\|, \|r_{i,k}+s_{i,k}\|] \ge \phi$ to define

$$L(\phi) = \max\big[|\pi(a-p-1)|\phi^{a-p-1}, L_{\max}\big].$$

We then observe that $\|s_{i,k}\| = \|r_{i,k}+s_{i,k}-r_{i,k}\| \ge \big|\|r_{i,k}+s_{i,k}\| - \|r_{i,k}\|\big|$ which finally yields that

$$\big\|\nabla^\ell_{\cdot}\|r_{i,k}+s_{i,k}\|^a - \nabla^\ell_{s_i} T_{m_i,p}(x_{i,k},s_{i,k})\big\|_{[\ell]} \le \frac{L(\phi)}{(p-\ell+1)!}\|s_{i,k}\|^{p-\ell+1}.$$

Combining this last inequality with (5.7) and the fact that $\nabla^\ell_{x_i}\|r_{i,k}+s_{i,k}\|^a = \nabla^\ell_{\cdot}\|r_{i,k}+s_{i,k}\|^a$ then ensures that (5.6) holds. □

Observe that the Lipschitz constant L is independent of ϕ whenever $\mathcal{H} = \emptyset$. Our model definition also implies the following bound.

Lemma 5.3 *For all $k \geq 0$ before termination, $s_k \neq 0$, (4.6) is well-defined and*

$$\Delta T_k \geq \frac{\sigma_{\min}\varsigma^{p+1}}{(p+1)!}\,\|s_k\|^{p+1}. \tag{5.10}$$

Proof. We immediately deduce that

$$\Delta T_k \geq \frac{\sigma_{\min}}{(p+1)!}\sum_{i\in\mathcal{N}}\|s_{i,k}\|^{p+1} \tag{5.11}$$

from (4.4), the observation that, at successful iterations, the algorithm enforces $\Delta m_k > 0$ and (4.11). As a consequence, $s_k \neq 0$. Hence at least one $\|s_{i,k}\|$ is strictly positive because of (5.2), and (5.11) therefore implies that (4.6) is well-defined. The inequality (5.10) then follows from Lemma 5.1. □

Following a now well-oiled track in convergence proofs for regularization methods, we derive an upper bound on the regularization parameters.

Lemma 5.4 *[17, Lemma 4.6] Suppose that AS.2 and AS.3 hold. Then, for all $i \in \mathcal{N}$ and all $k \geq 0$,*

$$\sigma_{i,k} \in [\sigma_{\min}, \sigma_{\max}], \tag{5.12}$$

where $\sigma_{\max} \stackrel{\rm def}{=} \gamma_2 L_{\max}$.

Proof. Assume that, for some $i \in \mathcal{N}$ and $k \geq 0$, $\sigma_{i,k} \geq L_i$. Then (5.5) gives that (4.7) must fail, ensuring (5.12) because of the mechanism of the algorithm. □

It is important to note that $\sigma_{\max}$ is independent of ϵ. We now verify that the trial step produced by Step 2 of the PSARp Algorithm either essentially fixes the residuals r_i to zero (their value being then fixed for the rest of the calculation), or is long enough (i.e. (4.1) holds), or maintains these residuals safely away from zero in the sense that their norm exceeds an ϵ-independent constant.

Lemma 5.5 *Suppose that AS.1, AS.2, AS.3 and AS.5 hold, that $\mathcal{H} \neq \emptyset$ and that (4.1) fails. Let*

$$\omega \stackrel{\rm def}{=} \min\left[\frac{a}{16}, \left(\frac{a}{12|\mathcal{N}|\left(\kappa_{\mathcal{N}} + \frac{\sigma_{\max}}{(p-q+1)!}\right)}\right)^{\frac{1}{1-a}}\right]. \tag{5.13}$$

Then, if, for some $i \in \mathcal{H}$,

$$\|r_{i,k}\| < \omega, \tag{5.14}$$

we have that

$$\|r_{i,k} + s_{i,k}\| \leq \epsilon \quad \textit{or} \quad \|r_{i,k} + s_{i,k}\| \geq \omega. \tag{5.15}$$

Proof. The conclusion is obvious if $i \in \mathcal{C}_k^+ = \mathcal{H} \setminus \mathcal{A}_k^+$. Consider now $i \in \mathcal{A}_k^+$ and suppose, for the purpose of deriving a contradiction, that

$$\|r_{i,k} + s_{i,k}\| \in (\epsilon, \omega) \quad \text{for some} \quad i \in \mathcal{A}_k^+, \tag{5.16}$$

and immediately note that the failure of (4.1) and the orthonormality of the rows of U_i imply that

$$\|s_{i,k}\| \leq \|s_k\| < \varpi \epsilon^{\frac{1}{p-q+1}} \leq 1 \tag{5.17}$$

and also that (4.2) must hold. As a consequence, for some $\delta_k \in (0, 1]$,

$$a\|r_{i,k} + s_{i,k}\|\chi_q(\delta_k) \geq \psi_{m,q}^{\epsilon,\delta_k}(x_k, s_k). \tag{5.18}$$

Consider now the vector

$$d_k = -\min\left[\delta_k, \|r_{i,k} + s_{i,k}\|\right] v_{i,k}^+ \text{ with } v_{i,k}^+ = U_i^\dagger u_{i,k}^+ \stackrel{\text{def}}{=} U_i^\dagger \frac{r_{i,k} + s_{i,k}}{\|r_{i,k} + s_{i,k}\|}. \tag{5.19}$$

We now verify that d_k is admissible for problem (4.3). Clearly $\|d_k\| = \delta_k$ because the rows of U_i orthonormal. We also see that (1.2) and (1.11) imply that, since $i \in \mathcal{A}_k^+$,

$$d_k \in \mathcal{R}_{\{i\}} \subseteq \mathcal{R}_k^+. \tag{5.20}$$

Moreover, we have that

$$x_k + s_k + d_k \in [\![x_k + s_k, x_k + s_k - U_i^\dagger(r_{i,k} + s_{i,k})]\!], \tag{5.21}$$

where $[\![v, w]\!]$ denotes the line segment joining the vectors v and w. But

$$\begin{aligned} x_k + s_k - U_i^\dagger(r_{i,k} + s_{i,k}) &= x_k + s_k - U_i^\dagger U_i(x_k + s_k) + U_i^\dagger b_i \\ &= (I - U_i^\dagger U_i)(x_k + s_k) + U_i^\dagger b_i \\ &= P_{\ker(U_i)}[x_k + s_k] + U_i^\dagger b_i \\ &\in \mathcal{F}, \end{aligned}$$

where we have used (1.3) to deduce the last inclusion. Since $\mathcal{F}$ is convex and $x_k + s_k \in \mathcal{F}$, we deduce from (5.21) that $x_k + s_k + d_k \in \mathcal{F}$. As a consequence, d_k is admissible for problem (4.3) and hence, using (5.18),

$$a\|r_{i,k} + s_{i,k}\|\chi_q(\delta_k) \geq \psi_{m,q}^{\epsilon,\delta_k}(x_k, s_k) \geq \max\left[0, m(x_k, s_k) - T_{m,q}(x_k, s_k - d_k)\right]. \tag{5.22}$$

Moreover (5.20) and (3.13) imply that

$$\begin{aligned} m(x_k, s_k) &- T_{m,q}(x_k, s_k - d_k) \\ &= m_{\mathcal{N}}(x_k, s_k) - T_{m_{\mathcal{N}},q}(x_k, s_k - d_k) + m_i(x_{i,k}, s_{i,k}) - m_i(x_{i,k}, s_{i,k} - U_i d_k) \\ &\geq -|m_{\mathcal{N}}(x_k, s_k) - T_{m_{\mathcal{N}},q}(x_k, s_k - d_k)| + m_i(x_{i,k}, s_{i,k}) - m_i(x_{i,k}, s_{i,k} - U_i d_k). \end{aligned} \tag{5.23}$$

We start by considering the first term in the right-hand side of this inequality. Observe now that (5.14) ensures that $x_k \in \mathcal{V}$ (as defined in (5.1)). Hence AS.5, (5.14) and (5.17) together

imply that, for each $i \in \mathcal{N}$,

$$\begin{aligned}
&|m_i(x_k, s_k) - T_{m_i,q}(x_k, s_k - d_k)| \\
&\quad \leq \left| \sum_{\ell=1}^{q} \frac{1}{\ell!} \nabla_x^\ell T_{m_i,q}(x_{i,k}, s_{i,k})[-U_i d_k]^\ell \right| \\
&\quad = \left| \sum_{\ell=1}^{q} \frac{1}{\ell!} \left(\sum_{t=\ell}^{p} \frac{1}{(t-\ell)!} \nabla_x^t f_i(x_{i,k})[s_{i,k}]^{t-\ell} + \frac{\sigma_{i,k}}{(p+1)!} \left\| \nabla_\cdot^\ell \|s_{i,k}\|^{p+1} \right\| \right) [-U_i d_k]^\ell \right|.
\end{aligned} \tag{5.24}$$

Using now the identity $\|U_i d_k\| = \|d_k\| = \delta_k$ and the fact that

$$\sum_{t=\ell}^{p} \frac{1}{(t-\ell)!} \leq 1 + \chi_{p-\ell}(1) < 3,$$

we obtain from (5.24), the triangle inequality and (5.17) that

$$|m_i(x_k, s_k) - T_{m_i,q}(x_k, s_k - d_k)| < \sum_{\ell=1}^{q} \frac{1}{\ell!} \left(3 \|\nabla_x^t f_i(x_{i,k})\|_{[t]} + \frac{\sigma_{i,k}}{(p+1)!} \left\| \nabla_\cdot^\ell \|s_{i,k}\|^{p+1} \right\| \right) \delta_k^\ell. \tag{5.25}$$

But we have from Lemma 3.1 and (5.17) that, for $\ell \in \{1, \ldots, q\}$,

$$\left\| \nabla_\cdot^\ell \|s_{i,k}\|^{p+1} \right\| = |\pi(p-\ell+1)| \, \|s_{i,k}\|^{p+1-\ell} \leq \frac{(p+1)!}{(p-q+1)!}, \tag{5.26}$$

and therefore that

$$\begin{aligned}
\left| m_{\mathcal{N}}(x_k, s_k) - T_{m_{\mathcal{N}},q}(x_k, s_k - \delta_k \|r_{i,k} + s_{i,k}\| v_{i,k}^+) \right| &< 3|\mathcal{N}| \left(\kappa_{\mathcal{N}} + \tfrac{1}{(p-q+1)!} \sigma_{\max} \right) \chi_q(\delta_k) \\
&\leq \tfrac{1}{4} a \omega^{a-1} \chi_q(\delta_k),
\end{aligned} \tag{5.27}$$

where we have used (5.13) to derive the last inequality. Let us now consider the second term in the right-hand side of (5.23). Applying Lemma 3.2, we obtain that $\mu(\|r_{i,k} + s_{i,k}\|, \cdot)$ is concave between 0 and $-\|r_{i,k} + s_{i,k}\|$ and $\mu(\|r_{i,k}\|, \cdot)$ is concave between 0 and $-\|r_{i,k}\|$. Therefore, because (5.14), we may deduce that

$$\begin{aligned}
m_i(x_{i,k}, s_{i,k}) - m_i(x_{i,k}, s_{i,k} - U_i d_k) &= \mu(\|r_{i,k} + s_{i,k}\|, 0) - \mu(\|r_{i,k} + s_{i,k}\|, \|U_i d_k\|) \\
&\geq \nabla_\zeta^1 \mu(\|r_{i,k} + s_{i,k}\|, 0) \|U_i d_k\| \\
&\geq \nabla_\zeta^1 \mu(\|r_{i,k}\|, \|r_{i,k} + s_{i,k}\| - \|r_{i,k}\|) \|U_i d_k\| \\
&\geq a \|r_{i,k}\|^{a-1} \delta_k \\
&\geq \tfrac{1}{2} a \omega^{a-1} \chi_q(\delta_k),
\end{aligned}$$

where the second and third inequalities result from (3.15). Combining now this inequality with (5.22), (5.23) and (5.27), we deduce that

$$a \|r_{i,k} + s_{i,k}\| \chi_q(\delta_k) > \tfrac{1}{2} a \omega^{a-1} \chi_q(\delta_k) - \tfrac{1}{4} a \omega^{a-1} \chi_q(\delta_k) = \tfrac{1}{4} a \omega^{a-1} \chi_q(\delta_k).$$

Finally, we obtain using (5.16) that

$$\omega > \tfrac{1}{4} \omega^{a-1},$$

which impossible in view of (5.13). Hence (5.16) cannot hold and the proof is complete. □

This last result is crucial in that it shows that there is a "forbidden" interval (ϵ, ω) for the residual's norms $\|r_i(x_k + s_k)\|$, where ω only depends on the problem and is independent of ϵ. This in turn allows to partition the successful iterates into subsets, distinguishing iterates which "fix" a residual to a near zero value, iterates with long steps and iterates with possibly short steps in regions where the considered objective function's p-th derivative tensor is safely bounded independently of ϵ. Our analysis now follows the broad outline of [17] while simplifying some arguments. Focusing on the case where $\mathcal{H} \neq \emptyset$, we first isolate the set of successful iterations which "deactivate" a residual, that is

$$\mathcal{S}_\epsilon \stackrel{\rm def}{=} \{k \in \mathcal{S} \mid \|r_{i,k} + s_{i,k}\| \leq \epsilon \ \text{ and } \ \|r_{i,k}\| > \epsilon \ \text{ for some } \ i \in \mathcal{H}\},$$

and notice that, by construction

$$|\mathcal{S}_\epsilon| \leq |\mathcal{H}|. \tag{5.28}$$

We next define the ϵ-independent constant

$$\alpha = \tfrac{3}{4}\omega$$

and

$$\mathcal{S}_{\|s\|} \stackrel{\rm def}{=} \{k \in \mathcal{S} \mid \|s_k\| \geq \tfrac{1}{4}\omega\}. \tag{5.29}$$

Moreover, for an iteration $k \in \mathcal{S} \setminus (\mathcal{S}_\epsilon \cup \mathcal{S}_{\|s\|})$, we verify that $\mathcal{A}_k$ can be partitioned into

$$\begin{aligned}
\mathcal{I}_{\heartsuit,k} &\stackrel{\rm def}{=} \{i \in \mathcal{A}_k \mid \|r_{i,k}\| \in [\alpha, +\infty) \ \text{ and } \ \|r_{i,k} + s_{i,k}\| \in [\alpha, +\infty)\} \\
\mathcal{I}_{\diamondsuit,k} &\stackrel{\rm def}{=} \{i \in \mathcal{A}_k \mid \big(\|r_{i,k}\| \in [\omega, +\infty) \ \text{ and } \ \|r_{i,k} + s_{i,k}\| \in (\epsilon, \alpha)\big) \\
&\qquad\qquad \text{or} \quad \big(\|r_{i,k}\| \in (\epsilon, \alpha) \ \text{ and } \ \|r_{i,k} + s_{i,k}\| \in [\omega, \infty)\big)\} \\
\mathcal{I}_{\clubsuit,k} &\stackrel{\rm def}{=} \{i \in \mathcal{A}_k \mid \|r_{i,k}\| \in (\epsilon, \omega) \ \text{ and } \ \|r_{i,k} + s_{i,k}\| \in (\epsilon, \omega)\}.
\end{aligned}$$

Morever, Lemma 5.5 shows that $\mathcal{I}_{\clubsuit,k}$ is always empty and one additionally has that, if $i \in \mathcal{I}_{\diamondsuit,k}$, then

$$\|s_k\| \geq \|s_{i,k}\| \geq \big|\|r_{i,k} + s_{i,k}\| - \|r_{i,k}\|\big| \geq \omega - \alpha = \tfrac{1}{4}\omega,$$

implies that $k \in \mathcal{S}_{\|s\|}$. Hence $\mathcal{I}_{\diamondsuit,k}$ is also empty and

$$\mathcal{A}_k = \mathcal{I}_{\heartsuit,k} \quad \text{for} \quad k \in \mathcal{S} \setminus (\mathcal{S}_\epsilon \cup \mathcal{S}_{\|s\|}) \stackrel{\rm def}{=} \mathcal{S}_\heartsuit. \tag{5.30}$$

The next important result shows that steps at iteration belonging to $\mathcal{S}_\heartsuit$ are long enough, because they are taken over region where a good ϵ-independent Lipschitz bounds holds. Indeed, if $\mathcal{H} \neq \emptyset$ and assuming that $\epsilon \leq \alpha$, we have, for $k \in \mathcal{S}_\heartsuit$, that $\mathcal{A}_k^+ = \mathcal{A}_k$ and thus that $\mathcal{W}_k^+ = \mathcal{W}_k$ and $\mathcal{R}_k^+ = \mathcal{R}_k$. Moreover, the definition of $I_{\heartsuit,k} = \mathcal{A}_k$ ensures that $\mathcal{A}_k \subseteq \mathcal{O}_{k,\alpha}$ and thus that Lemma 5.2 (and in particular (5.6)) guarantees that $f_{\mathcal{W}_k}$ satisfies standard derivative error bounds for functions with Lipschitz continuous p-th derivative (with corresponding Lipschitz constant $L(\alpha)$). We may therefore apply known results for such functions to $f_{\mathcal{W}_k}$. The following lemma is extracted from [11], by specializing Lemma 3.3 in that reference to optimization of $f_{\mathcal{W}_k}$ over $\mathcal{R}_k$ for functions with Lipschitz continuous p-th derivative (i.e. $\beta = 1$ in [11]).

Lemma 5.6 *Suppose that AS.1 – AS.3 and AS.5 hold, that*

$$\epsilon \leq \alpha \quad \textit{if} \quad \mathcal{H} \neq \emptyset \tag{5.31}$$

and consider $k \in \mathcal{S}_\heartsuit$ such that the PSARp Algorithm does not terminate at iteration $k+1$. Then

$$\|s_k\| \geq \kappa_\heartsuit \epsilon^{\frac{1}{p-q+1}} \quad \textit{with} \quad \kappa_\heartsuit \stackrel{\rm def}{=} \left(\frac{(p-q+1)!}{L(\alpha)+\theta+\sigma_{\max}} \right)^{\frac{1}{p-q+1}}. \tag{5.32}$$

We may finally establish our final evaluation complexity bound by combining our results so far.

Theorem 5.7 *Suppose that AS.1–AS.5 and (5.31) hold. Then the PSARp Algorithm requires at most*

$$\left\lfloor \kappa_{\mathcal{S}} (f(x_0) - f_{\rm low}) \epsilon^{-\frac{p+1}{p-q+1}} \right\rfloor + |\mathcal{H}| \tag{5.33}$$

successful iterations and at most

$$\left\lfloor \left\lfloor \kappa_S (f(x_0) - f_{\rm low}) \left(\epsilon^{-\frac{p+1}{p-q+1}} \right) \right\rfloor \left(1 + \frac{|\log \gamma_1|}{\log \gamma_2} \right) + \frac{1}{\log \gamma_2} \log \left(\frac{\sigma_{\max}}{\sigma_0} \right) \right\rfloor + |\mathcal{H}| + 1 \tag{5.34}$$

evaluations of f and its p first derivatives to return an (ϵ,δ)-approximate q-th-order-necessary minimizer for problem (1.1), where

$$\kappa_S \stackrel{\rm def}{=} \frac{(p+1)!}{\eta \sigma_{\min} \varsigma^{p+1}} \left(\frac{(p-q+1)!}{L(\alpha)+\theta+\sigma_{\max}} \right)^{-\frac{1}{p-q+1}}. \tag{5.35}$$

Proof. Consider $k \in \mathcal{S}$ before termination. Because the iteration is successful, we obtain from AS.4, Step 3 of the algorithm and Lemma 5.3 that

$$f(x_0) - f_{\rm low} \geq f(x_0) - f(x_{k+1}) = \sum_{j \in \mathcal{S}_k} \Delta f_k \geq \eta \sum_{j \in \mathcal{S}_k} \Delta T_k \geq \frac{\eta \sigma_{\min} \varsigma^{p+1}}{(p+1)!} \sum_{j \in \mathcal{S}_k} \|s_k\|^{p+1}. \tag{5.36}$$

Defining now

$$\mathcal{S}_{\epsilon,k} \stackrel{\rm def}{=} \mathcal{S}_\epsilon \cap \{0,\ldots,k\}, \quad \mathcal{S}_{\|s\|,k} \stackrel{\rm def}{=} \mathcal{S}_{\|s\|} \cap \{0,\ldots,k\} \text{ and } \mathcal{S}_{\heartsuit,k} \stackrel{\rm def}{=} \mathcal{S}_\heartsuit \cap \{0,\ldots,k\},$$

we verify that $\mathcal{S}_{\|s\|,k}$ and $\mathcal{S}_{\heartsuit,k}$ form a partition of $\mathcal{S}_k \setminus \mathcal{S}_{\epsilon,k}$. As a consequence, we have that

$$\begin{aligned}
f(x_0) - f_{\rm low} &\geq \frac{\eta \sigma_{\min} \varsigma^{p+1}}{(p+1)!} \left\{ |\mathcal{S}_{\|s\|,k}| \min_{j \in \mathcal{S}_{\|s\|,k}} \|s_k\|^{p+1} + |\mathcal{S}_{\heartsuit,k}| \min_{j \in \mathcal{S}_{\heartsuit,k}} \|s_k\|^{p+1} \right\} \\
&\geq \frac{\eta \sigma_{\min} \varsigma^{p+1}}{(p+1)!} \left\{ |\mathcal{S}_{\|s\|,k}| (\tfrac{1}{4}\omega)^{p+1} + |\mathcal{S}_{\heartsuit,k}| \left(\kappa_\heartsuit \epsilon^{\frac{1}{p-q+1}} \right)^{p+1} \right\} \\
&\geq \frac{\eta \sigma_{\min} \varsigma^{p+1}}{(p+1)!} \left\{ |\mathcal{S}_{\|s\|,k}| + |\mathcal{S}_{\heartsuit,k}| \right\} \min \left[(\tfrac{1}{4}\omega)^{p+1}, \left(\kappa_\heartsuit \epsilon^{\frac{1}{p-q+1}} \right)^{p+1} \right] \\
&\geq \frac{\eta \sigma_{\min} \varsigma^{p+1}}{(p+1)!} |\mathcal{S}_k \setminus \mathcal{S}_{\epsilon,k}| \, \kappa_\heartsuit^{\frac{1}{p-q+1}} \, \epsilon^{\frac{1}{p-q+1}},
\end{aligned}$$

where we have used (5.36), Lemma 5.1, (5.29) and (5.32) to deduce the second inequality, and the assumption that (without loss of generality in view of (5.32)) $\kappa_\heartsuit \leq \frac{1}{4}\omega$ to deduce the last. The above inequality yields that

$$|\mathcal{S}_k| = |\mathcal{S}_k \setminus \mathcal{S}_{\epsilon,k}| + |\mathcal{S}_{\epsilon,k}| \leq \kappa_S (f(x_0) - f_{\rm low}) \epsilon^{-\frac{p+1}{p-q+1}} + |\mathcal{S}_{\epsilon,k}|,$$

where κ_S is given by (5.35). Since $|\mathcal{S}_{\epsilon,k}| \leq |\mathcal{S}_\epsilon| \leq |\mathcal{H}|$, we finally deduce that the bound (5.33) holds. The bound (5.34) then follows by applying Lemma 4.1 and observing that f and its first p derivatives are evaluated at most once per iteration, plus once at termination. □

We conclude our development by recalling that the above result is valid for $\mathcal{H} = \emptyset$, in which case the problem is a smooth convexly-constrained partially-separable problem. Note that the norm-equivalence constant ς occurs in (5.35), which indicate that the underlying geometry of the problem's invariants subspaces $\ker(U_i)$ may have a significant impact on complexity.

6 Conclusions

We have shown that an (ϵ, δ)-approximate q-th-order critical point of partially-separable convexly-constrained optimization with non-Lipschitzian singularities can be found at most $O(\epsilon^{-(p+1)/(p-q+1)})$ evaluations of the objective function and its first p derivatives for any $q \in \{1, 2, \ldots, p\}$ whenever the smooth element functions f_i, $i \in \mathcal{N}$ of the objective function are p times differentiable. This worst-case complexity is obtained via our Algorithm 4.1 (PSAR$_p$) with an p-th order Taylor model which uses the underlying rotational symmetry of the Euclidean norm function for $f_{\mathcal{H}}$ and the first p derivatives (whenever they exist) of the "element functions" f_i, for $i \in \mathcal{M}$.

Several observations are of interest. A first one is that the results remain valid if Lipschitz continuity is not assumed on the whole of the feasible set, but restricted to the segments of the "path of iterates", that is $\cup_k [\![x_k, x_{k+1}]\!]$. While this might in general be difficult to ensure a priori, there may be case where problem structure could help. A second observation is that convexity of the feasible set is only used on the segments $\cup_{i,k} [\![x_{i,k}, U_i^\dagger b_i]\!]$. Again this might be exploitable in some cases. The third observation is that, in line with [11], it is possible to replace the Lipschitz continuity assumption by a weaker Hölder continuity.

While it may be possible to handle non-kernel-centered feasible sets (maybe along the lines of the discussion in [17]), this remains open at this stage. Another interesting perspective is a more generic exploitation of geometric symmetries inherent to optimization problems: our treatment here focuses on a specific case of rotational symmetry, but this should not, one hopes, be limitative.

Acknowledgements

Xiaojun Chen would like to thank Hong Kong Research Grant Council for grant PolyU153000/17p. Philippe Toint would like to thank the Hong Kong Polytechnic University for its support while this research was being conducted.

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Appendix

Proof Lemma 3.1 The proof of (3.3) is essentially borrowed from [11, Lemma 2.4], although details differ because the present version covers $a \in (0,1)$. We first observe that $\nabla_{\cdot}^{j}\|r\|^a$ is a j-th order tensor, whose norm is defined using (1.7). Moreover, using the relationships

$$\nabla_{\cdot}^{1}\|r\|^{\tau} = \tau\,\|r\|^{\tau-2}r \quad \text{and} \quad \nabla_{\cdot}^{1}\big(r^{\tau\otimes}\big) = \tau\, r^{(\tau-1)\otimes}\otimes I, \quad (\tau \in \mathbb{R}), \tag{A.1}$$

defining

$$\nu_0 \stackrel{\text{def}}{=} 1, \quad \text{and} \quad \nu_i \stackrel{\text{def}}{=} \prod_{\ell=1}^{i}(a+2-2\ell), \tag{A.2}$$

and proceeding by induction, we obtain that, for some $\mu_{j,i} \geq 0$ with $\mu_{1,1} = 1$,

$$\begin{aligned}
&\nabla^1_{\cdot}\left[\nabla^{j-1}_{\cdot}\|r\|^a\right] \\
&\quad= \nabla^1_{\cdot}\left[\sum_{i=2}^{j} \mu_{j-1,i-1}\nu_{i-1}\|r\|^{a-2(i-1)}\, r^{(2(i-1)-(j-1))\otimes} \otimes I^{((j-1)-(i-1))\otimes}\right] \\
&\quad= \sum_{i=2}^{j} \mu_{j-1,i-1}\nu_{i-1}\Big[(a-2(i-1))\|r\|^{a-2(i-1)-2}\, r^{(2(i-1)-(j-1)+1)\otimes} \otimes I^{(j-i)\otimes} \\
&\qquad\quad +((2(i-1)-(j-1))\|r\|^{a-2(i-1)}\, r^{(2(i-1)-(j-1)-1)\otimes} \otimes I^{(j-1)-(i-1)+1)\otimes}\Big] \\
&\quad= \sum_{i=2}^{j} \mu_{j-1,i-1}\nu_{i-1}\Big[(a+2-2i)\|r\|^{a-2i}\, r^{(2i-j)\otimes} \otimes I^{(j-i)\otimes} \\
&\qquad\quad +(2(i-1)-j+1)\|r\|^{a-2(i-1)}\, r^{(2(i-1)-j)\otimes} \otimes I^{(j-(i-1))\otimes}\Big] \\
&\quad= \sum_{i=2}^{j} \mu_{j-1,i-1}\nu_{i-1}(a+2-2i)\|r\|^{a-2i}\, r^{(2i-j)\otimes} \otimes I^{(j-i)\otimes} \\
&\qquad\quad + \sum_{i=1}^{j-1}(2i-j+1)\mu_{j-1,i}\nu_i\|r\|^{a-2i}\, r^{(2i-j)\otimes} \otimes I^{(j-i)\otimes} \\
&\quad= \sum_{i=1}^{j} \big((a+2-2i)\mu_{j-1,i-1}\nu_{i-1} + (2i-j+1)\mu_{j-1,i}\nu_i\big)\|r\|^{a-2i}\, r^{(2i-j)\otimes} \otimes I^{(j-i)\otimes},
\end{aligned}$$

where the last equation uses the convention that $\mu_{j,0} = 0$ and $\mu_{j-1,j} = 0$ for all j. Thus we may write

$$\nabla^j_{\cdot}\|r\|^a = \nabla^1_{\cdot}\left[\nabla^{j-1}_{\cdot}\|r\|^a\right] = \sum_{i=1}^{j} \mu_{j,i}\nu_i\, \|r\|^{a-2i}\, r^{(2i-j)\otimes} \otimes I^{(j-i)\otimes} \tag{A.3}$$

with

$$\begin{aligned}
\mu_{j,i}\nu_i &= (a+2-2i)\mu_{j-1,i-1}\nu_{i-1} + (2i-j+1)\mu_{j-1,i}\nu_i \\
&= \big[\mu_{j-1,i-1} + (2i-j+1)\mu_{j-1,i}\big]\nu_i,
\end{aligned} \tag{A.4}$$

where we used the identity

$$\nu_i = (a+2-2i)\nu_{i-1} \quad \text{for} \quad i = 1, \ldots, j \tag{A.5}$$

to deduce the second equality. Now (A.3) gives that

$$\nabla^j_{\cdot}\|r\|^a[v]^j = \sum_{i=1}^{j} \mu_{j,i}\nu_i\|r\|^{a-j}\, \left(\frac{r^T v}{\|r\|}\right)^{2i-j} (v^T v)^{j-i}.$$

It is then easy to see that the maximum in (1.7) is achieved for $v = r/\|r\|$, so that

$$\|\, \nabla^j_{\cdot}\|r\|^a\, \|_{[j]} = \left|\sum_{i=1}^{j} \mu_{j,i}\nu_i\right| \|r\|^{a-j} = |\pi_j|\, \|r\|^{a-j} \tag{A.6}$$

with

$$\pi_j \stackrel{\text{def}}{=} \sum_{i=1}^{j} \mu_{j,i}\, \nu_i. \tag{A.7}$$

Successively using this definition, (A.4), (A.5) (twice), the identity $\mu_{j-1,j} = 0$ and (A.7) again, we then deduce that

$$\begin{aligned}
\pi_j &= \sum_{i=1}^{j} \mu_{j-1,i-1}\nu_i + \sum_{i=1}^{j}(2i-j+1)\mu_{j-1,i}\nu_i \\
&= \sum_{i=1}^{j-1} \mu_{j-1,i}\nu_{i+1} + \sum_{i=1}^{j}(2i-j+1)\mu_{j-1,i}\nu_i \\
&= \sum_{i=1}^{j-1} \mu_{j-1,i}\big[\nu_{i+1} + (2i-j+1)\nu_i\big] \\
&= \sum_{i=1}^{j-1} \mu_{j-1,i}\big[(a+2-2(i+1))\nu_i + (2i-j+1)\nu_i\big] \\
&= (a+1-j)\sum_{i=1}^{j-1} \mu_{j-1,i}\,\nu_i \\
&= (a+1-j)\pi_{j-1}.
\end{aligned} \tag{A.8}$$

Since $\pi_1 = a$ from the first part of (A.1), we obtain from (A.8) that

$$\pi_j = \pi(a-j), \tag{A.9}$$

which, combined with (A.6) and (A.7), gives (3.3). Moreover, (A.9), (A.7) and (A.3) give (3.2) with $\phi_{i,j} = \mu_{j,i}\,\nu_i$. In order to prove (3.4) (where now $\|r\| = 1$), we use (A.3), (A.7), (A.9) and obtain that

$$\begin{aligned}
\nabla_x^j\|\beta_1 r\|^a - \nabla_x^j\|\beta_2 r\|^a &= \sum_{i=1}^{j} \mu_{j,i}\nu_i\,\|\beta_1 r\|^{a-2i}\,\beta_1^{(2i-j)} r^{(2i-j)\otimes} \otimes I^{(j-i)\otimes} \\
&\qquad - \sum_{i=1}^{j} \mu_{j,i}\nu_i\,\|\beta_2 r\|^{a-2i}\,\beta_2^{(2i-j)} r^{(2i-j)\otimes} \otimes I^{(j-i)\otimes} \\
&= \pi(a-j)\left[\beta_1^{a-j} - \beta_2^{a-j}\right]\,\|r\|^{a-2i}\,r^{(2i-j)\otimes} \otimes I^{(j-i)\otimes} \\
&= \pi(a-j)\left[\beta_1^{a-j} - \beta_2^{a-j}\right]\,r^{(2i-j)\otimes} \otimes I^{(j-i)\otimes}.
\end{aligned}$$

Using (1.7) again, it is easy to verify that the maximum defining the norm is achieved for $v = r$ and (3.4) then follows from $\|r\| = 1$.