On Electricity Market Equilibria with Storage: Modeling, Uniqueness, and a Distributed ADMM

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Abstract. We consider spot-market trading of electricity including storage operators as additional agents besides producers and consumers. Storage devices allow for shifting produced electricity from one time period to a later one. Due to this, multiple market equilibria may occur even if classical uniqueness assumptions for the case without storage systems are satisfied. For models containing storage operators, we derive sufficient conditions that ensure uniqueness of generation and demand. We also prove uniqueness of the market equilibrium for the special case of a single storage operator. Nevertheless, in case of multiple storage operators, uniqueness fails to hold in general, which we show by illustrative examples. We conclude the theoretical discussion with a general ex-post condition for proving the uniqueness of a given solution. In contrast to classical settings without storage systems, the computation of market equilibria is much more challenging since storage operations couple all trading events over time. For this reason, we propose a tailored parallel and distributed alternating direction method of multipliers (ADMM) for efficiently computing spot-market equilibria over long time horizons. We first analyze the parallel performance of the method itself. Finally, we show that the parallel ADMM clearly outperforms solving the respective problems directly and that it is capable of solving instances with more than 42 million variables in less than 13 minutes.

1. Introduction

Energy storage systems play an important role in modern electricity systems of today and the future for at least three reasons. First, many countries like, e.g., Germany with its energy turnaround, try to make a transition towards an energy supply free of fossil fuels and nuclear power; see, e.g., Babrowski et al. (2016), Hassler (2017), and Zerrahn et al. (2018) for recent discussions with a view towards techniques from optimization and operations research. To realize this, these countries are striving for a system with a large share of renewable energy sources like wind and solar power. In principle, these energy resources might deliver enough energy in total but are highly volatile. Additionally, renewable energy production cannot be controlled like it is the case for conventional power production. Consequently and besides, e.g., investment in network infrastructure or energy efficiency, large-scale storage systems become a key ingredient for the success of the transition towards an energy system mainly based on renewables. Second, modern energy systems change more and more from the principle of central generation and transport towards systems of distributed generation for local demand—often embedded in or in the context of smart grid systems. To make these decentralized and distributed smart grid systems work in practice, storage systems again are important to handle asynchronous peak patterns of (renewable) generation and

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demand. See, e.g., Dai and Charkhgard (2018) for a recent article on storage control in the context of smart grid infrastructure. Third, and often in connection with the second point, electricity storage systems can be used to provide additional services like load following (Muzhikyan et al. 2016), spinning reserve (Banshwar et al. 2019), peak shaving (Zheng et al. 2015), or black-start capability (Li et al. 2018). By doing so, they can also be used to increase the reliability and stability of the entire electricity system. For an overview and specific discussions of the additional services of energy storages see also, e.g., Aneke and Wang (2016), Divya and Østergaard (2009), and Ferreira et al. (2013).

From an economic point of view, (large-scale) storage devices are typically used for generating profit between time periods, i.e., they are charged if the price of electricity is low and are discharged if the price is high. If these storage systems achieve sufficiently large capacities, the market clearing price itself will depend on the actions of the storage operators as well. Hence, a well thought-out market integration is required; see, e.g., Cruise, Flatley, Gibbens, et al. (2019), Cruise, Flatley, and Zachary (2018), and Szabó et al. (2018) regarding the impact of storage systems on energy markets. This is especially the case in market environments that rely on investments of the private sector for the development and implementation of the required technical solutions. Hence, market integration of storage systems also need to be carefully designed so that proper investment incentives and business models may arise; see, e.g., Heredia et al. (2018) for a specific discussion of market integration of wind power plants with tailored storage solutions. Altogether, many new challenges for energy market design emerge.

In this paper, we contribute in three ways to the understanding of energy market equilibrium models including storage systems. First, we derive a market equilibrium model including storage operators. At this stage we need to make some important modeling decisions: For setting up an equilibrium model one has to decide who owns and controls the storage devices. We discuss different possible setups later in our literature survey. In our model, we consider the setting in which storage systems are operated standalone, i.e., independently of the generators and consumers. Moreover, we consider the situation of perfectly competitive markets in which both generators and storage operators act as price takers. This corresponds to the assumption that both generators and storages are not large enough to affect market prices. Second, we present a thorough equilibrium analysis. It is shown that economic efficiency is obtained under the assumption of perfect competition by proving the 1-1 correspondence of competitive market equilibria and welfare-maximal solutions. We then analyze the existence and uniqueness of equilibria. The study of existence and uniqueness is of key importance since it paves the way for tackling more complicated bi- or multilevel models that, e.g., deal with the long-run integration of storage systems into energy markets. We discuss this aspect in more detail at the beginning of Section 3, where we also give the respective references. In our setup, existence of equilibria is easily obtained, while uniqueness cannot be achieved in general. The main result is the following: Under certain assumptions on the problem's data, both demand and production as well as the total difference of charging and discharging of storage devices are unique, whereas the operations of single storage operators are not unique. In contrast to the standard case without storage devices, the equilibrium analysis is much more complicated in case of market integration of storage systems since their control couples all trading periods of the considered time horizon. This leads to a challenging time-dependent equilibrium model. Third and finally, we address the issue that the—now time-dependent—equilibrium models are also much harder to solve from a computational point of view. Here, we use the property that consecutive blocks of trading periods are only weakly coupled by the states of charge of the storage devices, yielding a so-called quasi block-separable structure of the corresponding optimization problem. This allows to develop tailored alternating direction methods of multipliers (ADMM); see, e.g., Bertsekas (2015), Boyd et al. (2011), and Parikh and Boyd (2014). Due to the large-scale nature of many practical instances, we apply a reformulation of the problem so that a scaled consensus form for the ADMM is achieved that allows for an entirely parallel and distributed solution framework. We show that this method clearly outperforms the standard approach of solving the respective quadratic programs (QPs) directly and that it enables us to solve instances with more than 42 million variables in less than 13 minutes.

Although the topic of (large-scale) energy storage systems is rather new, there already has evolved an enormous amount of literature in this area. Consequently, we can only review a certain amount of related literature. The modeling framework of mixed complementarity problems (MCPs) for electricity markets including storage operators is also used in Awad et al. (2014). For a general overview on complementarity modeling of energy systems see Gabriel et al. (2012) and the references therein. In addition to the setting considered in this paper, the authors also consider a transport network and can thus study the locational effects of storage placement. As in our context, perfect competition is assumed. The computationally studied instance is a rather small 9-bus system solved with the PATH solver (Ferris and Munson 2000) in GAMS (GAMS Development Corporation 2015) and there is no theoretical study regarding existence or uniqueness of equilibria. Discretely-constrained MCPs are also used in Fomeni et al. (2018) for power networks with storage operators, where it is shown that storage systems may be beneficial in terms of stabilizing the price of electricity during peak periods. However, this result requires the modeling of storage operators as service providers instead of standalone competitors of producers. In Sioshansi (2014), the equilibrium setup is more sophisticated. The paper considers welfare effects of storages under (i) different market structures like perfect competition and strategic interaction as well as under (ii) different ownership setups of storage devices like standalone and generator-owned storage devices. It is shown that in most market environments, market power in the generation sector, i.e., strategic behavior, is a prerequisite for storage systems to reduce the social welfare compared to the case without storages. If the generation sector is perfectly competitive, standalone storage devices that are not owned by generators cannot be welfare-diminishing. The analysis is carried out on a two-time-period model and the author also shows the uniqueness of the equilibrium but relies on strictly convex models.

Storages are also of high importance in studies using stochastic demand and renewable production models. In this context, in Gast et al. (2013), a two-stage model of a competitive market is analyzed. As before, different ownership structures (like supplier-owned, consumer-owned, and stand-alone) are considered. Existence of equilibria are shown under perfect competition but uniqueness or multiplicity of equilibria is not considered.

Long-term investment problems regarding the optimal capacities for generation, charging, discharging, and energy storage is considered in Lamont (2012). Here, a Lagrangian optimization approach is taken that also delivers notions of marginal values of charging and discharging capacities of storages, of total energy storage capacity, and generation capacity. For an explicit consideration of storages in the situation excess production of renewables see Steffen and Weber (2013).

The integration of storage systems in pricing models of production and demand can also be seen as an extension of the classical peak-load-pricing literature; see, e.g., Crew et al. (1995) for a survey on peak-load pricing and Grimm, Schewe, et al.

(2017) and Schewe and Schmidt (2019) for recent papers on peak-load pricing in the context of energy markets. The integration of storage systems in such pricing models is discussed, e.g., in Gravelle (1976), where the focus is on the study of analytical solutions in a stylized two-time-periods setting.

Let us also briefly comment on a practical example where storage devices are operated in a standalone way, which is exactly the situation that we consider in this paper. Multiple decentralized energy storage systems can be linked to form a large virtual storage system that participates independently in energy markets via an independent operator. In Germany, for example, such virtual storage systems have already participated successfully in the reserve power market; cf. Schlund and German (2017), Schlund, Steber, et al. (2017), and Steber et al. (2016). Independent storage operators in the context of energy markets are also addressed in, e.g., Cruise, Flatley, Gibbens, et al. (2019), Cruise, Flatley, and Zachary (2018), and Steffen and Weber (2016).

Finally, the problem studied in this paper is a large-scale quadratic program that we solve using a parallel and distributed ADMM. In this context, let us mention that there are also other parallel approaches to large and structured QP; see, e.g., the tailored interior-point methods in Gondzio and Grothey (2006, 2007a,b, 2009) and Hübner et al. (2017a,b).

The remainder of the paper is structured as follows. In Section 2, we present the equilibrium model as a mixed complementarity problem (MCP) for which we show that it is equivalent to a properly chosen welfare maximization problem. Afterward, in Section 3, we use this relationship to analyze the existence and uniqueness of equilibria. Section 4 then derives an ADMM-tailored reformulation of the problem that explicitly highlights the quasi block-separable structure of the original problem. This reformulation is then used in Section 5, where we present results of an entirely parallel and distributed ADMM in scaled consensus form. We discuss different configurations of the distribution of the algorithm, computationally analyze the convergence of the method, and finally present a comparison with a standard solution method applied to large-scale instances. The paper closes with a conclusion and a discussion of future research directions in Section 6.

2. An Electricity Market Equilibrium Model with Storages

In this section, we describe the market equilibrium model that we study throughout the paper. This market model is made up of the optimization problems of different types of agents: producers, consumers, and storage operators. In many of such models, the optimization problems of producers and consumers do not contain aspects that couple different trading periods over time. This situation changes if one considers storage operators within the equilibrium model since their actions couple successive trading periods. As a consequence, we need to set up a time-depending equilibrium model over a set $T := \{0, 1, \dots, t^e\}$ of trading periods. Since this time-dependence alone already complicates the analysis of equilibria, we consider the basic situation of a perfectly competitive market; i.e., all agents act as price takers. See also Grimm, Martin, et al. (2016) or Boucher and Smeers (2001) and Daxhelet and Smeers (2007) for other contributions using the assumption of perfect competition for modeling energy markets.

We now describe the optimization problems of all market players. The notation used in these optimization problems is summarized in Table 1. To simplify notation, we use quantities without an index to denote the vector of indexed quantities. For example, $y_j := (y_{t,j})_{t \in T}$ is the vector of all electricity generation of producer j and $y := (y_j)_{j \in J}$ is the vector of the overall electricity generation.

Table 1. Technical and economic quantities (sets and parameters above, variables below).

Symbol	Explanation
I	Set of consumers
J	Set of producers
K	Set of storage operators
T	Set of time periods ($T := \{0, 1, \ldots, t^e\}$)
π_t	Market price of electricity in time period $t \in T$
c_j	Variable production costs of producer $j \in J$
$\bar{y}_j$	Generation capacity of producer $j \in J$
$p_{t,i}(\cdot)$	Inverse market demand function of consumer $i \in I$ in $t \in T$
$\bar{s}_k$	Maximum charging/discharging of storage $k \in K$
e_k	Charging efficiency of storage $k \in K$
$\bar{\ell}_k$	Capacity of storage $k \in K$
$\ell_{-1,k}$	Initial state of charge of storage $k \in K$
$y_{t,j}$	Electricity generation of producer $j \in J$ in $t \in T$
$d_{t,i}$	Demand of consumer $i \in I$ in $t \in T$
$s^{\mathrm{c}}_{t,k}$	Charging of storage $k \in K$ in $t \in T$
$s^{\mathrm{d}}_{t,k}$	Discharging of storage $k \in K$ in $t \in T$
$\ell_{t,k}$	State of charge of storage $k \in K$ at the end of $t \in T$

The model of a producer $j \in J$ reads

$$\max_{y_j} \quad \sum_{t \in T} \pi_t y_{t,j} - \sum_{t \in T} c_j y_{t,j} \tag{1a}$$

$$\text{s.t.} \quad 0 \le y_{t,j} \le \bar{y}_j, \quad t \in T. \qquad [\beta^{\pm}_{t,j}] \tag{1b}$$

This means, the producer maximizes his profit (income less costs) by choosing his production level within the prescribed bounds. Here and in what follows, Greek letters behind constraints denote the corresponding dual variables. The solution of (1) is characterized by the Karush–Kuhn–Tucker (KKT) conditions

$$\pi_t - c_j + \beta^-_{t,j} - \beta^+_{t,j} = 0, \quad t \in T, \tag{2a}$$

$$0 \le \beta^-_{t,j} \perp y_{t,j} \ge 0, \quad t \in T, \tag{2b}$$

$$0 \le \beta^+_{t,j} \perp \bar{y}_j - y_{t,j} \ge 0, \quad t \in T. \tag{2c}$$

The optimization problem of the consumer $i \in I$ is given by

$$\max_{d_i} \quad \sum_{t \in T} \int_0^{d_{t,i}} p_{t,i}(x) \,\mathrm{d}x - \sum_{t \in T} \pi_t d_{t,i} \tag{3a}$$

$$\text{s.t.} \quad 0 \le d_{t,i}, \quad t \in T, \qquad [\gamma_{t,i}] \tag{3b}$$

where $p_{t,i} : \mathbb{R}^+_0 \to \mathbb{R}$ is the inverse demand function of consumer $i \in I$ in time period t. Here, the objective function models gross consumer surplus less costs. For the remainder of the paper we make the following assumption.

Assumption 1. *All inverse demand functions $p_{t,i}$, $t \in T$, $i \in I$, are strictly decreasing and continuously differentiable.*

The (again necessary and sufficient) KKT conditions of (3) read

$$p_{t,i}(d_{t,i}) - \pi_t + \gamma_{t,i} = 0, \quad t \in T, \tag{4a}$$

$$0 \le \gamma_{t,i} \perp d_{t,i} \ge 0, \quad t \in T. \tag{4b}$$

For what follows, the market model of a storage operator $k \in K$ with a given initial state of charge $\ell_{-1,k} \geq 0$ reads

$$\max_{s^{\mathrm{c}}_k, s^{\mathrm{d}}_k, \ell_k} \quad \sum_{t \in T} \pi_t (s^{\mathrm{d}}_{t,k} - s^{\mathrm{c}}_{t,k}) \tag{5a}$$

$$\text{s.t.} \quad \ell_{t,k} = \ell_{t-1,k} - (s^{\mathrm{d}}_{t,k} - e_k s^{\mathrm{c}}_{t,k}), \quad t \in T, \qquad [\delta_{t,k}] \tag{5b}$$

$$0 \leq \ell_{t,k} \leq \bar{\ell}_k, \quad t \in T, \qquad [\zeta^{\pm}_{t,k}] \tag{5c}$$

$$0 \leq s^{\mathrm{d}}_{t,k} \leq \bar{s}_k, \quad t \in T, \qquad [\alpha^{\pm}_{t,k}] \tag{5d}$$

$$0 \leq s^{\mathrm{c}}_{t,k} \leq \bar{s}_k, \quad t \in T. \qquad [\nu^{\pm}_{t,k}] \tag{5e}$$

The storage operator pays for charging ($s^{\mathrm{c}}_{t,k} \geq 0$) and earns via discharging ($s^{\mathrm{d}}_{t,k} \geq 0$) electricity at the current market price π_t. In addition, charging and discharging losses are considered via the storage's efficiency $e_k \in (0,1)$. For the ease of presentation, all losses, w.l.o.g., are assumed to occur while charging.[1] Additionally, the storage's state of charge is bounded by its capacity $\bar{\ell}_k > 0$ and we have a maximum charging and discharging rate $\bar{s}_k > 0$ per period of time.

The corresponding KKT conditions are given by

$$\pi_t + \alpha^-_{t,k} - \alpha^+_{t,k} - \delta_{t,k} = 0, \quad t \in T, \tag{6a}$$

$$-\pi_t + \nu^-_{t,k} - \nu^+_{t,k} + e_k \delta_{t,k} = 0, \quad t \in T, \tag{6b}$$

$$\delta_{t+1,k} - \delta_{t,k} + \zeta^-_{t,k} - \zeta^+_{t,k} = 0, \quad t \in T \setminus \{t^e\}, \tag{6c}$$

$$-\delta_{t^e,k} + \zeta^-_{t^e,k} - \zeta^+_{t^e,k} = 0, \tag{6d}$$

$$\ell_{t,k} = \ell_{t-1,k} - (s^{\mathrm{d}}_{t,k} - e_k s^{\mathrm{c}}_{t,k}), \quad t \in T, \tag{6e}$$

$$0 \leq \zeta^-_{t,k} \perp \ell_{t,k} \geq 0, \quad t \in T, \tag{6f}$$

$$0 \leq \zeta^+_{t,k} \perp \bar{\ell}_k - \ell_{t,k} \geq 0, \quad t \in T, \tag{6g}$$

$$0 \leq \alpha^-_{t,k} \perp s^{\mathrm{d}}_{t,k} \geq 0, \quad t \in T, \tag{6h}$$

$$0 \leq \alpha^+_{t,k} \perp \bar{s}_k - s^{\mathrm{d}}_{t,k} \geq 0, \quad t \in T, \tag{6i}$$

$$0 \leq \nu^-_{t,k} \perp s^{\mathrm{c}}_{t,k} \geq 0, \quad t \in T, \tag{6j}$$

$$0 \leq \nu^+_{t,k} \perp \bar{s}_k - s^{\mathrm{c}}_{t,k} \geq 0, \quad t \in T. \tag{6k}$$

Since Model (5) is a linear maximization problem, its KKT conditions (6) are necessary and sufficient.

Putting all separate KKT conditions (2), (4), and (6), as well as the market clearing conditions

$$\sum_{i \in I} d_{t,i} - \sum_{j \in J} y_{t,j} = \sum_{k \in K} (s^{\mathrm{d}}_{t,k} - s^{\mathrm{c}}_{t,k}), \quad t \in T, \tag{7}$$

together, we obtain the following mixed complementarity problem (MCP) of the electricity market equilibrium with storages:

$$\text{Producers: (2)} \quad \text{for all } j \in J, \tag{8a}$$

$$\text{Consumers: (4)} \quad \text{for all } i \in I, \tag{8b}$$

$$\text{Storages: (6)} \quad \text{for all } k \in K, \tag{8c}$$

$$\text{Market clearing: (7).} \tag{8d}$$

[1] Due to optimality, everything charged will also be discharged. The only difference to a model with charging and discharging efficiencies thus is that we might be able to charge slightly more into a storage because all losses are realized during charging. However, this can be addressed by choosing smaller storage capacities appropriately.

In what follows, we consider the relationship between the competitive equilibrium model (8) and a corresponding single-level welfare maximization problem. In analogy to the fundamental theorems of welfare economics as established in Walras (1900) and Arrow and Debreu (1954), we prove the 1-1 correspondence of competitive market equilibria and welfare maxima. Indeed, our case can be directly linked to the classical setup in Arrow and Debreu (1954): Consider electricity in different time periods as different goods. Then, the storage operator acts as a producer that transfers electricity of one time period into electricity of another time period in accordance with its "generation" technology. For a comprehensive overview of the fundamental welfare theorems see, e.g., Mas-Colell et al. (1995).

Theorem 2.1. *Suppose Assumption 1 holds. Let* (y, d, s^c, s^d, ℓ) *be a solution of the welfare maximization problem*

$$\max_{y,d,s^c,s^d,\ell} \quad \sum_{t\in T}\sum_{i\in I}\int_0^{d_{t,i}} p_{t,i}(x)\,\mathrm{d}x - \sum_{t\in T}\sum_{j\in J} c_j y_{t,j} \tag{9a}$$

$$\text{s.t.} \quad 0 \le y_{t,j} \le \bar{y}_j, \quad t\in T,\ j\in J, \qquad [\beta^{\pm}_{t,j}] \tag{9b}$$

$$0 \le d_{t,i}, \quad t\in T,\ i\in I, \qquad [\gamma_{t,i}] \tag{9c}$$

$$\ell_{t,k} = \ell_{t-1,k} - (s^d_{t,k} - e_k s^c_{t,k}), \quad t\in T,\ k\in K, \qquad [\delta_{t,k}] \tag{9d}$$

$$0 \le \ell_{t,k} \le \bar{\ell}_k, \quad t\in T,\ k\in K, \qquad [\zeta^{\pm}_{t,k}] \tag{9e}$$

$$0 \le s^d_{t,k} \le \bar{s}_k, \quad t\in T,\ k\in K, \qquad [\alpha^{\pm}_{t,k}] \tag{9f}$$

$$0 \le s^c_{t,k} \le \bar{s}_k, \quad t\in T,\ k\in K, \qquad [\nu^{\pm}_{t,k}] \tag{9g}$$

$$\sum_{i\in I} d_{t,i} - \sum_{j\in J} y_{t,j} = \sum_{k\in K}(s^d_{t,k} - s^c_{t,k}), \quad t\in T. \qquad [\lambda_t] \tag{9h}$$

Then, (y, d, s^c, s^d, ℓ) *is also part of a solution of the equilibrium problem (8) for* $\pi_t = \lambda_t$.

Proof. The KKT conditions of Problem (9) are given by the system

$$\lambda_t - c_j + \beta^-_{t,j} - \beta^+_{t,j} = 0, \quad t\in T,\ j\in J, \tag{10a}$$

$$p_{t,i}(d_{t,i}) - \lambda_t + \gamma_{t,i} = 0, \quad t\in T,\ i\in I, \tag{10b}$$

$$\lambda_t + \alpha^-_{t,k} - \alpha^+_{t,k} - \delta_{t,k} = 0, \quad t\in T,\ k\in K, \tag{10c}$$

$$-\lambda_t + \nu^-_{t,k} - \nu^+_{t,k} + e_k\delta_{t,k} = 0, \quad t\in T,\ k\in K, \tag{10d}$$

$$\delta_{t+1,k} - \delta_{t,k} + \zeta^-_{t,k} - \zeta^+_{t,k} = 0, \quad t\in T\setminus\{t^e\},\ k\in K, \tag{10e}$$

$$-\delta_{t^e,k} + \zeta^-_{t^e,k} - \zeta^+_{t^e,k} = 0, \quad k\in K, \tag{10f}$$

$$\ell_{t,k} = \ell_{t-1,k} - (s^{\mathrm{d}}_{t,k} - e_k s^{\mathrm{c}}_{t,k}), \quad t\in T,\ k\in K, \tag{10g}$$

$$0 \le \beta^-_{t,j} \perp y_{t,j} \ge 0, \quad t\in T,\ j\in J, \tag{10h}$$

$$0 \le \beta^+_{t,j} \perp \bar{y}_j - y_{t,j} \ge 0, \quad t\in T,\ j\in J, \tag{10i}$$

$$0 \le \gamma_{t,i} \perp d_{t,i} \ge 0, \quad t\in T,\ i\in I, \tag{10j}$$

$$0 \le \zeta^-_{t,k} \perp \ell_{t,k} \ge 0, \quad t\in T,\ k\in K, \tag{10k}$$

$$0 \le \zeta^+_{t,k} \perp \bar{\ell}_k - \ell_{t,k} \ge 0, \quad t\in T,\ k\in K, \tag{10l}$$

$$0 \le \alpha^-_{t,k} \perp s^{\mathrm{d}}_{t,k} \ge 0, \quad t\in T,\ k\in K, \tag{10m}$$

$$0 \le \alpha^+_{t,k} \perp \bar{s}_k - s^{\mathrm{d}}_{t,k} \ge 0, \quad t\in T,\ k\in K, \tag{10n}$$

$$0 \le \nu^-_{t,k} \perp s^{\mathrm{c}}_{t,k} \ge 0, \quad t\in T,\ k\in K, \tag{10o}$$

$$0 \le \nu^+_{t,k} \perp \bar{s}_k - s^{\mathrm{c}}_{t,k} \ge 0, \quad t\in T,\ k\in K, \tag{10p}$$

$$\sum_{i \in I} d_{t,i} - \sum_{j \in J} y_{t,j} = \sum_{k \in K} (s^{\mathrm{d}}_{t,k} - s^{\mathrm{c}}_{t,k}), \quad t \in T. \tag{10q}$$

As all constraints of (9) are linear and since Assumption 1 holds, these KKT conditions are necessary and sufficient optimality conditions. The assertion follows from the equivalence of the KKT conditions (10) and the MCP (8) for $\pi_t = \lambda_t$. □

Theorem 2.2. *Suppose Assumption 1 holds. Let* $(y, d, s^c, s^d, \ell; \beta^\pm, \gamma, \delta, \zeta^\pm, \alpha^\pm, \nu^\pm)$ *be a solution of the MCP (8) for given* π_t, $t \in T$. *Then,* (y, d, s^c, s^d, ℓ) *is a solution of the welfare maximization problem (9).*

Proof. By Assumption 1, Problem (9) is a concave maximization problem with linear constraints. Thus, its KKT conditions are necessary and sufficient. In addition, the KKT conditions (10) exactly correspond to System (8) if we identify π_t with the dual variable λ_t of the market clearing condition (7). □

Theorem 2.1 together with Theorem 2.2 yield the desired 1-1 correspondence of competitive market equilibria and welfare maxima.

3. Uniqueness and Multiplicity of Equilibria

If one considers the equilibrium model introduced in the last section without storage operators, establishing the existence and uniqueness of equilibria is rather easy. In this section, we study the existence and uniqueness of market equilibria if storage operators are considered as well.

The existence of equilibria of (8) is easy to see and follows directly by applying the classical theorem of Weierstraß to the equivalent welfare maximization problem (9). However, uniqueness of equilibria can, in general, not be established as we will show in this section. Nevertheless, the study of uniqueness is of key importance due to the following reasons. Short-run models as the one formulated in Section 2 are often a central component of more complicated bi- or multilevel models; see, e.g., Daxhelet and Smeers (2001), Hobbs et al. (2000), and Hu and Ralph (2007). As soon as, e.g., long-run decisions are considered, these multilevel models are typically of the following form: investment decisions are determined beforehand by different players on the upper levels in anticipation of the resulting short-run market outcomes of subsequent periods on the lower levels; see, e.g., Grimm, Martin, et al. (2016) and Jenabi et al. (2013). Thus, multiplicities of short-run equilibria may directly lead to different decisions w.r.t. long-run investment and, consequently, to different investment costs incurred. This has to be considered if long-run investment decisions are analyzed. Therefore, studying uniqueness of the presented model paves the way for analyzing and understanding more complicated multilevel models that, e.g., deal with the long-run integration of storages into energy markets. Moreover, the existence of uniqueness may also be helpful for developing more efficient algorithms for solving such multilevel models; see, e.g., Chapter 4 and 7 of Dempe (2002) for more information about the importance of lower-level uniqueness in the context of multilevel optimization problems.

The study of uniqueness is also of importance w.r.t. the energy market design. As argued above, multiplicities of short-run equilibria may lead to different long-run investment incentives. This has to be considered when political decisions are taken on the energy market design to avoid inefficient investment incentives. Furthermore, if, e.g., the efficiency potentials of market designs are evaluated, uniqueness may—due to multilevel frameworks—again be a prerequisite for comparing different market designs; see, e.g., Grimm, Rückel, et al. (2019) and the references therein.

For our setup, we prove that ambiguous equilibria differ in single storage operations while demand, generation, and the difference of total discharging and charging are unique for each time period. In order to be able to obtain these results we need to make certain assumptions. For all of these assumptions, we (i) show their necessity by presenting examples that illustrate the consequences of a violation of the assumptions and (ii) discuss the relevance of the assumptions in practice.

Let us start with the first one.

Assumption 2. *The initial and final states of charge are zero, i.e.,* $\ell_{-1,k} = \ell_{t^e,k} = 0$ *for all* $k \in K$.

Due to optimality, every storage will be empty after the final time period in an optimal solution, i.e., $\ell_{t^e,k} = 0$ always holds for all $k \in K$ in an equilibrium. Since a positive state of charge at the beginning of the time horizon thus would lead to a profit without invest, we also set $\ell_{-1,k} = 0$ for all $k \in K$.

In addition, we assume that market prices are positive for all periods. Note that this assumption is violated from time to time in current electricity markets.[2] However, in the context of large-scale storage systems, non-positive prices are not to be expected at real-world spot markets. In the presence of negative prices, the storage operators would naturally charge the production surplus, since this automatically leads to positive profits.

Assumption 3. *The market price* π_t *is positive in each period* $t \in T$.

First, we analyze the actions of a single storage operator and show that a storage is never charged and discharged in the same time period.

Lemma 3.1. *Suppose Assumption 3 holds. Let* (s^c_k, s^d_k, ℓ_k) *be optimal for the storage operator* $k \in K$. *Then,* $s^c_{t,k} s^d_{t,k} = 0$ *holds for all periods* $t \in T$.

Proof. Assume that there exists a period $t \in T$ such that $s^{\mathrm{c}}_{t,k} s^{\mathrm{d}}_{t,k} \neq 0$, i.e., $s^{\mathrm{c}}_{t,k} > 0$ and $s^{\mathrm{d}}_{t,k} > 0$ holds. From (6h) and (6j), $\alpha^-_{t,k} = \nu^-_{t,k} = 0$ follows. Moreover, $\delta_{t,k} = \pi_t - \alpha^+_{t,k}$ follows from (6a). With (6b), we have

$$-\pi_t - \nu^+_{t,k} + e_k(\pi_t - \alpha^+_{t,k}) = 0 \iff (e_k - 1)\pi_t = \nu^+_{t,k} + e_k \alpha^+_{t,k}.$$

Since $e_k - 1 < 0$ due to $e_k \in (0, 1)$ and $\nu^+_{t,k} + e_k \alpha^+_{t,k} \geq 0$ due to (6i) and (6k), we have $\pi_t \leq 0$. This contradicts Assumption 3. □

This result cannot be transferred to the combined actions of storage operators. There may exist scenarios in which some storage devices are charging while other storage devices are discharging. This is shown by the following example.

Example 3.2. For four time periods $T = \{0, 1, 2, 3\}$, we consider one producer with variable production costs $c = 2$ and a generation capacity of $\bar{y} = 4$. Furthermore, we consider a single consumer with demand function $p_0(d) = p_1(d) = 4 - d$ in periods 0 and 1, $p_2(d) = 8 - d$ in period 2, and $p_3(d) = 20 - d$ in period 3 as well as two storage operators $K = \{1, 2\}$ with efficiencies $e_1 = 0.75$ and $e_2 = 0.4$. The initial states of the storages are given by $\ell_{-1,1} = \ell_{-1,2} = 0$, the capacities are $\bar{\ell}_1 = \bar{\ell}_2 = 2$, and the maximum charging and discharging rates are $\bar{s}_1 = \bar{s}_2 = 1$. In this scenario, we get the solution with welfare value 106 listed in Table 2. In period 2, storage 1 discharges while storage 2 charges.

While uniqueness of demands is proven later without any further assumptions, we next make another assumption, which is a necessary prerequisite for obtaining uniqueness of generation. For being able to state the assumption we now turn to variable production costs that also depend on time.

[2]See, e.g., the negative day-ahead prices at the EPEX SPOT (DE-LU) on 04-22-2019 and 06-08-2019 (https://www.epexspot.com/de/marktdaten) or the negative real-time hourly prices of ComEd on 04-26-2019 and 05-22-2019 (https://hourlypricing.comed.com/live-prices/).

TABLE 2. Solution of Problem (9) for the scenario of Example 3.2.

d_0	d_1	d_2	d_3	y_0	y_1	y_2	y_3	$s^{\text{c}}_{0,1}$	$s^{\text{c}}_{0,2}$	$s^{\text{c}}_{1,1}$	$s^{\text{c}}_{1,2}$	$s^{\text{d}}_{2,1}$	$s^{\text{c}}_{2,2}$	$s^{\text{d}}_{3,1}$	$s^{\text{d}}_{3,2}$
2	2	4	6	4	4	4	4	1	1	1	1	0.5	0.5	1	1

TABLE 3. Two different solutions of Problem (9) for the scenario of Example 3.3.

Solution	d_0	d_1	d_2	$y_{0,1}$	$y_{1,1}$	$y_{2,1}$	$y_{0,2}$	$y_{1,2}$	$y_{2,2}$	s^{c}_0	s^{c}_1	s^{d}_2
1	2	2	4	1	1	1	2	1.5	2	1	0.5	1
2	2	2	4	1	1	1	1.5	2	2	0.5	1	1

Assumption 4. *All variable production costs satisfy the following two properties.*

(i) All variable costs are pairwise distinct, i.e.,

$$c_{t,j} \neq c_{t',j'} \quad \textit{for all } (t,j) \neq (t',j') \in T \times J.$$

(ii) Furthermore, it holds

$$c_{t,j} \prod_{i=1}^{m} e_{k_i} \neq c_{t',j'} \prod_{i=m+1}^{m'} e_{k_i}$$

for all $(t,j) \neq (t',j') \in T \times J$ *and* $k_1, \ldots, k_{m'} \in K$ *with* $1 \leq m' \leq |T| - 1$.

In a market with producers and consumers and without storage operators, pairwise distinct variable production costs are necessary in each time period to obtain unique equilibria; cf. Grimm, Schewe, et al. (2017), Krebs, Schewe, et al. (2018), and Krebs and Schmidt (2018). This condition is included in Assumption 4 (i) by the case $t = t'$. In the setting with storages, we also need to ensure pairwise distinct variable production costs for every pair of time periods, i.e., the case $t \neq t'$ in Assumption 4 (i). Thereby it is guaranteed that a storage operator is not indifferent with respect to, e.g., when a specific amount is charged before the respective discharging takes place. The necessity of this case for obtaining uniqueness of generation is illustrated in Example 3.3.

Example 3.3. For three time periods $T = \{0, 1, 2\}$, we consider two producers $J = \{1, 2\}$ with variable production costs $c_{0,1} = c_{1,1} = c_{2,1} = 1$ and $c_{0,2} = c_{1,2} = c_{2,2} = 2$, and capacities $\bar{y}_1 = 1$ and $\bar{y}_2 = 2$. Furthermore, we consider a single consumer with demand function $p_0(d) = p_1(d) = 4 - d$ in period 0 and 1, and $p_2(d) = 7 - d$ in period 2 as well as a single storage operator with efficiency $e = 2/3$. The initial state of the storage is given by $\ell_{-1} = 0$, its capacity is $\bar{\ell} = 4$, and the maximum charging and discharging rate is $\bar{s} = 4$. In this scenario, the optimal generation for Problem (9) is not unique. We get, e.g., the two solutions with welfare value 18 listed in Table 3.

Moreover, storages allow to shift the produced electricity from one time period to another, but with additional storage losses. The costs of the shifted electricity are equal to the respective variable production costs divided by the efficiency of the storage. In particular, these costs are not allowed to meet the variable production costs of any producer of the subsequent time periods since this would contradict the classical uniqueness assumption. Thus, Assumption 4 (ii) for $m' = 1$ is necessary for obtaining uniqueness of generation. We illustrate this fact with the following example.

Table 4. Three different solutions of Problem (9) for the scenario of Example 3.4.

Solution	d_0	d_1	$y_{0,1}$	$y_{1,1}$	$y_{0,2}$	$y_{1,2}$	s_0^{c}	s_1^{d}
1	0	4	0	2	0	2	0	0
2	0	4	1	2	0	9/7	1	5/7
3	0	4	2	2	0	4/7	2	10/7

Table 5. Three different solutions of Problem (9) for the scenario of Example 3.5.

Solution	d_0	d_1	d_2	d_3	y_1	y_2	$s_{1,1}^{\mathrm{c}}$	$s_{3,1}^{\mathrm{d}}$	$s_{0,2}^{\mathrm{c}}$	$s_{1,2}^{\mathrm{c}}$
1, 2, 3	1	1.625	4.1	4.875	3.5	4	1.125	1.125	0.75	0.75

Solution	y_0	y_3	$s_{0,1}^{\mathrm{c}}$	$s_{2,1}^{\mathrm{d}}$	$s_{2,2}^{\mathrm{c}}$	$s_{3,2}^{\mathrm{d}}$
1	2.15625	3.15	0.40625	0.1	0	0.6
2	2.46875	3.05	0.71875	0.35	0.25	0.7
3	2.625	3	0.875	0.475	0.375	0.75

Example 3.4. For two time periods $T = \{0, 1\}$, we consider two producers $J = \{1, 2\}$ with variable production costs $c_{0,1} = 5$, $c_{1,1} = 5.1$ and $c_{0,2} = 6.9$, $c_{1,2} = 7$, and generation capacities of $\bar{y}_1 = \bar{y}_2 = 2$. Furthermore, we consider a single consumer with demand function $p_0(d) = 5 - d$ in period 0 and $p_1(d) = 11 - d$ in period 1 as well as a single storage operator with efficiency $e = 5/7$. The initial state of the storage is given by $\ell_{-1} = 0$, the capacity is $\bar{\ell} = 2$, and its maximum charging and discharging rate is $\bar{s} = 2$. The variable costs satisfy Condition (i) of Assumption 4 but not Condition (ii) since $ec_{1,2} = c_{0,1}$ holds. The optimal generation for Problem (9) is not unique; see Table 4 for three solutions with an optimal welfare value of 11.8.

Since we already know from Example 3.2 that there may exist scenarios in which one storage charges from another, it follows directly that also the costs of combined shifts of multiple storages are not allowed to meet the variable production costs of any producer of the subsequent periods. For this reason, the cases $m' > 1$ are included in Assumption 4 (ii). The next example illustrates the necessity of considering these cases in order to avoid multiplicities of generation.

Example 3.5. For four time periods $T = \{0, 1, 2, 3\}$, we consider a single producer with variable production costs $c_0 = 2$, $c_1 = 1.5$, $c_2 = 0.1$, and $c_3 = 6.25$, and a generation capacity of $\bar{y} = 4$. Furthermore, we consider a single consumer with demand function $p_0(d) = 3 - d$ in period 0, $p_1(d) = 3.125 - d$ in period 1, $p_2(d) = 6.6 - d$ in period 2, and $p_3(d) = 11.125 - d$ in period 3. There are two storage devices $K = \{1, 2\}$ with efficiencies $e_1 = 0.8$ and $e_2 = 0.4$. The initial states of the storage devices are given by $\ell_{-1,1} = \ell_{-1,2} = 0$, the capacities are $\bar{\ell}_1 = 3$ and $\bar{\ell}_2 = 2$, and the maximum charging and discharging rates are $\bar{s}_1 = 1.125$ and $\bar{s}_2 = 0.75$. The variable costs do not satisfy Condition (ii) of Assumption 4 since $e_1 e_2 c_3 = c_0$ holds. It follows that the shift "storage 1 charges in period 0 and discharges in period 2 while storage 2 charges in period 2 and discharges in period 3" is profit-neutral for both storage operators. As a consequence, the optimal generation for Problem (9) is not unique; see Table 5 for three solutions with an optimal welfare value of 37.614375.

In summary, Assumption 4 (ii) prevents all shifts from being profit-neutral, i.e., storage operators are no longer indifferent with regard to carrying out such shifts or not. Next, we establish uniqueness of the demands.

Theorem 3.6. *Suppose Assumption 1 holds. Let* (y, d, s^c, s^d, ℓ) *and* $(\tilde{y}, \tilde{d}, \tilde{s}^c, \tilde{s}^d, \tilde{\ell})$ *be two solutions of Problem (9). Then,* $d = \tilde{d}$ *holds.*

Proof. This follows from Theorem 1a in Mangasarian (1988). □

With Lemma 1 of Grimm, Schewe, et al. (2017), we obtain the following result.

Lemma 3.7. *Suppose Assumption 1 holds. Then, exactly one of the following two cases is true:*

(i) There exist demands d^* *and generations* y^* *such that every solution of Problem (9) is of the form* $(y^*, d^*, s^c, s^d, \ell)$ *for some* s^c, s^d*, and* ℓ*.*

(ii) There exist two solutions $z = (y, d^*, s^c, s^d, \ell)$ *and* $\tilde{z} = (\tilde{y}, d^*, \tilde{s}^c, \tilde{s}^d, \tilde{\ell})$ *of Problem (9) with* $y \neq \tilde{y}$ *and*

$$\begin{aligned}
\{t \in T \colon y_{t,j} = 0\} &= \{t \in T \colon \tilde{y}_{t,j} = 0\},\\
\{t \in T \colon y_{t,j} = \bar{y}_j\} &= \{t \in T \colon \tilde{y}_{t,j} = \bar{y}_j\},\\
\{t \in T \colon s^c_{t,k} = 0\} &= \{t \in T \colon \tilde{s}^c_{t,k} = 0\},\\
\{t \in T \colon s^c_{t,k} = \bar{s}_k\} &= \{t \in T \colon \tilde{s}^c_{t,k} = \bar{s}_k\},\\
\{t \in T \colon s^d_{t,k} = 0\} &= \{t \in T \colon \tilde{s}^d_{t,k} = 0\},\\
\{t \in T \colon s^d_{t,k} = \bar{s}_k\} &= \{t \in T \colon \tilde{s}^d_{t,k} = \bar{s}_k\},\\
\{t \in T \colon \ell_{t,k} = 0\} &= \{t \in T \colon \tilde{\ell}_{t,k} = 0\},\\
\{t \in T \colon \ell_{t,k} = \bar{\ell}_k\} &= \{t \in T \colon \tilde{\ell}_{t,k} = \bar{\ell}_k\},\\
\{t \in T \colon 0 < s^c_{t,k} < \bar{s}_k\} &= \{t \in T \colon 0 < \tilde{s}^c_{t,k} < \bar{s}_k\},\\
\{t \in T \colon 0 < s^d_{t,k} < \bar{s}_k\} &= \{t \in T \colon 0 < \tilde{s}^d_{t,k} < \bar{s}_k\},
\end{aligned}$$

for all $j \in J$ *and* $k \in K$*.*

We want to point out that storage quantities do not appear in Lemma 1 of Grimm, Schewe, et al. (2017). However, in Grimm, Schewe, et al. (2017), the proof of Lemma 1 is generally based on the vector of all variables and therefore, the extension to the case with storage quantities is straightforward. In what follows, our goal is to establish that only Case (i) of Lemma 3.7 occurs. Therefore, we need one more assumption.

Assumption 5. *The total demand* $\sum_{i \in I} d_{t,i}$ *is positive for all time periods* $t \in T$*.*

Note that the latter assumption is not very restrictive for real-world applications. Even in case of large-scale storage systems, there should always be consumers who are willing to pay and, more precisely, whose willingness to pay is higher than the willingness to pay of the storage operators.

Assumption 5 allows to prove uniqueness of market prices. From this point on, the uniqueness of market prices is important for guaranteeing that the revenues of the storage operators are the same in all welfare-maximal solutions.

Lemma 3.8. *Suppose Assumptions 1 and 5 hold. Then, the market prices* π_t *are unique for all* $t \in T$*.*

Proof. For periods $t \in T$ with $\sum_{i \in I} d_{t,i} > 0$, there exists at least one $i \in I$ with $d_{t,i} > 0$. Thus, $\gamma_{t,i} = 0$ follows from (4b). Moreover, we have $p_{t,i}(d_{t,i}) = \pi_t$ from (4a). Since demands are unique by Theorem 3.6, we then obtain unique market prices in periods $t \in T$ with $\sum_{i \in I} d_{t,i} > 0$ under Assumption 1. The case $\sum_{i \in I} d_{t,i} = 0$ is excluded by Assumption 5. □

Lemma 3.9. *Suppose Assumption 1 holds. Let $z = (y, d, s^c, s^d, \ell)$ be a solution of Problem (9). Then, in periods $t \in T$ with $c_{t,j} \neq \pi_t$ for all $j \in J$, the generation $y_{t,j}$, $j \in J$, as well as the difference $\sum_{k \in K}(s^d_{t,k} - s^c_{t,k})$ of total discharging and charging are unique.*

Proof. For $j \in J$ with $c_{t,j} < \pi_t$, $\beta^+_{t,j} > 0$ follows from (2a) and thus, $y_{t,j} = \bar{y}_j$ from (2c). For $j \in J$ with $c_{t,j} > \pi_t$, $\beta^-_{t,j} > 0$ follows from (2a) and hence, $y_{t,j} = 0$ from (2b). Since $c_{t,j} \neq \pi_t$ for all $j \in J$, all generations $y_{t,j}$, $j \in J$, and thus the total generation

$$\sum_{j \in J} y_{t,j} = \sum_{j \in J : c_{t,j} < \pi_t} \bar{y}_j$$

is unique. According to Theorem 3.6, $\sum_{i \in I} d_{t,i}$ is unique. Then, the uniqueness of $\sum_{k \in K}(s^{\mathrm{d}}_{t,k} - s^{\mathrm{c}}_{t,k})$ directly follows from the market clearing condition (7). ☐

It remains to consider the case where $c_{t,j} = \pi_t$ holds for at least one producer $j \in J$. For this we need some more notation. We denote by $W(z)$ the welfare value of a feasible point $z = (y, d, s^{\mathrm{c}}, s^{\mathrm{d}}, \ell)$ of Problem (9). Furthermore, we denote by $W_t(z)$ the welfare value in time period t. The welfare difference of two feasible points $\tilde{z}$ and z is denoted by $\Delta W(\tilde{z}, z) := W(\tilde{z}) - W(z)$. Analogously, we denote by $S(z)$ the objective value of a feasible storage operation $z = (s^{\mathrm{c}}_k, s^{\mathrm{d}}_k, \ell_k)$ of Problem (5), by $S_t(z)$ the corresponding objective value in time period t, and by $\Delta S(\tilde{z}, z) := S(\tilde{z}) - S(z)$ the difference of objective function values. Moreover, we introduce the sets $\mathring{T}^{\mathrm{c}}_k$ and $\mathring{T}^{\mathrm{d}}_k$ of time periods, in which charging and discharging of an individual storage $k \in K$ are not binding, i.e.,

$$\mathring{T}^{\mathrm{c}}_k := \{t \in T : 0 < s^{\mathrm{c}}_{t,k} < \bar{s}_k\}, \quad \mathring{T}^{\mathrm{d}}_k := \{t \in T : 0 < s^{\mathrm{d}}_{t,k} < \bar{s}_k\}.$$

Next, we call two feasible storage operations $z = (s^{\mathrm{c}}_k, s^{\mathrm{d}}_k, \ell_k)$ and $\tilde{z} = (\tilde{s}^{\mathrm{c}}_k, \tilde{s}^{\mathrm{d}}_k, \tilde{\ell}_k)$ interchangeable in time period $t \in T$ if $\hat{z} = (\hat{s}^{\mathrm{c}}_k, \hat{s}^{\mathrm{d}}_k, \hat{\ell}_k)$ with $\hat{z}_{t'} := z_{t'}$ for all $t' \leq t$ and $\hat{z}_{t'} := \tilde{z}_{t'}$ for all $t' > t$ is a feasible storage operation. In particular, storage operations are interchangeable in time periods $t \in T$ with $\ell_{t,k} = \tilde{\ell}_{t,k}$. Finally, we may also divide the time horizon $T = [t^e] = \{0, 1, \ldots, t^e\}$ into $N+1$ disjoint and consecutive blocks, i.e., $T = \bigcup_{n=0}^N T_n$ with $T_n = \{t^s_n, \ldots, t^e_n\}$ and $t^s_0 = 0$, $t^e_N = t^e$.

Lemma 3.10. *Suppose Assumptions 1–5 hold. Let $z = (y, d, s^c, s^d, \ell)$ be a solution of Problem (9). Then, in periods $t \in T$ with $c_{t,j'} = \pi_t$ for at least one $j' \in J$, the generation $y_{t,j}$, $j \in J$, as well as the difference $\sum_{k \in K}(s^d_{t,k} - s^c_{t,k})$ of total discharging and charging are unique.*

Proof. Let $t \in T$ with $c_{t,j'} = \pi_t$ for at least one $j' \in J$ be given. Since all variable costs $c_{t,j}$, $j \in J$, are pairwise distinct, there exists exactly one producer $j' \in J$ with $c_{t,j'} = \pi_t$. The same argument as in the proof of Lemma 3.9 yields $y_{t,j} = 0$ for all producers $j \in J$ with $c_{t,j} > c_{t,j'} = \pi_t$. Analogously, $y_{t,j} = \bar{y}_j$ holds for all producers $j \in J$ with $c_{t,j} < c_{t,j'} = \pi_t$. Consequently, it remains to show that $y_{t,j'}$ is unique. The uniqueness of the difference of total discharging and charging then again follows from the market clearing condition (7).

Now, suppose for the sake of contradiction that there exist two welfare-maximal solutions $z = (y, d^*, s^{\mathrm{c}}, s^{\mathrm{d}}, \ell)$ and $\tilde{z} = (\tilde{y}, d^*, \tilde{s}^{\mathrm{c}}, \tilde{s}^{\mathrm{d}}, \tilde{\ell})$ as described in Case (ii) of Lemma 3.7, for which $y_{t,j'} \neq \tilde{y}_{t,j'}$ holds. Without loss of generality, we assume that $\tilde{y}_{t,j'} = y_{t,j'} + \Delta y_{t,j'}$ with $0 < \Delta y_{t,j'} < \bar{y}_{j'} - y_{t,j'}$, i.e., we have

$$\sum_{k \in K} \tilde{s}^{\mathrm{d}}_{t,k} = \sum_{k \in K} s^{\mathrm{d}}_{t,k} - \Delta y^{\mathrm{d}}_{t,j'} \quad \text{and} \quad \sum_{k \in K} \tilde{s}^{\mathrm{c}}_{t,k} = \sum_{k \in K} s^{\mathrm{c}}_{t,k} + \Delta y^{\mathrm{c}}_{t,j'}$$

with $\Delta y_{t,j'} = \Delta y^{\mathrm{d}}_{t,j'} + \Delta y^{\mathrm{c}}_{t,j'}$, $\Delta y^{\mathrm{d}}_{t,j'} > 0$ or $\Delta y^{\mathrm{c}}_{t,j'} > 0$. From now on, we assume that $\Delta y^{\mathrm{d}}_{t,j'} > 0$ and derive a contradiction. The same arguments can be applied to the case $\Delta y^{\mathrm{c}}_{t,j'} > 0$.

Since $\Delta W_t(\tilde{z}, z) = -\pi_t \Delta y_{t,j'}$ and $W(z) = W(\tilde{z})$ holds, it follows that $\Delta W_{T\setminus\{t\}}(\tilde{z}, z) = \pi_t \Delta y_{t,j'}$. As the demand d^* is unique, there exists at least one time period $t' \in T \setminus \{t\}$ such that $\sum_{j\in J} \tilde{y}_{t',j} = \sum_{j\in J} y_{t',j} - \Delta y_{t'}$ for $\Delta y_{t'} \in \mathbb{R}_{>0}$. Hence, generation in period t' is not unique. This implies $\pi_{t'} = c_{t',j}$ for some $j \in J$ by Lemma 3.9. For the storages, we have

$$\sum_{k\in K} \tilde{s}^{\mathrm{d}}_{t',k} = \sum_{k\in K} s^{\mathrm{d}}_{t',k} + \Delta y^{\mathrm{d}}_{t'} \quad \text{and} \quad \sum_{k\in K} \tilde{s}^{\mathrm{c}}_{t',k} = \sum_{k\in K} s^{\mathrm{c}}_{t',k} - \Delta y^{\mathrm{c}}_{t'}$$

with $\Delta y_{t'} = \Delta y^{\mathrm{d}}_{t'} + \Delta y^{\mathrm{c}}_{t'}$, $\Delta y^{\mathrm{d}}_{t'} \in \mathbb{R}_{>0}$ or $\Delta y^{\mathrm{c}}_{t'} \in \mathbb{R}_{>0}$.

Now, we consider the individual storage operations. Let $\Delta s^{\mathrm{d}}_{t,k} := s^{\mathrm{d}}_{t,k} - \tilde{s}^{\mathrm{d}}_{t,k}$ be the difference in discharging of the storage $k \in K$ in time period t. The value $\Delta s^{\mathrm{c}}_{t,k}$ for charging is defined analogously. Since $\Delta y^{\mathrm{d}}_{t,j'} = \sum_{k\in K} \Delta s^{\mathrm{d}}_{t,k} > 0$, $\Delta s^{\mathrm{d}}_{t,k}$ must be greater than zero for at least one storage. We restrict our argumentation to one of these storages, i.e., to $k' \in K$ with

$$\tilde{s}^{\mathrm{d}}_{t,k'} = s^{\mathrm{d}}_{t,k'} - \Delta s^{\mathrm{d}}_{t,k'}, \quad \Delta s^{\mathrm{d}}_{t,k'} > 0. \tag{11}$$

Theorems 2.1 and 2.2 imply that welfare-maximal solutions also yield optimal storage operations. By Lemma 3.8, $S(z) = S(\tilde{z})$ follows. Hence, we have $\Delta S_t(\tilde{z}, z) = -\pi_t \Delta s^{\mathrm{d}}_{t,k'} < 0$ and $\Delta S_{T\setminus\{t\}}(\tilde{z}, z) > 0$. Again, to compensate this difference in objective function values caused in period t, at least one of the following two cases must occur

(i) $\tilde{s}^{\mathrm{d}}_{t'',k'} = s^{\mathrm{d}}_{t'',k'} + \Delta s^{\mathrm{d}}_{t'',k'}$ for a period $t'' \in \mathring{T}^{\mathrm{d}}_{k'} \setminus \{t\}$ with $\Delta s^{\mathrm{d}}_{t'',k'} > 0$ or
(ii) $\tilde{s}^{\mathrm{c}}_{t'',k'} = s^{\mathrm{c}}_{t'',k'} - \Delta s^{\mathrm{c}}_{t'',k'}$ for a period $t'' \in \mathring{T}^{\mathrm{c}}_{k'} \setminus \{t\}$ with $\Delta s^{\mathrm{c}}_{t'',k'} > 0$.

Equation (11) implies that the storage k' discharges less in solution $\tilde{z}$ than in solution z in period t. The Cases (i) and (ii) express that there must exist at least one other time period where either there is more discharging in solution $\tilde{z}$ than in z or there is less charging in $\tilde{z}$ than in z in order to balance the storage operators objective function. Note that the storage operations in z and $\tilde{z}$ are interchangeable at all time periods $t \in T \setminus (\mathring{T}^{\mathrm{c}}_{k'} \cup \mathring{T}^{\mathrm{d}}_{k'})$ since

$$\{t \in T : \ell_{t,k'} = 0\} = \{t \in T : \tilde{\ell}_{t,k'} = 0\},$$
$$\{t \in T : \ell_{t,k'} = \bar{\ell}_{k'}\} = \{t \in T : \tilde{\ell}_{t,k'} = \bar{\ell}_{k'}\},$$

holds due to Case (ii) of Lemma 3.7. Consequently, the difference in objective function values caused in time period t must be compensated within a time block T_n such that the storage is never empty or full over the periods in this time block. In particular, there exist time periods $t_1, \dots, t_m \in T_n \cap \mathring{T}^{\mathrm{c}}_{k'}$ and $t_{m+1}, \dots, t_{m'} \in T_n \cap \mathring{T}^{\mathrm{d}}_{k'} \setminus \{t\}$, in which the imbalance of the storage operators objective function, which is caused by discharging $\Delta s^{\mathrm{d}}_{t,k'} > 0$, is leveled, i.e.,

$$\Delta s^{\mathrm{d}}_{t,k'} = \sum_{i=1}^{m} e_{k'} \Delta s^{\mathrm{c}}_{t_i,k'} + \sum_{i=m+1}^{m'} \Delta s^{\mathrm{d}}_{t_i,k'}$$

with $\Delta s^{\mathrm{c}}_{t_i,k'}, \Delta s^{\mathrm{d}}_{t_i,k'} > 0$; cf. the Cases (i) and (ii). Since furthermore

$$\{t \in T : s^{\mathrm{d}}_{t,k'} = 0\} = \{t \in T : \tilde{s}^{\mathrm{d}}_{t,k'} = 0\},$$
$$\{t \in T : s^{\mathrm{d}}_{t,k'} = \bar{s}_{k'}\} = \{t \in T : \tilde{s}^{\mathrm{d}}_{t,k'} = \bar{s}_{k'}\},$$
$$\{t \in T : s^{\mathrm{c}}_{t,k'} = 0\} = \{t \in T : \tilde{s}^{\mathrm{c}}_{t,k'} = 0\},$$
$$\{t \in T : s^{\mathrm{c}}_{t,k'} = \bar{s}_{k'}\} = \{t \in T : \tilde{s}^{\mathrm{c}}_{t,k'} = \bar{s}_{k'}\},$$

holds, both charging and discharging can be modified by sufficiently small amounts in the time periods $t_1, \ldots, t_{m'}$. Thus, if one of the differences $\Delta s^{\mathrm{c}}_{t_i,k'}, \Delta s^{\mathrm{d}}_{t_i,k'}$ in storage operations would be profitable, this could be used to slightly increase profits of the storage operation $(s^{\mathrm{c}}_{k'}, s^{\mathrm{d}}_{k'}, \ell_{k'})$ or the storage operation $(\tilde{s}^{\mathrm{c}}_{k'}, \tilde{s}^{\mathrm{d}}_{k'}, \tilde{\ell}_{k'})$. This contradicts optimality of both operations. Consequently, $\pi_{t_1} = \cdots = \pi_{t_m} = e_{k'} c_{t,j'}$ and $\pi_{t_{m+1}} = \cdots = \pi_{t_{m'}} = c_{t,j'}$ hold. From Assumption 4, $\pi_{t_i} \neq c_{t_i,j}$ results for all $i \in \{1, 2, \ldots, m'\}$ and $j \in J$. Thus,

$$\sum_{k \in K} \left(s^{\mathrm{d}}_{t_i,k} - s^{\mathrm{c}}_{t_i,k}\right) = \sum_{k \in K} \left(\tilde{s}^{\mathrm{d}}_{t_i,k} - \tilde{s}^{\mathrm{c}}_{t_i,k}\right) \tag{12}$$

follows from Lemma 3.9 for all $i \in \{1, 2, \ldots, m'\}$.

Now, we combine the results that we obtained from the welfare perspective and the point of view of the storage operator. To achieve (12), all differences $\Delta s^{\mathrm{c}}_{t_i,k'}, \Delta s^{\mathrm{d}}_{t_i,k'}$ in storage operations must be compensated by other storages. Since $\Delta s^{\mathrm{d}}_{t,k'} > 0$ is part of the surplus generation $\Delta y_{t,j'}$ in period t, there must exist other storages that, in the end, compensate this surplus generation in periods like $t' \in T \setminus \{t\}$ with $\sum_{j \in J} \tilde{y}_{t',j} = \sum_{j \in J} y_{t',j} - \Delta y_{t'}$ for $\Delta y_{t'} \in \mathbb{R}_{>0}$ and with $\pi_{t'} = c_{t',j}$ for some $j \in J$. The same argument as for storage k' yields that all of these subsequent storages could also charge and discharge sufficiently small additional units in time blocks in which their storage operations are unbalanced. Furthermore, as storage operations are optimal, the differences in storage operations cannot be profitable. Thus, $c_{t,j'} \prod_{i=1}^{\tilde{m}} e_{k_i} = c_{t',j} \prod_{i=\tilde{m}+1}^{\hat{m}} e_{k_i}$ follows for $k_1, \ldots, k_{\hat{m}} \in K$ with $0 \leq \hat{m} \leq |T| - 1$. This contradicts Assumption 4. □

If we now combine Theorem 3.6 as well as Lemma 3.9 and 3.10, we obtain our main uniqueness result.

Theorem 3.11. *Suppose Assumptions 1–5 hold. Then, there exist demands d^* and generations y^* such that every solution of Problem (9) is of the form $(y^*, d^*, s^c, s^d, \ell)$ for some s^c, s^d, and ℓ. Furthermore, the differences $\sum_{k \in K}(s^d_{t,k} - s^c_{t,k})$ of total discharging and charging are unique for all time periods $t \in T$.*

In particular, Theorem 3.11 implies the uniqueness of the market equilibrium in case of a single storage operator.

Corollary 3.12. *Suppose Assumptions 1–5 hold. Furthermore, suppose that only one storage operates at the spot market, i.e., $|K| = 1$. Then, the solution of Problem (9) is unique.*

The next examples show that this may no longer be true in case of multiple storage operators. All of these examples satisfy Assumptions 1–5 and illustrate different types of multiplicities in discharging and charging although, according to Theorem 3.11, the differences of total discharging and charging are unique for all time periods. In the following setting, the storage operators are indifferent w.r.t. the specific time period in which they discharge.

Example 3.13. For three time periods $T = \{0, 1, 2\}$, we consider two producers $J = \{1, 2\}$ with variable production costs $c_{0,1} = 1$, $c_{1,1} = 1.1$, $c_{2,1} = 1.2$, and $c_{0,2} = 2$, $c_{1,2} = 2.1$, $c_{2,2} = 2.2$, as well as generation capacities $\bar{y}_1 = 3$ and $\bar{y}_2 = 2$. Furthermore, we consider a single consumer with demand function $p_0(d) = 2 - d$ in period 0, $p_1(d) = 8 - d$ in period 1, and $p_2(d) = 8.1 - d$ in period 2 as well as two storage operators $K = \{1, 2\}$ with efficiencies $e_1 = 0.95$ and $e_2 = 0.85$. The initial states of the storage devices are given by $\ell_{-1,1} = \ell_{-1,2} = 0$, the capacities are $\bar{\ell}_1 = 2$ and $\bar{\ell}_2 = 4$, and the maximum charging and discharging rates are $\bar{s}_1 = 1.4$ and $\bar{s}_2 = 1$. Then, the solution of Problem (9) is not unique; see Table 6 for two solutions with optimal welfare value of 43.541.

TABLE 6. Two different solutions of Problem (9) for the scenario of Example 3.13.

Solution	d_0	d_1	d_2	$y_{0,1}$	$y_{1,1}$	$y_{2,1}$	$y_{0,2}$	$y_{1,2}$	$y_{2,2}$
1, 2	0.6	5.8	5.9	3	3	3	0	2	1.52

Solution	$s^{\mathrm{c}}_{0,1}$	$s^{\mathrm{d}}_{1,1}$	$s^{\mathrm{d}}_{2,1}$	$s^{\mathrm{c}}_{0,2}$	$s^{\mathrm{d}}_{1,2}$	$s^{\mathrm{d}}_{2,2}$
1	1.4	0	1.33	1	0.8	0.05
2	1.4	0.8	0.53	1	0	0.85

TABLE 7. Two different solutions of Problem (9) for the scenario of Example 3.14.

Solution	d_0	d_1	d_2	$y_{0,1}$	$y_{1,1}$	$y_{2,1}$	$y_{0,2}$	$y_{1,2}$	$y_{2,2}$
1, 2	2.9	2.0	3.3	1	1	1	2	41/30	2

Solution	$s^{\mathrm{c}}_{0,1}$	$s^{\mathrm{c}}_{1,1}$	$s^{\mathrm{d}}_{2,1}$	$s^{\mathrm{c}}_{0,2}$	$s^{\mathrm{c}}_{1,2}$	$s^{\mathrm{d}}_{2,2}$
1	0.1	1/6	0.2	0	0.2	0.1
2	1/15	0.2	0.2	1/30	1/6	0.1

Note that the multiplicity of the individual storage operations in Example 3.13 occurs although the efficiencies of the two storage devices are distinct. Multiplicities arise not only in the case of discharging but also in the case of charging as it is shown by the following example.

Example 3.14. For three time periods $T = \{0, 1, 2\}$, we consider two producers $J = \{1, 2\}$ with variable production costs as in the last example and generation capacities $\bar{y}_1 = 1$ and $\bar{y}_2 = 2$. Furthermore, we consider a single consumer with demand function $p_0(d) = 5 - d$ in period 0, $p_1(d) = 4.1 - d$ in period 1, and $p_2(d) = 7.5 - d$ in period 2 as well as two storage operators $K = \{1, 2\}$ with efficiencies $e_1 = 0.75$ and $e_2 = 0.5$. The initial states of the storage devices are given by $\ell_{-1,1} = \ell_{-1,2} = 0$, the capacities are $\bar{\ell}_1 = 1$ and $\bar{\ell}_2 = 2$, and the maximum charging and discharging rates are $\bar{s}_1 = 0.2$ and $\bar{s}_2 = 0.5$. Then, the solution of Problem (9) is not unique; see Table 7 for two solutions with optimal welfare value of 21.23.

All scenarios with multiple solutions shown so far have two things in common. First, the sum of total charging as well as the sum of total discharging are unique in each time period. Second, multiplicity in storage operations is caused by the same price over multiple periods. In Example 3.15, we show that there also exist scenarios with multiplicities in which the sum of total charging and the sum of total discharging are not unique in each time period. Still, this is due to the same price over multiple periods. Moreover, we demonstrate in Example 3.16 that multiplicity in storage operations may also occur if the price of one period multiplied by a storage efficiency equals the price of another period.

Example 3.15. For five time periods $T = \{0, 1, 2, 3, 4\}$, we consider a single producer with variable production costs $c_0 = 2$, $c_1 = 2.1$, $c_2 = 2.2$, $c_3 = 2.3$, and $c_4 = 2.4$, and a generation capacity of $\bar{y} = 4$. Furthermore, we consider a single consumer with demand function $p_0(d) = 3.5 - d$ in period 0, $p_1(d) = 3.6 - d$ in period 1, $p_2(d) = 8.3 - d$ in period 2, $p_3(d) = 8.4 - d$ in period 3, and $p_4(d) = 25.4 - d$ in period 4. There are two storage operators $K = \{1, 2\}$ with efficiencies $e_1 = 0.6$

TABLE 8. Two different solutions of Problem (9) for the scenario of Example 3.15.

Solution	d_0	d_1	d_2	d_3	d_4	y_0	y_1	y_2	y_3	y_4
1, 2	1.5	1.5	103/30	106/30	6.5	4	4	4	4	4

Solution	$s^{\mathrm{c}}_{0,1}$	$s^{\mathrm{c}}_{1,1}$	$s^{\mathrm{d}}_{2,1}$	$s^{\mathrm{d}}_{3,1}$	$s^{\mathrm{d}}_{4,1}$	$s^{\mathrm{c}}_{0,2}$	$s^{\mathrm{c}}_{1,2}$	$s^{\mathrm{c}}_{2,2}$	$s^{\mathrm{c}}_{3,2}$	$s^{\mathrm{d}}_{4,2}$
1	1.5	1.5	0.2	0.1	1.5	1	1	23/30	17/30	1
2	1.5	1.5	0.1	0.2	1.5	1	1	2/3	2/3	1

TABLE 9. Three different solutions of Problem (9) for the scenario of Example 3.16.

Solution	d_0	d_1	d_2	d_3	y_0	y_1	y_2	y_3	$s^{\mathrm{d}}_{3,1}$	$s^{\mathrm{c}}_{0,2}$	$s^{\mathrm{c}}_{1,2}$
1, 2, 3	1	1.125	4.1	4.875	4	4	4	2.24	1.125	0.75	0.75

Solution	$s^{\mathrm{c}}_{0,1}$	$s^{\mathrm{c}}_{1,1}$	$s^{\mathrm{d}}_{2,1}$	$s^{\mathrm{c}}_{2,2}$	$s^{\mathrm{d}}_{3,2}$	$s^{\mathrm{c}}_{0,3}$	$s^{\mathrm{c}}_{1,3}$	$s^{\mathrm{d}}_{3,3}$
1	0.90625	0.625	0.1	0	0.6	1.34375	1.5	0.91
2	1.015625	0.75	0.2875	0.1875	0.675	1.234375	1.375	0.835
3	1.125	0.875	0.475	0.375	0.75	1.125	1.25	0.76

and $e_2 = 0.3$. The initial states of the storage devices are given by $\ell_{-1,1} = \ell_{-1,2} = 0$, the capacities are $\bar{\ell}_1 = 3$ and $\bar{\ell}_2 = 2$, and the maximum charging and discharging rates are $\bar{s}_1 = 1.5$ and $\bar{s}_2 = 1$. Then, the solution of Problem (9) is not unique; see Table 8 for two solutions with optimal welfare value of 154.42 (rounded to the second decimal).

Example 3.16. For four time periods $T = \{0, 1, 2, 3\}$, we consider the same consumer and producer as in Example 3.5 except for the variable production costs $c_0 = 1.9$ in period 0. Furthermore, we consider three storage operators $K = \{1, 2, 3\}$. The storage operators 1 and 2 are also the same as in Example 3.5. The third storage has an efficiency of $e_3 = 0.32$, an initial state of $\ell_{-1,3} = 0$, and a capacity of $\bar{\ell}_3 = 4$. The maximum charging and discharging rate is $\bar{s}_3 = 1.5$. Then, e.g., $\pi_0 = \pi_2 e_1$ and $\pi_2 = \pi_3 e_2$ hold. As a consequence, the solution of Problem (9) is not unique; see Table 9 for three solutions with optimal welfare value of 38.389375.

Finally, we establish an ex-post uniqueness condition. To this end, we introduce the following assumption.

Assumption 6. *The efficiencies of the storage devices are pairwise distinct, i.e.,* $e_k \neq e_{k'}$ *for all* $k, k' \in K$.

Assumption 6 is necessary, as otherwise there are exchange possibilities between the storage operations of the storages with the same efficiencies.

Theorem 3.17. *Suppose Assumptions 1–6 hold. Furthermore, assume that* $\pi_t \neq \pi_{t'}$ *and* $\pi_t \neq \pi_{t'} e_k$ *for all* $t, t' \in T$, $t \neq t'$, *and* $k \in K$. *Let* $z = (y^*, d^*, s^{\mathrm{c}}, s^{d}, \ell)$ *and* $\tilde{z} = (y^*, d^*, \tilde{s}^{\mathrm{c}}, \tilde{s}^{d}, \tilde{\ell})$ *be solutions of Problem* (9). *Then,* $z = \tilde{z}$ *holds.*

Proof. Lemma 1 of Grimm, Schewe, et al. (2017) implies that if there are two solutions of Problem (9), then, there are also two solutions with the same binding patterns; cf. also Case (ii) of Lemma 3.7. Thus, we assume that the binding patterns in z and $\tilde{z}$ coincide. If $s^{\mathrm{c}} = \tilde{s}^{\mathrm{c}}$ and $s^{\mathrm{d}} = \tilde{s}^{\mathrm{d}}$ hold, $z = \tilde{z}$ follows obviously. Hence, we first assume for the sake of contradiction that $s^{\mathrm{c}} \neq \tilde{s}^{\mathrm{c}}$. In particular, $s_k^{\mathrm{c}} \neq \tilde{s}_k^{\mathrm{c}}$ holds for some $k \in K$. Let $t \in T$ be the first time period in which $s_{t,k}^{\mathrm{c}} \neq \tilde{s}_{t,k}^{\mathrm{c}}$. Without loss of generality, we assume that $s_{t,k}^{\mathrm{c}} < \tilde{s}_{t,k}^{\mathrm{c}}$, i.e., $S_t(z) > S_t(\tilde{z})$. Since profits are the same in all welfare-maximal solutions (which follows from Lemma 3.8), this difference in the objective values has to be compensated in other periods by more charging or less discharging in z compared to $\tilde{z}$. The same argument as in the proof of Lemma 3.10 yields that such differences in storage operations must be profit-neutral. To charge profit-neutral less in one period and more in another, the prices of the respective periods must be the same. This is not possible due to $\pi_t \neq \pi_{t'}$ for all $t, t' \in T$. Consequently, it is also not possible to discharge profit-neutral less in one period and more in another. Hence, under the assumption that $\pi_t \neq \pi_{t'}$ for all $t, t' \in T$, $s^{\mathrm{d}} \neq \tilde{s}^{\mathrm{d}}$ follows directly from $s^{\mathrm{c}} \neq \tilde{s}^{\mathrm{c}}$. In particular, $s_{t,k}^{\mathrm{c}} < \tilde{s}_{t,k}^{\mathrm{c}}$ is at least partly compensated by $s_{t',k}^{\mathrm{d}} < \tilde{s}_{t',k}^{\mathrm{d}}$ for some time period $t' \in T$. Again, to guarantee profit-neutrality of this difference in storage operations, $\pi_t = \pi_{t'} e_k$ must hold but that contradicts the assumption. □

Let us close this section with some comments on the results. In countries like, e.g., Germany, it is believed that the market integration of electricity storage systems is an important prerequisite for the success of the energy turnaround. Thus, appropriate models need to be set up and studied. In this section, we showed that one needs to be very careful when designing such models since it is easy to lose the desired property of unique market equilibria. This is also important for more complicated models compared to the one that we consider here. First, long-run models that also take into account the investment decisions in, e.g., storage capacities, may yield unwanted solutions when made based on short-run models with multiple equilibria; see, e.g., Grimm, Schewe, et al. (2017) for a respective analysis of long-run equilibria in the context without storages. Second, uniqueness of market equilibria is of key importance in more complicated multilevel models that, e.g., consider optimal network expansion or study regulatory decisions; see, e.g., Ambrosius et al. (2018), Grimm, Kleinert, et al. (2019), Grimm, Martin, et al. (2016), and Kleinert and Schmidt (2019).

The assumptions required to guarantee uniqueness a priorily or to check uniqueness ex post are hard to satisfy in practice. One way that we see to resolve the issue of multiple equilibria is to implement suitable equilibrium selection procedures as, e.g., in Huppmann and Egerer (2015), Ralph and Xu (2011), and Ruiz and Conejo (2015). However, these procedures are discussed very controversially in the literature. Another way is to consider strictly convex optimization problems to model the behavior of storage operators. This would directly ensure uniqueness of equilibria. However, the specific choice of these strictly convex models is not clear in advance; see also Section 6.

4. An ADMM Tailored Reformulation

Equilibrium models as discussed in this paper, but without storage operators, often are computationally rather easy to solve because they have a time-separable structure. This means that, in principle, the equilibrium problem over a long time horizon can be solved by solving separate small equilibrium problems for the respective trading periods standalone. However, the market integration of storage systems leads to a coupling of all trading periods of the entire time horizon and thus makes the problem much harder to solve.

In order to compute market equilibria efficiently also for large-scale instances, we now develop a reformulation that is tailored for a parallel and distributed alternating direction method of multipliers (ADMM). To this end, recall the welfare optimization problem given in (9):

$$\max_{y,d,s^{\mathrm{c}},s^{\mathrm{d}},\ell} \quad \sum_{t\in T}\sum_{i\in I}\int_0^{d_{t,i}} p_{t,i}(x)\,\mathrm{d}x - \sum_{t\in T}\sum_{j\in J} c_j y_{t,j} \tag{13a}$$

$$\text{s.t.} \quad 0 \le y_{t,j} \le \bar{y}_j, \quad t\in T,\ j\in J, \tag{13b}$$

$$0 \le d_{t,i}, \quad t\in T,\ i\in I, \tag{13c}$$

$$\ell_{t,k} = \ell_{t-1,k} - (s^{\mathrm{d}}_{t,k} - e_k s^{\mathrm{c}}_{t,k}), \quad t\in T,\ k\in K, \tag{13d}$$

$$0 \le \ell_{t,k} \le \bar{\ell}_k, \quad t\in T,\ k\in K, \tag{13e}$$

$$0 \le s^{\mathrm{d}}_{t,k} \le \bar{s}_k, \quad t\in T,\ k\in K, \tag{13f}$$

$$0 \le s^{\mathrm{c}}_{t,k} \le \bar{s}_k, \quad t\in T,\ k\in K, \tag{13g}$$

$$\sum_{i\in I} d_{t,i} - \sum_{j\in J} y_{t,j} = \sum_{k\in K} (s^{\mathrm{d}}_{t,k} - s^{\mathrm{c}}_{t,k}), \quad t\in T. \tag{13h}$$

Note that we now consider time-independent variable costs again for the ease of presentation and implementation. In this section, we divide the time horizon into time blocks as introduced before Lemma 3.10. That is, we divide the time horizon $T = [t^e] = \{0, 1, \dots, t^e\}$ into $N+1$ disjoint and consecutive blocks, i.e., $T = \bigcup_{n=0}^N T_n$ with $T_n = \{t_n^s, \dots, t_n^e\}$ and $t_0^s = 0$, $t_N^e = t^e$. Let $z = (y, d, s^{\mathrm{c}}, s^{\mathrm{d}}, \ell)$ denote the entire variable vector of Problem (13) and let $z_n = (y_n, d_n, s_n^{\mathrm{c}}, s_n^{\mathrm{d}}, \ell_n)$ be the variable vector of time block T_n. This means that, e.g., $y_n = (y_t)_{t\in T_n}$. Note that the objective function of (13) is time-separable, i.e.,

$$f(z) = \sum_{t\in T}\sum_{i\in I}\int_0^{d_{t,i}} p_{t,i}(x)\,\mathrm{d}x - \sum_{t\in T}\sum_{j\in J} c_j y_{t,j} = \sum_{n\in[N]} f_n(z_n)$$

with

$$f_n(z_n) = \sum_{t\in T_n}\sum_{i\in I}\int_0^{d_{t,i}} p_{t,i}(x)\,\mathrm{d}x - \sum_{t\in T_n}\sum_{j\in J} c_j y_{t,j}$$

holds. We now split the feasible region Ω of Problem (13) accordingly into $N+1$ disjoint feasible sets

$$\begin{aligned}\Omega_n = \Big\{ z_n \ :\ & 0 \le y_{t,j} \le y_j, \quad t\in T_n,\ j\in J,\\ & 0 \le d_{t,i}, \quad t\in T_n,\ i\in I,\\ & \ell_{t,k} = \ell_{t-1,k} - (s^{\mathrm{d}}_{t,k} - e_k s^{\mathrm{c}}_{t,k}), \quad t\in T_n\setminus\{t_n^s\},\ k\in K,\\ & 0 \le \ell_{t,k} \le \bar{\ell}_k, \quad t\in T_n,\ k\in K,\\ & 0 \le s^{\mathrm{d}}_{t,k} \le \bar{s}_k, \quad t\in T_n,\ k\in K,\\ & 0 \le s^{\mathrm{c}}_{t,k} \le \bar{s}_k, \quad t\in T_n,\ k\in K,\\ & \sum_{i\in I} d_{t,i} - \sum_{j\in J} y_{t,j} = \sum_{k\in K}(s^{\mathrm{d}}_{t,k} - s^{\mathrm{c}}_{t,k}), \quad t\in T_n \Big\}\end{aligned}$$

of the separate time blocks. Note that initial and terminal conditions for storages must be added to the first and last block, i.e., to Ω_0 and Ω_N. Possible conditions are $\ell_{-1,k} = \ell_{t^e,k} = 0$ for all $k\in K$; see Assumption 2 in Section 3. In addition, Ω_0 contains the equation $\ell_{0,k} = \ell_{-1,k} - (s^{\mathrm{d}}_{0,k} - e_k s^{\mathrm{c}}_{0,k})$ for all $k\in K$. This yields the time-block-decomposed formulation

$$\max_{(z_n)_{n\in[N]}} \quad \sum_{n\in[N]} f_n(z_n) \tag{15a}$$

$$\text{s.t.} \quad z_n \in \Omega_n, \quad n \in [N], \tag{15b}$$

$$\ell_{t_n^s,k} = \ell_{t_{n-1}^e,k} - (s^{\mathrm{d}}_{t_n^s,k} - e_k s^{\mathrm{c}}_{t_n^s,k}), \quad n \in [N] \setminus \{0\},\ k \in K. \tag{15c}$$

Thus, each time block can be optimized independently of the other blocks as long as the coupling constraints (15c) for the state of charge of the storages are satisfied. For each block T_n, we define the variables

$$l_n^s := (l_{n,k}^s)_{k\in K}, \quad l_n^e := (l_{n,k}^e)_{k\in K}, \quad n \in [N], \tag{16}$$

modeling the state of charge of the storages at the beginning and the end of the nth block, i.e.,

$$l_{n,k}^s = \ell_{t_n^s,k} + (s^{\mathrm{d}}_{t_n^s,k} - e_k s^{\mathrm{c}}_{t_n^s,k}), \quad l_{n,k}^e = \ell_{t_n^e,k}, \quad n \in [N],\ k \in K. \tag{17}$$

Hence, Problem (15) is equivalent to

$$\max_{(z_n, l_n^s, l_n^e)_{n\in[N]}} \quad \sum_{n\in[N]} f_n(z_n) \tag{18a}$$

$$\text{s.t.} \quad (z_n, l_n^s, l_n^e) \in \hat{\Omega}_n, \quad n \in [N], \tag{18b}$$

$$l_{n-1,k}^e = l_{n,k}^s, \quad n \in [N] \setminus \{0\},\ k \in K, \tag{18c}$$

with

$$\hat{\Omega}_n := \Big\{ \hat{z}_n := (z_n, l_n^s, l_n^e) : z_n \in \Omega_n, \\ l_{n,k}^s = \ell_{t_n^s,k} + (s^{\mathrm{d}}_{t_n^s,k} - e_k s^{\mathrm{c}}_{t_n^s,k}), \quad k \in K, \\ l_{n,k}^e = \ell_{t_n^e,k}, \quad k \in K \Big\},$$

where $\hat{\Omega}_0$ and $\hat{\Omega}_N$ must again be adapted as described above for Ω_0 and Ω_N.

Up to now, we have reformulated the original model as a quasi block-separable problem, i.e., a block-decomposed problem that is only coupled by the constraints in (18c). Using the notation $\eta_n = (\eta_{n,k})_{k\in K}$, $n \in [N] \setminus \{0\}$, for the dual variables of the coupling constraints (18c), a standard iteration of an ADMM reads

$$\hat{z}_n^{r+1} := \arg\max_{\hat{z}_n} \left\{ L_\rho((\hat{z}_{\hat{n}})_{\hat{n}\in[N]}, (\eta_{\hat{n}}^r)_{\hat{n}\in[N]\setminus\{0\}}) \colon \hat{z}_n \in \hat{\Omega}_n,\ \hat{z}_{n'} = \hat{z}_{n'}^r,\ n' \neq n \right\}$$

for all $n \in [N]$ and

$$\eta_n^{r+1} := \eta_n^r + \rho \left((l_{n-1}^e)^{r+1} - (l_n^s)^{r+1} \right), \quad n \in [N] \setminus \{0\}.$$

Here,

$$L_\rho((\hat{z}_n)_{n\in[N]}, (\eta_n)_{n\in[N]\setminus\{0\}}) \\ := \sum_{n\in[N]} f_n(z_n) + \sum_{n\in[N]\setminus\{0\}} \eta_n^\top (l_{n-1}^e - l_n^s) - \frac{\rho}{2} \left\| (l_{n-1}^e - l_n^s)_{n\in[N]\setminus\{0\}} \right\|_2^2$$

is the augmented Lagrangian of Problem (18) and $\rho/2$ is the corresponding penalty parameter. The advantage of this setting is that the original large-scale problem can now be iteratively tackled by solving a series of smaller problems in every iteration. However, the first $N + 1$ primal problems need to be solved consecutively. For being able to solve them in a parallel and distributed way, we also provide a consensus version of the above setting in scaled form; see, e.g., Chapter 7 of Boyd et al. (2011). To this end, we introduce further copies of the variables in (16) via

$$l_{n-1,k} = l^s_{n,k}, \quad n \in [N] \setminus \{0\},\ k \in K, \tag{20a}$$
$$l_{n,k} = l^e_{n,k}, \quad n \in [N-1],\ k \in K, \tag{20b}$$

and for $k \in K$, let $\theta^s_{n,k}$, $n \in [N]\setminus\{0\}$, and $\theta^e_{n,k}$, $n \in [N-1]$, denote the corresponding dual variables of the newly introduced equality constraints (20). Thus, we can again rewrite Problem (18) in the consensus form

$$\begin{aligned}
\max_{\hat{z},l} \quad & \sum_{n\in[N]} f_n(z_n) \\
\text{s.t.} \quad & \hat{z}_n \in \hat{\Omega}_n, \quad n \in [N], \\
& l_{n-1,k} = l^s_{n,k}, \quad n \in [N] \setminus \{0\},\ k \in K, \\
& l_{n,k} = l^e_{n,k}, \quad n \in [N-1],\ k \in K,
\end{aligned}$$

where we abbreviate $\hat{z} = (\hat{z}_n)_{n\in[N]}$, $l = (l_n)_{n\in[N-1]}$, and $l_n = (l_{n,k})_{k\in K}$. In this consensus form of the problem, the variables $\hat{z}_n$, $i \in [N]$, are called local variables and the variables l are so-called global variables. We further denote the ρ-scaled dual variables of (20) by κ_n, i.e.,

$$\begin{aligned}
\kappa^s_{n,k} &:= \frac{1}{\rho}\theta^s_{n,k}, \quad n \in [N] \setminus \{0\},\ k \in K, \\
\kappa^e_{n,k} &:= \frac{1}{\rho}\theta^e_{n,k}, \quad n \in [N-1],\ k \in K,
\end{aligned}$$

and

$$\begin{aligned}
\kappa^s_n &:= \left(\kappa^s_{n,k}\right)_{k\in K}, \quad n \in [N] \setminus \{0\}, \\
\kappa^e_n &:= \left(\kappa^e_{n,k}\right)_{k\in K}, \quad n \in [N-1].
\end{aligned}$$

Finally, the scaled ADMM for the consensus problem in iteration r is given by the three primal and decoupled blocks

$$\begin{aligned}
\hat{z}^{r+1}_0 &= \arg\max_{\hat{z}_0} \left\{ f_0(z_0) - \frac{\rho}{2} \|l^e_0 - l^r_0 + \kappa^{e,r}_0\|^2_2 \right\}, \\
\hat{z}^{r+1}_N &= \arg\max_{\hat{z}_N} \left\{ f_N(z_N) - \frac{\rho}{2} \left\|l^s_N - l^r_{N-1} + \kappa^{s,r}_N\right\|^2_2 \right\},
\end{aligned}$$

and

$$\hat{z}^{r+1}_n = \arg\max_{\hat{z}_n} \left\{ f_n(z_n) - \frac{\rho}{2} \left\|l^s_n - l^r_{n-1} + \kappa^{s,r}_n\right\|^2_2 - \frac{\rho}{2} \|l^e_n - l^r_n + \kappa^{e,r}_n\|^2_2 \right\}$$

for $n \in [N-1] \setminus \{0\}$. Note that these problems can now be solved in parallel. Afterward, we update the consensus variables via

$$l^{r+1}_{n,k} = \frac{1}{2}\left(l^{s,r+1}_{n+1,k} + l^{e,r+1}_{n,k} + \kappa^{s,r}_{n+1,k} + \kappa^{e,r}_{n,k} \right), \quad n \in [N-1],\ k \in K,$$

and update the scaled dual variables by

$$\begin{aligned}
\kappa^{s,r+1}_{n,k} &= \kappa^{s,r}_{n,k} + l^{s,r+1}_{n,k} - l^{r+1}_{n-1,k}, && n \in [N] \setminus \{0\},\ k \in K, \\
\kappa^{e,r+1}_{n,k} &= \kappa^{e,r}_{n,k} + l^{e,r+1}_{n,k} - l^{r+1}_{n,k}, && n \in [N-1],\ k \in K.
\end{aligned}$$

For a more detailed discussion of ADMMs, we refer to Boyd et al. (2011) and the references therein.

Let us close this section with a brief discussion on why we choose an ADMM for the problem at hand. First, it is folklore knowledge that almost separable optimization problems can be tackled with ADMMs in practice very well; see Boyd et al. (2011) for a general survey and, e.g., Geißler et al. (2015, 2017, 2018) for

TABLE 10. Original instances from the literature with the respective number of consumers $|I|$, of producers $|J|$, and of scenarios $|T|$.

Network	$\|I\|$	$\|J\|$	$\|T\|$	Reference
Grimm-et-al-2016-3	1	2	4	Grimm, Martin, et al. (2016)
Chao-Peck-1998	3	3	4	Chao and Peck (1998)
Grimm-et-al-2016-6	3	3	52	Grimm, Martin, et al. (2016)
Haefner-2017	6	3	52	Häfner (2017)
Kleinert-Schmidt-2018-9	4	5	52	Kleinert and Schmidt (2019)
Kleinert-Schmidt-2018-12	6	6	52	Kleinert and Schmidt (2019)
DE-28	28	47	8760	Grimm, Kleinert, et al. (2019)

energy-specific applications where alternating methods have shown to be highly effective on models with quasi block-separable structure. In general, ADMMs are known to reach almost optimal points rather quickly but tend to require relatively many iterations to finally reach an optimal point of high precision. However, in the case that is discussed at the end of Section 3 in which the respective quadratic model possesses a unique solution it is known that one obtains local linear convergence if the solution satisfies strict complementarity; see Boley (2013). Moreover, linear convergence of ADMMs is shown in Giselsson and Boyd (2017).

5. Numerical Results

In this section, we present numerical results for the ADMM introduced in Section 4 applied to large-scale academic and real-world instances. We first give some details on the computational setup and our test instances. Then, we analyze different parallel setups and determine the best configuration of the algorithm applied to the considered electricity market instances. Finally, we compare the ADMM with the standard approach of solving the problem directly as a single quadratic problem (QP).

5.1. **Computational Setup and Test Instances.** We implemented the ADMM in Python 2.7.15. As mentioned in Section 4, the $N+1$ time blocks can be solved in parallel. The parallelization framework is realized with the Python package Dask.distributed in version 1.24.0; see Dask Development Team (2016). It allows for the distribution of code and data to several compute nodes as well as using several processes per compute node. We further used Gurobi 7.5.1, see Gurobi Optimization (2018), with its default settings and a thread limit of 4 for solving the QPs. All computations have been carried out on a compute cluster; see Regionales Rechenzentrum Erlangen (2016) for details on the installed hardware. If not specified otherwise, we used compute nodes with two Xeon 2660v2 "Ivy Bridge" chips (10 cores per chip and simultaneous multi-threading) running at 2.2 GHz with 25 MB shared cache per chip and 64 GB of RAM for the ADMM computations. Since the benchmark QP computations (i.e., standard Gurobi directly applied to the original problem formulation) are highly memory consuming for the larger instances of our test set, we solved them on compute nodes with two Intel Xeon E5-2680v4 "Broadwell" chips running at 2.4 GHz and 512 GB of RAM. Note that this is not an issue for ADMM since all time-dependent data is distributed across the compute nodes, i.e., every node only holds the data required for the ADMM blocks assigned to this node.

For our numerical tests we adopted instances from the literature. In Table 10 we give an overview of the original instances we used. We extended these instances by (i) adding storage devices with the respective data and by (ii) enlarging the scenario set via copying and randomly perturbing the original scenarios k times.

TABLE 11. Extended instances with number of storages $|K|$, scaling factor k, and number of scenarios $|T|$. All numbers for k and $|T|$ are given in thousands.

Instance	$\lvert K\rvert$	k	$\lvert T\rvert$
Grimm-et-al-2016-3	2	$\{13, 65, 130\}$	$\{52, 260, 520\}$
Chao-Peck-1998	2	$\{13, 65, 130\}$	$\{52, 260, 520\}$
Grimm-et-al-2016-6	2	$\{10, 15, 20\}$	$\{520, 780, 1040\}$
Haefner-2017	4	$\{4, 8, 12\}$	$\{208, 416, 624\}$
Kleinert-Schmidt-2018-9	4	$\{4, 8, 12\}$	$\{208, 416, 624\}$
Kleinert-Schmidt-2018-12	6	$\{1, 3, 9\}$	$\{52, 156, 468\}$
DE-28	16	$\{0.01, 0.02, 0.03\}$	$\{87.6, 175.2, 262.8\}$

TABLE 12. Intervals of capacities and maximum (dis-)charging rates of the introduced storages.

Instance	Capacity $\bar{\ell}$	Max. (dis-)charging rate $\bar{s}$
Grimm-et-al-2016-3	$[80.0, 120.0]$	$[12.0, 25.0]$
Chao-Peck-1998	$[100.0, 120.0]$	$[25.0, 50.0]$
Grimm-et-al-2016-6	$[100.0, 120.0]$	$[25.0, 50.0]$
Haefner-2017	$[70.0, 120.0]$	$[25.0, 50.0]$
Kleinert-Schmidt-2018-9	$[70.0, 120.0]$	$[25.0, 50.0]$
Kleinert-Schmidt-2018-12	$[70.0, 120.0]$	$[25.0, 50.0]$
DE-28	$[100.0]$	$[50.0]$

The resulting instances are displayed in Table 11. Details on the introduced storages can be found in Table 12. Note that we set the efficiency of all storages to $e = 1$. In total, our test set contains 21 instances (seven original instances with three different scenario scalings each). In the following, we refer to the instances by their original name extended with the scaling factor k. The instance with the largest scenario set is Grimm-et-al-2016-6-20000 with $|T| = 1\,040\,000$ scenarios. This corresponds to approximately 30 years in a quarter-hourly resolution; see, e.g., Babrowski et al. (2016) for problem definitions of comparable scale.

To get a better understanding of the instance sizes, we give some statistics of the sizes of the corresponding QPs. The smallest instance, Grimm-et-al-2016-3-13000, leads to a QP with 364 002 constraints, 624 000 variables, 1 196 000 constraint matrix nonzeros, and 52 000 quadratic objective terms. The largest instance, DE-28-30, leads to a QP with 12 877 216 constraints, 42 573 600 variables, 61 758 000 constraint matrix nonzeros, and 7 358 400 quadratic objective terms. The average (by means of the median) instance of our test set has 3 640 002 constraints, 10 400 000 variables, 13 520 000 constraint matrix nonzeros, and 1 560 000 quadratic objective terms. Note that, due to our model structure, every QP has exactly $|I||T|$ quadratic objective terms.

5.2. **Analysis of the Parallel ADMM.** As mentioned earlier, Dask.distributed allows to use several compute nodes and several workers (i.e., processes) per compute node. Thus, we tested our parallel setup with different combinations of compute nodes n and workers per node w. In particular, we tried any combination $(n, w) \in \{1, 2, 4, 8, 16, 32\} \times \{1, 2, 4\}$. This means, we decompose Problem (13) into $W = nw = N + 1$ many problems like in (15). Now, two questions arise:

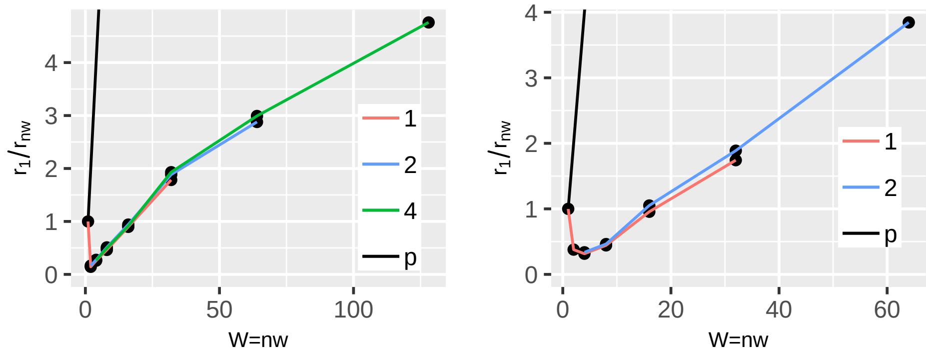

FIGURE 1. Speed-ups for the instances Grimm-et-al-2016-6-10000 (left) and DE-28-30 (right).

(i) What is the impact of the total number of workers W on running times, i.e., what is the speed-up?
(ii) Is there a general "best-practice" setting for the considered instances?

We first analyze speed-ups, i.e., ratios of the form r_1/r_{nw}. Here, r_1 denotes the running time of the ADMM benchmark setting with $n = w = 1$ and r_{nw} denotes the running time for n compute nodes with w workers per node.[3] A parallelization is considered perfect if $r_1/r_{nw} = nw$ holds. In practice, a perfect speed-up is typically not achievable due to, e.g., communication overhead, idle times, or additional algorithmic effort required for the parallelization of the method. We emphasize that we are not interested in implementation-specific performance (resp. overhead) but want to analyze the parallelization capabilities of the mathematical method applied to the considered models. Thus, in this section, all stated running times disregard CPU time to build the models and the overhead generated by Dask.distributed. To be more specific, we compute an idealized running time per iteration by adding up the maximum Gurobi solution time over all parallel blocks and the additional time consumed for central update steps; e.g., the dual update within every ADMM iteration.

Figure 1 shows the speed-ups of two representative instances, namely the mid-sized instance Grimm-et-al-2016-6-10000 (left) and the largest instance DE-28-30 (right). In both figures we plot the speed-up versus the total number of workers W for every considered w separately. Note that for DE-28-30 we only consider $w \in \{1, 2\}$ since we ran into memory issues for $w = 4$.[4] The plots also show the perfect speed-up indicated by a black line and labeled with p. The first observation is that the actual distribution of the total number of workers on different compute nodes has no real impact on the speed-up. This can be seen, e.g., in Figure 1 (left). For every $W \in \{4, 8, 16, 32\}$, the speed-up for all combinations of compute nodes n and workers per node w with $nw = W$ is almost the same. This was to be expected since we analyze idealized running times.

Both plots also reveal that the speed-up is almost linear in the total number of workers W. This indicates that the parallelization scales well, even for many

[3]Note that we carried out the computations for $n = w = 1$ on the same machines as the QP benchmarks in order to prevent memory issues.

[4]When increasing the number of workers per node the amount of time-dependent data per node is the same, since the ADMM block size is halved when the number of blocks is doubled. Thus, in theory, the memory overhead related to a finer decomposition should consist only of additional coupling variables. However, we observe unproportionally large memory overhead when switching from $w = 2$ to $w = 4$ caused by Dask.distributed, such that we exceeded the available memory of 64 GB.

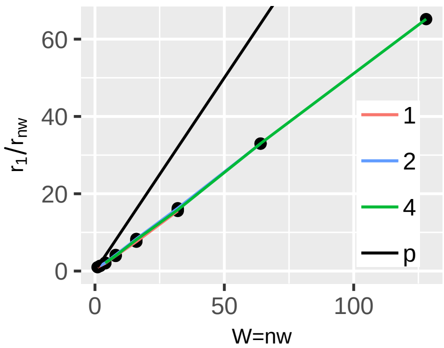

FIGURE 2. Speed-ups for the instance Haefner-9-8000.

decomposed blocks. A finer decomposition, i.e., smaller subproblems, is not overcompensated by additional algorithmic overhead like, e.g., slower convergence and thus more iterations.

However, the plots also show that the obtained speed-up still leaves significant space for further improvement. For example, for instance Grimm-et-al-2016-6-10000, a total number of workers $W = 64$ leads to a moderate speed-up of 3. The same pattern can be observed in Figure 1 (right).

Let us discuss possible reasons for these results. A first explanation are idle times. The ADMM, as it is implemented, solves all time-separated blocks in parallel and then applies a central update step. Thus, different block computation times would result in idle times of the algorithm. One possible remedy is an asynchronous implementation that performs an update step whenever one block is ready; see Eckstein (2017) and Eckstein et al. (2018) for more details. However, in our case, the computation times of all blocks are rather similar so that asynchronous methods do not seem to be a reasonable remedy.

The reported speed-ups may also result from slow convergence due to very tight optimality tolerances, inappropriate penalty parameters, or bad initial points. We discuss the impact of these factors in the following.

It is well known (Boyd et al. 2011) that ADMMs converge rather fast to approximate solutions of moderate quality whereas the convergence to high accuracy may be slow. In our computations, we terminate the ADMM if both the squared primal and squared dual residual are below 10^{-4}. We also considered to use relaxed tolerances. In the particular case of our previous example Grimm-et-al-2016-6-10000 using a tolerance of 10^{-2} yields a speed-up of roughly 4.5, which is an improvement of around 50 % compared to a tolerance of 10^{-4}. This behavior can be observed across all tested instances. However, in order to have a fair comparison with state-of-the-art techniques in Subsection 5.3, we stayed with a tolerance of 10^{-4}.

The convergence behavior is also strongly affected by the choice of the penalty parameter and its update strategy. For the instances at hand, we figured out reasonable penalty parameters via systematic preliminary testing. For some instances, we are able to find parameters that perform significantly better than the best parameters we determined for other instances. This is the case, e.g., for all realizations of the Haefner-9 instances. Figure 2 shows the speed-up for the specific instance Haefner-9-8000. One observes that a total number $W = 64$ of workers yields a speed-up of approximately 32, which is way better than the speed-up to workers ratio for the instance Grimm-et-al-2016-6-10000 discussed before ($\frac{32}{64}$ vs. $\frac{3}{64}$). Again, we observe linear scaling, i.e., for $W = 128$ the speed-up is approximately 65. These numbers indicate that a well-chosen penalty parameter is crucial for the efficiency

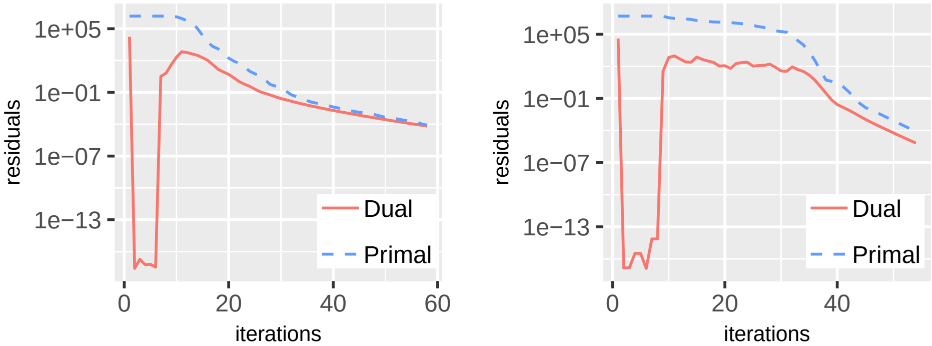

FIGURE 3. Squared primal and dual residuals for Grimm-et-al-2016-6-10000 (left) with $n = 32$, $w = 4$ and DE-28-30 (right) with $n = 32$, $w = 4$.

of the ADMM. We thus tested an adaptive penalty parameter using the update rule proposed in Boyd et al. (2011). However, our implementation did not benefit from this adaptive penalty strategy in general and we therefore stayed with fixed penalty parameters as described above.

Finally, convergence may also benefit from good initial points. To discuss this in more detail, we briefly analyze the course of the squared primal and dual residuals over the iterations of the ADMM. We do this again for the two representative instances Grimm-et-al-2016-6-10000 and DE-28-30 with $n = 32$ as well as $w = 4$ and $w = 2$, respectively. In Figure 3 (left), we see that the ADMM needs slightly less than 60 iterations to converge. However, up to iteration 10, the squared primal residual stays almost constant. The squared dual residual drops below 10^{-16} and then increases again up to its initial level. After iteration 10, one can observe a convergence of both squared residuals. Thus, the entire performance of the algorithm could be greatly improved by a good initial point that avoids this behavior during the early iterations. The situation is even more drastic in Figure 3 (right). Again, in the first 10 iterations, the ADMM does not make any progress w.r.t. the squared primal residual. After that, for another 20 iterations, the squared primal and dual residuals make only little progress and then, approximately after iteration 30, linear convergence takes place. For this behavior, both a tailored penalty parameter update rule and improved initial values might be a remedy. Both algorithmic developments are out of scope of this paper and thus postponed for future work; see also Section 6.

5.3. **Comparison of the Parallel ADMM and Direct QP Solution.** We now finally come to the most important numerical study—namely the comparison of the parallel ADMM and the standard approach of directly solving the original QP (13) using Gurobi. Note that in this section we do not consider idealized running times anymore but simply compare total times including communication overhead, idle times, etc. caused by Dask.distributed. We first analyze total running times that include the required time to build the models. All running times can be found in Table 13 in Appendix A. Figure 4 shows performance profiles according to Dolan and Moré (2002). In particular, for every instance $i \in I$ we compute ratios $r_{i,s} = t_{i,s} / \min\{t_{i,s} \colon s \in S\}$, where $S := \{\text{ADMM, QP}\}$ is the set of the two tested approaches ADMM and QP and $t_{i,s}$ the respective running time. Then, Figure 4 shows the percentage of instances for which the performance ratio $r_{i,s}$ of approach s is within a factor $\tau \geq 1$ of the best possible ratio. The left profile compares the best ADMM result across all (n, w) combinations with the QP. The parallel ADMM clearly outperforms the direct solution of the QP and is the faster

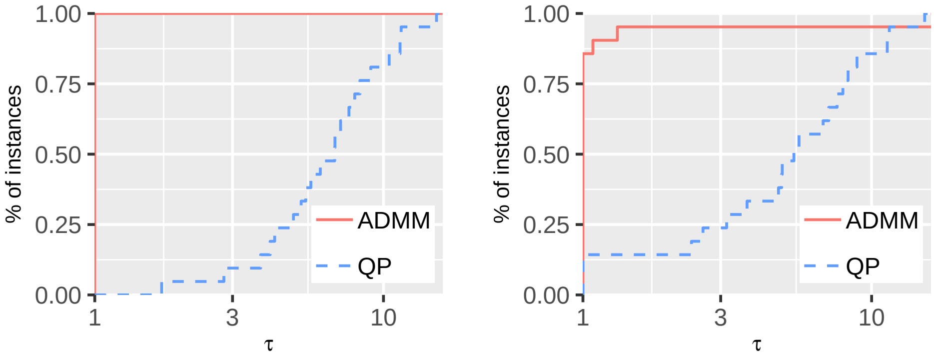

FIGURE 4. Performance profiles for the best ADMM results (left) and ADMM with $(n, w) = (32, 4)$ (right), both for total running times (including building the model).

method for every instance in our test set. This is in particular true for the larger instances of the test set. In fact, the ADMM is capable of solving the largest instance of the test set, namely DE-28-30 with more than 42 million variables, in less than 13 minutes—compared to almost 1 hour for the QP approach. Given the case that enough memory is available, both approaches are fully reliable and solve all tested instances. The discussion about the linear scaling of the ADMM in the previous section indicates that it is beneficial to use as many workers as possible. Thus, Figure 4 (right) benchmarks the results of the ADMM parameterized with $n = 32$ and $n = 4$, i.e., our largest available setting. It can be seen that the parallel ADMM does not solve all instances using this setting since it fails to solve the largest instance DE-28-30 due to the mentioned memory overhead generated by the used parallelization framework. For almost every other instance, the ADMM is again the faster approach, which supports the suggestion to use as many workers as possible. In fact, only for the smallest instances in the test set, this maximum worker setting harms the solution process (see Table 13) since the parallelization overhead dominates the plain solution time for these "small" instances.

We observed that a significant amount of computation time of the benchmark approach is related to the model construction, whereas the ADMM strongly benefits from building smaller model blocks in parallel. Although building the model belongs to the entire solution process, we additionally compare actual solution times without model building times in Figure 5. All running times (without model building times) are presented in Table 14 in Appendix A. In the left figure we compare again the best combination of nodes and workers per node of each instance with the benchmark. It turns out that the parallel ADMM still clearly dominates the direct solution of the QPs. The same holds for the ADMM parameterization $n = 32$ and $w = 4$ (right figure), albeit not as distinct as for the best parameterizations.

To sum up, it is obvious that the parallel ADMM clearly outperforms the standard approach of a direct solution of the respective QP.

6. Conclusion

In this paper we studied an electricity market equilibrium model including storage systems that couple all trading periods over time. This makes both the theoretical analysis as well as the computational treatment much harder compared to the standard setting without storage operators. For this extended setting, our results are the following: We

(i) develop an inter-temporal market equilibrium model,

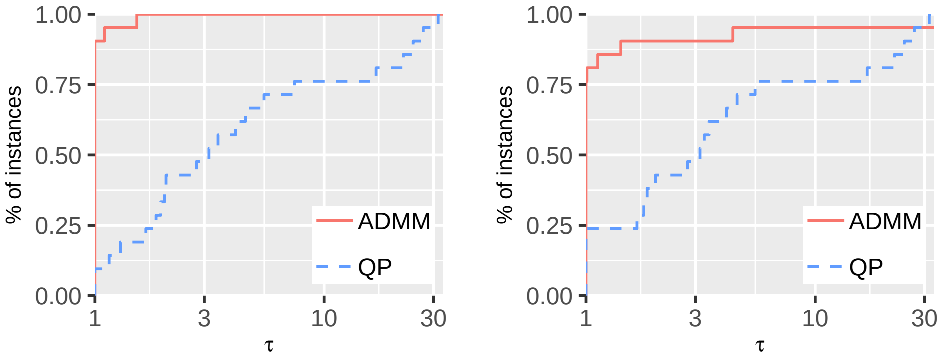

FIGURE 5. Performance profiles for the best ADMM results (left) and ADMM with $(n, w) = (32, 4)$ (right), both for running times after model construction.

(ii) demonstrate the economic efficiency of its equilibria by proving its equivalence to a suitably chosen welfare maximization problem,
(iii) derive sufficient conditions for the uniqueness of certain parts of the entire solution and provide illustrative examples for that, in general, uniqueness is not obtained; and, finally,
(iv) present a parallel ADMM that allows for solving huge-scale instances.

Besides these contributions, many questions remain open and many algorithmic improvements can be studied in future research.

Let us first comment on the modeling and the theoretical results. First of all, also sufficient conditions for the nonexistence of uniqueness would be desirable. Second and most prominently, the current legislature in many countries leaves open how market integration of storages should be realized. In our opinion, one of the main questions is who owns and controls the storages if they are embedded in today's markets. Here, we studied the setting in which storage operators optimize their actions independently of, e.g., producers, consumers, or the transmission system operator. Other ownership settings might lead to different theoretical results and might also pave the way for different algorithmic techniques. Going further, one of our next research routes is the consideration of strategic interaction between producers and storage operators; e.g., in the setup of classical Nash–Cournot models. Again, both theoretical and algorithmic insights need to be reconsidered. Furthermore, market integration of storages is of special importance w.r.t. the analysis of resulting long-run investment incentives, which we completely abstracted from in this paper. In this context, many other practical questions arise that need to be answered: (i) How do large-scale storage systems influence market prices? (ii) How do these systems influence the operations of existing power plants and the investment in conventional and renewable capacities? (iii) How can storage operators be incentivized to operate in a useful way w.r.t. the underlying power grid? The latter question also relates to the issue of locational placement of large-scale storage systems, which we did not consider due to the absence of a transport network infrastructure in our model.

Last but not least, the algorithmic approach developed in this contribution is able to solve huge-scale instances—but also leaves enough space for further contributions. At this point, let us only sketch a few. First, the convergence behavior analyzed in Section 5.2 strongly suggests to elaborate two things in more detail: (i) Can we propose penalty parameter updates and/or (ii) problem-specific initial value heuristics that allow to significantly reduce the overall required ADMM iterations?

Finally, a well-known characteristic of ADMMs also showed up in our numerical results: Fast decent to moderate approximate solutions but slow convergence to solutions of high precision. An algorithmic development, either for the specific instances under consideration or for general convex problems, would be great.

Acknowledgments

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We further thank the RRZE for their support and patience in setting up the parallelization framework on their machines.

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Appendix A. Detailed Result Tables

TABLE 13. Comparison of absolute running times (including model building times; in s). Direct QP solution and ADMM on different numbers of compute nodes and processes $(1, 2, 4)$ per node. The shortest running time per row is printed bold.

Instance	QP	ADMM																	
		1 node			2 nodes			4 nodes			8 nodes			16 nodes			32 nodes		
		1	2	4	1	2	4	1	2	4	1	2	4	1	2	4	1	2	4
Chao-Peck-1998-13000	68.31	89.17	247.00	136.58	288.36	132.80	74.21	139.99	75.05	51.00	88.13	53.94	**40.06**	56.92	40.81	40.45	60.52	66.05	73.96
Chao-Peck-1998-65000	342.47	447.15	1719.11	939.44	1714.37	900.09	484.10	991.75	481.55	254.54	511.82	250.15	143.95	268.92	151.75	**91.24**	181.13	96.84	131.46
Chao-Peck-1998-130000	715.14	924.16	2691.79	1800.27	2610.16	1737.39	968.11	1834.93	935.53	548.38	955.28	536.60	288.35	627.03	283.93	169.45	290.61	195.86	**146.61**
DE-28-10	1080.84	1483.11	1795.47	2606.20	1782.82	2549.15	1295.94	2701.50	1239.47	664.64	1327.87	676.76	403.68	673.46	375.32	**159.51**	397.68	162.54	227.23
DE-28-20	2216.73	2945.56	–	–	3938.01	1949.44	2754.17	1985.58	2720.26	1534.80	2836.87	1486.79	808.50	1502.96	760.72	536.01	863.92	554.39	**278.78**
DE-28-30	3260.55	4427.43	–	–	6146.25	6551.18	–	6841.33	4665.77	–	4818.52	2272.48	–	2342.75	1348.82	–	1375.43	**776.59**	–
Grimm-et-al-2016-3-13000	50.40	66.96	33.70	31.80	34.96	23.04	14.76	30.38	15.14	10.89	15.57	11.96	**8.34**	12.34	11.36	9.97	62.37	58.96	66.33
Grimm-et-al-2016-3-65000	253.68	309.74	182.83	96.53	169.47	97.97	56.55	98.90	58.73	37.50	58.43	36.94	26.93	38.53	27.17	**22.23**	33.37	74.82	22.40
Grimm-et-al-2016-3-130000	507.44	626.46	375.05	195.55	346.17	199.21	111.70	179.69	108.88	67.91	106.44	68.04	55.16	70.16	50.59	37.69	52.20	85.45	**33.24**
Grimm-et-al-2016-6-10000	723.49	906.79	1758.08	1083.50	1948.99	1049.92	605.41	1064.21	592.38	350.74	634.54	340.02	194.63	351.85	199.90	149.51	211.93	215.60	**134.49**
Grimm-et-al-2016-6-15000	1091.94	1356.38	3010.00	–	3065.23	1489.78	898.91	1493.57	856.23	490.92	867.62	479.07	427.43	492.58	380.03	198.96	443.40	202.54	**153.74**
Grimm-et-al-2016-6-20000	1446.09	1842.29	3371.81	2101.72	3847.53	2121.89	1463.71	2123.49	1439.84	786.93	1474.35	776.56	450.85	862.33	436.46	280.93	516.55	278.39	**212.96**
Haefner-9-node-4000	474.18	607.48	314.31	209.15	302.93	193.88	115.02	199.77	117.15	80.96	114.72	75.94	56.11	73.62	58.88	53.73	62.01	**52.48**	96.73
Haefner-9-node-8000	963.99	1222.50	669.27	382.64	625.25	387.77	237.49	375.30	228.97	146.12	212.35	146.84	110.38	146.91	111.50	91.69	106.34	129.70	**83.73**
Haefner-9-node-12000	1394.72	1817.06	1002.58	597.42	994.39	590.77	335.57	555.56	339.27	229.68	314.28	226.45	211.87	211.00	153.18	133.29	206.18	**133.26**	156.74
Kleinert-Schmidt-2018-9-node-4000	461.45	557.69	997.29	564.52	989.21	555.26	357.11	586.18	358.01	192.18	367.15	190.58	130.20	242.73	125.31	**88.98**	131.53	91.26	124.65
Kleinert-Schmidt-2018-9-node-8000	915.86	1152.41	2158.89	1157.33	2039.18	1146.15	724.55	1213.17	709.95	440.97	740.88	409.83	240.51	489.33	238.55	149.87	307.41	159.49	**120.66**
Kleinert-Schmidt-2018-9-node-12000	1399.51	1730.29	3803.81	–	4061.72	2092.56	1101.47	2226.30	1080.38	699.41	1072.41	672.41	393.72	693.90	381.68	237.77	452.26	240.31	**168.85**
Kleinert-Schmidt-2018-12-node-1000	132.74	166.40	434.70	236.59	487.14	228.15	126.76	247.26	124.40	78.76	147.54	76.08	80.42	88.46	119.34	**47.40**	141.77	48.14	55.86
Kleinert-Schmidt-2018-12-node-3000	400.43	512.08	1127.60	631.91	1187.90	624.82	355.04	630.30	349.24	228.29	364.25	226.81	146.53	242.16	142.43	**98.86**	200.16	101.69	127.28
Kleinert-Schmidt-2018-12-node-9000	1225.79	1555.80	3585.02	–	3646.23	2228.11	1167.64	2248.93	1166.88	632.20	1134.22	616.41	464.95	634.66	500.26	248.23	513.21	252.46	**218.74**

TABLE 14. Comparison of absolute running times (without model building times in s). Direct QP solution and ADMM on different numbers of compute nodes and processes $(1, 2, 4)$ per node. The shortest running time per row is printed bold.

Instance	QP	ADMM																	
		1 node			2 nodes			4 nodes			8 nodes			16 nodes			32 nodes		
		1	2	4	1	2	4	1	2	4	1	2	4	1	2	4	1	2	4
Chao-Peck-1998-13000	**13.86**	21.12	215.26	117.18	237.91	112.40	59.25	117.06	59.13	38.35	61.44	41.28	28.90	41.90	28.66	28.98	41.15	44.13	60.61
Chao-Peck-1998-65000	69.06	111.13	1559.42	850.48	1586.20	810.64	427.38	900.71	422.48	213.70	453.17	203.58	108.47	220.23	115.91	**59.88**	123.39	60.42	69.71
Chao-Peck-1998-130000	146.19	222.88	2382.02	1622.22	2297.23	1558.18	856.90	1650.36	825.13	471.90	858.88	458.22	220.87	499.09	218.72	115.23	226.63	132.06	**87.60**
DE-28-10	398.13	607.86	1414.40	2390.31	1468.74	2334.79	1155.17	2517.03	1099.53	566.30	1194.90	533.00	319.17	583.59	295.79	**87.69**	320.06	88.29	121.38
DE-28-20	808.53	1188.82	–	–	3195.81	1518.49	2491.14	1602.73	2438.66	1339.23	2581.38	1247.65	652.66	1320.78	605.81	399.10	700.95	377.18	**147.91**
DE-28-30	1121.62	1830.21	–	–	5143.16	5900.06	–	6249.60	4269.08	–	4414.91	1921.52	–	2085.65	1082.60	–	1156.62	**549.06**	–
Grimm-et-al-2016-3-13000	10.63	13.83	10.47	7.69	11.67	6.84	3.65	7.44	3.85	2.18	3.65	2.28	1.60	2.28	1.56	**1.43**	1.80	1.62	2.33
Grimm-et-al-2016-3-65000	52.08	76.49	71.33	36.30	77.09	34.90	18.58	39.72	18.48	9.59	19.68	9.23	4.99	9.85	5.19	3.37	5.32	3.40	**3.09**
Grimm-et-al-2016-3-130000	105.64	142.88	154.81	77.52	160.72	73.14	37.62	77.50	34.57	19.06	41.06	18.76	9.64	20.08	9.56	5.59	10.16	5.63	**4.32**
Grimm-et-al-2016-6-10000	144.59	218.55	1451.38	904.74	1635.23	874.92	491.16	911.06	476.68	267.96	523.09	259.74	131.01	275.25	136.11	92.10	142.41	98.49	**78.23**
Grimm-et-al-2016-6-15000	231.00	339.45	2532.59	–	2602.97	1219.82	730.29	1266.34	692.00	374.14	713.26	361.74	285.53	380.26	283.76	115.39	299.93	117.74	**73.48**
Grimm-et-al-2016-6-20000	301.16	472.63	2745.88	1747.93	3201.98	1762.52	1249.92	1787.62	1221.48	632.30	1261.58	617.11	320.74	656.56	310.80	170.13	350.60	166.81	**108.72**
Haefner-9-node-4000	96.15	146.69	103.78	75.00	116.40	71.70	36.77	80.13	35.61	18.52	39.10	18.39	9.38	20.65	9.78	5.43	11.57	5.75	**4.34**
Haefner-9-node-8000	181.07	284.44	228.40	147.27	233.93	142.93	79.35	153.67	73.82	38.49	79.51	37.54	20.20	41.28	19.52	10.17	21.59	10.29	**6.69**
Haefner-9-node-12000	294.89	449.10	344.03	236.13	351.44	226.21	112.30	228.88	114.17	59.83	111.86	55.07	29.17	58.44	28.96	15.35	32.04	15.36	**9.38**
Kleinert-Schmidt-2018-9-node-4000	109.45	148.74	802.46	457.87	825.76	444.83	289.60	487.99	279.83	144.39	300.22	141.60	90.70	154.67	89.97	**56.48**	95.14	58.89	61.23
Kleinert-Schmidt-2018-9-node-8000	222.56	321.23	1767.88	941.68	1732.83	923.95	596.48	986.08	579.22	335.08	622.38	319.63	171.77	340.32	165.89	91.17	184.76	92.19	**64.62**
Kleinert-Schmidt-2018-9-node-12000	369.87	431.13	3213.97	–	3524.10	1778.16	911.10	1920.30	883.82	564.35	905.99	542.78	290.57	568.85	277.47	151.17	306.06	151.21	**90.04**
Kleinert-Schmidt-2018-12-node-1000	**28.61**	41.93	376.99	203.10	426.14	191.44	102.36	212.46	99.08	60.69	108.23	57.16	64.39	63.12	62.11	31.74	63.27	31.52	40.60
Kleinert-Schmidt-2018-12-node-3000	84.41	135.14	956.16	535.96	1015.41	522.43	291.07	539.10	285.85	183.63	302.60	179.06	106.32	191.26	103.02	**65.38**	107.34	65.85	94.99
Kleinert-Schmidt-2018-12-node-9000	270.76	388.10	3062.85	–	3223.89	1939.36	981.34	2007.21	980.35	504.50	971.73	484.30	359.57	516.04	342.13	157.99	365.60	162.78	**134.62**

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