

# Doubly nonnegative relaxations are equivalent to completely positive reformulations of quadratic optimization problems with block-clique graph structures

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## Abstract

We study the equivalence among a nonconvex QOP, its CPP and DNN relaxations under the assumption that the aggregated and correlative sparsity of the data matrices of the CPP relaxation is represented by a block-clique graph  $G$ . By exploiting the correlative sparsity, we decompose the CPP relaxation problem into a clique-tree structured family of smaller subproblems. Each subproblem is associated with a node of a clique tree of  $G$ . The optimal value can be obtained by applying an algorithm that we propose for solving the subproblems recursively from leaf nodes to the root node of the clique-tree. We establish the equivalence between the QOP and its DNN relaxation from the equivalence between the reduced family of subproblems and their DNN relaxations by applying the known results on: (i) CPP and DNN reformulation of a class of QOPs with linear equality, complementarity and binary constraints in 4 nonnegative variables. (ii) DNN reformulation of a class of quadratically constrained convex QOPs with any size. (iii) DNN reformulation of LPs with any size. As a result, we show that a QOP whose subproblems are the QOPs mentioned in (i), (ii) and (iii) is equivalent to its DNN relaxation, if the subproblems form a clique-tree structured family induced from a block-clique graph.

**Key words.** Equivalence of doubly nonnegative relaxations and completely positive programs, sparsity of completely positive reformulations, aggregated and correlative sparsity, block clique graphs, completely positive and doubly nonnegative matrix completion, exact optimal values of nonconvex QOPs.

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# 1 Introduction

Completely positive programming (CPP) relaxations of a class of quadratic optimization problems (QOPs) have received a great deal of attention as they can provide the optimal value of the original QOP [2, 7, 8, 17, etc.] even with their computational intractability. They are also referred as CPP reformulations of QOPs. As for computationally tractable alternatives, further relaxations of the CPP reformulations to doubly nonnegative programming (DNN) relaxations have been studied in [4, 16], and their effectiveness in obtaining good lower bounds for the optimal value of the QOP have been demonstrated in [14, 16]. Since the equivalence among QOPs, their CPP and DNN relaxations cannot be obtained in general, in this paper, we explore structured QOPs for which the equivalence to their DNN reformulations can be established. In particular, we focus on some structured sparsity characterized by block-clique graphs [9] and partial convexity to establish their equivalence for a class of QOPs. As far as we are aware of, this is the first time that the equivalence of nonconvex QOPs and their DNN relaxations are studied for this class of structured QOPs.

We start by introducing a conic optimization problem (COP) model to simultaneously represent QOPs, CPP problems and DNN problems. Let  $\mathbb{R}^n$  denote the  $n$ -dimensional Euclidean space and  $\mathbb{R}_+^n$  the corresponding nonnegative orthant. We assume that each  $\mathbf{x} \in \mathbb{R}^n$  is a column vector, and  $\mathbf{x}^T$  denotes its transpose. Let  $\mathbb{S}^n$  denote the space of  $n \times n$  symmetric matrices. For every pair of  $\mathbf{A} \in \mathbb{S}^n$  and  $\mathbf{X} \in \mathbb{S}^n$ ,  $\langle \mathbf{A}, \mathbf{X} \rangle$  stands for their inner product defined as the trace of  $\mathbf{A}\mathbf{X}$ . Let  $I_{\text{eq}}$  and  $I_{\text{ineq}}$  be disjoint finite subsets of positive integers and  $\mathbf{Q}^p \in \mathbb{S}^n$  ( $p \in \{0\} \cup I_{\text{eq}} \cup I_{\text{ineq}}$ ). Given a closed (possibly nonconvex) cone  $\mathbb{K} \subset \mathbb{S}^n$ , we consider the following general COP:

$$\text{COP}(\mathbb{K}): \zeta(\mathbb{K}) = \inf \left\{ \langle \mathbf{Q}^0, \mathbf{X} \rangle : \begin{array}{l} \mathbf{X} \in \mathbb{K}, X_{11} = 1, \\ \langle \mathbf{Q}^p, \mathbf{X} \rangle = 0 \ (p \in I_{\text{eq}}), \\ \langle \mathbf{Q}^p, \mathbf{X} \rangle \leq 0 \ (p \in I_{\text{ineq}}) \end{array} \right\}$$

The distinctive feature of  $\text{COP}(\mathbb{K})$  is that it only involves *homogeneous* equality and inequality constraints except  $X_{11} = 1$ . A general equality standard form COP can be transformed to  $\text{COP}(\mathbb{K})$  with  $I_{\text{ineq}} = \emptyset$  in a straightforward manner (see Section 2.2). If  $\mathbf{\Gamma}^n = \{\mathbf{x}\mathbf{x}^T \in \mathbb{S}^n : \mathbf{x} \in \mathbb{R}_+^n\}$  is chosen as the closed cone  $\mathbb{K} \subset \mathbb{S}^n$ , then  $\text{COP}(\mathbf{\Gamma}^n)$  represents a QOP with quadratic equality and inequality constraints in  $\mathbf{x} \in \mathbb{R}_+^n$  (see Section 2.5). Notice that the inequality constraints  $\langle \mathbf{Q}^p, \mathbf{X} \rangle \leq 0$  ( $p \in I_{\text{ineq}}$ ) are dealt with separately from the equality constraints  $\langle \mathbf{Q}^p, \mathbf{X} \rangle = 0$  ( $p \in I_{\text{eq}}$ ) without introducing slack variables. In particular, the inequality constraints  $\langle \mathbf{Q}^p, \mathbf{X} \rangle \leq 0$  ( $p \in I_{\text{ineq}}$ ) with  $\mathbf{X} \in \mathbf{\Gamma}^n$  correspond to convex quadratic inequality constraints in Section 2.5. If  $\mathbb{K}$  is the DNN cone of size  $n$  (denoted as  $\mathbb{DNN}^n$ ) or the CPP cone of size  $n$  (denoted as  $\mathbb{CPP}^n$ ), then  $\text{COP}(\mathbb{K})$  represents a general DNN or CPP problem, respectively (see Section 2.2). They are known to serve as convex relaxations of the nonconvex QOP,  $\text{COP}(\mathbf{\Gamma}^n)$ . In general,  $\zeta(\mathbb{DNN}^n) \leq \zeta(\mathbb{CPP}^n) \leq \zeta(\mathbf{\Gamma}^n)$  holds since  $\mathbf{\Gamma}^n \subset \mathbb{CPP}^n$  (= the convex hull of  $\mathbf{\Gamma}^n$ )  $\subset \mathbb{DNN}^n$ .

The main purpose of this paper is to investigate structured sparsity, in particular, the aggregated sparsity [12] and the correlative sparsity [18] characterized by block-clique graphs [9], and partial convexity of the data matrices of  $\text{COP}(\mathbb{K})$  to establish the equivalence among the three types of problems,  $\text{COP}(\mathbf{\Gamma}^n)$ ,  $\text{COP}(\mathbb{CPP}^n)$  and  $\text{COP}(\mathbb{DNN}^n)$ .

Sparsity, especially the chordal graph sparsity, has been heavily used to improve the computational efficiency of solving semidefinite programming (SDP) problems. The sparsity exploitation technique [12, 18, 20, 21, etc.] for SDP problems was based on the semidefinite matrix completion in order to reduce the size of the positive semidefinite variable matrix. More precisely, it replaces a large but sparse variable matrix of a given SDP problem with smaller positive semidefinite variable matrices whose sizes are determined by the maximal cliques of the extended chordal graph characterizing the aggregated sparsity of the data matrices of the SDP problem. On the other hand, exploiting sparsity in CPP problems has not been studied in the literature, to the best of our knowledge, as the studies on CPP problems have been mainly for theoretical interests.

We discuss the equivalence of  $\text{COP}(\mathbf{\Gamma}^n)$ ,  $\text{COP}(\text{CPP}^n)$  and  $\text{COP}(\text{DNN}^n)$  based on the following techniques and/or facts.

- (a) The CPP and DNN matrix completion in [10].
- (b) Exploiting (aggregated and correlative) sparsity in chordal graph structured SDPs [12, 18].
- (c)  $\text{CPP}^n = \text{DNN}^n$  if  $n \leq 4$ .
- (d) CPP reformulation of a class of QOPs with linear equality, binary and complementarity constraints [2, 7, 8, 17, etc.].
- (e) DNN reformulation of quadratically constrained convex QOPs in nonnegative variables. See Lemma 2.5 in Section 2.6.

We note that (c) is well-known. A block-clique graph  $G$  is a chordal graph in which any two maximal cliques intersect in at most one vertex [9]. It was shown in [10] that every partial CPP (or DNN) matrix whose specified entries are determined by an undirected graph  $G$  has a CPP (or DNN) completion if and only if  $G$  is a block-clique graph.

Let  $\mathbb{K} \in \{\mathbf{\Gamma}^n, \text{CPP}^n, \text{DNN}^n\}$ . In our method, the basic idea developed in (b) combined with (a), instead of positive semidefinite matrix completion [13], is applied to  $\text{COP}(\mathbb{K})$  with  $\mathbf{Q}^p \in \mathbb{S}^n$  ( $p \in \{0\} \cup I_{\text{eq}} \cup I_{\text{ineq}}$ ). The data structure of  $\text{COP}(\mathbb{K})$  is characterized by a block-clique graph  $G$  with the maximal cliques  $C_r$  ( $r = 1, \dots, \ell$ ).  $\text{COP}(\mathbb{K})$  is then decomposed into a family of  $\ell$  smaller size subproblems according to a clique tree structure induced from  $G$ . The family of  $\ell$  subproblems inherit the clique tree structure of the block-clique graph  $G$ . More precisely, they are associated with the  $\ell$  nodes of the clique tree. Two distinct subproblems are almost independent but weakly connected in the sense that they share one scalar variable if they are adjacent in the clique tree and no common variable otherwise. In addition, each problem associated with a clique  $C_r$  in the family is of the same form as  $\text{COP}(\mathbb{K})$ , but the size of its matrix variable is decreased to the size of  $C_r$ . It is important to note that the decomposition is independent of the choice of  $\mathbb{K} \in \{\mathbf{\Gamma}^n, \text{CPP}^n, \text{DNN}^n\}$ .

We utilize the aforementioned decomposition of  $\text{COP}(\mathbb{K})$  for two purposes: to efficiently solve large scale COPs and to show the equivalence among  $\text{COP}(\mathbf{\Gamma}^n)$ ,  $\text{COP}(\text{CPP}^n)$  and  $\text{COP}(\text{DNN}^n)$ . For the first purpose of efficiently solving large scale COPs, the optimal value  $\zeta(\mathbb{K})$  of  $\text{COP}(\mathbb{K})$  is computed by solving the  $\ell$  small decomposed subproblems in the family. We propose an algorithm for sequentially solving the decomposed subproblems. As a result, the proposed algorithm is more efficient than directly solving the

original  $\text{COP}(\mathbb{K})$  when its size becomes increasingly large. Here we implicitly assume that  $\mathbb{K} = \text{DNN}^n$ , although all the results would remain valid even when the decomposed subproblems are not numerically tractable. We should emphasize that the decomposition into smaller subproblems is certainly beneficial computationally. For example, the decomposition of a DNN problem of size 1000 into 200 DNN subproblems of size at most 10 is certainly much more numerically tractable than the original DNN problem.

For the second purpose of showing the equivalence among  $\text{COP}(\mathbf{\Gamma}^n)$ ,  $\text{COP}(\text{CPP}^n)$  and  $\text{COP}(\text{DNN}^n)$ , the decomposition is applied to a pair of  $\text{COP}(\mathbb{K}_1)$  and  $\text{COP}(\mathbb{K}_2)$  with two distinct  $\mathbb{K}_1, \mathbb{K}_2 \in \{\mathbf{\Gamma}^n, \text{CPP}^n, \text{DNN}^n\}$ . Then, two families of subproblems, say family 1 from  $\text{COP}(\mathbb{K}_1)$  and family 2 from  $\text{COP}(\mathbb{K}_2)$ , are obtained. The equivalence between  $\text{COP}(\mathbb{K}_1)$  and  $\text{COP}(\mathbb{K}_2)$  is reduced to the equivalence of family 1 and family 2. We note that a pair of decomposed subproblems, one from family 1 and the other from family 2, associated with a common clique  $C_r$  share the common objective function and constraints except for their cone constraints. Thus, (c), (d) and/or (e) can be applied for the equivalence of each pair. If all pairs of subproblems from families 1 and 2 are equivalent, then  $\text{COP}(\mathbb{K}_1)$  and  $\text{COP}(\mathbb{K}_2)$  are equivalent. In particular, all of nonconvex QOPs with linear and complementarity constraints in 4 variables, quadratically constrained convex QOPs with any size and LPs with any size can be included as subproblems in a single QOP formulated as  $\text{COP}(\mathbf{\Gamma})$ , which can then be equivalently reformulated as its DNN relaxation. We should mention that a structured sparsity, *i.e.*, the aggregate and correlative sparsity characterized by a block-clique graph, needs to be imposed on the QOP. Such a QOP may not appear frequently in practice, but our study here has theoretical importance as nonconvex QOPs are NP hard to solve in general.

This paper is organized as follows: In Section 2, some basics on block-clique graphs, (a), (c), (d) and (e) are described. We also define the aggregate and correlative sparsity which are represented by an undirected graph. Sections 3 and 4 include the main results of this paper. In Section 3, the equivalence among  $\text{COP}(\mathbf{\Gamma}^n)$ ,  $\text{COP}(\text{CPP}^n)$  and  $\text{COP}(\text{DNN}^n)$  based on (c) and (d) is established by exploiting the aggregated sparsity characterized by block-clique graphs. In Section 4, we show how  $\text{COP}(\mathbb{K})$  with  $\mathbb{K} \in \{\mathbf{\Gamma}^n, \text{CPP}^n, \text{DNN}^n\}$  can be decomposed into smaller subproblems by exploiting the correlative sparsity. Then the equivalence results given in Section 3 as well as (e) are applied to a pair of subproblems induced from distinct  $\text{COP}(\mathbb{K}_1)$  and  $\text{COP}(\mathbb{K}_2)$  to verify their equivalence as mentioned above. We also present an algorithm to sequentially solve smaller decomposed subproblems. In Section 5, we illustrate two types of QOPs, block-clique graph structured QOPs with linear equality and complementarity constraints and partially convex QOPs, which can be formulated as the equivalent DNN problems. We conclude the paper in Section 6.

## 2 Preliminaries

### 2.1 Notation and symbols

We will consider the following cones in  $\mathbb{S}^n$ .

$$\begin{aligned}
\mathbb{S}_+^n &= \text{the cone of } n \times n \text{ symmetric positive semidefinite matrices,} \\
\mathbb{N}^n &= \text{the cone of } n \times n \text{ symmetric nonnegative matrices} \\
&= \{\mathbf{X} \in \mathbb{S}^n : X_{ij} \geq 0 \ (i = 1, \dots, n, \ j = 1, \dots, n)\}, \\
\mathbb{DNN}^n &= \text{DNN cone} = \mathbb{S}_+^n \cap \mathbb{N}^n, \\
\mathbf{\Gamma}^n &= \{\mathbf{x}\mathbf{x}^T \in \mathbb{S}^n : \mathbf{x} \in \mathbb{R}_+^n\}, \\
\mathbb{CPP}^n &= \text{CPP cone} = \text{the convex hull of } \mathbf{\Gamma}^n.
\end{aligned}$$

The following relation is well-known [5]:

$$\begin{aligned}
\mathbb{CPP}^n &= \mathbb{DNN}^n && \text{if } n \leq 4, \\
\mathbb{CPP}^n &\text{ is a proper subset of } \mathbb{DNN}^n && \text{otherwise.}
\end{aligned} \tag{1}$$

Let  $N = \{1, \dots, n\}$ . For each nonempty subset  $C$  of  $N$ ,  $\mathbb{R}^C$  denotes the  $|C|$ -dimensional Euclidean space of column vectors of  $x_i$  ( $i \in C$ ), and  $\mathbb{S}^C$  the linear space of  $|C| \times |C|$  symmetric matrices consisting of the elements  $X_{ij}$  ( $(i, j) \in C \times C$ ). A vector in  $\mathbb{R}^C$  denoted by  $\mathbf{x}_C$  is regarded as a subvector of  $\mathbf{x} \in \mathbb{R}^n$ , and a matrix in  $\mathbb{S}^C$  denoted by  $\mathbf{X}_C$  as a principal submatrix of  $\mathbf{X} \in \mathbb{S}^n$ . We define  $\mathbf{\Gamma}^C = \{\mathbf{x}_C\mathbf{x}_C^T : \mathbf{x}_C \in \mathbb{R}_+^C\}$ , and  $\mathbb{DNN}^C$  and  $\mathbb{CPP}^C$  to be the DNN cone and the CPP cone in  $\mathbb{S}^C$ , respectively.

We say a subset  $F$  of  $N \times N$  symmetric if it satisfies  $(i, j) \in F \Leftrightarrow (j, i) \in F$ . For every symmetric subset  $F$  of  $N \times N$ , we denote  $\{(i, j) \in F : i \neq j\}$  and  $F \cup \{(i, i) : i \in N\}$  by  $F^\circ$  and  $\bar{F}$ , respectively.

### 2.2 Conversion of general COPs to COP( $\mathbb{K}$ )

Let  $d_p \in \mathbb{R}$  and  $\bar{\mathbf{Q}}^p \in \mathbb{S}^n$  ( $p \in \{0\} \cup I_{\text{eq}} \cup I_{\text{ineq}}$ ). We assume  $d_0 = 0$ . For each  $\mathbb{K} \in \{\mathbf{\Gamma}^n, \mathbb{CPP}^n, \mathbb{DNN}^n\}$ , consider a general COP in the standard equality form:

$$\inf \left\{ \langle \bar{\mathbf{Q}}^0, \mathbf{Y} \rangle : \mathbf{Y} \in \mathbb{K}, \langle \bar{\mathbf{Q}}^p, \mathbf{Y} \rangle = d_p \ (p = 1, \dots, m) \right\}.$$

Let  $I_{\text{eq}} = \{1, \dots, m\}$ ,  $I_{\text{ineq}} = \emptyset$ , and  $\mathbf{Q}^p = \begin{pmatrix} -d_p & \mathbf{0}^T \\ \mathbf{0} & \bar{\mathbf{Q}}^p \end{pmatrix} \in \mathbb{S}^{1+n}$  ( $p = 0, 1, \dots, m$ ).

Then  $\langle \mathbf{Q}^p, \mathbf{X} \rangle = \langle \bar{\mathbf{Q}}^p, \mathbf{Y} \rangle - d_p X_{11}$  for every  $\mathbf{X} = \begin{pmatrix} X_{11} & \mathbf{x}^T \\ \mathbf{x} & \mathbf{Y} \end{pmatrix} \in \mathbb{S}^{1+n}$  ( $p = 0, 1, \dots, m$ ).

Therefore, the above standard equality form COP with  $\mathbb{K} = \mathbf{\Gamma}^n$ ,  $\mathbb{K} = \mathbb{CPP}^n$  and  $\mathbb{K} = \mathbb{DNN}^n$  can be rewritten as COP( $\mathbf{\Gamma}^{1+n}$ ), COP( $\mathbb{CPP}^{1+n}$ ) and COP( $\mathbb{DNN}^{1+n}$ ), respectively. Conversely, if slack variables are introduced for the inequality constraints  $\langle \mathbf{Q}^p, \mathbf{X} \rangle \leq 0$  ( $p \in I_{\text{ineq}}$ ) in COP( $\mathbb{K}$ ), COP( $\mathbb{K}$ ) can be converted into the standard equality form COP in a straightforward fashion.

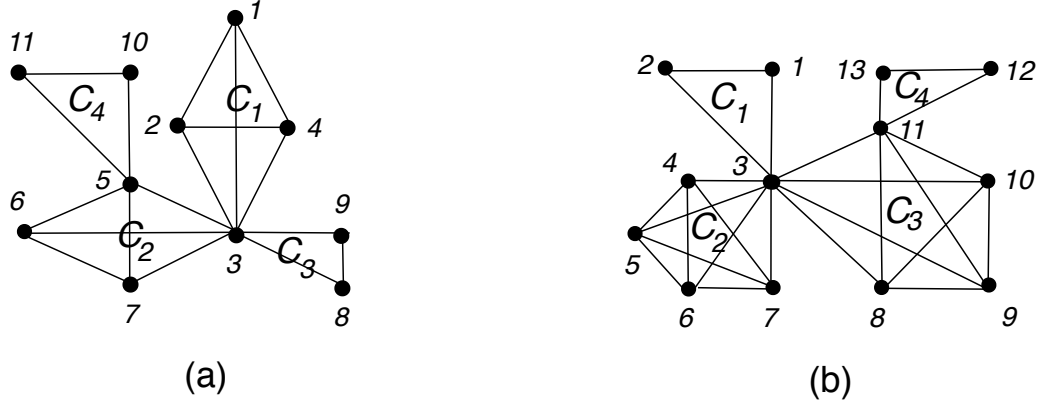


Figure 1: Illustration of block-clique graphs. In case (a), the maximal cliques are  $C_1 = \{1, 2, 3, 4\}$ ,  $C_2 = \{3, 5, 6, 7\}$ ,  $C_3 = \{3, 8, 9\}$  and  $C_4 = \{5, 10, 11\}$ . In this case, every clique  $C_q$  ( $q = 1, 2, 3, 4$ ) contains at most 4 nodes, so that the graph  $G(N, E)$  satisfies the assumption of (ii) and (iii) of Theorem 3.1. In case (b), the maximal cliques are  $C_1 = \{1, 2, 3\}$ ,  $C_2 = \{3, 4, 5, 6, 7\}$ ,  $C_3 = \{3, 8, 9, 10, 11\}$  and  $C_4 = \{11, 12, 13\}$ . If a new edge (10, 12) is added, then a new clique  $C_5 = \{10, 11, 12\}$  is created. The resulting graph is no longer block-clique, but it remains to be chordal.

## 2.3 Chordal and block-clique graphs

We consider an undirected graph  $G(N, E)$  with the node set  $N = \{1, \dots, n\}$  and the edge set  $E$ . Here  $E$  is a symmetric subset of  $\{(i, j) \in N \times N : i \neq j\}$  (hence  $E^o = E$ ) and  $(i, j) \in E$  is identified with  $(j, i) \in E$ . A graph  $G(N, E)$  is called *chordal* if every cycle in  $G(N, E)$  of length 4 or more has a chord, and *block-clique* [9] if it is a chordal graph and any pair of two maximal cliques of  $G(N, E)$  intersects in at most one node. See Figure 1 for examples of block-clique graphs.

Let  $G(N, E)$  be a chordal graph with the maximal cliques  $C_q$  ( $q = 1, \dots, \ell$ ). We assume that the graph is connected. If it is not, the subsequent discussion can be applied to each connected component. Consider an undirected graph on the maximal cliques,  $G(\mathcal{N}, \mathcal{E})$  with the node set  $\{C_q : q = 1, \dots, \ell\}$  and the edge set  $\mathcal{E} = \{(C_q, C_r) : C_q \cap C_r \neq \emptyset\}$ . Since  $G(N, E)$  is assumed to be connected,  $G(\mathcal{N}, \mathcal{E})$  is connected. Then, add the weight  $|C_q \cap C_r|$  to each edge  $(C_q, C_r) \in \mathcal{E}$ . Here  $|C_q \cap C_r|$  denotes the number of nodes contained in the clique  $C_q \cap C_r$ . It is known that every maximum weight spanning tree  $G(\mathcal{N}, \mathcal{T})$  of  $G(\mathcal{N}, \mathcal{E})$  satisfies the following clique intersection property:

$$\left. \begin{array}{l} \text{for every pair of distinct cliques } C_q \text{ and } C_r, C_q \cap C_r \text{ is a subset of} \\ \text{every clique on the (unique) path connecting } C_q \text{ and } C_r \text{ in the tree.} \end{array} \right\} \quad (2)$$

Such a tree is called as a *clique tree* of  $G(N, E)$ . We refer to [6] for the fundamental properties of chordal graphs and clique trees including the clique intersection property and the running intersection property described below.

Now suppose that  $G(N, E)$  is a connected block-clique graph. Then we know that  $|C_q \cap C_r| = 1$  if  $(C_q, C_r) \in \mathcal{E}$ . Hence, every spanning tree of  $G(\mathcal{N}, \mathcal{E})$  is a clique tree. See Figure 2. Let  $G(\mathcal{N}, \mathcal{T})$  be a clique tree of  $G(N, E)$ . Choose an arbitrary maximal clique

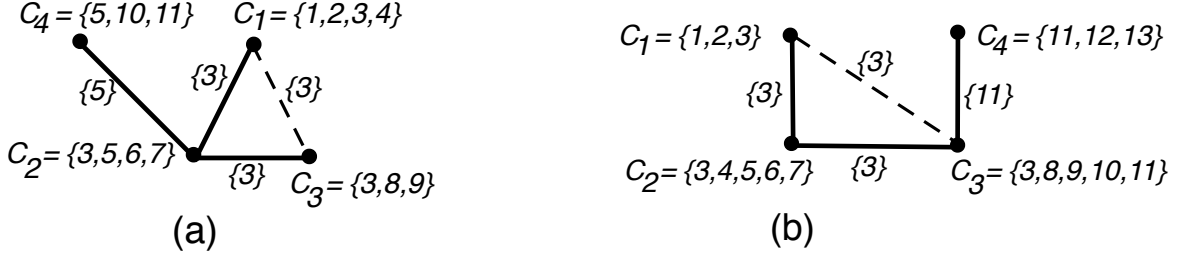


Figure 2: Illustration of clique trees. (a) and (b) are clique trees  $G(\mathcal{N}, \mathcal{T})$  of the block-clique graphs  $G(N, E)$  shown in (a) and (b) of Figure 1, respectively. The notation  $\{i\}$  added on each edge  $(C_q, C_r)$  denotes the intersection of the two cliques  $C_q$  and  $C_r$  of  $G(N, E)$ ; for example,  $(C_1, C_2)$  is an edge of the clique tree  $G(\mathcal{N}, \mathcal{T})$  in (a), and  $C_1 \cap C_2 = \{3\}$ .

as a root node, say  $C_1$ . The rest of the maximal cliques  $C_2, \dots, C_\ell$  can be renumbered such that for any pair of distinct  $C_q$  and  $C_r$ , if  $C_q$  is on the (unique) path from the root node  $C_1$  to  $C_r$  then  $q < r$  holds by applying a topological sorting (ordering). In this way, the renumbered sequence of maximal cliques  $C_1, C_2, \dots, C_\ell$  satisfies *the running intersection property*:

$$\forall r \in \{2, \dots, \ell\}, \exists q \in \{1, \dots, r-1\} \text{ such that } (C_1 \cup \dots \cup C_{r-1}) \cap C_r \subset C_q.$$

Let  $r \in \{2, \dots, \ell\}$ , and  $C_s$  the (unique) parent of  $C_r$ . Then  $s \in \{1, \dots, r-1\}$  and

$$\{k\} = C_s \cap C_r \subset (C_1 \cup \dots \cup C_{r-1}) \cap C_r.$$

for some  $k \in N$ . If  $q \in \{1, \dots, r-1\}$  satisfies  $(C_1 \cup \dots \cup C_{r-1}) \cap C_r \subset C_q$  in the running intersection property above, then

$$\{k\} = C_s \cap C_r \subset (C_1 \cup \dots \cup C_{r-1}) \cap C_r \subset C_q \cap C_r = \{k\}.$$

Here, the last equality follows from the assumption that  $|C_q \cap C_r| \leq 1$ . Therefore, we have shown the following result.

**Lemma 2.1.** *(The running intersection property applied to a block-clique graph.) Let  $G(N, E)$  be a connected block-clique graph with the maximal cliques  $C_1, C_2, \dots, C_\ell$ . Choose one of the maximal cliques arbitrary, say  $C_1$ . Then, the rest of the maximal cliques can be renumbered such that*

$$\left. \begin{array}{l} \forall r \in \{2, \dots, \ell\}, \exists q \in \{1, \dots, r-1\}, \exists k_r \in C_r \text{ such that} \\ (C_1 \cup \dots \cup C_{r-1}) \cap C_r = C_q \cap C_r = \{k_r\} \end{array} \right\} \quad (3)$$

holds.

## 2.4 Matrix completion

We call an  $n \times n$  matrix array  $X_{ij} = X_{ji}$  ( $(i, j) \in N \times N$ ) a *partial symmetric matrix* if a part of its elements  $X_{ij} = X_{ji}$  ( $(i, j) \in F$ ) are specified for some symmetric  $F \subset N \times N$

and the other elements are not specified. We denote a partial symmetric matrix with specified elements  $\bar{X}_{ij} = \bar{X}_{ji}$  ( $(i, j) \in F$ ) by  $[\bar{X}_{ij} : F]$ . Given a property P characterizing a symmetric matrix in  $\mathbb{S}^n$  and a partial symmetric matrix  $[\bar{X}_{ij} : F]$  for some symmetric  $F \subset N \times N$ , the matrix completion problem with property P is to find values  $\bar{X}_{ij} = \bar{X}_{ji}$  ( $(i, j) \notin F$ ) of unspecified elements  $X_{ij} = X_{ji}$  ( $(i, j) \notin F$ ) such that the resulting  $n \times n$  symmetric matrix  $\bar{\mathbf{X}}$  has property P. We say that the partial symmetric matrix  $[\bar{X}_{ij} : F]$  has a *completion*  $\bar{\mathbf{X}}$  with property P.

We mainly consider CPP and DNN matrices in the subsequent discussions. In these matrices, we may assume without loss of generality that  $(i, i) \in F$  ( $i \in N$ ) since unspecified diagonal elements can be given as sufficiently large positive values to realize the property. Each partial symmetric matrix  $[\bar{X}_{ij} : F]$  can be associated with a graph  $G(N, F^o)$ . (Recall that  $F^o = \{(i, j) \in F : i \neq j\}$ .) Let  $C_q$  ( $q = 1, \dots, \ell$ ) be the maximal cliques of  $G(N, F^o)$ . Then the partial symmetric matrix  $[\bar{X}_{ij} : F]$  is decomposed into partial symmetric matrices  $[\bar{X}_{ij} : C_q]$ , which can be consistently described as  $\bar{\mathbf{X}}_{C_q}$ , ( $q = 1, \dots, \ell$ ). We say that a partial symmetric matrix  $[\bar{X}_{ij} : F]$  is *partially DNN* (*partially CPP*) if every  $\bar{\mathbf{X}}_{C_q}$  is DNN (CPP, respectively) in  $\mathbb{S}^{C_q}$  ( $q = 1, \dots, \ell$ ).

**Lemma 2.2.** [10] *Let  $F$  be a symmetric subset of  $N \times N$  such that  $(i, i) \in F$  for every  $i \in N$ . Then every partial CPP (DNN) matrix  $[\bar{X}_{ij} : F]$  has a CPP (DNN, respectively) completion, i.e., a CPP (DNN, respectively) matrix  $\mathbf{X}$  such that  $X_{ij} = \bar{X}_{ij}$  ( $(i, j) \in F$ ), iff  $G(N, F^o)$  is a block clique graph.*

## 2.5 A class of QOPs and their CPP and DNN relaxations

Any quadratic function in nonnegative variables  $x_2, \dots, x_n$  can be represented as  $\langle \mathbf{Q}, \mathbf{x}\mathbf{x}^T \rangle$  with  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$  and  $x_1^2 = 1$  for some  $\mathbf{Q} \in \mathbb{S}^n$ . By introducing a variable matrix  $\mathbf{X} \in \Gamma^n \equiv \{\mathbf{x}\mathbf{x}^T : \mathbf{x} \in \mathbb{R}_+^n\}$ , the function can be rewritten as  $\langle \mathbf{Q}, \mathbf{X} \rangle$  with  $X_{11} = 1$ . As a result, a general quadratically constrained QOP in nonnegative variables  $x_2, \dots, x_n$  can also be represented as COP( $\Gamma^n$ ) introduced in Section 1. Since  $\Gamma^n \subset \text{CPP}^n \subset \text{DNN}^n$ ,  $\zeta(\text{DNN}^n) \leq \zeta(\text{CPP}^n) \leq \zeta(\Gamma^n)$  holds in general.

On the equivalence between QOPs and their CPP relaxations, Burer's reformulation [8] for a class of QOPs with linear constraints in nonnegative and binary variables is well-known. In this paper, we employ the following result, which is essentially equivalent to Burer's reformulation. See [17] for CPP reformulations of more general class of QOPs.

**Lemma 2.3.** [3, Theorem 3.1] *For COP( $\Gamma^n$ ), assume that*

$$\begin{aligned} I_{\text{ineq}} &= \emptyset, \text{ COP}(\Gamma^n) \text{ is feasible,} \\ \langle \mathbf{Q}^0, \mathbf{X} \rangle &\geq 0 \text{ if } \mathbf{X} \in \Gamma^n, X_{11} = 0 \text{ and } \langle \mathbf{Q}^p, \mathbf{X} \rangle = 0 \text{ (} p \in I_{\text{eq}}\text{),} \\ \mathbf{Q}^p &\in \mathbb{S}_+^n + \mathbb{N}^n \text{ (the dual of } \text{DNN}^n\text{) (} p \in I_{\text{eq}}\text{).} \end{aligned}$$

*Then,  $\zeta(\Gamma^n) = \zeta(\text{CPP}^n)$ .*

**Example 2.4.** Let  $\mathbf{A}$  be a  $k \times n$  matrix,  $I_{\text{comp}} \subset \{(i, j) \in N \times N : 1 < i < j\}$  and  $\mathbf{Q}^0 \in \mathbb{S}^n$ . Consider a QOP with linear equality and complementarity constraints in nonnegative variables  $\mathbf{x} \in \mathbb{R}_+^n$ .

$$\zeta_{\text{QOP}} = \inf \{ \mathbf{x}^T \mathbf{Q}^0 \mathbf{x} : \mathbf{x} \in \mathbb{R}_+^n, x_1 = 1, \mathbf{A}\mathbf{x} = \mathbf{0}, x_i x_j = 0 \text{ ((} i, j \text{) } \in I_{\text{comp}}) \}.$$



Define

$$\begin{aligned} \mathbf{Q}^1 &= \mathbf{A}^T \mathbf{A} \in \mathbb{S}^n, \\ \mathbf{Q}^{ij} &= \text{the } n \times n \text{ matrix with 1 at the } (i, j)\text{th and } (j, i)\text{th elements,} \\ &\quad \text{and 0 elsewhere.} \end{aligned}$$

Then, the QOP can be rewritten as

$$\zeta_{\text{QOP}} = \inf \left\{ \langle \mathbf{Q}^0, \mathbf{X} \rangle : \begin{array}{l} \mathbf{X} \in \mathbf{\Gamma}^n, X_{11} = 1, \langle \mathbf{Q}^1, \mathbf{X} \rangle = 0, \\ \langle \mathbf{Q}^{ij}, \mathbf{X} \rangle = 0 \ ((i, j) \in I_{\text{comp}}) \end{array} \right\}.$$

Enumerate  $(i, j) \in I_{\text{comp}}$  from 2 through some integer  $m$ , and let  $I_{\text{eq}} = \{1, \dots, m\}$  and  $I_{\text{ineq}} = \emptyset$ . Then QOP can be rewritten as  $\text{COP}(\mathbf{\Gamma}^n)$ . Obviously  $\mathbf{Q}^1 \in \mathbb{S}_+^n$  and  $\mathbf{Q}^{ij} \in \mathbb{N}^n$  ( $(i, j) \in I_{\text{comp}}$ ). Hence, if the QOP is feasible and  $\mathbf{O} = \{\mathbf{X} \in \mathbf{\Gamma}^n : X_{11} = 0, \langle \mathbf{Q}^1, \mathbf{X} \rangle = 0\} = \{\mathbf{x}\mathbf{x}^T : \mathbf{x} \in \mathbb{R}_+^n, x_1 = 0, \mathbf{A}\mathbf{x} = \mathbf{0}\}$ , then all the assumptions in Lemma 2.3 are satisfied. Consequently,  $\zeta(\mathbf{\Gamma}^n) = \zeta(\text{CPP}^n)$  holds. We note that the binary condition on variable  $x$  can be represented as  $x, y \geq 0, x + y = 1$  and  $xy = 0$  with a slack variable  $y$ . Thus, binary variables can be included in the QOP above. This QOP model covers various combinatorial optimization problems. See [3, 16] for more details.

It is well-known that a convex QOP can be reformulated as an SDP (see, for example, [11]). The following result may be regarded as a variation of the SDP reformulation to CPP and DNN reformulations of a convex QOP in nonnegative variables.

**Lemma 2.5.** *Let  $\tilde{N} = N \setminus \{1\}$ . Assume that  $\mathbf{Q}_{\tilde{N}}^0 \in \mathbb{S}_+^{n-1}$ ,  $\mathbf{Q}^p \in \mathbb{S}_+^n$  ( $p \in I_{\text{eq}}$ ) and  $\mathbf{Q}_{\tilde{N}}^p \in \mathbb{S}_+^{n-1}$  ( $p \in I_{\text{ineq}}$ ). Then  $\zeta(\text{DNN}^n) = \zeta(\text{CPP}^n) = \zeta(\mathbf{\Gamma}^n)$ . Furthermore, if  $\mathbf{X} = \begin{pmatrix} 1 & \mathbf{y}^T \\ \mathbf{y} & \mathbf{Y} \end{pmatrix} \in \text{DNN}^n$  with some  $\mathbf{y} \in \mathbb{R}_+^{n-1}$  and  $\mathbf{Y} \in \text{DNN}^{n-1}$  is an optimal solution of  $\text{COP}(\text{DNN}^n)$ , then  $\begin{pmatrix} 1 & \mathbf{y}^T \\ \mathbf{y} & \mathbf{y}\mathbf{y}^T \end{pmatrix}$  is a common optimal solution of  $\text{COP}(\mathbf{\Gamma}^n)$ ,  $\text{COP}(\text{CPP}^n)$  and  $\text{COP}(\text{DNN}^n)$  with the objective value  $\zeta(\text{DNN}^n) = \zeta(\text{CPP}^n) = \zeta(\mathbf{\Gamma}^n)$ .*

*Proof.* The inequality  $\zeta(\text{DNN}^n) \leq \zeta(\text{CPP}^n) \leq \zeta(\mathbf{\Gamma}^n)$  follows from  $\mathbf{\Gamma}^n \subset \text{CPP}^n \subseteq \text{DNN}^n$ . It suffices to show  $\zeta(\mathbf{\Gamma}^n) \leq \zeta(\text{DNN}^n)$ . Let  $\mathbf{X} = \begin{pmatrix} 1 & \mathbf{y}^T \\ \mathbf{y} & \mathbf{Y} \end{pmatrix} \in \text{DNN}^n$  with some  $\mathbf{y} \in \mathbb{R}_+^{n-1}$  and  $\mathbf{Y} \in \text{DNN}^{n-1}$  be an arbitrary feasible solution of  $\text{COP}(\text{DNN}^n)$ . Then,

$$\begin{aligned} \begin{pmatrix} 1 \\ \mathbf{y} \end{pmatrix} &\in \mathbb{R}_+^n, \bar{\mathbf{X}} \equiv \mathbf{X} - \begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{Y} - \mathbf{y}\mathbf{y}^T \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{y}^T \\ \mathbf{y} & \mathbf{y}\mathbf{y}^T \end{pmatrix} \in \mathbf{\Gamma}^n, \\ \langle \mathbf{H}^0, \bar{\mathbf{X}} \rangle &= 1, \mathbf{Q}_{\tilde{N}}^p \in \mathbb{S}_+^{n-1} \ (p \in \{0\} \cup I_{\text{eq}} \cup I_{\text{ineq}}), \mathbf{Y} - \mathbf{y}\mathbf{y}^T \in \mathbb{S}_+^{n-1}, \\ \langle \mathbf{Q}^p, \bar{\mathbf{X}} \rangle &= \langle \mathbf{Q}^p, \mathbf{X} \rangle - \langle \mathbf{Q}_{\tilde{N}}^p, \mathbf{Y} - \mathbf{y}\mathbf{y}^T \rangle \leq \langle \mathbf{Q}^p, \mathbf{X} \rangle \ (p \in \{0\} \cup I_{\text{eq}} \cup I_{\text{ineq}}). \end{aligned}$$

From the last inequality, the assumptions and  $\bar{\mathbf{X}} \in \text{CPP}^n \subset \mathbb{S}_+^n$ , we see that

$$\begin{aligned} 0 &\leq \langle \mathbf{Q}^p, \bar{\mathbf{X}} \rangle \leq \langle \mathbf{Q}^p, \mathbf{X} \rangle = 0 \ (\text{hence } \langle \bar{\mathbf{Q}}^p, \mathbf{X} \rangle = 0) \ (p \in I_{\text{eq}}), \\ \langle \mathbf{Q}^p, \bar{\mathbf{X}} \rangle &\leq \langle \mathbf{Q}^p, \mathbf{X} \rangle \leq 0 \ (p \in I_{\text{ineq}}), \langle \mathbf{Q}^0, \bar{\mathbf{X}} \rangle \leq \langle \mathbf{Q}^0, \mathbf{X} \rangle. \end{aligned}$$

Thus we have shown that  $\overline{\mathbf{X}}$  is a feasible solution of  $\text{COP}(\mathbf{\Gamma}^n)$  whose objective value is not greater than that of the feasible solution  $\mathbf{X}$  of  $\text{COP}(\mathbb{DNN}^n)$ . Therefore  $\zeta(\mathbf{\Gamma}^n) \leq \zeta(\mathbb{DNN}^n)$  has been shown. The second assertion follows by choosing an optimal solution of  $\text{COP}(\mathbb{DNN}^n)$  for  $\mathbf{X}$  in the proof above.  $\square$

**Example 2.6.** (Quadratically constrained convex QOPs) Let  $\tilde{N} = \{2, \dots, n\}$  and  $I_{\text{ineq}} = \{2, \dots, m\}$ . Let  $\mathbf{A}$  be a  $k \times n$  matrix and  $\mathbf{Q}^p \in \mathbb{S}^n$  be such that  $\mathbf{Q}_{\tilde{N}}^0 \in \mathbb{S}_+^{\tilde{N}}$  ( $p \in \{0\} \cup I_{\text{ineq}}$ ). Consider a QOP:

$$\zeta_{\text{QOP}} = \inf \{ \mathbf{x}^T \mathbf{Q}^0 \mathbf{x} : \mathbf{x} \in \mathbb{R}_+^n, x_1 = 1, \mathbf{A}\mathbf{x} = 0, \mathbf{x}^T \mathbf{Q}^p \mathbf{x} \leq 0 \ (p \in I_{\text{ineq}}) \},$$

which represents a general quadratically constrained convex QOP in nonnegative variables  $x_i$  ( $i = 2, \dots, n$ ). If we let  $\mathbf{Q}^1 = \mathbf{A}^T \mathbf{A} \in \mathbb{S}_+^n$  and  $I_{\text{eq}} = \{1\}$ , we can represent the QOP as  $\text{COP}(\mathbf{\Gamma}^n)$ . By Lemma 2.5, not only  $\zeta(\mathbb{DNN}^n) = \zeta(\mathbb{CPP}^n) = \zeta(\mathbf{\Gamma}^n)$  holds, but also the DNN relaxation provides an optimal solution of the QOP.

## 2.6 Two types of sparsity

We consider two types of sparsity of the data matrices  $\mathbf{Q}^p$  ( $p \in \{0\} \cup I_{\text{eq}} \cup I_{\text{ineq}}$ ) of  $\text{COP}(\mathbb{K})$  with  $\mathbb{K} \in \{\mathbf{\Gamma}^n, \mathbb{CPP}^n, \mathbb{DNN}^n\}$ , the *aggregated sparsity* [12] and the *correlative sparsity* [18]. We say that the aggregated sparsity of matrices  $\mathbf{A}^p \in \mathbb{S}^n$  ( $p = 1, \dots, m$ ) is represented by a graph  $G(N, E)$  if  $\{(i, j) \in N \times N : A_{ij}^p \neq 0\} \subset \overline{E} \equiv E \cup \{(i, i) : i \in N\}$  for every  $p = 1, \dots, m$ , and that their correlative sparsity is represented by a graph  $G(N, E)$  with the maximal cliques  $C_q$  ( $q = 1, \dots, \ell$ ) if

$$\forall p \in \{1, \dots, m\}, \exists q \in \{q = 1, \dots, \ell\} \text{ such that} \\ \{(i, j) \in N \times N : A_{ij}^p \neq 0\} \subset C_q \times C_q. \quad (4)$$

Note that if  $C_1, \dots, C_\ell$  are the maximal cliques of  $G(N, E)$ , then  $\overline{E} = \cup_{q=1}^{\ell} C_q$ . We also note that a graph  $G(N, E)$  which represents the aggregate (correlative) sparsity of  $\mathbf{A}^p \in \mathbb{S}^n$  ( $p = 1, \dots, m$ ) is not unique.

Our main interest in the subsequent discussion is a block-clique graph  $G(N, E)$  which represents the aggregate (correlative) sparsity of  $\mathbf{A}^p \in \mathbb{S}^n$  ( $p = 1, \dots, m$ ). If we let  $F = \{(i, j) \in N \times N : A_{ij}^p \neq 0 \text{ for some } p\}$ , then  $G(N, F^o)$  is “the smallest” graph which represents their aggregate sparsity. Since it is not block-clique in general, a block-clique extension  $G(N, E)$  of  $G(N, F^o)$  is necessary. Assume that  $G(N, F^o)$  is connected. It is straightforward to verify that a node of a connected block-clique graph is a cut node iff it is contained in at least two distinct maximal cliques of the graph. Therefore, if there exists no cut node in  $G(N, F^o)$ , then the complete graph  $G(N, (N \times N)^o)$  is the only block-clique extension of  $G(N, F^o)$ . Otherwise, take the maximal edge set  $E \subset (N \times N)^o$  such that the graph  $G(N, E)$  has the same cut nodes as  $G(N, F^o)$ . Then  $G(N, E)$  forms the smallest block-clique extension of  $G(N, F^o)$ .

Obviously, if a graph  $G(N, E)$  represents the correlative sparsity of matrices  $\mathbf{A}^p \in \mathbb{S}^n$  ( $p = 1, \dots, m$ ), then it also represents their aggregate sparsity. But the converse is not true as we see in the following example.

**Example 2.7.** Let  $n = 3$ ,  $N = \{1, 2, 3\}$  and

$$\mathbf{A}^1 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad \mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad \overline{\mathbf{A}} = \begin{pmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}$$

where  $*$  denotes a nonzero element. We see that the aggregate sparsity pattern of the two matrices  $\mathbf{A}^1$  and  $\mathbf{A}^2$  corresponds to  $\overline{\mathbf{A}}$ . Thus their aggregated sparsity is represented by the graph with the maximal cliques  $C_1 = \{1, 2\}$  and  $C_2 = \{2, 3\}$ . But neither  $C_1 \times C_1$  nor  $C_2 \times C_2$  covers the nonzero elements of  $\mathbf{A}^1$ . We need to take the complete graph with the single maximal clique  $N = \{1, 2, 3\}$  to represent the correlative sparsity of  $\mathbf{A}^1$  and  $\mathbf{A}^2$ .

### 3 Exploiting the aggregated sparsity characterized by block-clique graphs

Throughout this section, we assume that the aggregate sparsity of the data matrices  $\mathbf{Q}^p$  ( $p \in \{0\} \cup I_{\text{eq}} \cup I_{\text{ineq}}$ ) of  $\text{COP}(\mathbb{K})$  with  $\mathbb{K} \in \{\mathbf{\Gamma}^n, \text{CPP}^n, \text{DNN}^n\}$  is represented by a block-clique graph  $G(N, E)$ ;

$$\{(i, j) \in N \times N : Q_{ij}^p \neq 0\} \subset \overline{E} \quad (p \in \{0\} \cup I_{\text{eq}} \cup I_{\text{ineq}}). \quad (5)$$

Under this assumption, we provide sufficient conditions for  $\zeta(\text{DNN}^n) = \zeta(\text{CPP}^n)$  and  $\zeta(\mathbf{\Gamma}^n) = \zeta(\text{DNN}^n) = \zeta(\text{CPP}^n)$ .

Note that the values of elements  $X_{ij}$  ( $(i, j) \in \overline{E}$ ) determine the value of  $\langle \mathbf{Q}^p, \mathbf{X} \rangle$ , *i.e.*,  $\langle \mathbf{Q}^p, \mathbf{X} \rangle = \sum_{(i,j) \in \overline{E}} Q_{ij}^p X_{ij}$  ( $p \in \{0\} \cup I_{\text{eq}} \cup I_{\text{ineq}}$ ). All other elements  $X_{ij}$  ( $(i, j) \notin \overline{E}$ ) affect only the cone constraint  $\mathbf{X} \in \mathbb{K}$  in  $\text{COP}(\mathbb{K})$ . Thus, the cone constraint can be replaced by “[ $X_{ij} : \overline{E}$ ] has a completion  $\mathbf{X} \in \mathbb{K}$ ”. More precisely,  $\text{COP}(\mathbb{K})$  can be written as

$$\zeta(\mathbb{K}) = \inf \left\{ \sum_{(i,j) \in \overline{E}} Q_{ij}^0 X_{ij} : \begin{array}{l} [X_{ij} : \overline{E}] \text{ has a completion } \mathbf{X} \in \mathbb{K}, X_{11} = 1, \\ \sum_{(i,j) \in \overline{E}} Q_{ij}^p X_{ij} = 0 \quad (p \in I_{\text{eq}}), \\ \sum_{(i,j) \in \overline{E}} Q_{ij}^p X_{ij} \leq 0 \quad (p \in I_{\text{ineq}}) \end{array} \right\}. \quad (6)$$

Let  $C_q$  ( $q = 1, \dots, \ell$ ) be the maximal cliques of  $G(N, E)$ . We now introduce a (sparse) relaxation of  $\text{COP}$  (6) (=  $\text{COP}(\mathbb{K})$ ).

$$\eta(\mathbb{K}) = \inf \left\{ \sum_{(i,j) \in \overline{E}} Q_{ij}^0 X_{ij} : \begin{array}{l} \mathbf{X}_{C_q} \in \mathbb{K}^{C_q} \quad (q = 1, \dots, \ell), X_{11} = 1, \\ \sum_{(i,j) \in \overline{E}} Q_{ij}^p X_{ij} = 0 \quad (p \in I_{\text{eq}}), \\ \sum_{(i,j) \in \overline{E}} Q_{ij}^p X_{ij} \leq 0 \quad (p \in I_{\text{ineq}}) \end{array} \right\}. \quad (7)$$

Since we have been dealing with the case where  $\mathbb{K} \in \{\mathbf{\Gamma}^n, \text{CPP}^n, \text{DNN}^n\}$ ,  $\mathbb{K}^{C_q}$  stands for  $\mathbf{\Gamma}^{C_q}$ ,  $\text{DNN}^{C_q}$  or  $\text{CPP}^{C_q}$  ( $q = 1, \dots, \ell$ ). In either case,  $\mathbf{X} \in \mathbb{K}$  implies  $\mathbf{X}_{C_q} \in \mathbb{K}^{C_q}$  ( $q = 1, \dots, \ell$ ). Thus  $\text{COP}$  (7) serves as a relaxation of  $\text{COP}$  (6);  $\eta(\mathbb{K}) \leq \zeta(\mathbb{K})$ .

**Theorem 3.1.** Let  $\mathbb{K} \in \{\Gamma^n, \text{CPP}^n, \text{DNN}^n\}$ . Assume that (5) holds for a block-clique graph  $G(N, E)$  with the maximal cliques  $C_q$  ( $q = 1, \dots, \ell$ ). Then,

- (i)  $\eta(\mathbb{K}) = \zeta(\mathbb{K})$ .
- (ii) If every clique  $C_q$  contains at most 4 nodes ( $q = 1, \dots, \ell$ ), then COP (7) with  $\mathbb{K} = \text{CPP}^n$  and COP (7) with  $\mathbb{K} = \text{DNN}^n$  are equivalent; more precisely they share the same feasible solutions, the optimal solutions and the optimal value  $\eta(\text{CPP}^n) = \eta(\text{DNN}^n)$ .
- (iii) If every clique  $C_q$  contains at most 4 nodes ( $q = 1, \dots, \ell$ ) and the assumptions of Lemma 2.3 are satisfied, then  $\eta(\Gamma^n) = \zeta(\Gamma^n) = \zeta(\text{CPP}^n) = \eta(\text{CPP}^n) = \eta(\text{DNN}^n)$ .

*Proof.* (i) Suppose that  $\mathbb{K} \in \{\text{DNN}^n, \text{CPP}^n\}$ . By Lemma 2.2, the condition “[ $X_{ij} : (i, j) \in \bar{E}$ ] has a completion  $\mathbf{X} \in \mathbb{K}$ ” in COP (6) is equivalent to the condition “[ $\mathbf{X}_{C_q} \in \mathbb{K}^{C_q}$  ( $q = 1, \dots, \ell$ )” in COP (7). Hence COPs (6) and (7) are equivalent, which implies that  $\eta(\mathbb{K}) = \zeta(\mathbb{K})$ . Now suppose that  $\mathbb{K} = \Gamma^n$ . Since we already know that  $\eta(\Gamma^n) \leq \zeta(\Gamma^n)$ , it suffices to show that  $\zeta(\Gamma^n) \leq \eta(\Gamma^n)$ . Choose a maximal clique that contains node 1 among  $C_1, \dots, C_\ell$ , say  $C_1$ . By Lemma 2.1, we can renumber the rest of the maximal cliques such that the running intersection property (3) holds. Assume that  $(\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_\ell})$  is a feasible solution of COP (7) with  $\mathbb{K} = \Gamma^n$ . We will construct an  $\bar{\mathbf{x}} \in \mathbb{R}_+^n$  such that  $\bar{\mathbf{X}} \equiv \bar{\mathbf{x}}\bar{\mathbf{x}}^T$  is a feasible solution of COP( $\Gamma^n$ ) with the same objective value of COP (7) with  $\mathbb{K} = \Gamma^n$  at its feasible solution  $(\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_\ell})$ . Then  $\zeta(\Gamma^n) \leq \eta(\Gamma^n)$  follows. For each  $r \in \{1, \dots, \ell\}$ , there exists an  $\mathbf{y}_{C_r} \in \mathbb{R}_+^{C_r}$  such that  $\mathbf{X}_{C_r} = \mathbf{y}_{C_r}\mathbf{y}_{C_r}^T$ . For  $r = 1$ , let  $\bar{x}_i = [\mathbf{y}_{C_1}]_i$  ( $i \in C_1$ ). By (3), for each  $r \in \{2, \dots, \ell\}$ , there exists a  $k_r \in C_r$  such that  $(C_1 \cup \dots \cup C_{r-1}) \cap C_r = \{k_r\}$ . Hence, if we define  $\bar{x}_i = [\mathbf{y}_{C_r}]_i$  ( $i \in C_r \setminus \{k_r\}$ ) for  $r = 2, \dots, \ell$ , then  $\bar{\mathbf{x}} \in \mathbb{R}_+^n$  satisfies  $\bar{x}_i = [\mathbf{y}_{C_r}]_i$  ( $i \in C_r, r \in \{1, \dots, \ell\}$ ) and  $\bar{\mathbf{X}} = \bar{\mathbf{x}}\bar{\mathbf{x}}^T \in \Gamma^n$  is a completion of [ $X : \bar{E}$ ]. By construction,  $\bar{\mathbf{X}}$  is a feasible solution of COP( $\Gamma^n$ ), and attains the same objective value of COP (7) with  $\mathbb{K} = \Gamma^n$  at its feasible solution  $(\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_\ell})$ .

(ii) By (1),  $\text{CPP}^{C_q} = \text{DNN}^{C_q}$  ( $q = 1, \dots, \ell$ ), which implies the equivalence of COP (7) with  $\mathbb{K} = \text{CPP}^n$  and COP (7) with  $\mathbb{K} = \text{DNN}^n$ .

(iii) The identity  $\zeta(\Gamma^n) = \zeta(\text{CPP}^n)$  follows from Lemma 2.3, and all other identities from (i) and (ii).  $\square$

To decompose the COP (7) based on the maximal cliques  $C_1, \dots, C_\ell$ , we first decompose each  $\mathbf{Q}^p$  ( $p \in \{0\} \cup I_{\text{eq}} \cup I_{\text{ineq}}$ ) such that

$$\mathbf{Q}^p = \sum_{q=1}^{\ell} \widehat{\mathbf{Q}}^{pq}, \quad \widehat{\mathbf{Q}}_{ij}^{pq} = 0 \text{ if } (i, j) \notin C_q \times C_q \text{ (} q = 1, \dots, \ell\text{)}.$$

(We note that the decomposition of  $\mathbf{Q}^p$  above is not unique.) Then COP (7) can be represented as

$$\eta(\mathbb{K}) = \inf \left\{ \sum_{q=1}^{\ell} \langle \widehat{\mathbf{Q}}^{0q}, \mathbf{X}_{C_q} \rangle : \begin{array}{l} \mathbf{X}_{C_q} \in \mathbb{K}^{C_q} \text{ (} q = 1, \dots, \ell\text{), } X_{11} = 1, \\ \sum_{q=1}^{\ell} \langle \widehat{\mathbf{Q}}^{pq}, \mathbf{X}_{C_q} \rangle = 0 \text{ (} p \in I_{\text{eq}}\text{),} \\ \sum_{q=1}^{\ell} \langle \widehat{\mathbf{Q}}^{pq}, \mathbf{X}_{C_q} \rangle \leq 0 \text{ (} p \in I_{\text{ineq}}\text{)} \end{array} \right\}. \quad (8)$$

**Remark 3.2.** (a) Note that under the assumption (5), if  $\mathbf{Q}^p \in \mathbb{S}_+^n$  (or  $\mathbf{Q}^p \in \mathbb{S}_+^n + \mathbb{N}^n$ ), then we can take  $\widehat{\mathbf{Q}}^{pq} \in \mathbb{S}_+^n$  (or  $\widehat{\mathbf{Q}}^{pq} \in \mathbb{S}_+^n + \mathbb{N}^n$ ) for the decomposition; see [1, Theorem 2.3]. In this case, the equality constraints  $\sum_{q=1}^{\ell} \langle \widehat{\mathbf{Q}}_{C_q}^{pq}, \mathbf{X}_{C_q} \rangle = 0$  ( $p \in I_{\text{eq}}$ ) can be replaced by  $\langle \widehat{\mathbf{Q}}_{C_q}^{pq}, \mathbf{X}_{C_q} \rangle = 0$  ( $q = 1, \dots, \ell, p \in I_{\text{eq}}$ ) since  $\langle \widehat{\mathbf{Q}}_{C_q}^{pq}, \mathbf{X}_{C_q} \rangle \geq 0$  for every  $\mathbf{X} \in \mathbb{K} \in \{\mathbf{\Gamma}^n, \mathbb{C}\text{PP}^n, \mathbb{D}\text{NN}^n\}$  is known.

(b) Suppose that  $C_q$  ( $q = 1, \dots, \ell$ ) form a partition of  $N$ ;  $\cup_{q=1}^{\ell} C_q = N$  and  $C_q \cap C_r = \emptyset$  ( $q \neq r$ ). Then we have  $\widehat{\mathbf{Q}}_{C_q}^{pq} = \mathbf{Q}_{C_q}^p$  ( $q = 1, \dots, \ell, p \in \{0\} \cup I_{\text{eq}} \cup I_{\text{ineq}}$ ). In particular, if  $C_q = \{q\}$  ( $q = 1, \dots, \ell = n$ ), then COP (8) turns out to be an LP of the form

$$\eta(\mathbb{K}) = \inf \left\{ \sum_{q=1}^{\ell} Q_{ii}^0 X_{ii} : \begin{array}{l} X_{ii} \in \mathbb{R}_+, (i = 1, \dots, n), X_{11} = 1, \\ \sum_{i=1}^n Q_{ii}^p X_{ii} = 0 (p \in I_{\text{eq}}), \\ \sum_{i=1}^n Q_{ii}^p X_{ii} \leq 0 (p \in I_{\text{ineq}}) \end{array} \right\}.$$

Here  $\mathbf{\Gamma}^1 = \mathbb{C}\text{PP}^1 = \mathbb{D}\text{NN}^1 = \mathbb{R}_+$ . Although this observation itself is trivial, it is important in the sense that LPs can be embedded as subproblems in a QOP that can be reformulated as its DNN relaxations, as we will see in (v) of Theorem 4.6 and Section 5.2.

## 4 Exploiting correlative sparsity characterized by block-clique graphs

We consider  $\text{COP}(\mathbb{K})$  with  $\mathbb{K} \in \{\mathbf{\Gamma}^n, \mathbb{C}\text{PP}^n, \mathbb{D}\text{NN}^n\}$ . Throughout this section, we assume that the correlative sparsity [18] of the data matrices  $\mathbf{Q}^p$  ( $p \in I_{\text{eq}} \cup I_{\text{ineq}}$ ) of the linear equality and inequality constraints of  $\text{COP}(\mathbb{K})$  is represented by a connected block-clique graph  $G(N, E)$  with the maximal cliques  $C_q$  ( $q = 1, \dots, \ell$ ) and that the aggregate sparsity of  $\mathbf{Q}^0$  is represented by the same block-clique graph  $G(N, E)$ , i.e.,

$$\forall p \in I_{\text{eq}} \cup I_{\text{ineq}}, \exists q \in \{1, \dots, \ell\} \text{ such that} \\ \{(i, j) \in N \times N : Q_{ij}^p \neq 0\} \subset C_q \times C_q, \quad (9)$$

$$\{(i, j) \in N \times N : Q_{ij}^0 \neq 0\} \subset \overline{E} \equiv \cup_{i=1}^{\ell} C_q \times C_q. \quad (10)$$

In addition, we assume that  $\text{COP}(\mathbb{K})$  has an optimal solution. It should be noted that (9) and (10) imply (5), thus, all the results discussed in the previous section remain valid. In particular, by (i) of Theorem 3.1,  $\text{COP}(\mathbb{K})$  is equivalent to COP (7).

Choose a clique that contains node 1 among  $C_q$  ( $q = 1, \dots, \ell$ ), say  $C_1$ . By Lemma 2.1, the maximal cliques  $C_2, \dots, C_{\ell}$  can be renumbered such that the running intersection property (3) holds. Let  $F_r = \cup_{q=1}^r C_q \times C_q$  ( $r = 1, \dots, \ell$ ). We then obtain from (3) that

$$\forall r \in \{2, \dots, \ell\}, \exists k_r \in C_r \text{ such that} \\ F_{r-1} \cap (C_r \times C_r) = (\cup_{q=1}^{r-1} C_q \times C_q) \cap (C_r \times C_r) = \{(k_r, k_r)\}. \quad (11)$$

By (9), there exists a partition  $L_1, \dots, L_{\ell}$  of  $I_{\text{eq}} \cup I_{\text{ineq}}$  (i.e.,  $\cup_{q=1}^{\ell} L_q = I_{\text{eq}} \cup I_{\text{ineq}}$  and  $L_q \cap L_r = \emptyset$  if  $1 \leq q < r \leq \ell$ ) such that  $\{(i, j) \in N \times N : Q_{ij}^p \neq 0\} \subset C_q \times C_q$  if  $p \in L_q$  ( $q =$

$1, \dots, \ell$ ). Hence, we can rewrite COP (7), which is equivalent to  $\text{COP}(\mathbb{K})$ , as

$$\overline{\text{COP}}(\mathbb{K}): \bar{\zeta}(\mathbb{K}) = \inf \left\{ \begin{array}{l} \mathbf{X}_{C_q} \in \mathbb{K}^{C_q} \ (q = 1, \dots, \ell), \ X_{11} = 1, \\ \sum_{(i,j) \in C_q \times C_q} Q_{ij}^p X_{ij} = 0 \\ \sum_{(i,j) \in F_\ell} Q_{ij}^0 X_{ij} : \quad (p \in I_{\text{eq}}^q, \ q = 1, \dots, \ell), \\ \sum_{(i,j) \in C_q \times C_q} Q_{ij}^p X_{ij} \leq 0 \\ (p \in I_{\text{ineq}}^q, \ q = 1, \dots, \ell) \end{array} \right\}. \quad (12)$$

Here  $I_{\text{eq}}^q = L_q \cap I_{\text{eq}}$  and  $I_{\text{ineq}}^q = L_q \cap I_{\text{ineq}}$  ( $q = 1, \dots, \ell$ ). In particular,  $(\bar{\mathbf{X}}_{C_1}, \dots, \bar{\mathbf{X}}_{C_\ell})$  is an optimal solution of  $\overline{\text{COP}}(\mathbb{K})$  with the objective value  $\bar{\zeta}(\mathbb{K}) = \zeta(\mathbb{K}) > -\infty$  if  $\bar{\mathbf{X}}$  is an optimal solution of  $\text{COP}(\mathbb{K})$ .

#### 4.1 Recursive reduction of $\overline{\text{COP}}(\mathbb{K})$ to a sequence of smaller-sized COPs

We first describe the basic idea on how to reduce  $\overline{\text{COP}}(\mathbb{K})$  to smaller COPs by recursively eliminating variable matrices  $\mathbf{X}_{C_r}$  ( $r = \ell, \dots, 2$ ). The constraints of  $\overline{\text{COP}}(\mathbb{K})$  except  $X_{11} = 1$  can be decomposed into  $\ell$  families of constraints

$$\sum_{(i,j) \in C_q \times C_q} Q_{ij}^p X_{ij} = 0 \ (p \in I_{\text{eq}}^q) \text{ and } \sum_{(i,j) \in C_q \times C_q} Q_{ij}^p X_{ij} \leq 0 \ (p \in I_{\text{ineq}}^q) \quad (13)$$

in the variable matrix  $\mathbf{X}_{C_q} \in \mathbb{K}^{C_q}$  ( $q = \ell, \dots, 1$ ). Although they may look independent, they are weakly connected in the sense that

$$(\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_{r-1}}) \text{ and } \mathbf{X}_{C_r} \text{ share only one scalar variable } X_{k_r k_r}$$

( $r = \ell, \dots, 2$ ). This relation follows from (11).

Let  $r = \ell$ . Then, the linear objective function of  $\overline{\text{COP}}(\mathbb{K})$  can be decomposed into two non-interactive terms such that

$$\sum_{(i,j) \in F_{\ell-1}} Q_{ij}^0 X_{ij} + \sum_{(i,j) \in C_\ell \times C_\ell \setminus \{(k_\ell, k_\ell)\}} Q_{ij}^0 X_{ij}. \quad (14)$$

Hence, if the value of  $X_{k_\ell k_\ell} \geq 0$  is specified, the subproblem of minimizing the second term  $\sum_{(i,j) \in C_\ell \times C_\ell \setminus \{(k_\ell, k_\ell)\}} Q_{ij}^0 X_{ij}$  over the decomposed constraint (13) with  $q = \ell$  in  $\mathbf{X}_{C_\ell} \in \mathbb{K}^{C_\ell}$  can be solved independently from  $\overline{\text{COP}}(\mathbb{K})$ . (This subproblem corresponds to  $\bar{\text{P}}_\ell(\mathbb{K}^{C_\ell}, X_{k_r k_r})$  defined later in (19)). Since all equalities and inequalities in the constraint (13) are homogeneous, the optimal value is proportional to the specified value  $X_{k_\ell k_\ell} \geq 0$ , i.e., equal to  $\tilde{\eta}_\ell(C_\ell, 1) X_{k_\ell k_\ell}$ , where  $\tilde{\eta}_\ell(C_\ell, 1)$  denotes the optimal value of the subproblem with  $X_{k_\ell k_\ell}$  specified to 1. Thus, the constraint (13) with  $q = \ell$  in the variable matrix  $\mathbf{X}_{C_\ell}$  can be eliminated from  $\overline{\text{COP}}(\mathbb{K})$  by replacing the objective function (14) with

$$\sum_{(i,j) \in F_{\ell-1}} Q_{ij}^0 X_{ij} + \tilde{\eta}_\ell(C_\ell, 1) X_{k_r k_r} = \sum_{(i,j) \in F_{\ell-1}} Q_{ij}^{0\ell-1} X_{ij},$$

where  $Q_{ij}^{0\ell-1} = Q_{ij}^0 + \tilde{\eta}_\ell(C_\ell, 1)$  if  $i = j = k_r$  and  $Q_{ij}^{0\ell-1} = Q_{ij}^0$  otherwise. (The resulting problem will be represented as  $P_{\ell-1}(\mathbb{K})$  in later discussion). We continue this elimination process and updating  $Q_{ij}^{0r}$  to  $Q_{ij}^{0r-1}$  for  $r = \ell - 1, \dots, 2$ , till we obtain

$$\inf \left\{ \begin{array}{l} \mathbf{X}_{C_1} \in \mathbb{K}^{C_1}, X_{11} = 1, \\ \sum_{(i,j) \in C_1 \times C_1} Q_{ij}^{01} X_{ij} : \begin{array}{l} \sum_{(i,j) \in C_1 \times C_1} Q_{ij}^p X_{ij} = 0 \ (p \in I_{\text{eq}}^1), \\ \sum_{(i,j) \in C_1 \times C_1} Q_{ij}^p X_{ij} \leq 0 \ (p \in I_{\text{ineq}}^1) \end{array} \end{array} \right\},$$

which has the same optimal value as  $\overline{\text{COP}}(\mathbb{K})$ . (The above problem corresponds to  $P_1(\mathbb{K})$  to be defined later).

In the above brief description of the elimination process, we have implicitly assumed that the subproblem with  $X_{k_r k_r}$  specified to 1 has an optimal solution, but it may be infeasible or unbounded. We need to deal with such cases. In the subsequent discussion, we also show how an optimal solution of  $\overline{\text{COP}}(\mathbb{K})$  is retrieved in detail.

To embed a recursive structure in  $\overline{\text{COP}}(\mathbb{K})$ , we introduce some notation. Let  $k_1 = 1$ , and let

$$\Phi_r(\mathbb{K}) = \left\{ \begin{array}{l} \mathbf{X}_{C_q} \in \mathbb{K}^{C_q} \ (q = 1, \dots, r), X_{11} = 1, \\ (\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_r}) : \begin{array}{l} \sum_{(i,j) \in C_q \times C_q} Q_{ij}^p X_{ij} = 0 \ (p \in I_{\text{eq}}^q, q = 1, \dots, r) \\ \sum_{(i,j) \in C_q \times C_q} Q_{ij}^p X_{ij} \leq 0 \ (p \in I_{\text{ineq}}^q, q = 1, \dots, r) \end{array} \end{array} \right\},$$

$$\Psi_r(\mathbb{K}^{C_r}, \lambda) = \left\{ \begin{array}{l} X_{k_r k_r} = \lambda, \\ \mathbf{X}_{C_r} \in \mathbb{K}^{C_r} : \begin{array}{l} \sum_{(i,j) \in C_r \times C_r} Q_{ij}^p X_{ij} = 0 \ (p \in I_{\text{eq}}^r) \\ \sum_{(i,j) \in C_r \times C_r} Q_{ij}^p X_{ij} \leq 0 \ (p \in I_{\text{ineq}}^r) \end{array} \end{array} \right\},$$

( $r = \ell, \dots, 1$ ). By (11), each  $\Phi_r(\mathbb{K})$  ( $r = \ell, \dots, 2$ ) can be represented as

$$\Phi_r(\mathbb{K}) = \left\{ (\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_r}) : \begin{array}{l} (\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_{r-1}}) \in \Phi_{r-1}(\mathbb{K}), \\ \mathbf{X}_{C_r} \in \Psi_r(\mathbb{K}^{C_r}, X_{k_r k_r}) \end{array} \right\}. \quad (15)$$

This recursive representation of  $\Phi_r(\mathbb{K})$  ( $r = \ell, \dots, 2$ ) plays an essential role in the discussions below.

We now construct a sequence of COPs:

$$P_r(\mathbb{K}): \quad \eta_r(\mathbb{K}) = \inf \left\{ \sum_{(i,j) \in F_r} Q_{ij}^{0r} X_{ij} : (\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_r}) \in \Phi_r(\mathbb{K}) \right\} \quad (16)$$

( $r = \ell, \dots, 1$ ) such that

$$\left. \begin{array}{l} \text{if } \overline{\mathbf{X}} \text{ is an optimal solution of } \overline{\text{COP}}(\mathbb{K}), \text{ then } (\overline{\mathbf{X}}_{C_1}, \dots, \overline{\mathbf{X}}_{C_q}) \text{ is} \\ \text{an optimal solution of } P_q(\mathbb{K}) \text{ with the optimal value } \eta_q(\mathbb{K}) = \zeta(\mathbb{K}) > -\infty \end{array} \right\}, \quad (17)$$

where  $Q_{ij}^{0r} \in \mathbb{R} \cup \{\infty\}$  ( $(i, j) \in F_q$ ) ( $r = \ell, \dots, 1$ ). We note that every  $Q_{ij}^{0q}$  is fixed to  $Q_{ij}^0 \in \mathbb{R}$  for  $i \neq j$ , but  $Q_{ii}^{0q} \in \mathbb{R} \cup \{\infty\}$  ( $(i, i) \in F_q$ ) ( $q = \ell, \dots, 1$ ) are updated in the sequence. If we assign  $Q_{ii}^{0q} = +\infty$  for some  $(i, i) \in F_q$ , then the objective quadratic function  $\sum_{(i,i) \in F_q} Q_{ij}^{0q} X_{ij}$  takes  $\infty$  at  $(\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_q}) \in \Phi_q(\mathbb{K})$  unless  $X_{ii} = 0$ . Thus, if  $Q_{ii}^{0q} = +\infty$ , then we must take  $X_{ii} = 0$  in  $P_q(\mathbb{K})$ .

As  $r$  decreases from  $\ell$  to 1 in the sequence, we obtain at the final iteration:

$$P_1(\mathbb{K}): \eta_1(\mathbb{K}) = \inf \left\{ \sum_{(i,j) \in C_1 \times C_1} Q_{ij}^{0r} X_{ij} : \mathbf{X}_{C_1} \in \Phi_1(\mathbb{K}) \right\},$$

which is equivalent to  $\overline{\text{COP}}(\mathbb{K})$ , *i.e.*, (17) holds for  $q = 1$ .

To construct the sequence  $P_r(\mathbb{K})$  ( $r = \ell, \dots, 1$ ), we first set

$$Q_{ij}^{0\ell} = Q_{ij}^0 \quad ((i, j) \in F_\ell). \quad (18)$$

Obviously,  $P_\ell(\mathbb{K})$  coincides with  $\overline{\text{COP}}(\mathbb{K})$ . Hence (17) holds for  $q = \ell$ . As the induction hypothesis, we assume that for  $r \in \{\ell, \dots, 2\}$ , the objective coefficients  $Q_{ij}^{0q} \in \mathbb{R}$  ( $(i, j) \in F_q$ ) of  $P_q(\mathbb{K})$  ( $q = \ell, \ell - 1, \dots, r$ ) have been computed so that (17) holds for  $q = \ell, \dots, r$ . To show how  $Q_{ij}^{0(r-1)} \in \mathbb{R}$  ( $(i, j) \in F_{r-1}$ ) of  $P_{r-1}(\mathbb{K})$  are computed so that (17) holds for  $q = r - 1$ , we consider the following subproblem of  $P_r(\mathbb{K})$ :

$$\tilde{P}_r(\mathbb{K}^{C_r}, \lambda): \tilde{\eta}_r(\mathbb{K}^{C_r}, \lambda) = \inf \{ G_r(\mathbf{X}_{C_r}) : \mathbf{X}_{C_r} \in \Psi_r(\mathbb{K}^{C_r}, \lambda) \}, \quad (19)$$

where  $G_r(\mathbf{X}_{C_r}) := \langle \mathbf{Q}_{C_r}^{0r}, \mathbf{X}_{C_r} \rangle - Q_{k_r k_r}^{0r} X_{k_r k_r} = \sum_{(i,j) \in (C_r \times C_r) \setminus \{(k_r, k_r)\}} Q_{ij}^{0r} X_{ij}$ , and  $\lambda \geq 0$

denotes a parameter. By (15), we observe that

$$\begin{aligned} \eta_r(\mathbb{K}) &= \inf \left\{ \sum_{(i,j) \in F_{r-1}} Q_{ij}^{0r} X_{ij} + G_r(\mathbf{X}_{C_r}) : \begin{array}{l} (\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_{r-1}}) \in \Phi_{r-1}(\mathbb{K}), \\ \mathbf{X}_{C_r} \in \Psi_r(\mathbb{K}^{C_r}, X_{k_r k_r}) \end{array} \right\} \\ &= \inf \left\{ \sum_{(i,j) \in F_{r-1}} Q_{ij}^{0r} X_{ij} + \inf \{ G_r(\mathbf{X}_{C_r}) : \mathbf{X}_{C_r} \in \Psi_r(\mathbb{K}^{C_r}, X_{k_r k_r}) \} : \right. \\ &\quad \left. (\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_{r-1}}) \in \Phi_{r-1}(\mathbb{K}) \right\} \\ &= \inf \left\{ \sum_{(i,j) \in F_{r-1}} Q_{ij}^{0r} X_{ij} + \tilde{\eta}_r(\mathbb{K}^{C_r}, X_{k_r k_r}) : (\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_{r-1}}) \in \Phi_{r-1}(\mathbb{K}) \right\} \quad (20) \\ &= \sum_{(i,j) \in F_{r-1}} Q_{ij}^{0r} \bar{X}_{ij} + \tilde{\eta}_r(\mathbb{K}^{C_r}, \bar{X}_{k_r k_r}) \\ &\quad \text{for every optimal solution } \bar{\mathbf{X}} \text{ of } \text{COP}(\mathbb{K}). \quad (21) \end{aligned}$$

Here the last equality follows from the induction assumption.

We now focus on the inner problem  $\tilde{P}_r(\mathbb{K}^{C_r}, X_{k_r k_r})$  with the optimal value  $\tilde{\eta}_r(\mathbb{K}^{C_r}, X_{k_r k_r})$ . Define

$$\tilde{\mathbf{X}}^{C_r} = \begin{cases} \text{an optimal solution of } \tilde{P}_r(\mathbb{K}^{C_r}, 1) & \text{if it exists,} \\ \mathbf{0} \in \mathbb{S}^{C_r} & \text{otherwise.} \end{cases} \quad (22)$$



**Lemma 4.1.** For a fixed  $r$ , assume that (17) holds for  $q = \ell, \dots, r$ . Let  $(\mathbf{X}_{C_1}^*, \dots, \mathbf{X}_{C_{r-1}}^*) \in \Phi_{r-1}(\mathbb{K})$  be an optimal solution of COP (20), i.e., there exists an optimal solution  $\mathbf{X}_{C_r}^*$  of  $\tilde{P}(\mathbb{K}^{C_r}, X_{k_r k_r}^*)$  with the optimal value  $\tilde{\eta}(\mathbb{K}^{C_r}, X_{k_r k_r}^*)$  such that

$$\sum_{(i,j) \in F_{r-1}} Q_{ij}^{0r} X_{ij}^* + \tilde{\eta}_r(\mathbb{K}^{C_r}, X_{k_r k_r}^*) = \eta_r(\mathbb{K}).$$

Then  $\mathbf{X}_{C_r} = X_{k_r k_r}^* \tilde{\mathbf{X}}_{C_r}$  is an optimal solution of  $\tilde{P}_r(\mathbb{K}^{C_r}, X_{k_r k_r}^*)$  with the optimal value  $X_{k_r k_r}^* \tilde{\eta}_r(\mathbb{K}^{C_r}, 1)$ , where  $0 \times \tilde{\eta}_r(\mathbb{K}^{C_r}, 1) = 0$  is assumed even when  $\tilde{\eta}_r(\mathbb{K}^{C_r}, 1) = \infty$ , and  $(\mathbf{X}_{C_1}^*, \dots, \mathbf{X}_{C_{r-1}}^*, X_{k_r k_r}^* \tilde{\mathbf{X}}_{C_r})$  is an optimal solution of  $P_r(\mathbb{K})$  with the optimal value  $\eta_r(\mathbb{K})$ .

*Proof.* We consider two cases separately:  $X_{k_r k_r}^* = 0$  and  $X_{k_r k_r}^* > 0$ . First, assume that  $X_{k_r k_r}^* = 0$ . In this case,  $\mathbf{X}'_{C_r}$  is a feasible solution of  $\tilde{P}_r(\mathbb{K}^{C_r}, 0)$  iff  $\lambda \mathbf{X}'_{C_r}$  is a feasible solution of  $\tilde{P}_r(\mathbb{K}^{C_r}, 0)$  for every  $\lambda > 0$ . This implies that  $\tilde{\eta}_r(\mathbb{K}^{C_r}, 0)$  is either 0 or  $-\infty$ . It follows from  $\eta_r(\mathbb{K}) > -\infty$  that  $\tilde{\eta}_r(\mathbb{K}^{C_r}, 0) = -\infty$  cannot occur. Hence  $\tilde{\eta}_r(\mathbb{K}^{C_r}, 0) = 0 = X_{k_r k_r}^* \tilde{\eta}_r(\mathbb{K}^{C_r}, 1)$  follows (even when  $\tilde{\eta}_r(\mathbb{K}^{C_r}, 1) = \infty$ ). We also see that  $\mathbf{X}_{C_r} = \mathbf{O} = X_{k_r k_r}^* \tilde{\mathbf{X}}_{C_r}$  is a trivial optimal solution with the objective value 0. Now assume that  $X_{k_r k_r}^* > 0$ . By assumption,  $\tilde{P}_r(\mathbb{K}^{C_r}, X_{k_r k_r}^*)$  has an optimal solution  $\mathbf{X}_{C_r}^*$ . It is easy to see that  $\mathbf{X}'_{C_r}$  is an optimal solution of  $\tilde{P}_r(\mathbb{K}^{C_r}, X_{k_r k_r}^*)$  with the objective value  $\xi$  iff  $\mathbf{X}'_{C_r} / X_{k_r k_r}^*$  is an optimal solution of  $\tilde{P}_r(\mathbb{K}^{C_r}, 1)$  with the objective value  $\xi / X_{k_r k_r}^*$ . We also see from the definition that  $\tilde{\mathbf{X}}_{C_r}$  is an optimal solution of  $\tilde{P}_r(\mathbb{K}^{C_r}, 1)$  with the objective value  $\tilde{\eta}_r(\mathbb{K}^{C_r}, 1)$ . Hence  $X_{k_r k_r}^* \tilde{\mathbf{X}}_{C_r}$  is an optimal solution of  $\tilde{P}_r(\mathbb{K}^{C_r}, X_{k_r k_r}^*)$  with the objective value  $X_{k_r k_r}^* \tilde{\eta}_r(\mathbb{K}^{C_r}, 1)$ . Finally, we observe that  $(\mathbf{X}_{C_1}^*, \dots, \mathbf{X}_{C_{r-1}}^*, X_{k_r k_r}^* \tilde{\mathbf{X}}_{C_r})$  is a feasible solution of  $P_r(\mathbb{K})$  with the objective value  $\eta_r(\mathbb{K})$ . Therefore it is an optimal solution of  $P_r(\mathbb{K})$ .  $\square$

By Lemma 4.1,  $\tilde{\eta}_r(\mathbb{K}^{C_r}, X_{k_r k_r}^*)$  in (20) can be replaced with  $X_{k_r k_r}^* \tilde{\eta}_r(\mathbb{K}^{C_r}, 1)$  and  $\tilde{\eta}_r(\mathbb{K}^{C_r}, \bar{X}_{k_r k_r})$  in (21) with  $\bar{X}_{k_r k_r} \tilde{\eta}_r(\mathbb{K}^{C_r}, 1)$ . Therefore, by defining

$$Q_{ij}^{0r-1} = \begin{cases} Q_{k_r k_r}^{0r} + \tilde{\eta}_r(\mathbb{K}^{C_r}, 1) & \text{if } (i, j) = (k_r, k_r), \\ Q_{ij}^{0r} & \text{otherwise,} \end{cases} \quad (23)$$

we obtain

$$\begin{aligned} \eta_r(\mathbb{K}) &= \inf \left\{ \sum_{(i,j) \in F_{r-1}} Q_{ij}^{0r-1} X_{ij} : (\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_{r-1}}) \in \Phi_{r-1}(\mathbb{K}) \right\} \\ &= \eta_{r-1}(\mathbb{K}) = \sum_{(i,j) \in F_{r-1}} Q_{ij}^{0r-1} \bar{X}_{ij} \text{ for every optimal solution } \bar{\mathbf{X}} \text{ of COP}(\mathbb{K}). \end{aligned}$$

(We note that if  $Q_{kk}^{0r-1} + \tilde{\eta}_r(\mathbb{K}^{C_r}, 1) = \infty$  occurs in (23), then  $\tilde{\mathbf{X}}_{C_r}$  is set to be  $\mathbf{O} \in \mathbb{S}^{C_r}$ ). Thus we have shown that (17) holds for  $q = r - 1$  under the assumption that (17) holds for  $q = \ell, \dots, r$ .

Consequently, we obtain the following theorem by induction with decreasing  $r$  from  $\ell$  to 2.

**Theorem 4.2.** Let  $\mathbb{K} \in \{\Gamma^n, \text{CPP}^n, \text{DNN}^n\}$ . Initialize the objective coefficients  $Q_{ij}^{0r}$  ( $(i, j) \in F_r$ ) of the sequence  $P_r(\mathbb{K})$  ( $r = \ell, \dots, 1$ ) by (18) for  $r = \ell$ , and update them by (23) for  $r = \ell, \dots, 2$ . Then each  $P_r(\mathbb{K})$  in the sequence is equivalent to  $\overline{\text{COP}}(\mathbb{K})$ , more precisely, (17) holds for  $q = \ell, \dots, 1$ .

## 4.2 An algorithm for solving $\overline{\text{COP}}(\mathbb{K})$

By Theorem 4.2, we know that  $\overline{\text{COP}}(\mathbb{K})$  in (12) is equivalent to  $P_1(\mathbb{K})$ , i.e.,  $\eta_1(\mathbb{K}) = \bar{\zeta}(\mathbb{K})$ . Hence, the optimal value  $\bar{\zeta}(\mathbb{K})$  of  $\overline{\text{COP}}(\mathbb{K})$  can be obtained by solving  $P_1(\mathbb{K})$ .

Lemma 4.1 suggests how to retrieve an optimal solution of  $\overline{\text{COP}}(\mathbb{K})$ . Let  $\mathbf{X}_{C_1}^*$  be an optimal solution of  $P_1(\mathbb{K})$ . For  $r \in \{2, \dots, \ell\}$ , assume that an optimal solution  $(\mathbf{X}_{C_1}^*, \dots, \mathbf{X}_{C_{r-1}}^*)$  of  $P_{r-1}(\mathbb{K})$  has been computed. Let  $\mathbf{X}_{C_r}^* = \widetilde{\mathbf{X}}_{C_r} X_{k_r k_r}^*$ , which is an optimal solution of  $\widetilde{P}_r(\mathbb{K}^{C_r}, X_{k_r k_r}^*)$  with the objective value  $\tilde{\eta}(\mathbb{K}^{C_r}, 1) X_{k_r k_r}^*$ . By Lemma 4.1,  $(\mathbf{X}_{C_1}^*, \dots, \mathbf{X}_{C_r}^*)$  is an optimal solution of  $P_r(\mathbb{K})$ . We can continue this procedure until an optimal solution of  $P_\ell(\mathbb{K})$  is obtained. Note that  $P_\ell(\mathbb{K})$  is equivalent to  $\overline{\text{COP}}(\mathbb{K})$ .

### Algorithm 4.3.

- Step 1: (Initialization for computing  $Q_{ij}^{0r}$  ( $(i, j) \in F_r, r = \ell, \dots, 1$ ) and  $\widetilde{\mathbf{X}}_{C_r}$  ( $r = \ell, \dots, 2$ )) Let  $r = \ell$  and  $Q_{ij}^{0r} = Q_{ij}^0$  ( $(i, j) \in F_\ell$ ).
- Step 2: If  $r = 1$ , go to Step 4. Otherwise, choose  $k_r \in C_r$  such that  $(C_1 \cup \dots \cup C_{r-1}) \cap C_r = \{k_r\}$ . Solve  $\widetilde{P}_r(\mathbb{K}^{C_r}, 1)$ . If  $\widetilde{P}_r(\mathbb{K}^{C_r}, 1)$  is infeasible, let  $\widetilde{\mathbf{X}}_{C_r} = \mathbf{O}$  and  $\tilde{\eta}_r(\mathbb{K}^{C_r}, 1) = \infty$ . Otherwise, let  $\widetilde{\mathbf{X}}_{C_r}$  be an optimal solution of  $\widetilde{P}_r(\mathbb{K}^{C_r}, 1)$  with the optimal value  $\tilde{\eta}_r(\mathbb{K}^{C_r}, 1)$ . Define  $Q_{ij}^{0r-1}$  ( $(i, j) \in F_{r-1}$ ) by (23).
- Step 3: Replace  $r$  by  $r - 1$  and go to Step 2.
- Step 4: (Initialization for computing an optimal solution of  $\overline{\text{COP}}(\mathbb{K})$ ) Solve  $P_1(\mathbb{K})$ . Let  $\bar{\zeta}(\mathbb{K}) = \eta_1(\mathbb{K})$  and  $\mathbf{X}_{C_1}^*$  be an optimal solution of  $P_1(\mathbb{K})$ . Let  $r = 1$ .
- Step 5: If  $r = \ell$ , then output the optimal value  $\bar{\zeta}(\mathbb{K})$  and an optimal solution  $(\mathbf{X}_{C_1}^*, \dots, \mathbf{X}_{C_\ell}^*)$  of  $\overline{\text{COP}}(\mathbb{K})$ . Otherwise go to Step 6.
- Step 6: Replace  $r$  by  $r + 1$ . Let  $\mathbf{X}_{C_r}^* = X_{k_r k_r}^* \widetilde{\mathbf{X}}_{C_r}$ . Go to Step 5.

**Remark 4.4.** In Steps 1 through 3 of Algorithm 4.3, the problems  $\widetilde{P}_r(\mathbb{K}^{C_r}, 1)$  are solved sequentially from  $\ell$  to 2. This sequential order has been determined by the running intersection property (11), which is induced from a clique tree of the block-clique graph  $G(N, E)$ . Recall that  $G(N, E)$  represents the aggregated and correlative sparsity of  $\text{COP}(\mathbb{K})$ . If  $C_r$  other than  $C_\ell$  is also a leaf node of the clique-tree, then the problem  $\widetilde{P}_r(\mathbb{K}^{C_r}, 1)$  can be solved independently from the other problems and the objective coefficient  $Q_{k_r k_r}^{0r}$  can be updated to  $Q_{k_r k_r}^{0r-1}$  by (23), where it is assumed that having chosen  $C_1$  as the root node of the clique tree naturally determines all leaf nodes. In fact, the problems associated with leaf nodes of the clique-tree can be solved in parallel. Furthermore, if we remove (some of) those nodes from the clique tree after solving their associated problems and updating the coefficients of the corresponding objective function by (23), then new leaf nodes may appear. Then, the procedure of solving the problems associated with those new leaf nodes in parallel and updating the objective coefficients can be repeatedly applied until we solve  $P_1(\mathbb{K})$ .

### 4.3 Equivalence of $\overline{\text{COP}}(\mathbb{K}_1)$ and $\overline{\text{COP}}(\mathbb{K}_2)$ for $\mathbb{K}_1, \mathbb{K}_2 \in \{\Gamma^n, \text{CPP}^n, \text{DNN}^n\}$

Let  $\mathbb{K}_1, \mathbb{K}_2 \in \{\Gamma^n, \text{CPP}^n, \text{DNN}^n\}$ . For each of  $s = \{1, 2\}$ , initialize the objective coefficients  $Q_{ij}^{0r}$  ( $(i, j) \in F_r$ ) of the sequence of  $P_r(\mathbb{K}_s)$  ( $r = \ell, \dots, 1$ ) by (18) for  $r = \ell$ , and update them by (23) for  $r = \ell, \dots, 2$ . Then  $\tilde{P}_r(\mathbb{K}_1^{C_r}, 1)$  and  $\tilde{P}_r(\mathbb{K}_2^{C_r}, 1)$  share the same linear equality and inequality constraints, although the cone constraints  $\mathbf{X}_{C_r} \in \mathbb{K}_1^{C_r}$  and  $\mathbf{X}_{C_r} \in \mathbb{K}_2^{C_r}$  differ. Moreover, if  $r = \ell$ , the objective coefficients of  $\tilde{P}_r(\mathbb{K}_1^{C_r}, 1)$  and  $\tilde{P}_r(\mathbb{K}_2^{C_r}, 1)$  are the same as  $\mathbf{Q}_{C_\ell}^0$ . Thus, it is natural to investigate the equivalence between  $\tilde{P}_\ell(\mathbb{K}_1^{C_\ell}, 1)$  and  $\tilde{P}_\ell(\mathbb{K}_2^{C_\ell}, 1)$  or whether  $\tilde{\eta}_\ell(\mathbb{K}_1^{C_\ell}) = \tilde{\eta}_\ell(\mathbb{K}_2^{C_\ell})$  holds. Let  $r \in \{\ell, \dots, 2\}$ . Assume that the objective coefficients of  $\tilde{P}_q(\mathbb{K}_1^{C_q}, 1)$  and  $\tilde{P}_q(\mathbb{K}_2^{C_q}, 1)$  are the same, and they are equivalent, *i.e.*,  $\tilde{\eta}_q(\mathbb{K}_1^{C_q}) = \tilde{\eta}_q(\mathbb{K}_2^{C_q})$  holds ( $q = \ell, \dots, r$ ). Then  $\tilde{P}_{r-1}(\mathbb{K}_1^{C_{r-1}}, 1)$  and  $\tilde{P}_{r-1}(\mathbb{K}_2^{C_{r-1}}, 1)$  share common objective coefficients and constraints except for the cone constraints  $\mathbf{X}_{C_{r-1}} \in \mathbb{K}_1^{C_{r-1}}$  and  $\mathbf{X}_{C_{r-1}} \in \mathbb{K}_2^{C_{r-1}}$ . Thus, the pair of  $\tilde{P}_r(\mathbb{K}_1^{C_r}, 1)$  and  $\tilde{P}_r(\mathbb{K}_2^{C_r}, 1)$  can be compared for their equivalence recursively from  $r = \ell$  to  $r = 2$ . When all the pairs are equivalent, we can conclude by Theorem 4.2 that  $\eta_1(\mathbb{K}_1^{C_1}) = \bar{\zeta}(\mathbb{K}_1)$  and  $\eta_1(\mathbb{K}_2^{C_1}) = \bar{\zeta}(\mathbb{K}_2)$ . We also see that  $P_1(\mathbb{K}_1)$  and  $P_1(\mathbb{K}_2)$  share common objective coefficients. Consequently, the question on whether  $\overline{\text{COP}}(\mathbb{K}_1)$  and  $\overline{\text{COP}}(\mathbb{K}_2)$  are equivalent is reduced to the question on whether the pair of  $\tilde{P}_r(\mathbb{K}_1^{C_r}, 1)$  and  $\tilde{P}_r(\mathbb{K}_2^{C_r}, 1)$  is equivalent ( $r = \ell, \dots, 2$ ) and whether the pair of  $P_1(\mathbb{K}_1)$  and  $P_1(\mathbb{K}_2)$  is equivalent.

For the convenience of the subsequent discussion, we introduce the sequence of the following COPs:

$$\begin{aligned} \tilde{P}'_r(\mathbb{K}^{C_r}) : \quad \tilde{\eta}'_r(\mathbb{K}^{C_r}) &= \inf \{ \langle \mathbf{Q}_{C_r}^{0r}, \mathbf{X} \rangle : \mathbf{X} \in \Psi_r(\mathbb{K}^{C_r}, 1) \} \\ &= \inf \left\{ \langle \mathbf{Q}_{C_r}^{0r}, \mathbf{X} \rangle : \begin{array}{l} \mathbf{X} \in \mathbb{K}^{C_r}, X_{k_r k_r} = 1, \\ \langle \mathbf{Q}_{C_r}^p, \mathbf{X}_{C_r} \rangle = 0 \quad (p \in I_{\text{eq}}^r), \\ \langle \mathbf{Q}_{C_r}^p, \mathbf{X}_{C_r} \rangle \leq 0 \quad (p \in I_{\text{ineq}}^r) \end{array} \right\}. \end{aligned}$$

( $r = \ell, \dots, 1$ ). For each  $r \in \{\ell, \dots, 2\}$ ,  $\tilde{P}'_r(\mathbb{K}^{C_r})$  and  $\tilde{P}_r(\mathbb{K}^{C_r}, 1)$  share a common feasible region  $\Psi_r(\mathbb{K}^{C_r}, 1)$  and their objective values differ by a constant  $Q_{k_r k_r}^{0r}$  for every  $\mathbf{X}_{C_r} \in \Psi_r(\mathbb{K}^{C_r}, 1)$ , which shows that they are essentially the same. For  $r = 1$ ,  $\tilde{P}'_1(\mathbb{K}^{C_1})$  coincides with  $P_1(\mathbb{K})$ . Thus,  $\tilde{P}_r(\mathbb{K}^{C_r}, 1)$  ( $r = \ell, \dots, 2$ ) and  $P_1(\mathbb{K})$  can be dealt with as  $\tilde{P}'_r(\mathbb{K}^{C_r})$  ( $r = \ell, \dots, 1$ ). In particular, the question on the equivalence of  $\overline{\text{COP}}(\mathbb{K}_1)$  and  $\overline{\text{COP}}(\mathbb{K}_2)$  can be stated as whether the pair of  $\tilde{P}'_r(\mathbb{K}_1^{C_r})$  and  $\tilde{P}'_r(\mathbb{K}_2^{C_r})$  are equivalent for  $r = \ell, \dots, 1$ .

Summarizing the discussions above, we obtain the following results.

**Theorem 4.5.** *Let  $\mathbb{K}_s \in \{\Gamma^n, \text{CPP}^n, \text{DNN}^n\}$  ( $s = 1, 2$ ) and  $\mathbb{K}_1 \neq \mathbb{K}_2$ . For each  $s = 1, 2$ , initialize the objective coefficients  $Q_{ij}^{0r}$  ( $(i, j) \in F_r$ ) of the sequence  $P_r(\mathbb{K}_s)$  ( $r = \ell, \dots, 1$ ) by (18) for  $r = \ell$ , and update them by (23) for  $r = \ell, \dots, 2$ . Assume that  $\tilde{\eta}'_r(\mathbb{K}_1^{C_r}) = \tilde{\eta}'_r(\mathbb{K}_2^{C_r})$  ( $r = \ell, \dots, 1$ ). Then  $\bar{\zeta}(\mathbb{K}_1) = \bar{\zeta}(\mathbb{K}_2)$ .*

### 4.4 Equivalence of $\tilde{P}'_r(\mathbb{K}_1)$ and $\tilde{P}'_r(\mathbb{K}_2)$ for $\mathbb{K}_1, \mathbb{K}_2 \in \{\Gamma^n, \text{CPP}^n, \text{DNN}^n\}$

**Theorem 4.6.** *Let  $r \in \ell, \dots, 1$  be fixed.*

- (i) Assume that the aggregated sparsity of the data matrices  $\mathbf{Q}_{C_r}^{0r}$  and  $\mathbf{Q}_{C_r}^p$  ( $p \in I_{\text{eq}}^r \cup I_{\text{ineq}}^r$ ) is represented by a block-clique graph  $G(C, E_C)$  with the maximal cliques of size at most 4. Then  $\tilde{\eta}'_r(\text{CPP}^{C_r}) = \tilde{\eta}'_r(\text{DNN}^{C_r})$ .
- (ii) Assume that  $I_{\text{ineq}}^r = \emptyset$ ,  $\Psi_r(\mathbf{\Gamma}^{C_r}, 1) \neq \emptyset$ ,  $\langle \mathbf{Q}_{C_r}^{0r}, \mathbf{X} \rangle \geq 0$  if  $\mathbf{X}_{C_r} \in \Psi_r(\mathbf{\Gamma}^{C_r}, 0)$  and  $\mathbf{Q}_{C_r}^p \in \mathbb{S}_+^{C_r} + \mathbb{N}^{C_r}$  (the dual of  $\text{DNN}^{C_r}$ ). Then  $\tilde{\eta}'_r(\mathbf{\Gamma}^{C_r}) = \tilde{\eta}'_r(\text{CPP}^{C_r})$ .
- (iii) If the assumptions in (i) and (ii) above are satisfied, then  $\tilde{\eta}'_r(\mathbf{\Gamma}^{C_r}) = \tilde{\eta}'_r(\text{CPP}^{C_r}) = \tilde{\eta}'_r(\text{DNN}^{C_r})$ .
- (iv) Let  $\tilde{C}_r = C_r \setminus \{k_r k_r\}$ . Assume that  $\mathbf{Q}_{\tilde{C}_r}^{0r} \in \mathbb{S}_+^{\tilde{C}_r}$ ,  $\mathbf{Q}_{C_r}^p \in \mathbb{S}_+^{C_r}$  ( $p \in I_{\text{eq}}^r$ ) and  $\mathbf{Q}_{\tilde{C}_r}^p \in \mathbb{S}_+^{\tilde{C}_r}$  ( $p \in I_{\text{ineq}}^r$ ). Then  $\tilde{\eta}'_r(\mathbf{\Gamma}^{C_r}) = \tilde{\eta}'_r(\text{CPP}^{C_r}) = \tilde{\eta}'_r(\text{DNN}^{C_r})$ .
- (v) Assume that the data matrices  $\mathbf{Q}_{C_r}^{0r}$  and  $\mathbf{Q}_{C_r}^p$  ( $p \in I_{\text{eq}}^r \cup I_{\text{ineq}}^r$ ) are all diagonal. Then  $\tilde{\eta}'_r(\mathbf{\Gamma}^{C_r}) = \tilde{\eta}'_r(\text{CPP}^{C_r}) = \tilde{\eta}'_r(\text{DNN}^{C_r})$ .

*Proof.* (i), (ii) and (iv) follow directly from (ii) of Theorem 3.1, Lemma 2.3 and Lemma 2.5, respectively. (i) and (ii) imply (iii). (v) follows from the discussion in Remark 3.2 (b).  $\square$

We note that the aggregated sparsity of the updated objective coefficient  $\mathbf{Q}_{C_r}^{0r}$  is the same as the original objective coefficient  $\mathbf{Q}_{C_r}^0$  since only the diagonal elements  $Q_{k_r k_r}^0$  ( $r = \ell, \dots, 2$ ) are updated by (23). Thus the assumption in (i) can be verified from the original data matrices  $\mathbf{Q}_{C_r}^0$  and  $\mathbf{Q}_{C_r}^p$  ( $p \in I_{\text{eq}}^r \cup I_{\text{ineq}}^r$ ) before solving  $\overline{\text{COP}}(\text{DNN}^n)(\mathbb{K})$  by Algorithm 4.3. On the other hand, the assumptions “ $\langle \mathbf{Q}_{C_r}^{0r}, \mathbf{X} \rangle \geq 0$  if  $\mathbf{X}_{C_r} \in \Psi_r(\mathbf{\Gamma}^{C_r}, 0)$ ” in (ii) and  $\mathbf{Q}_{\tilde{C}_r}^{0r} \in \mathbb{S}_+^{\tilde{C}_r}$  in (iv) depend on  $\mathbf{Q}_{C_r}^{0r}$  ( $r = \ell, \dots, 1$ ) in general. If we know that  $\Psi_r(\mathbb{K}^{C_r}, 0) = \{\mathbf{O}\}$ , then “ $\langle \mathbf{Q}_{C_r}^{0r}, \mathbf{X} \rangle \geq 0$  if  $\mathbf{X}_{C_r} \in \Psi_r(\mathbf{\Gamma}^{C_r}, 0)$ ” obviously holds. In this case, the assumptions in (ii) can be verified from the original data matrices. On the other hand, each element  $Q_{ij}^{0r}$  of the matrix  $\mathbf{Q}_{\tilde{C}_r}^{0r}$  is determined recursively by (23) such that

$$\left. \begin{aligned} Q_{ij}^{0\ell} &= Q_{ij}^0, \\ Q_{ij}^{0q-1} &= \begin{cases} Q_{k_q k_q}^{0q} + \tilde{\eta}_q(\mathbb{K}^{C_q}, 1) & \text{if } (i, j) = (k_q, k_q), \\ Q_{ij}^{0q} & \text{otherwise} \end{cases} \end{aligned} \right\} \quad (q = \ell, \dots, r). \quad (24)$$

As a result, only some of the diagonal elements of  $\mathbf{Q}_{\tilde{C}_r}^{0r}$  can differ from  $\mathbf{Q}_{\tilde{C}_r}^0$ . For example, if  $\mathbf{Q}_{\tilde{C}_r}^0$  is positive semidefinite and  $\mathbf{Q}_{C_q}^0 \in \mathbb{N}^{C_q}$  for all  $q = \ell, \dots, r+1$ , then  $\mathbf{Q}_{\tilde{C}_r}^{0r}$  is guaranteed to be positive semidefinite, since then  $\tilde{\eta}_q(\mathbb{K}^{C_q}) \geq 0$  ( $q = \ell, \dots, r+1$ ). In such a case, the assumptions in (iv) can be verified from the original data matrices. By (24), if the data matrices  $\mathbf{Q}_{C_r}^0$  and  $\mathbf{Q}_{C_r}^p$  ( $p \in I_{\text{eq}}^r \cup I_{\text{ineq}}^r$ ) are all diagonal, then the assumption of (v) is satisfied.

## 5 Examples of QOPs

We present two examples of QOPs that can be reformulated as their DNN relaxations. The problem in Section 5.1 is a nonconvex QOP with linear and complementarity constraints, and the one in Section 5.2 is a partially convex QOP with quadratic inequality constraints. They are constructed as follows. First, choose a block-clique graph  $G(N, E)$

with the maximal cliques  $C_q$  ( $q = 1, \dots, \ell$ ). For the first example in Section 5, we use the block-clique graph in Figure 1 (a), and for the second example in Section 5.2, the one in Figure 1 (b). As the block-clique graph  $G(N, E)$  induces a clique tree (see Figure 2 (a) and see Figure 2 (b), respectively), we renumber its  $\ell$  maximal cliques so that they can satisfy the running intersection property (3).

Using the definitions of  $\Phi_r(\mathbb{K})$ ,  $\Psi_r(\mathbb{K}, \lambda)$  ( $r = \ell, \dots, 1$ ) and (15), we can rewrite  $\overline{\text{COP}}(\mathbb{K})$  as

$$\begin{aligned} \bar{\zeta}(\mathbb{K}) &= \eta_\ell(\mathbb{K}) = \inf \left\{ \sum_{(i,j) \in F_\ell} Q_{ij}^0 X_{ij} : (\mathbf{X}_{C_1}, \dots, \mathbf{X}_{C_\ell}) \in \Phi_\ell(\mathbb{K}) \right\} \\ &= \inf \left\{ \sum_{(i,j) \in F_\ell} Q_{ij}^0 X_{ij} : \mathbf{X}_{C_1} \in \Phi_1(\mathbb{K}), \mathbf{X}_{C_r} \in \Psi(\mathbb{K}^{C_r}, X_{k_r k_r}) \ (r = \ell, \dots, 2) \right\} \\ &= \inf \left\{ \sum_{(i,j) \in F_\ell} Q_{ij}^0 X_{ij} : \mathbf{X}_{C_r} \in \Psi_r(\mathbb{K}^{C_r}, X_{k_r k_r}) \ (r = \ell, \dots, 1), X_{11} = 1 \right\} \end{aligned}$$

Instead of  $\overline{\text{COP}}(\mathbb{K})$  itself, we describe the problem with its subproblems

$$\hat{P}_r(\mathbb{K}^{C_r}, \lambda) : \hat{\eta}_r(\mathbb{K}^{C_r}, \lambda) = \inf \{ \langle \mathbf{Q}_{C_r}^0, \mathbf{X}_{C_r} \rangle : \mathbf{X}_{C_r} \in \Psi_r(\mathbb{K}^{C_r}, \lambda) \}$$

using

$$C_r : k_r \in C_r, \mathbf{Q}_{C_r}^0 \in \mathbb{S}^{C_r} \text{ (and its property if any) and } \Psi_r(\mathbb{K}^{C_r}, \lambda)$$

for ( $r = 1, \dots, \ell$ ), where  $\lambda \geq 0$  denotes a parameter. Notice that  $\hat{P}(\mathbb{K}^{C_r}, \lambda)$  is similar to the problem  $\hat{P}_r(\mathbb{K}^{C_r}, \lambda)$  introduced in Section 4.1. We also note that the objective coefficient  $\mathbf{Q}_{C_r}^{0r}$  is updated from  $\mathbf{Q}_{C_r}^0$  in the construction of the sequence  $P_r(\mathbb{K})$  ( $r = \ell, \dots, 1$ ) by (23), while the objective coefficient of  $\hat{P}(\mathbb{K}^{C_r}, \lambda)$  is fixed to  $\mathbf{Q}_{C_r}^0$ . Indeed, the description  $\hat{P}(\mathbb{K}^{C_r}, \lambda)$  is sufficient and more convenient to execute Algorithm 4.3 for computing the optimal value  $\bar{\zeta}(\mathbb{K})$  of  $\overline{\text{COP}}(\mathbb{K})$  and also to apply Theorems 4.5 and 4.6 to establish  $\zeta(\mathbf{\Gamma}^n) = \zeta(\text{CPP}^n) = \zeta(\text{DNN}^n)$ .

## 5.1 A QOP with linear and complementarity constraints

Consider the block-clique graph  $G(N, E)$  given in Figure 1 (a). In this case,  $n = 11$  and  $N = \{1, \dots, 11\}$ . To represent a QOP as  $\overline{\text{COP}}(\mathbf{\Gamma}^n)$ , let

$$\begin{aligned} C_1 = \{1, 2, 3, 4\} & : k_1 = 1, \mathbf{Q}_{C_1}^0 \in \mathbb{S}^{C_1}, \\ & \Psi_1(\mathbf{\Gamma}^{C_1}, \lambda) = \left\{ \mathbf{x}_{C_1} \mathbf{x}_{C_1}^T : \begin{array}{l} \mathbf{x}_{C_1} \in \mathbb{R}_+^{C_1}, x_1 = \lambda, \\ -8x_1 + x_2 + x_3 + 2x_4 = 0 \end{array} \right\}, \\ C_2 = \{3, 5, 6, 7\} & : k_2 = 3, \mathbf{Q}_{C_2}^0 \in \mathbb{S}^{C_2}, \\ & \Psi_1(\mathbf{\Gamma}^{C_2}, \lambda) = \left\{ \mathbf{x}_{C_2} \mathbf{x}_{C_2}^T : \begin{array}{l} \mathbf{x}_{C_2} \in \mathbb{R}_+^{C_2}, x_3 = \lambda, x_5 x_6 = 0, \\ -x_3 + x_5 + 2x_6 + x_7 = 0 \end{array} \right\}, \end{aligned}$$

$$\begin{aligned}
C_3 = \{3, 8, 9\} & : k_3 = 3, \mathbf{Q}_{C_3}^0 \in \mathbb{S}^{C_3}, \\
\Psi_1(\mathbf{\Gamma}^{C_3}, \lambda) & = \left\{ \mathbf{x}_{C_3} \mathbf{x}_{C_3}^T : \begin{array}{l} \mathbf{x}_{C_3} \in \mathbb{R}_+^{C_3}, x_3 = \lambda, \\ -2x_3 + 3x_8 + x_9 = 0 \end{array} \right\}, \\
C_4 = \{5, 10, 11\} & : k_4 = 5, \mathbf{Q}_{C_4}^0 \in \mathbb{S}^{C_4}, \\
\Psi_1(\mathbf{\Gamma}^{C_4}, \lambda) & = \left\{ \mathbf{x}_{C_4} \mathbf{x}_{C_4}^T : \begin{array}{l} \mathbf{x}_{C_4} \in \mathbb{R}_+^{C_4}, x_5 = \lambda, \\ -x_5 + x_{10} + x_{11} = 0 \end{array} \right\}.
\end{aligned}$$

The assumption in (i) of Theorem 4.6 is satisfied since  $C_r$  ( $r = 1, 2, 3, 4$ ) are of size at most 4. All assumptions in (ii) of Theorem 4.6 are also satisfied. In fact,  $I_{\text{ineq}}^r = \emptyset$ ,  $\Psi_r(\mathbf{\Gamma}^{C_r}, 1) \neq \emptyset$  and  $\Psi_r(\mathbf{\Gamma}^{C_r}, 0) = \{\mathbf{O}\}$ , which implies that  $\langle \mathbf{Q}_{C_r}^{0r}, \mathbf{X} \rangle \geq 0$  if  $\mathbf{X}_{C_r} \in \Psi_r(\mathbf{\Gamma}^{C_r}, 0)$ , ( $r = 1, 2, 3, 4$ ). As shown in Example 2.4, each linear equality constraint can be rewritten in  $\Psi_r(\mathbf{\Gamma}^{C_r}, \lambda)$  ( $r = 1, 2, 3, 4$ ) as  $\langle \mathbf{Q}_{C_r}, \mathbf{x}_{C_r} \mathbf{x}_{C_r}^T \rangle = 0$  for some  $\mathbf{Q}_{C_r} \in \mathbb{S}_+^{C_r}$  and the complementarity constraint  $x_5 x_6 = 0$  in  $\Psi_2(\mathbf{\Gamma}^{C_2}, \lambda)$  as  $\langle \mathbf{Q}_{C_2}, \mathbf{x}_{C_2} \mathbf{x}_{C_2}^T \rangle = 0$  for some  $\mathbf{Q}_{C_2} \in \mathbb{N}^{C_2}$ . By (i) and (ii) of Theorem 4.6,  $\tilde{\eta}'_r(\mathbf{\Gamma}^{C_r}) = \tilde{\eta}'_r(\text{CPP}^{C_r}) = \tilde{\eta}'_r(\text{DNN}^{C_r})$  ( $r = 1, 2, 3, 4$ ). Therefore, by Theorem 4.5, it follows that  $\bar{\zeta}(\mathbf{\Gamma}^n) = \bar{\zeta}(\text{CPP}^n) = \bar{\zeta}(\text{DNN}^n)$  holds.

## 5.2 A partially convex QOP

Consider the block-clique graph  $G(N, E)$  given in Figure 1 (b). In this case,  $n = 13$  and  $N = \{1, \dots, 13\}$ . To represent a QOP as  $\overline{\text{COP}}(\mathbf{\Gamma}^n)$ , let

$$\begin{aligned}
C_1 = \{1, 2, 3\} & : k_1 = 1, \mathbf{Q}_{C_1}^0 \in \mathbb{S}^{C_1}, \\
\Psi_1(\mathbf{\Gamma}^{C_1}, \lambda) & = \left\{ \mathbf{x}_{C_1} \mathbf{x}_{C_1}^T : \begin{array}{l} \mathbf{x}_{C_1} \in \mathbb{R}_+^{C_1}, x_1 = \lambda, \\ -x_1 + x_2 + x_3 = 0 \end{array} \right\}, \\
C_2 = \{3, 4, 5, 6, 7\} & : k_2 = 3, \mathbf{Q}_{C_2}^p \in \mathbb{S}^{C_2} : \text{diagonal } (p = 0, 1, 2), \\
\Psi_2(\mathbf{\Gamma}^{C_2}, \lambda) & = \left\{ \begin{array}{l} \mathbf{x}_{C_2} \mathbf{x}_{C_2}^T : \begin{array}{l} \mathbf{x}_{C_2} \in \mathbb{R}_+^{C_2}, x_3 = \lambda, \\ \langle \mathbf{Q}_{C_2}^1, \mathbf{x}_{C_2} \mathbf{x}_{C_2}^T \rangle = 0 \\ \langle \mathbf{Q}_{C_2}^2, \mathbf{x}_{C_2} \mathbf{x}_{C_2}^T \rangle \leq 0 \end{array} \end{array} \right\} \neq \emptyset, \\
C_3 = \{3, 8, 9, 10, 11\} & : k_3 = 3, \mathbf{Q}_{C_3}^p \in \mathbb{S}^{C_3}, \tilde{C}_3 = \{8, 9, 10, 11\}, \mathbf{Q}_{\tilde{C}_3}^p \in \mathbb{S}_+^{\tilde{C}_3} (p = 0, 3, 4), \\
\Psi_3(\mathbf{\Gamma}^{C_3}, \lambda) & = \left\{ \begin{array}{l} \mathbf{x}_{C_3} \mathbf{x}_{C_3}^T : \begin{array}{l} \mathbf{x}_{C_3} \in \mathbb{R}_+^{C_3}, x_3 = \lambda, \\ -x_3 - 2x_8 + x_9 + x_{10} = 0, \\ \langle \mathbf{Q}_{C_3}^p, \mathbf{x}_{C_3} \mathbf{x}_{C_3}^T \rangle \leq 0 (p = 3, 4) \end{array} \end{array} \right\} \neq \emptyset, \\
C_4 = \{11, 12, 13\} & : k_4 = 11, \mathbf{Q}_{C_4}^0 \in \mathbb{S}^{C_4}, \\
\Psi_4(\mathbf{\Gamma}^{C_4}, \lambda) & = \left\{ \mathbf{x}_{C_4} \mathbf{x}_{C_4}^T : \begin{array}{l} \mathbf{x}_{C_4} \in \mathbb{R}_+^{C_4}, x_{11} = \lambda, \\ -x_{11} + x_{12} + 2x_{13} = 0 \end{array} \right\}.
\end{aligned}$$

For  $r = 1, 4$ , we similarly see that all assumptions in (i) and (ii) of Theorem 4.6 are satisfied as in the example in Section 5.1. Thus, the identity  $\tilde{\eta}_r(\mathbf{\Gamma}^{C_r}) = \tilde{\eta}_r(\text{CPP}^{C_r}) = \tilde{\eta}_r(\text{DNN}^{C_r})$  holds ( $r = 1, 4$ ).

Since the data matrices  $\mathbf{Q}_{C_2}^p \in \mathbb{S}^{C_2}$  ( $p = 0, 1, 2$ ) are diagonal,  $\widehat{\text{P}}_2(\mathbb{K}^{C_2}, \lambda)$  becomes an

LP of the form

$$\hat{\eta}_r(\mathbb{K}^{C_2}, \lambda) = \inf \left\{ \sum_{i \in C_2} Q_{ii}^0 X_{ii} : \begin{array}{l} X_{ii} \in \mathbb{R}_+ \ (i \in C_2), \ X_{33} = \lambda, \\ \sum_{i \in C_2} Q_{ii}^1 X_{ii} = 0, \ \sum_{i \in C_2} Q_{ii}^1 X_{ii} \leq 0 \end{array} \right\}.$$

Hence (v) of Theorem 4.6,  $\tilde{\eta}_2(\mathbf{\Gamma}^{C_2}) = \tilde{\eta}_2(\mathbb{C}\mathbb{P}\mathbb{P}^{C_2}) = \tilde{\eta}_2(\mathbb{D}\mathbb{N}\mathbb{N}^{C_2})$  holds.

To see whether the identity holds for  $r = 3$ , we need to apply (iv) of Theorem 4.6 since  $\Psi_3(\mathbb{K}^{C_3}, \lambda)$  involves convex quadratic inequality. It suffices to check whether  $\mathbf{Q}_{\tilde{C}_3}^{03}$  is positive semidefinite. By the assumption and the updating formula (24), we know that

$$Q_{ij}^{03} = \begin{cases} Q_{11,11}^0 + \tilde{\eta}_4(\mathbf{\Gamma}^{C_4}) & \text{if } (i, j) = (11, 11), \\ Q_{ij}^0 & \text{otherwise.} \end{cases}$$

For example, if  $\tilde{\eta}_4(\mathbf{\Gamma}^{C_4})$  is nonnegative, then  $\mathbf{Q}_{\tilde{C}_3}^{03} \in \mathbb{S}_+^{\tilde{C}_3}$  and the identity  $\tilde{\eta}_3(\mathbf{\Gamma}^{C_3}) = \tilde{\eta}_3(\mathbb{C}\mathbb{P}\mathbb{P}^{C_3}) = \tilde{\eta}_3(\mathbb{D}\mathbb{N}\mathbb{N}^{C_3})$  holds. Therefore, by Theorem 4.5, it follows that  $\bar{\zeta}(\mathbf{\Gamma}^n) = \bar{\zeta}(\mathbb{C}\mathbb{P}\mathbb{P}^n) = \bar{\zeta}(\mathbb{D}\mathbb{N}\mathbb{N}^n)$  holds.

## 6 Concluding remarks

Nonconvex QOPs and CPP problems are known to be NP hard and/or numerically intractable in general, as opposed to computationally tractable DNN problems. Thus, finding some classes of QOPs or CPP problems that are equivalent to DNN problems is an essential problem in the study of the theory and applications of nonconvex QOPs. Two major obstacles to finding such classes are: (A)  $\mathbb{C}\mathbb{P}\mathbb{P}^n$  is a proper subset of  $\mathbb{D}\mathbb{N}\mathbb{N}^n$  if  $n \geq 5$  and (B) general quadratically constrained nonconvex QOPs are NP hard and numerically intractable. As a result, CPP reformulations of a class of QOPs with linear equality, binary and complementarity constraints in nonnegative variables still remain numerically intractable. One way to overcome these obstacles is to “decompose” the cone  $\mathbb{C}\mathbb{P}\mathbb{P}^n$  into cones with size at most 4 and/or to “decompose” a QOP into convex QOPs of any size and linearly constrained nonconvex QOPs with variables at most 4. To obtain such decompositions, a fundamental method is exploiting structured sparsity.

To describe a QOP, its CPP and DNN relaxations,  $\text{COP}(\mathbb{K})$  with  $\mathbb{K} \in \{\mathbf{\Gamma}^n, \mathbb{C}\mathbb{P}\mathbb{P}^n, \mathbb{D}\mathbb{N}\mathbb{N}^n\}$  has been introduced in Section 1. In Section 3, we have provided a method to decompose the cone  $\mathbb{C}\mathbb{P}\mathbb{P}^n$  of the CPP relaxation,  $\text{COP}(\mathbb{C}\mathbb{P}\mathbb{P}^n)$  of the QOP described as  $\text{COP}(\mathbf{\Gamma}^n)$  into cones with size at most 4 by exploiting the aggregated sparsity of the data matrices  $\mathbf{Q}^p$  ( $p \in \{\mathbf{0}\} \cup I_{rmeq} \cup I_{ineq}$ ) of  $\text{COP}(\mathbb{C}\mathbb{P}\mathbb{P}^n)$  which have been represented by a block-clique graph  $G(N, E)$ . In Section 4, a method to decompose the QOP itself into convex QOPs of any size and linearly constrained nonconvex QOPs with variables at most 4 has been presented by exploiting their correlative sparsity.

As for the structured sparsity that leads to the equivalence among  $\text{COP}(\mathbf{\Gamma}^n)$ ,  $\text{COP}(\mathbb{C}\mathbb{P}\mathbb{P}^n)$  and  $\text{COP}(\mathbb{D}\mathbb{N}\mathbb{N}^n)$ , the aggregated and/or correlative sparsity of the data matrices represented with a block-clique graph have played a crucial role in our discussion. We should mention that block-clique graphs may not be frequently observed in a wide class of QOPs.

It is interesting, however, to construct a new optimization model based on the structure provided by a block-clique graph.

For further development of such an optimization model, we emphasize that if a QOP described as  $\text{COP}(\Gamma^n)$  can be solved exactly then it can be incorporated in the model as a subproblem. In [15], it was shown that the exact solutions of nonconvex QOPs with nonpositive off-diagonal data matrices can be found. The result has been applied to the optimal power flow problems [19]. More precisely, assume that  $I_{\text{eq}} = \emptyset$  and that all off-diagonal elements of  $\mathbf{Q}^p$  ( $p \in \{0\} \cup I_{\text{ineq}}$ ) are nonpositive. Then the QOP described as  $\text{COP}(\Gamma^n)$  can be solved exactly by its SDP and SOCP relaxation [15, Theorem 3.1]. Thus, the QOP can be incorporated in our model as a subproblem. The details are omitted here.

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