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Multi-objective optimization models for many-to-one matching problems

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Abstract This paper is concerned with many-to-one matching problems for assigning residents to hospitals according to their preferences. The stable matching model aims at finding a stable matching, and the assignment game model involves maximizing the total utility; however, these two objectives are incompatible in general. We also focus on a situation where there are predetermined groups of residents, and they want to be matched in groups. To pursue these conflicting objectives simultaneously, we propose multi-objective optimization models for many-to-one matching problems. We firstly derive a bi-objective optimization model for maximizing the total utility and minimizing the number of blocking pairs for stability purpose. We next introduce the small-subgroup penalty to be minimized as the third objective for the purpose of matching in groups. Our optimization models are formulated as mixed-integer optimization problems, which can be solved to optimality using optimization software. The efficacy of our method is assessed through simulation experiments by comparison with the common matching algorithms, namely, deferred acceptance algorithm and Gales's top trading cycles algorithm. Our research results highlight the potential of optimization models for computing good-quality solutions to various difficult matching problems.

Keywords Many-to-one matching · Multi-objective optimization · Stable matching · Assignment game · Matching in groups

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1 Introduction

Many-to-one matching problems arise in a variety of situations like labor markets [26]. A well-known example is the National Resident Matching Program[1], which provides a mechanism for assigning residents (i.e., medical students) to hospitals according to their preferences [24]. Other applications can be found in the school admissions [31], student–project allocation [1], and house allocation [7]. To deal with these matching problems, two standard models have been proposed, namely, stable matching model [13] and assignment game model [30].

We consider the problem of matching residents with hospitals, where each resident can gain at most one residency position, and each hospital offers a limited number of such positions. A matching is called *stable*[2] if it does not have any blocking pairs, because such a resident–hospital pair can form a private arrangement outside of the matching. The stable matching model [13] aims at finding a matching that is stable. Gale and Shapley [13] devised the deferred acceptance (DA) algorithm [25] that generates a stable matching. Another commonly employed algorithm is Gales's top trading cycles (TTC) algorithm [29], which produces a matching that is Pareto-efficient but not necessarily stable.

Another standard matching model is the assignment game model proposed by Shapley and Shubik [30]. It involves maximizing the total utility gained by residents and hospitals on the assumption that the utility can be exchanged between a resident–hospital pair in the matching. Although the assignment game model reduces to a linear optimization problem, the obtained matching is not necessarily stable when the exchange of utility is not allowed. Additionally, the algorithms developed for the stable matching model often do not maximize the total utility. These facts suggest that the two objectives (i.e., stability and utility) are incompatible in general.

This paper is particularly focused on a situation where residents have preferences over colleagues [10,11]. More specifically, we assume that there are predetermined groups of residents, and they want to be matched in groups. For instance, a group of friends want to enter the same school. Another important example is temporary housing of disaster victims; those living in the neighborhood should move in a low-cost way to temporary houses in the same area so that their local community will not be destroyed. This special matching problem is known as *matching in groups*; see Hogan [16] for the details. This problem is also related to the roommates problem [7,13,23], whereas the roommates problem aims at forming a stable set of roommate pairs.

The matching problem with couples [5,21] is a special class of problems of matching in groups. Even in the couples case, however, there exists a problem instance that does not have a stable matching [8,17,18,24]. This fact clearly illustrates the theoretical difficulty of dealing with predetermined groups of residents in matching problems. In the case of matching with couples, several

[1] `http://www.nrmp.org/about-nrmp/`

[2] See Sect. 2 for the clear definition.

algorithms work well in practice [4,27], and various optimization models have been employed [2,6,9,15,22]. To the best of our knowledge, however, none of the existing studies have developed an effective algorithm for matching in groups.

The aim of this paper is to propose multi-objective optimization models to pursue several conflicting objectives simultaneously for many-to-one matching problems. Firstly, we derive a bi-objective optimization model for maximizing the total utility and minimizing the number of blocking pairs. The number of blocking pairs is minimized to find a nearly stable matching based on the characterization [3] of stable matchings. We next introduce the small-subgroup penalty to be minimized as the third objective in the multi-objective optimization model. This penalty increases when a resident group is divided into small subgroups, so minimizing the penalty leads to assigning many members of a group to the same hospital. Our optimization models are formulated as mixed-integer optimization problems, which can be solved to optimality using optimization software.

The efficacy of our method is assessed through simulation experiments following the previous studies [12,14,20]. The simulation results demonstrate that our method is capable of visualizing the optimal trade-offs between the total utility and the number of blocking pairs in many-to-one matching problems. In addition, our method clearly outperformed the DA and TTC algorithms in terms of the total utility. The simulation results also show that our method worked well for matching in groups because it found good matchings without dividing resident groups into too small subgroups.

2 Multi-objective optimization models

This section presents our multi-objective optimization models for matching problems.

2.1 Assignment game model

Let us consider assigning residents to hospitals according to their preferences. We denote by R and H the sets of residents and hospitals, respectively. To decide a matching between them, we introduce the binary decision variable $\boldsymbol{x} := (x_{rh})_{(r,h)\in R\times H}$ such that

$$x_{rh} = \begin{cases} 1 & \text{if resident } r \text{ is assigned to hospital } h, \\ 0 & \text{otherwise,} \end{cases}$$

for all $(r,h) \in R \times H$.

It is supposed that each resident can gain only one residency position, and that the number of residency positions offered by hospital h is $q_h \in \mathbb{Z}_+$. A matching is defined based on these conditions as follows.

Definition 1 We call $\boldsymbol{x} \in \{0,1\}^{|R|\times|H|}$ a *matching* if the following constraints are satisfied:

$$\sum_{h\in H} x_{rh} \leq 1 \quad (\forall r \in R),$$
$$\sum_{r\in R} x_{rh} \leq q_h \quad (\forall h \in H).$$

It is also supposed that when resident r is matched with hospital h, the resident and the hospital gain utilities of $u_r(h)$ and $u_h(r)$, respectively. For simplicity, we make the following assumptions throughout the paper.

Assumption 1 Utilities of residents and hospitals satisfy that

1. $u_r(h) \geq 0$ and $u_h(r) \geq 0$ for all $(r,h) \in R \times H$;
2. $u_r(h) \neq u_r(h')$ for all $(r,h,h') \in R \times H \times H$ with $h \neq h'$;
3. $u_h(r) \neq u_h(r')$ for all $(r,r',h) \in R \times R \times H$ with $r \neq r'$.

Accordingly, the resident–hospital pair (r,h) yields the utility of $u_{rh} := u_r(h) + u_h(r)$, and the *total utility* of a matching $\boldsymbol{x} \in \{0,1\}^{|R|\times|H|}$ is given by

$$\sum_{r\in R}\sum_{h\in H} u_{rh}x_{rh}. \tag{1}$$

The assignment game model [30] aims at finding a matching such that the total utility (1) is maximized. This model can be written as the following integer optimization problem:

$$\begin{aligned}
\text{maximize} \quad & \sum_{r\in R}\sum_{h\in H} u_{rh}x_{rh} && (2)\\
\text{subject to} \quad & \sum_{h\in H} x_{rh} \leq 1 \quad (\forall r \in R), && (3)\\
& \sum_{r\in R} x_{rh} \leq q_h \quad (\forall h \in H), && (4)\\
& x_{rh} \in \{0,1\} \quad (\forall r \in R, \forall h \in H). && (5)
\end{aligned}$$

Note that the binary constraint (5) is imposed for the sake of consistency throughout the paper. Even if it is relaxed to a non-negativity constraint (i.e., $x_{rh} \geq 0$), all optimal basic solutions to the assignment game model are binary valued (see, e.g., Korte and Vygen [19]).

2.2 Bi-objective optimization model for many-to-one matching

For a matching $\boldsymbol{x} \in \{0,1\}^{|R|\times|H|}$, let $m_{\boldsymbol{x}}(r) \in H$ be the hospital to which resident r is assigned, and $M_{\boldsymbol{x}}(h) \subseteq R$ be the set of residents that are assigned to hospital h; that is,

$$m_{\boldsymbol{x}}(r) = h \iff x_{rh} = 1,$$
$$M_{\boldsymbol{x}}(h) := \{r \in R \mid x_{rh} = 1\}.$$

We also define the preference orders $\succ_r$ and $\succ_h$ of resident r and hospital h based on their utilities as

$$h \succ_r h' \iff u_r(h) > u_r(h'),$$
$$r \succ_h r' \iff u_h(r) > u_h(r').$$

Note that these are strict linear orders from Assumption 1.

Suppose that there is a resident–hospital pair (r, h) such that resident r is not matched with hospital h, but they prefer each other. In this case, they can form a private arrangement outside of the matching. Such a pair is called a blocking pair, which is formally defined as follows.

Definition 2 Let $\boldsymbol{x} \in \{0,1\}^{|R|\times|H|}$ be a matching. A pair $(r, h) \in R \times H$ is a *blocking pair* for $\boldsymbol{x}$ if the following conditions are fulfilled:

1. resident r is not assigned to hospital h (i.e., $x_{rh} = 0$);
2. resident r is unassigned or prefers hospital h to $m_{\boldsymbol{x}}(r)$;
3. hospital h is undersubscribed or prefers resident r to at least one member of $M_{\boldsymbol{x}}(h)$.

Since the individual rationality condition is fulfilled from Assumption 1 for all $(r, h) \in R \times H$, a stable matching is defined based on the absence of blocking pairs as follows.

Definition 3 A matching $\boldsymbol{x} \in \{0,1\}^{|R|\times|H|}$ is *stable* if it does not have any blocking pairs.

Baïou and Balinski [3] mentioned (without proof) that stable matchings to a many-to-one matching problem can be characterized by the following linear inequalities:

$$q_h x_{rh} + q_h \sum_{j \succ_r h} x_{rj} + \sum_{i \succ_h r} x_{ih} \geq q_h \quad (\forall r \in R, \forall h \in H). \tag{6}$$

For the sake of completeness, we provide a proof of this statement.

Theorem 1 (Baïou and Balinski [3]) *Suppose that* $\boldsymbol{x} \in \{0,1\}^{|R|\times|H|}$ *is a matching. Then, it is stable if and only if constraint* (6) *is satisfied.*

Proof We prove the theorem by contraposition:

$$\begin{aligned}
&\text{constraint (6) is violated} \\
\iff\ & \exists (r,h) \in R \times H;\ x_{rh} = 0,\ \textstyle\sum_{j \succ_r h} x_{rj} = 0,\ \text{and} \ \textstyle\sum_{i \succ_h r} x_{ih} \leq q_h - 1 \\
\iff\ & \exists (r,h) \in R \times H;\ (r,h) \text{ is a blocking pair for } \boldsymbol{x} \quad (\because \text{Definition 2}) \\
\iff\ & \boldsymbol{x} \text{ is not stable} \quad (\because \text{Definition 3}) \quad \square
\end{aligned}$$

We shall use the characterization (6) of stability to minimize the number of blocking pairs. Let us introduce the binary decision variable $\boldsymbol{w} := (w_{rh})_{(r,h)\in R\times H} \in \{0,1\}^{|R|\times|H|}$. We then make use of the following constraint:

$$q_h x_{rh} + q_h \sum_{j \succ_r h} x_{rj} + \sum_{i \succ_h r} x_{ih} \geq q_h(1 - w_{rh}) \quad (\forall r \in R, \forall h \in H).$$

Note that this constraint is the same as the characterization (6) of stability when $w_{rh} = 0$. Therefore, $w_{rh} = 1$ indicates that (r, h) is a blocking pair, so the number of blocking pairs is given by

$$\sum_{r\in R}\sum_{h\in H} w_{rh}. \tag{7}$$

We consider maximizing the total utility and minimizing the number of blocking pairs at the same time. To accomplish this, we employ the weighted sum of them as an objective function. Our bi-objective optimization model for many-to-one matching is formulated as the following integer optimization problem:

$$\begin{aligned}
\text{maximize} \quad & \lambda_1 \sum_{r\in R}\sum_{h\in H} u_{rh}x_{rh} - \lambda_2 \sum_{r\in R}\sum_{h\in H} w_{rh} && (8)\\
\text{subject to} \quad & \sum_{h\in H} x_{rh} \leq 1 \quad (\forall r \in R), && (9)\\
& \sum_{r\in R} x_{rh} \leq q_h \quad (\forall h \in H), && (10)\\
& q_h x_{rh} + q_h \sum_{j \succ_r h} x_{rj} + \sum_{i \succ_h r} x_{ih} \geq q_h(1 - w_{rh}) \quad (\forall r \in R, \forall h \in H), && (11)\\
& x_{rh} \in \{0,1\} \quad (\forall r \in R, \forall h \in H), && (12)\\
& w_{rh} \in \{0,1\} \quad (\forall r \in R, \forall h \in H), && (13)
\end{aligned}$$

where $\lambda_1 \in \mathbb{R}_+$ and $\lambda_2 \in \mathbb{R}_+$ are user-defined weight parameters.

2.3 Tri-objective optimization model for matching in groups

We address the problem of matching in groups [16]. Let us suppose that there are some predetermined groups of residents, $R_g \subseteq R$ for $g \in G$ (e.g., couples, friends, and people living in the same area). Residents in the same group want to be assigned to the same hospital. To fulfill their hopes, we define a target number $\beta \in \mathbb{Z}_+$ of residents. We then attempt to ensure that even if resident group R_g is divided into some subgroups, the number of residents in each subgroup will be β or more.

To this end, we introduce the binary decision variable $\boldsymbol{z} := (z_{gh})_{(g,h)\in G\times H} \in \{0,1\}^{|G|\times|H|}$ and the nonnegative decision variable $\boldsymbol{s} := (s_{gh})_{(g,h)\in G\times H} \in$

$\mathbb{R}_+^{|G|\times|H|}$ that represents the shortage of subgroup members. For the purpose of matching in groups, we make use of the following constraint:

$$\beta z_{gh} - s_{gh} \leq \sum_{r \in R_g} x_{rh} \leq |R_g| z_{gh} \quad (\forall g \in G, \forall h \in H).$$

The right-hand inequality means that if at least one member $r \in R_g$ is assigned to hospital h (i.e., $\sum_{r \in R_g} x_{rh} \geq 1$), we must have $z_{gh} = 1$. From the left-hand inequality, the number of the corresponding subgroup members (i.e., $\sum_{r \in R_g} x_{rh}$) at hospital h must be $\beta - s_{gh}$ or more. We then minimize the *small-subgroup penalty* $\sum_{g \in G} \sum_{h \in H} (s_{gh})^2$, which is the sum of squared shortages of subgroup members in relation to the target number β.

Our tri-objective optimization model for matching in groups is formulated as the following mixed-integer optimization problem:

$$\text{maximize} \quad \lambda_1 \sum_{r \in R} \sum_{h \in H} u_{rh} x_{rh} - \lambda_2 \sum_{r \in R} \sum_{h \in H} w_{rh} - \lambda_3 \sum_{g \in G} \sum_{h \in H} (s_{gh})^2 \tag{14}$$

$$\text{subject to} \quad \sum_{h \in H} x_{rh} \leq 1 \quad (\forall r \in R), \tag{15}$$

$$\sum_{r \in R} x_{rh} \leq q_h \quad (\forall h \in H), \tag{16}$$

$$q_h x_{rh} + q_h \sum_{j \succ_r h} x_{rj} + \sum_{i \succ_h r} x_{ih} \geq q_h (1 - w_{rh}) \quad (\forall r \in R, \forall h \in H), \tag{17}$$

$$\beta z_{gh} - s_{gh} \leq \sum_{r \in R_g} x_{rh} \leq |R_g| z_{gh} \quad (\forall g \in G, \forall h \in H), \tag{18}$$

$$x_{rh} \in \{0, 1\} \quad (\forall r \in R, \forall h \in H), \tag{19}$$

$$w_{rh} \in \{0, 1\} \quad (\forall r \in R, \forall h \in H), \tag{20}$$

$$z_{gh} \in \{0, 1\} \quad (\forall g \in G, \forall h \in H), \tag{21}$$

$$s_{gh} \geq 0 \quad (\forall g \in G, \forall h \in H), \tag{22}$$

where $\lambda_3 \in \mathbb{R}_+$ is a user-defined weight parameter. If λ_3 is sufficiently large, every subgroup will consist of β or more residents.

3 Simulation experiments

This section evaluates the effectiveness of our method through simulation experiments.

3.1 Experimental design

For simulation purpose, we generated utilities of residents and hospitals by means of the linear model [12,14,20]. From a uniform distribution, we randomly produced a common vector $\boldsymbol{v} \in [0, 1]^{|H|}$ and individual vector $\boldsymbol{v}^{(r)} \in$

$[0,1]^{|H|}$ for each resident $r \in R$. We then calculated residents' utilities as

$$(u_r(h))_{h \in H} = \alpha \boldsymbol{v} + (1-\alpha)\boldsymbol{v}^{(r)} \quad (\forall r \in R),$$

where $\alpha \in [0,1]$ is a user-defined parameter. As α decreases, residents' preferences become more diverse. We tested $\alpha \in \{0.3, 0.6\}$ as in the previous studies [12,20]. We also randomly produced hospitals' utilities $u_h(r) \in [0,1]$ from a uniform distribution for all $(r,h) \in R \times H$.

We compare the performance of the following matching methods.

BO: our bi-objective optimization model (8)–(13);
TO(β, λ_3): our tri-objective optimization model (14)–(22), where $\beta \in \mathbb{Z}_+$ is the target number of subgroup members, and $\lambda_3 \in \mathbb{R}_+$ is the weight for the small-subgroup penalty;
DA: deferred acceptance algorithm [13,25];
TTC: Gales's top trading cycles algorithm [29].

Our optimization problems were solved using Gurobi Optimizer[3] 8.1.0. Note here that constraint (15) was replaced with an equality constraint (i.e., $\sum_{h \in H} x_{rh} = 1$) so that all residents can be matched even when λ_3 is very large. The resident-proposing versions of DA and TTC algorithms were implemented in the Python programming language.

3.2 Results of many-to-one matching

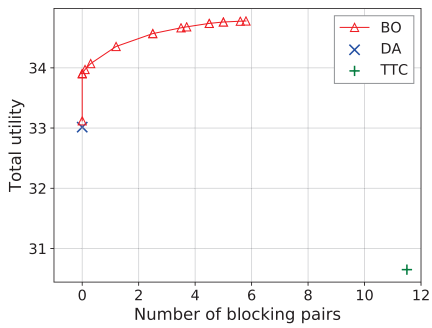
We compare the performance of our bi-objective optimization model (8)–(13) with those of the DA and TTC algorithms for many-to-one matching problems. We considered $(|R|,|H|) \in \{(24,6),(100,10)\}$ as the numbers of residents and hospitals. We set the number of residency positions as $q_h = |R|/|H|$ for all $h \in H$. We chose values of the weight parameters in the objective function (8) as

$$\lambda_1 \in \{0, 0.1, 0.2, \ldots, 0.9, 0.91, 0.92, \ldots, 0.99\}, \quad \lambda_2 = 1 - \lambda_1. \tag{23}$$

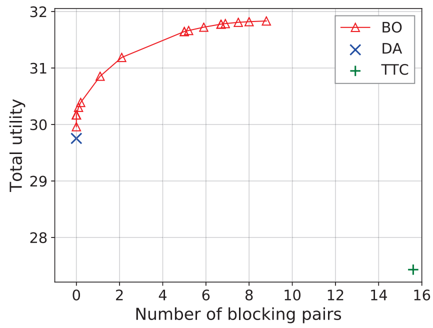
The computed results were averaged over ten repetitions.

Figure 1 shows the total utility (1) and the number (7) of blocking pairs of the matchings obtained by each method. We can see that our method BO provided a variety of good matchings depending on the weight parameter values. Specifically, we should notice that BO found stable matchings, which have no blocking pairs, for all the problem instances. Additionally, BO further improved the total utility at the expense of stability; Fig. 1d shows that the total utility was increased from 137.0 to 144.7 by admitting 59.0 blocking pairs. BO can also greatly reduce the number of blocking pairs by slightly decreasing the total utility; see the two points $(59.0, 144.7)$ and $(16.1, 143.0)$ in Fig. 1d. They imply that the number of blocking pairs decreased by about 73% from

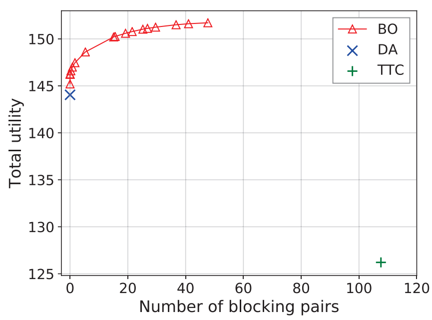
[3] `www.gurobi.com`



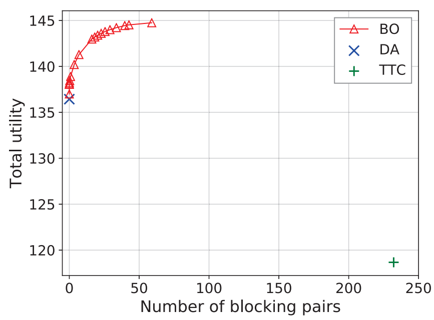
(a) $(|R|, |H|) = (24, 6)$, $\alpha = 0.3$



(b) $(|R|, |H|) = (24, 6)$, $\alpha = 0.6$



(c) $(|R|, |H|) = (100, 10)$, $\alpha = 0.3$



(d) $(|R|, |H|) = (100, 10)$, $\alpha = 0.6$

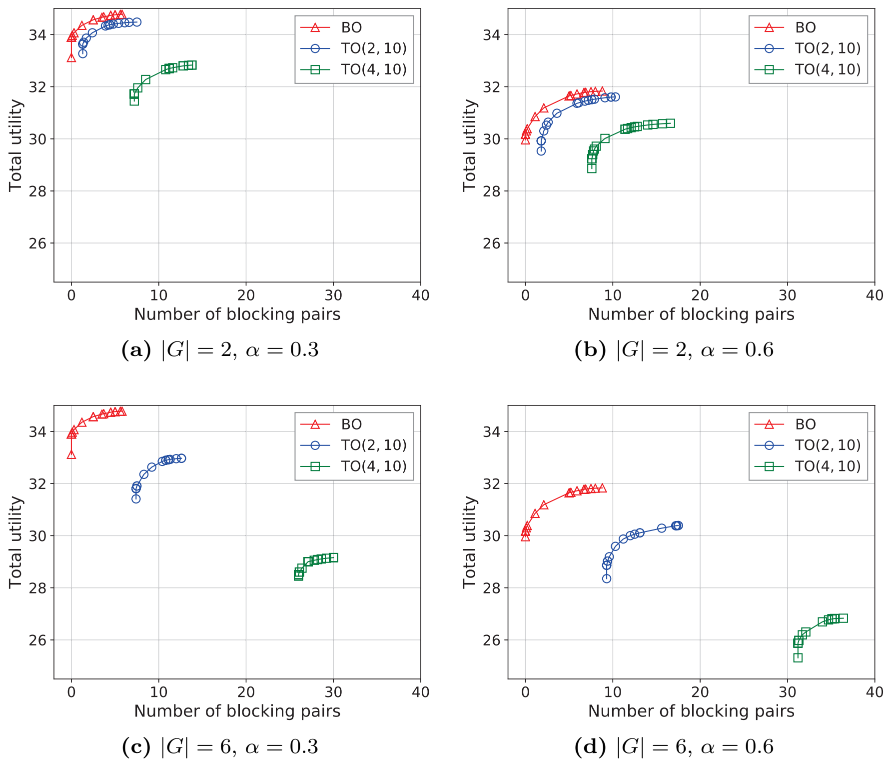
Fig. 1 Results of many-to-one matching

59.0 to 16.1, whereas the total utility decreased by about 1% from 144.7 to 143.0.

DA also found stable matchings for all the problem instances; however, BO further improved the total utility without increasing the number of blocking pairs from zero. Fig. 1a is a notable example; DA had the total utility of 33.0, but BO increased it to 33.9 without admitting any blocking pairs. On the other hand, TTC generated matchings of poor quality in both the total utility and the number of blocking pairs.

3.3 Results of matching in groups

We examine the effectiveness of our tri-objective optimization model (14)–(22) for matching in groups. We set the numbers of residents and hospitals as $(|R|, |H|) = (24, 6)$. We considered $|G| \in \{2, 6\}$ as the number of resident groups, where each group was composed of the same number of residents (i.e., $|R_g| = |R|/|G|$ for all $g \in G$). We set the number of residency positions as $q_h = |R|/|H| = 4$ for all $h \in H$. Similarly to Eq. (23), we chose values of the weight parameters in the objective functions (8) and (14). The computed results were averaged over ten repetitions.



(a) $|G| = 2$, $\alpha = 0.3$ **(b)** $|G| = 2$, $\alpha = 0.6$

(c) $|G| = 6$, $\alpha = 0.3$ **(d)** $|G| = 6$, $\alpha = 0.6$

Fig. 2 Results of matching in groups: $(|R|, |H|) = (24, 6)$

Figure 2 shows the total utility (1) and the number (7) of blocking pairs of the obtained matchings. As for our method $\text{TO}(\beta, \lambda_3)$, we employed $\beta \in \{2, 4\}$ as the target number of subgroup members. We also set the weight parameter as $\lambda_3 = 10$ so that the number of subgroup members will always be β or more.

We firstly focus on the two-group case $|G| = 2$ (i.e., Fig. 2a and Fig. 2b). We can see that $\text{TO}(2, 10)$ performed similarly to BO despite the existence of resident groups. Additionally, $\text{TO}(4, 10)$ verified that even when $\beta = 4$, the total utility can be increased to more than 30, and the number of blocking pairs can be reduced to less than 10. On the other hand, $\text{TO}(2, 10)$ and $\text{TO}(4, 10)$ did not reduce the number of blocking pairs to zero; this verifies that it was impossible to find a stable matching when residents had to be matched in groups.

We next move on to the six-group case $|G| = 6$ (i.e., Fig. 2c and Fig. 2d). Since the number of resident groups is increased, it becomes very difficult to fulfill their hopes about groups. Consequently, the difference between BO and $\text{TO}(2, 10)$ was larger in this case than in the two-group case. We should also notice that $\text{TO}(4, 10)$ does not divide any resident groups because $|R_g| = \beta = 4$ for all $g \in G$. As a result, $\text{TO}(4, 10)$ has a small number of options for

Table 1 Numbers of resident subgroup members: $(|R|, |H|) = (24, 6)$, $|G| = 6$, and $\alpha = 0.6$

$(\lambda_1, \lambda_2) = (0, 1)$																				
BO							TO(2, 10)							TO(4, 10)						
	Group g							Group g							Group g					
h	1	2	3	4	5	6	h	1	2	3	4	5	6	h	1	2	3	4	5	6
1	2	1	1	0	0	0	1	2	0	0	0	2	0	1	0	4	0	0	0	0
2	0	0	0	2	1	1	2	0	0	0	2	0	2	2	0	0	0	4	0	0
3	0	1	1	0	0	2	3	0	0	2	0	0	2	3	0	0	0	0	4	0
4	1	1	1	1	0	0	4	2	0	0	2	0	0	4	0	0	4	0	0	0
5	0	0	1	1	2	0	5	0	2	2	0	0	0	5	0	0	0	0	0	4
6	1	1	0	0	1	1	6	0	2	0	0	2	0	6	4	0	0	0	0	0
$(\lambda_1, \lambda_2) = (0.99, 0.01)$																				
BO							TO(2, 10)							TO(4, 10)						
	Group g							Group g							Group g					
h	1	2	3	4	5	6	h	1	2	3	4	5	6	h	1	2	3	4	5	6
1	2	1	1	0	0	0	1	2	2	0	0	0	0	1	4	0	0	0	0	0
2	0	0	0	1	1	2	2	0	0	0	2	0	2	2	0	0	0	0	0	4
3	0	0	1	1	1	1	3	0	0	2	0	0	2	3	0	0	4	0	0	0
4	2	2	0	0	0	0	4	2	0	0	2	0	0	4	0	4	0	0	0	0
5	0	0	1	2	1	0	5	0	0	2	0	2	0	5	0	0	0	4	0	0
6	0	1	1	0	1	1	6	0	2	0	0	2	0	6	0	0	0	0	4	0

matchings. For this reason, the variety of matchings obtained by TO$(4, 10)$ was smaller in this case than in the two-group case.

We conclude this section by giving typical examples of matching results obtained by our methods for $(\lambda_1, \lambda_2) \in \{(0, 1), (0.99, 0.01)\}$. Table 1 lists the numbers of resident subgroup members (i.e., $\sum_{r \in R_g} x_{rh}$) assigned to hospital $h \in H$ from group $g \in G$. Since BO takes no account of such resident groups, it developed many one-member subgroups. TO$(2, 10)$ aggregated these one-member subgroups appropriately to form two-member subgroups. TO$(4, 10)$ promoted further aggregation so that all subgroups were composed of four members.

4 Conclusion

This paper dealt with the many-to-one matching problems for assigning residents to hospitals according to their preferences. We proposed the bi-objective optimization model for maximizing the total utility and minimizing the number of blocking pairs. We also focused on the problem of matching in groups, where residents want to be matched in groups. For this purpose, we proposed the tri-objective optimization model that employs the small-subgroup penalty as the third objective. These multi-objective optimization models were posed as mixed-integer optimization problems.

We demonstrated through simulation experiments that for many-to-one matching problems, our method generated a variety of good matchings depending on weight parameter values of utility and stability. More importantly, our method visualized the optimal trade-offs between the total utility and the number of blocking pairs; this is the most remarkable feature that the DA and TTC algorithms lack. We also confirmed that our method worked well for matching in groups. These results support the efficacy of optimization models for computing good-quality solutions to various difficult matching problems.

A future direction of study will be to speed up the computation of solving our mixed-integer optimization problems. To this end, the stable admissions polytope [3, 28], which is the convex hull of stable matchings of a many-to-one matching problem, will be effective in minimizing the number of blocking pairs. We will also consider another optimization formulation for matching in groups. For instance, it is possible to incorporate residents' utilities of being matched in groups into optimization models.

Compliance with ethical standards

Conflict of interest: The authors declare that they have no conflict of interest.
Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.
Informed consent: Informed consent was obtained from all individual participants included in the study.

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