Planning Out-of-Hours Services for Pharmacies

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Abstract

The supply of pharmaceuticals is one important factor in a functioning health care system. In the German health care system, the chambers of pharmacists are legally obliged to ensure that every resident can find an open pharmacy at any day and night time within an appropriate distance. To that end, the chambers of pharmacists create an out-of-hours plan for a whole year in which every pharmacy has to take over some 24 hours shifts. These shifts are important for a reliable supply of pharmaceuticals in the case of an emergency but also unprofitable and stressful for the pharmacists. Therefore, an efficient planning that meets the needs of the residents and reduces the load of shifts on the pharmacists is crucial.

In this paper, we present a model for the assignment of out-of-hours services to pharmacies, which arises from a collaboration with the Chamber of Pharmacists North Rhine. Since the problem, which we formulate as an MILP, is very hard to solve for large-scale instances, we propose several tailored solution approaches. We aggregate mathematically equivalent pharmacies in order to reduce the size of the MILP and to break symmetries. Furthermore, we use a rolling horizon heuristic in which we decompose the planning horizon into a number of intervals on which we iteratively solve subproblems. The rolling horizon algorithm is also extended by an intermediate step in which we discard specific decisions made in the last iteration.

A case study based on real data reveals that our approaches provide nearly optimal solutions. The model is evaluated by a detailed analysis of the obtained out-of-hours plans.

Keywords: Pharmacies, Scheduling, Out-of-Hours Service, Rolling Horizon, Aggregation, MIP

1. Introduction

Germany is a welfare state and by law every resident is granted health care services [1]. These services include on the one hand medical treatment and on the other hand the supply of pharmaceuticals, which is provided primarily by pharmacies. To ensure that the population is supplied with pharmaceuticals at any time of the day,

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all pharmacies are obliged by law to be open 24/7 [2]. However, the responsible chambers of pharmacists have the possibility to daily exempt a part of the pharmacies from this duty outside the regular opening hours, provided that the supply of pharmaceuticals is guaranteed by the remaining pharmacies [2]. The pharmacies which are not exempted are called *out-of-hours pharmacies* and the task of being open all day is called *out-of-hours service*.

Since most pharmacists in Germany are self-employed, performing an out-of-hours service is a burden, as a highly qualified pharmacist must be present during the whole 24 hours shift, while the demand by the customers is usually relatively low outside the regular opening hours. Accordingly, the out-of-hours service on the one hand is rather unprofitable and stressful from the pharmacist's point of view, but on the other hand very important for the supply of the residents in emergency situations. This implies the need for an *out-of-hours plan,* which specifies the days a pharmacy has an out-of-hours service. The plan has to guarantee a good supply for the residents, but it should also be efficient in the sense that it assigns as few out-of-hours services as possible, out-of-hours pharmacies are not geographically close, and the burden on the pharmacists is evenly distributed.

Currently, most chambers of pharmacists in Germany divide their planning area into small districts in which they organize the out-of-hours service locally as a rotation of the included pharmacies [3]. This planning approach has the advantage that the coordinators of the districts have a good understanding of the local circumstances but it also has a number of drawbacks. Since the districts are planned independently, it is possible that two pharmacies that are close to each other but in different districts have an out-of-hours service on the same day. This leads to an oversupply for the corresponding area and many unnecessary out-of-hours services, as also pointed out in [4].

The chamber of pharmacists of the area North Rhine in Germany, formerly divided into 69 local districts [5], performs an algorithm-based planning since the beginning of 2014. This already resulted in a reduction of the total number of out-of-hours services of more than 20% [5]. Since the algorithms are not based on mathematical optimization techniques, the planning still has optimization potential. This potential lies not only in a further reduction of the number of out-of-hours services, but also in the satisfaction of all planning constraints. These are hard to meet altogether without mathematical optimization.

Related Work. The planning of out-of-hours services is at the intersection of covering and rostering problems. Considering only a single day of the time horizon, the planning can be seen as a facility location problem, where we choose a subset of pharmacies to cover the residents. Much work in the operations research literature focuses on the location of health care facilities like hospitals [6], health care facilities in general [7, 8], or the location of speciality care services within a health care network [9]. An analysis of various modeling approaches for such location problems can be found in [10]. A comprehensive survey is given in [11] and [12].

Considering that we have to assign out-of-hours services not only once, but for every day of the time horizon, the planning of out-of-hours pharmacies can be seen as a rostering problem. In the literature, rostering problems are extensively studied for hospital staff like nurses or physicians [13, 14, 15]. The problems are similar in the sense that we have to assign shifts while reducing the workload of the staff and guaranteeing the care of the residents. However, the planning of out-of-hours services has a different structure, as pharmacies are much more inhomogeneous compared to nurses or physicians. A pharmacy's capability to cover an area depends only on its location, which may differ vastly from pharmacy to pharmacy. In contrast, there are usually only few types of nurses or physicians considered for the respective rostering problems. Due to these structures, the modeling of the planning of out-of-hours services for pharmacies is much different.

Despite its importance for the health care sector, the planning of out-of-hours services for pharmacies has not yet gained much attention in the operations research literature. To the best of our knowledge, all contributions come from a group of researchers from Turkey [4, 16, 17], which introduced the planning as the *Pharmacy Duty Scheduling Problem* (PDS). The setting for the Turkish and German pharmacies is similar in the sense that the chambers of pharmacists organize the service decentrally in small local districts. However, the PDS differs considerably from the problem proposed in this paper, which is why we introduce a different name for the latter. In the PDS, the authors retain the historical structure given by the local districts and assign on each day in each district exactly one out-of-hours service to one of the pharmacies. The resulting model is described as a multi duty variation of the p-median problem. The objective is to minimize the sum of the distances between aggregated customer nodes and their nearest out-of-hours pharmacy. Most importantly, in contrast to our model, the number of services performed by the pharmacists is fixed and there are no regulations regarding periods of rest between two out-of-hours services of the same pharmacy. The PDS has been solved using variable neighborhood search [16], tabu search [17] and branch and price [4].

Our Contribution. In this paper, we present a model for the assignment of out-of-hours services to pharmacies, which arises from a collaboration with the Chamber of Pharmacists North Rhine. The problem of constructing an out-of-hours plan includes aspects of covering and independent set problems, among others. These subproblems are additionally expanded over a planning horizon and linked by time-based constraints. Since the planning of out-of-hours services includes many $\mathcal{NP}$-hard subproblems, it is not surprising that the problem itself is also $\mathcal{NP}$-hard. We will formally show that it is even $\mathcal{NP}$-hard to decide whether there exists a feasible plan, i.e., in contrast to the included subproblems, it is not trivial to even compute any solution.

In order to construct out-of-hours plans, we formulate the problem as a mixed integer linear program (MILP). Because the MILP becomes too big and too difficult to solve for large real-world instances by state-of-the-art solvers, we propose a reformulation via aggregation of mathematically equivalent pharmacies. Aggregation is a common technique for reducing the size of large-scale optimization problems [18]. In most cases, such an approach results in a more tractable model but also in an inexact simplification of the original problem. Therefore, an optimal solution to the aggregated model is in general of high quality, but not optimal for the original problem. In our case, the aggregation is exact in the sense that an optimal solution to the aggregated model corresponds to an optimal solution of the original model. The aggregation results not only in a reduction of the size of the MILP to be solved, but also breaks many symmetries, a typical challenge in large-scale mathematical programming [19].

In addition to the aggregation, we present a rolling horizon approach that uses a time-based decomposition of the planning horizon into smaller subproblems. We consider an iteratively increasing subinterval of the planning horizon and compute an out-of-hours plan for each subinterval. In each iteration, the services assigned in previous iterations are fixed in order to reduce the number of decisions and thus the complexity of the problem. We proceed until the subinterval equals the whole planning horizon, which results in a solution of the original planning problem. In the past, rolling horizon algorithms have already been successfully applied to large-scale optimization problems including planning horizons. In [20] and [21], rolling horizon algorithms are used in which the planning horizon is decomposed into two intervals. The first interval, which increases iteratively, is considered exactly, while the second interval is considered as a relaxed version via time-based aggregation and serves as a look-ahead. Our approach is different, as we do not consider a relaxation of the whole remaining planning horizon, but use a look-ahead of an interval of additional days which we also consider exactly. This is similar to the approach in [13] and fits the repetitive structure of our planning problem, due to which it is less beneficial to consider all remaining days. Another possibility to divide the planning of out-of-hours services into more tractable subproblems is via spatial decomposition, where the original problem is split with respect to the location and not the time [22]. Again, due to the repetitive structure of the planning, we think that the rolling horizon approach fits better than a spatial decomposition. The drawback of the rolling horizon approach is that we loose exactness and most likely do not obtain an optimal solution. This is compensated by the advantage of a significant speed up of the solution process and good practical results. Furthermore, we enhance the rolling horizon approach with an extension that reduces the impact of suboptimal decisions made in early iterations. In the extension, we free a subset of former fixed variables before each iteration. In contrast to most rolling horizon approaches, the subset of freed variables does not only include fractional variables, but also binary variables which are chosen by solving an intermediate optimization problem. The unfixing enlarges the solution space in the following iterations and usually allows for much better solutions. Additionally, the freed variables are chosen such that the complexity of the subsequent problems does not increase.

We test our approaches extensively on a real-world instance with more than 2000 pharmacies provided by the Chamber of Pharmacists North Rhine. We will see that both the aggregation and decomposition approaches are beneficial for the computation of out-of-hours services and provide us high quality solutions in short time. Especially the unfixing of variables improves the performance of our rolling horizon approach. This idea may also be used to enhance the performance of rolling horizon approaches for other optimization problems.

We test in detail the influence of input parameters on the performance of our algorithms as well as the resulting plans. Finally, we study the plans regarding their practicability, analyzing coverage properties from the customer's perspective and the distribution of the out-of-hours services from the perspective of pharmacists.

Although this paper focuses on pharmacies, our models and approaches can also be applied to other real-world problems where we need to cover an area with services that changing providers offer from decentral locations. In Germany, the out-of-hours service of general practitioners is generally performed in a central location. However, pediatricians and dentists partially offer an out-of-hours service that is similar to the one of the pharmacies. Other examples include on-call duties, e.g. of care services or janitorial services, which are performed from home.

Outline. The paper is structured as follows. We describe the model for the planning of out-of-hours services in Section 2. In Section 3, we formulate the MILP and introduce the decomposition approaches. The computational study using the real-world instance is presented in Section 4. Finally, we conclude the paper with a discussion on the model and the approaches.

2. Problem Definition and Notation

For the planning of the out-of-hours service, we consider a set of pharmacies $\mathcal{P}$ and a time horizon $\mathcal{T} = \{1, \ldots, T\}$, normally a whole year, consisting of the days to be planned. We identify an *out-of-hours plan* with a mapping $F : \mathcal{T} \to 2^{\mathcal{P}}$, where $F(t) \subseteq \mathcal{P}$ is the subset of pharmacies that have an out-of-hours service on day $t \in \mathcal{T}$. If pharmacy $p \in \mathcal{P}$ has an out-of-hours service on day $t \in \mathcal{T}$, i.e., $p \in F(t)$, then we say that p is an *out-of-hours pharmacy* on day t.

In consultation with the Chamber of Pharmacists North Rhine, an out-of-hours plan has to fulfill the following conditions regarding the coverage of residents and a fair distribution among pharmacies. We start with a short summary and then discuss the requirements in detail.

- *Covering municipality centers:* There is at least one out-of-hours pharmacy in the vicinity of each municipality center. The allowed distance depends on the municipality. (Figure 1a)
- *Meeting demand in larger cities:* Multiple out-of-hours pharmacies are needed to meet the demand of residents in larger cities. The number of pharmacies needed is city-specific and may vary for different days. (Figure 1b)
- *Minimum distances between out-of-hours pharmacies:* Distances between out-of-hours pharmacies ensure that they are geographically dispersed. (Figure 2a)
- *Periods of rest between services of the same pharmacy:* Periods of rest prevent that pharmacies are assigned out-of-hours services on consecutive days. (Figure 2b)
- *Equitable distribution within municipalities:* An equitable assignment of services within the municipalities creates fairness among and satisfaction of the pharmacists.
- *Minimum number of services per pharmacy:* Each pharmacy participates with a minimum number of out-of-hours services.

Under the above conditions, which we define in the next paragraphs in more detail, we are aiming to construct a plan that is efficient in the sense that we assign as few out-of-hours services as possible. Accordingly, we look for a plan $F : \mathcal{T} \to 2^{\mathcal{P}}$ that minimizes $\sum_{t\in\mathcal{T}} |F(t)|$. We call this problem the *Out-of-Hours Planning Problem* (OHP).

Covering Municipality Centers. In order to guarantee a comprehensive supply of pharmaceuticals, we consider the centers of the municipalities in the planning area as reference points for the location of the residents. Based on legal requirements for the planning of out-of-hours services in the region North Rhine, we demand that for every municipality center on every day there is an out-of-hours pharmacy within a given distance [23].

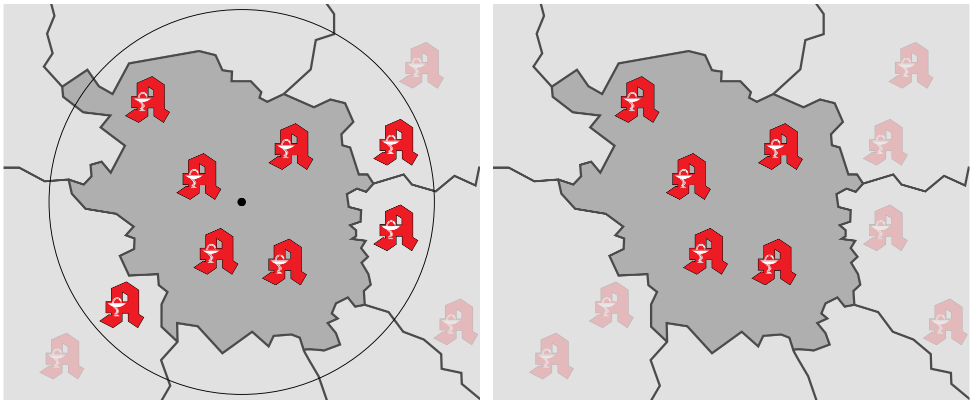
Note that by law every pharmacy needs to have a standardized inventory to guarantee the supply of pharmaceuticals during the out-of-hours service. Furthermore, there will always be only one pharmacist in an out-of-hours pharmacy, regardless of its size. Hence, only the location of a pharmacy is relevant for the planning of out-of-hours services and we consider all pharmacies to be equal except for their location.

More formally, let $\mathcal{M}$ be the set of municipalities to be covered. We define distances $\delta : (\mathcal{P} \cup \mathcal{M})^2 \to \mathbb{Q}_{\geq 0}$ between all pairs of municipalities and pharmacies, which represent the *distances of the shortest paths* between these locations in the road network. Note that the distances are metric, i.e., it holds $\delta(v_1, v_3) \leq \delta(v_1, v_2) + \delta(v_2, v_3)$ for all $v_1, v_2, v_3 \in \mathcal{V}$, but not necessarily symmetric due to one-way roads.

For all municipalities $m \in \mathcal{M}$, $\delta^{\text{cov}}(m) \in \mathbb{Q}_{\geq 0}$ is the *cover radius,* that is the maximum distance allowed from the municipality to the nearest out-of-hours pharmacy. We say that $p \in \mathcal{P}$ can *cover* $m \in \mathcal{M}$ if it holds $\delta(m, p) \leq \delta^{\text{cov}}(m)$ and define $C(m) = \{p \in \mathcal{P} \mid \delta(m, p) \leq \delta^{\text{cov}}(m)\}$ as the set of pharmacies that can cover m (highlighted in Figure 1a). Additionally, the set $C(p) = \{m \in \mathcal{M} \mid \delta(m, p) \leq \delta^{\text{cov}}(m)\}$ consists of all municipalities that are covered by p. Furthermore, we say that a municipality is *covered* in plan $F : \mathcal{T} \to 2^{\mathcal{P}}$ on day $t \in \mathcal{T}$ if at least one pharmacy $p \in C(m)$ has an out-of-hours service on day t, i.e., $C(m) \cap F(t) \neq \emptyset$. In a feasible out-of-hours plan F every municipality has to be covered every day, i.e., $C(m) \cap F(t) \neq \emptyset$ for all $m \in \mathcal{M}$ and $t \in \mathcal{T}$. Note that a pharmacy does not necessarily have to be within the boundaries of a municipality in order to cover it and that the cover radius may vary for different municipalities according to the number of residents, the density of pharmacies in the area, and other characteristics.

Meeting Demand in Larger Cities. For some municipalities which represent larger cities, one out-of-hours pharmacy cannot guarantee a sufficient supply of pharmaceuticals for the whole municipality. Every day we require a minimum number of out-of-hours pharmacies that are located within the boundaries of these municipalities. In contrast to the covering of municipality centers, the demand of larger cities cannot be met by pharmacies that are located outside the municipalities, because according to the Chamber of Pharmacists North Rhine, residents in cities are less willing to move to another municipality.

For the modeling, we denote with $P(m) \subseteq \mathcal{P}$ the set of all pharmacies within the boundaries of a municipality $m \in \mathcal{M}$ (highlighted in Figure 1b). Accordingly,



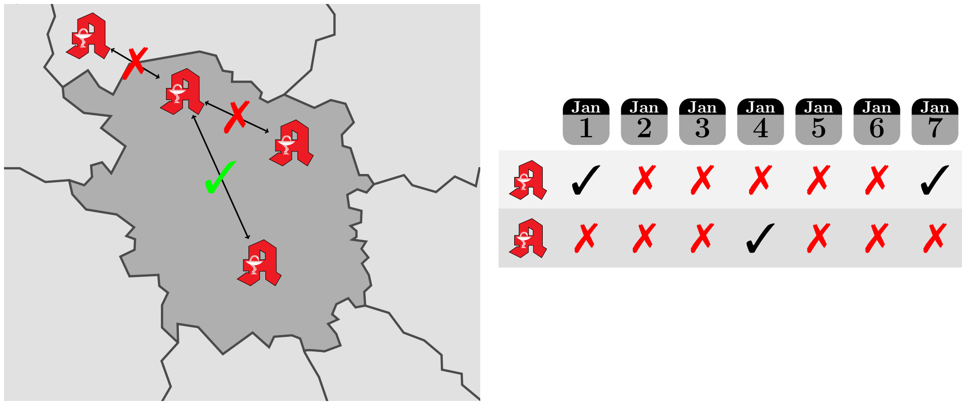
(a) (b)

Figure 1: Pharmacies that can cover the center of a municipality (a) and meet the demand of larger cities (b). For the sake of simplicity, the distances in (a) are shown as the crow flies.

$m(p) \in \mathcal{M}$ denotes the municipality a pharmacy $p \in \mathcal{P}$ is located in. Furthermore, we define for every municipality $m \in \mathcal{M}$ and day $t \in \mathcal{T}$ a *minimum demand* for out-of-hours pharmacies $d(m,t) \in \mathbb{Z}_{\geq 0}$. Note that the minimum demand is always zero for municipalities with few residents and pharmacies. Additionally, it may vary depending on the considered day, since we tend to need more out-of-hours pharmacies on Sundays and holidays when other pharmacies are regularly closed. A feasible out-of-hours plan $F : \mathcal{T} \to 2^{\mathcal{P}}$ assigns for all municipalities $m \in \mathcal{M}$ and all days $t \in \mathcal{T}$ at least $d(m,t)$ services to pharmacies within the municipality, i.e., $|P(m) \cap F(t)| \geq d(m,t)$ for all $m \in \mathcal{M}$ and $t \in \mathcal{T}$.

Minimum Distances Between Out-of-Hours Pharmacies. Assigning an out-of-hours service to two geographically close pharmacies on the same day usually has only a minor positive effect to the residents. Furthermore, it is undesirable for the pharmacists from an economical perspective, since they have to share their customers. A solution that minimizes the number of out-of-hours services tends to avoid assigning out-of-hours services to neighboring pharmacies, as they often cover the same municipalities. Nevertheless, it is possible that a pair of close pharmacies is with respect to the coverage conditions equal to a pair of pharmacies that are further apart. Especially in larger cities, where we assign several services in order to meet the demand, we need to make sure that the out-of-hours pharmacies are spread out. Therefore, we prohibit out-of-hours services on the same day for pharmacies that are too close to each other.

We define *conflicting* pharmacies that cannot have an out-of-hours service on the same day. To do so, we introduce for each municipality $m \in \mathcal{M}$ a value $\delta^{\text{con}}(m) \in \mathbb{Q}_{\geq 0}$ that is the *minimum distance* for an out-of-hours pharmacy $p \in P(m)$ to the next out-of-hours pharmacy $p' \in \mathcal{P} \setminus \{p\}$. Just like the cover radius, the minimum distances depend on the municipalities, since in general we have areas that differ in the density of pharmacies. We define two pharmacies $p \neq p' \in \mathcal{P}$ to be *conflicting*



(a) (b)

Figure 2: Minimum distance between two out-of-hours pharmacies on the same day (a). Period of rest of five days (b).

if $\delta(p, p') \leq \min\{\delta^{\text{con}}(m(p)), \delta^{\text{con}}(m(p'))\}$ and define the set of *conflicting pharmacy pairs* $\mathcal{C} = \{\{p, p'\} \subseteq \mathcal{P} \mid p \neq p',\ \delta(p, p') \leq \min\{\delta^{\text{con}}(m(p)), \delta^{\text{con}}(m(p'))\}\}$. For a pharmacy $p \in \mathcal{P}$, we define the set of conflicting pharmacies as $\mathcal{C}(p) = \{p' \in \mathcal{P} \mid \{p, p'\} \in \mathcal{C}\}$. In a feasible out-of-hours plan $F : \mathcal{T} \to 2^{\mathcal{P}}$, it must hold $\{p, p'\} \nsubseteq F(t)$ for all conflicting pairs $\{p, p'\} \in \mathcal{C}$ and all days $t \in \mathcal{T}$. Stated otherwise, $F(t)$ is an independent set in the graph $(\mathcal{P}, \mathcal{C})$.

Periods of Rest Between Services of the Same Pharmacy. Since an out-of-hours service implies a 24-hours shift that is often followed by normal opening times, a pharmacy should not be assigned a service on consecutive days or days close to each other. Therefore, we have for all municipalities $m \in \mathcal{M}$ a value $r(m) \in \mathbb{Z}_{\geq 0}$ which defines the *period of rest*, that is the minimum number of days between two out-of-hours services of the same pharmacy $p \in P(m)$. The period of rest depends on the municipality, as we may not be able to grant a large number of days of rest in areas with few pharmacies. A feasible out-of-hours plan $F : \mathcal{T} \to 2^{\mathcal{P}}$ has to guarantee these periods of rest, i.e., for all pharmacies $p \in \mathcal{P}$ and days $t \neq t' \in \mathcal{T} = \{1, \ldots, T\}$ with $|t - t'| \leq r(m(p))$ it must hold $p \notin F(t) \cap F(t')$.

Equitable distribution within municipalities. As pointed out in the legal commentary [24], based on the general principle of equality, the out-of-hours services that are unattractive to the pharmacists should be assigned in a fair manner. The definition of a fair plan strongly depends on the personal viewpoint of each pharmacist and is in general difficult to measure and not trivial to incorporate into optimization problems. A review of inequity averse optimization is given in [25]. According to the chamber of pharmacists of the area North Rhine, the pharmacists evaluate an out-of-hours plan based on the number of services assigned to them during the time horizon. They expect that another pharmacy within the same municipality is assigned not much fewer services. Therefore, the Chamber of Pharmacists North Rhine demands that all pharmacies within the same municipality are assigned a similar number of out-of-hours services.

For all municipalities $m \in \mathcal{M}$, we introduce values $e(m) \in \mathbb{Z}_{\geq 0}$ representing *equity coefficients* which bound the difference in the number of services that two pharmacies $p, p' \in P(m)$ belonging to municipality m can have. Furthermore, for a pharmacy $p \in \mathcal{P}$ and a plan $F : \mathcal{T} \to 2^{\mathcal{P}}$, we define $S_F(p) = \{t \in \mathcal{T} \mid p \in F(t)\}$ as the set of days on which p has an out-of-hours service. We say that an out-of-hours plan $F : \mathcal{T} \to 2^{\mathcal{P}}$ is *equitable* if it holds $||S_F(p)| - |S_F(p')|| \leq e(m(p))$ for all pairs $p, p' \in \mathcal{P}$ with $m(p) = m(p')$.

Minimum Number of Services per Pharmacy. Since the authorities designing an out-of-hours plan are legally obliged to assign out-of hours services to all pharmacies [24], we have to ensure that all pharmacies participate appropriately in the service.

We introduce a minimum number of services $n^{\min} \in \mathbb{Z}_{\geq 0}$ that all pharmacies should be assigned. We say that a plan $F : \mathcal{T} \to 2^{\mathcal{P}}$ fulfills the *minimum number of services condition* if it holds $|S_F(p)| \geq n^{\min}$ for all $p \in \mathcal{P}$. We will see in the case study in Section 4 that enforcing equity within municipalities is not sufficient to guarantee that all pharmacies perform out-of-hours services, as there can be whole municipalities in which no pharmacy is assgined a service. Conversely, setting a minimum number of services does not necessarily lead to an equitable distribution within municipalities.

The Out-of-Hours Planning Problem. The following definition summarizes the problem statement.

Definition 1. For the planning of out-of-hours services, we are given a time horizon $\mathcal{T} = \{1, \ldots, T\}$, a set of pharmacies $\mathcal{P}$, a set of municipalities $\mathcal{M}$ and metric distances $\delta : (\mathcal{P} \cup \mathcal{M})^2 \to \mathbb{Q}_{\geq 0}$ between these locations. Furthermore, we are given cover radii $\delta^{\text{cov}} : \mathcal{M} \to \mathbb{Q}_{\geq 0}$, municipality affiliations $m : \mathcal{P} \to \mathcal{M}$, demands $d : \mathcal{M} \times \mathcal{T} \to \mathbb{Z}_{\geq 0}$, minimum distances between out-of-hours pharmacies $\delta^{\text{con}} : \mathcal{M} \to \mathbb{Q}_{\geq 0}$, periods of rest $r : \mathcal{M} \to \mathbb{Z}_{\geq 0}$ as well as equity coefficients $e : \mathcal{M} \to \mathbb{Q}_{\geq 0}$ and a minimum number of services $n^{\min} \in \mathbb{Z}_{\geq 0}$. The *Out-of-Hours Planning Problem* (OHP) consists in finding an out-of-hours plan $F : \mathcal{T} \to 2^{\mathcal{P}}$ that fulfills the constraints defined above and minimizes the number of assigned services $\sum_{t \in \mathcal{T}} |F(t)|$.

The OHP is a combination of the classical $\mathcal{NP}$-hard set cover and independent set problems and also includes structures of scheduling problems. Hence, it is not surprising that the OHP itself is also hard to solve. The theorem below classifies the complexity of the OHP more precisely. The proof is given in Appendix A.

Theorem 2. *The problem of deciding whether there exists a solution to an OHP instance is strongly $\mathcal{NP}$-complete. This holds already for periods of rest $r \equiv 1$ and without considering demands, conflicts, equity, and minimum numbers of services, i.e.,* $d \equiv 0$, $\delta^{con} \equiv 0$, $e \equiv \infty$ *and* $n^{\min} = 0$.

In the next section, we propose exact and heuristic approaches for solving the OHP based on different MILP formulations, aggregation techniques as well as a time-based decomposition.

3. Solution Approaches

In this section, we state an MILP formulation to solve the OHP as well as two more approaches to handle the size of the OHP instance tested in our real-world case study in Section 4.

3.1. An MILP Formulation for the OHP

We present an MILP formulation that contains binary decision variables $x_{pt} \in \{0,1\}$ to indicate whether or not pharmacy $p \in \mathcal{P}$ is assigned an out-of-hours service on day $t \in \mathcal{T}$. Therefore, a vector $x \in \{0,1\}^{|\mathcal{P}||\mathcal{T}|}$ implies an out-of-hours plan $F : \mathcal{T} \to 2^{\mathcal{P}}$ with $F(t) = \{p \in \mathcal{P} \mid x_{pt} = 1\}$ for $t \in \mathcal{T}$. To model the equity in an easy way, we introduce two auxiliary variables $\bar{y}_m \geq 0$ and $\underline{y}_m \geq 0$ for each municipality $m \in \mathcal{M}$ that bound the maximum number (minimum number, respectively) of out-of-hours services assigned to a pharmacy $p \in P(m)$ within the municipality. The proposed MILP formulation reads as follows

$$
\begin{aligned}
\min \quad & \sum_{p \in \mathcal{P}} \sum_{t \in \mathcal{T}} x_{pt} && && \text{(1a)}\\
\text{s.t.} \quad & \sum_{p \in C(m)} x_{pt} \geq 1 && \forall m \in \mathcal{M}, t \in \mathcal{T}, && \text{(1b)}\\
& \sum_{p \in P(m)} x_{pt} \geq d(m,t) && \forall m \in \mathcal{M}, t \in \mathcal{T}, && \text{(1c)}\\
& x_{pt} + x_{p't} \leq 1 && \forall \{p, p'\} \in \mathcal{C}, t \in \mathcal{T}, && \text{(1d)}\\
& \sum_{t'=t}^{t+r(m(p))} x_{pt'} \leq 1 && \forall p \in \mathcal{P}, 1 \leq t \leq T - r(m(p)), && \text{(1e)}\\
& \underline{y}_{m(p)} \leq \sum_{t \in \mathcal{T}} x_{pt} \leq \bar{y}_{m(p)} && \forall p \in \mathcal{P}, && \text{(1f)}\\
& \bar{y}_m - \underline{y}_m \leq e(m) && \forall m \in \mathcal{M}, && \text{(1g)}\\
& \underline{y}_m \geq n^{\min} && \forall m \in \mathcal{M}, && \text{(1h)}\\
& x_{pt} \in \{0,1\} && \forall p \in \mathcal{P}, t \in \mathcal{T}, && \text{(1i)}\\
& \bar{y}_m, \underline{y}_m \geq 0 && \forall m \in \mathcal{M}. && \text{(1j)}
\end{aligned}
$$

The objective function (1a) sums over all decision variables and therefore counts the number of services assigned to the pharmacies over the time horizon. The constraints (1b) ensure that every municipality is covered every day. The inequalities (1c) guarantee that the demand of every municipality is met every day. Due to the constraints (1d), conflicting pharmacies cannot have an out-of-hours service on the same day. The inequalities (1e) ensure that the rest periods are respected. The constraints (1f) demand that the number of services assigned to a pharmacy $p \in P(m)$ is between $\underline{y}_m$ and $\bar{y}_m$. Together with the constraints (1g), this guarantees that the difference in the number of services assigned to two pharmacies of the same municipality is bounded as desired. Lastly, the constraints (1h) guarantee that every pharmacy is assigned at least the minimum number of out-of-hours services.

In the following, we say that $x \in \{0,1\}^{|\mathcal{P}||\mathcal{T}|}$ is a solution to the MILP above, since the remaining variables are only auxiliary variables and can be computed from the x_{pt} variables.

On the positive side, we will see that the above MILP provides a very good integrality gap for our real-world case study in Section 4. On the negative side, the model becomes large when considering all pharmacies and the whole time horizon of 365 days, such that we were not even able to solve the LP relaxation within a time limit of one day. Additionally, the planning of the out-of-hours services contains many symmetries that can complicate the solution process.

In the subsequent Sections 3.2 and 3.3, we will present a reformulation that reduces the symmetries as well as the size of the MILP and propose a heuristic for the construction of out-of-hours plans.

3.2. Aggregating Equivalent Pharmacies

We already mentioned that the planning of out-of-hours services is highly symmetrical. Kocatürk and Özpeynirci [16] pointed out that for the Pharmacy Duty Scheduling Problem, a feasible solution can be permuted to roughly $\mathcal{O}(T!)$ equivalent solutions by swapping the whole assignment for different days. In our case, this is not possible, as the period of rest constraints (1e) and the demand constraints (1c), which vary for different days, prohibit an arbitrary swapping of days. However, swapping services of pharmacies in similar locations may result in a feasible out-of-hours plan of the same solution value.

Consider two pharmacies within the same municipality that are located close to each other such that they can cover the same set of municipalities, are in conflict with each other and have the same conflicts with other pharmacies. Since all pharmacies are considered equal regarding the of out-of-hours service, except for their location, both pharmacies can be considered mathematically equivalent in the sense that if we manipulate a feasible plan by swapping all services of the two pharmacies, we again obtain a feasible plan. This symmetry can reduce the performance of MILP solvers dramatically, see [19] for more information. We will however use this structure to reduce the size of the MILP formulation given in Section 3.1 by replacing such sets of pharmacies by a single artificial pharmacy.

Definition 3. We call two pharmacies $p_1, p_2 \in \mathcal{P}$ *equivalent* and write $p_1 \sim p_2$ if they are in the same municipality $m(p_1) = m(p_2)$, cover the same municipalities $C(p_1) = C(p_2)$ and satisfy $\mathcal{C}(p_1) \cup \{p_1\} = \mathcal{C}(p_2) \cup \{p_2\}$. Let $p^{\mathrm{s}} = \{p_1, \ldots, p_n\}$ be a subset of pharmacies. We call p^{s} a *superpharmacy* if for all pairs $p_1, p_2 \in \mathcal{P}$ it holds $p_1 \sim p_2$.

For the coverage and conflicts, it is irrelevant which pharmacy of the superpharmacy we assign a service to. Therefore, we decompose the planning into two steps. First, we consider the superpharmacies as a single pharmacy to which we may assign more out-of-hours services, but due to the conflicts within the superpharmacy only one service per day. Then we split the services assigned to the superpharmacy among the corresponding pharmacies.

According to the definition, a single pharmacy $p \in \mathcal{P}$ also defines a superpharmacy $\{p\}$. Therefore, we can partition the set of pharmacies $\mathcal{P}$ into a *set of superpharmacies* $\mathcal{P}^{\mathrm{s}}$ with $\mathcal{P} = \biguplus_{p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}} p^{\mathrm{s}}$. For the sake of simplicity, in the first step we solely construct plans based on a partition of $\mathcal{P}$ into superpharmacies $\mathcal{P}^{\mathrm{s}}$. This way we do not have to distinguish between superpharmacies and pharmacies.

In order to plan based on a partition $\mathcal{P}^{\mathrm{s}}$ of $\mathcal{P}$ into superpharmacies, we have to update our notation. We say that a superpharmacy $p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}$ can cover a municipality $m \in \mathcal{M}$ if the pharmacies aggregated within p^{s} can cover m, i.e., $p^{\mathrm{s}} \subseteq C\left(m\right)$. Accordingly, we define the set of superpharmacies that can cover m as $C^{\mathrm{s}}\left(m\right) = \{p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}} \mid p^{\mathrm{s}} \subseteq C\left(m\right)\}$. Furthermore, we say that a superpharmacy $p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}$ lies within a municipality $m \in \mathcal{M}$ if the pharmacies aggregated within p^{s} lie within m, i.e., $p^{\mathrm{s}} \subseteq P\left(m\right)$. Then we define the set of superpharmacies that lie within m as $P^{\mathrm{s}}\left(m\right) = \{p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}} \mid p^{\mathrm{s}} \subseteq P\left(m\right)\}$ and conversely $m\left(p^{\mathrm{s}}\right)$ as the municipality in which $p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}$ lies. We consider two superpharmacies $p^{\mathrm{s}}_1, p^{\mathrm{s}}_2 \in \mathcal{P}^{\mathrm{s}}$ in conflict if the aggregated pharmacies are in conflict, i.e., $\{p_1, p_2\} \in \mathcal{C}$ for all $p_1 \in p^{\mathrm{s}}_1$ and $p_2 \in p^{\mathrm{s}}_2$. Accordingly, we define $\mathcal{C}^{\mathrm{s}} = \{\{p^{\mathrm{s}}_1, p^{\mathrm{s}}_2\} \subseteq \mathcal{P}^{\mathrm{s}} \mid \{p_1, p_2\} \in \mathcal{C} \text{ for } p_1 \in p^{\mathrm{s}}_1, p_2 \in p^{\mathrm{s}}_2\}$.

For a given partition into superpharmacies $\mathcal{P}^{\mathrm{s}}$, we include the superpharmacies into the MILP given in Section 3.1 by replacing the variables $x \in \{0,1\}^{|\mathcal{P}||\mathcal{T}|}$ with decision variables $x^{\mathrm{s}} \in \{0,1\}^{|\mathcal{P}^{\mathrm{s}}||\mathcal{T}|}$ indicating for every superpharmacy $p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}$ and day $t \in \mathcal{T}$ whether p^{s} is assigned an out-of-hours service on that day. Furthermore, we restrict the continuous variables $\bar{y}_m, \underline{y}_m \geq 0$ to integer variables $\bar{y}^{\mathrm{s}}_m, \underline{y}^{\mathrm{s}}_m \in \mathbb{Z}_{\geq 0}$, since, in contrast to the MILP given in Section 3.1, we cannot assume these variables to be integer solely due to the structure of the MILP. In the proof of the exactness of this approach, we will see that the integrality is crucial for the splitting of services assigned to the superpharmacies. Here, $\bar{y}^{\mathrm{s}}_m$ and $\underline{y}^{\mathrm{s}}_m$ bound the maximum number (minimum number, respectively) of out-of-hours services performed by a pharmacy $p \in P\left(m\right)$ after splitting the services assigned to the superpharmacies $p^{\mathrm{s}} \in P^{\mathrm{s}}\left(m\right)$ among the corresponding pharmacies. The resulting *superpharmacy MILP* reads

$$
\begin{aligned}
\min \quad & \sum_{p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}} \sum_{t \in \mathcal{T}} x^{\mathrm{s}}_{p^{\mathrm{s}} t} && && \text{(2a)} \\
\text{s.t.} \quad & \sum_{p^{\mathrm{s}} \in C^{\mathrm{s}}(m)} x^{\mathrm{s}}_{p^{\mathrm{s}} t} \geq 1 && \forall m \in \mathcal{M}, t \in \mathcal{T}, && \text{(2b)} \\
& \sum_{p^{\mathrm{s}} \in P^{\mathrm{s}}(m)} x^{\mathrm{s}}_{p^{\mathrm{s}} t} \geq d\left(m, t\right) && \forall m \in \mathcal{M}, t \in \mathcal{T}, && \text{(2c)} \\
& x^{\mathrm{s}}_{p^{\mathrm{s}}_1 t} + x^{\mathrm{s}}_{p^{\mathrm{s}}_2 t} \leq 1 && \forall \left\{p^{\mathrm{s}}_1, p^{\mathrm{s}}_2\right\} \in \mathcal{C}^{\mathrm{s}}, t \in \mathcal{T}, && \text{(2d)} \\
& \sum_{t'=t}^{t+r(m(p^{\mathrm{s}}))} x^{\mathrm{s}}_{p^{\mathrm{s}} t'} \leq \left|p^{\mathrm{s}}\right| && \forall p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}, 1 \leq t \leq T - r\left(m\left(p^{\mathrm{s}}\right)\right), && \text{(2e)} \\
& \underline{y}^{\mathrm{s}}_{m(p^{\mathrm{s}})} \leq \frac{1}{\left|p^{\mathrm{s}}\right|} \sum_{t \in \mathcal{T}} x^{\mathrm{s}}_{p^{\mathrm{s}} t} \leq \bar{y}^{\mathrm{s}}_{m(p^{\mathrm{s}})} && \forall p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}, && \text{(2f)} \\
& \bar{y}^{\mathrm{s}}_m - \underline{y}^{\mathrm{s}}_m \leq e\left(m\right) && \forall m \in \mathcal{M}, && \text{(2g)} \\
& \underline{y}^{\mathrm{s}}_m \geq n^{\min} && \forall m \in \mathcal{M}, && \text{(2h)} \\
& x^{\mathrm{s}}_{p^{\mathrm{s}} t} \in \{0, 1\} && \forall p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}, t \in \mathcal{T}, && \text{(2i)} \\
& \bar{y}^{\mathrm{s}}_m, \underline{y}^{\mathrm{s}}_m \in \mathbb{Z}_{\geq 0} && \forall m \in \mathcal{M}. && \text{(2j)}
\end{aligned}
$$

In the objective function, as well as in the constraints (2b) to (2d), we simply replace the pharmacy variables by the ones for the superpharmacies. The inequalities (2e) allow the assignment of up to $|p^{\mathrm{s}}|$ services to a superpharmacy $p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}$ in an interval of $r\left(m\left(p^{\mathrm{s}}\right)\right) + 1$ consecutive days, since we will later split the services among $|p^{\mathrm{s}}|$ pharmacies. With the constraints (2f), we again bound the number of services that we assign to superpharmacies within a municipality and take into account that we can assign a superpharmacy $p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}$ more services according to its size $|p^{\mathrm{s}}|$. The inclusion of the factor $\frac{1}{|p^{\mathrm{s}}|}$ is the reason why we cannot assume integrality for $\bar{y}^{\mathrm{s}}_m$ and $\underline{y}^{\mathrm{s}}_m$, in contrast to the variables $\bar{y}_m, \underline{y}_m$. The remaining inequalities (2g) and (2h) are the same as in the original MILP, but with replaced variables $\bar{y}^{\mathrm{s}}_m$ and $\underline{y}^{\mathrm{s}}_m$.

The following result shows that the superpharmacy MILP is equivalent to the original MILP in Section 3.1. Furthermore, the proof gives instructions on how to convert a solution of the superpharmacy MILP to a solution of the original MILP.

Proposition 4. *For every solution $\left(x^s, \bar{y}^s, \underline{y}^s\right)$ of the superpharmacy MILP, there exists a solution $\left(x, \bar{y}, \underline{y}\right)$ of the original MILP with the same objective value and vice versa.*

Proof. Let $\left(x^{\mathrm{s}}, \bar{y}^{\mathrm{s}}, \underline{y}^{\mathrm{s}}\right)$ be a solution to the superpharmacy MILP. For a superpharmacy $p^{\mathrm{s}} = \{p_1, \ldots, p_n\} \in \mathcal{P}^{\mathrm{s}}$ that consists of n pharmacies, let $\{t_1, \ldots, t_q\} = \left\{t \in \mathcal{T} \mid x^{\mathrm{s}}_{p^{\mathrm{s}} t} = 1\right\}$ with $t_1 < \cdots < t_q$ be the set of days on which we assign services to p^{s}. In order to meet the period of rest constraints, we split the services among the pharmacies included in p^{s} in a cyclic way, i.e., we reassign the days $t_1, t_{n+1}, t_{2n+1}, \ldots$ to pharmacy p_1, the days $t_2, t_{n+2}, t_{2n+2}, \ldots$ to pharmacy p_2 and so on. By construction, we reassign each service to exactly one included pharmacy and therefore obtain $x^{\mathrm{s}}_{p^{\mathrm{s}}t} = \sum_{p \in p^{\mathrm{s}}} x_{pt}$. One can easily see that both solutions have the same objective value and that the constraints (1b) and (1c) are satisfied by replacing $x^{\mathrm{s}}_{p^{\mathrm{s}}t}$ with $\sum_{p \in p^{\mathrm{s}}} x_{pt}$. Furthermore, we violate no conflicts, since, also by replacing $x^{\mathrm{s}}_{p^{\mathrm{s}}t}$ with $\sum_{p \in p^{\mathrm{s}}} x_{pt}$ in (2d), we obtain even stronger inequalities than the ones in (1d). Now, assume we violate a period of rest constraint (1e). Then there exists a pharmacy $p \in p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}$ and two days $t_i < t_j \in \mathcal{T}$ with $t_j - t_i \leq r\left(m\left(p\right)\right)$ and $x_{pt_i} = x_{pt_j} = 1$. But then there exist at least $n - 1 = |p^{\mathrm{s}}| - 1$ days t between t_i and t_j with $x^{\mathrm{s}}_{p^{\mathrm{s}}t} = 1$ and we have $\sum_{t'=t_i}^{t+r(m(p^{\mathrm{s}}))} x^{\mathrm{s}}_{p^{\mathrm{s}}t'} \geq |p^{\mathrm{s}}| + 1$ which contradicts the constraints (2e). For the equity, we simply define $\bar{y} = \bar{y}^{\mathrm{s}}$ and $\underline{y} = \underline{y}^{\mathrm{s}}$ and therefore satisfy the constraints (1g) to (1h). By construction, we have $\sum_{t \in \mathcal{T}} x_{pt} \in \left\{ \left\lfloor \frac{1}{|p^{\mathrm{s}}|} \sum_{t \in \mathcal{T}} x^{\mathrm{s}}_{p^{\mathrm{s}}t} \right\rfloor, \left\lceil \frac{1}{|p^{\mathrm{s}}|} \sum_{t \in \mathcal{T}} x^{\mathrm{s}}_{p^{\mathrm{s}}t} \right\rceil \right\}$ for all pharmacies $p \in p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}$. Then in combination with (2f) and the definition of $\bar{y}^{\mathrm{s}}, \underline{y}^{\mathrm{s}}$, which we restricted to integral values, it follows that we satisfy the constraints (1f).

Let $\left(x, \bar{y}, \underline{y}\right)$ be a solution to the original MILP. Then we define $x^{\mathrm{s}}_{p^{\mathrm{s}}t} = \sum_{p \in p^{\mathrm{s}}} x_{pt}$ for all $p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}$ and $t \in \mathcal{T}$. Since all pharmacies within a superpharmacy are in conflict, at most one of them can have an out-of-hours service on a fixed day. Hence, we have $x^{\mathrm{s}} \in \{0,1\}^{|\mathcal{P}^{\mathrm{s}}||\mathcal{T}|}$. Again, by replacing $x^{\mathrm{s}}_{p^{\mathrm{s}}t}$ with $\sum_{p \in p^{\mathrm{s}}} x_{pt}$, we see that both solutions have the same objective value and that the constraints (1b) and (1c) are met. Furthermore, the conflicts are respected since a violation of the constraints (2d) would imply that there exist two pharmacies $p_1 \in p^{\mathrm{s}}_1$ and $p_2 \in p^{\mathrm{s}}_2$ with $\{p_1, p_2\} \in \mathcal{C}$ and $x_{p_1t} + x_{p_2t} = 2$, which is a contradiction to the constraints (1d). For the rest periods we have

$$\sum_{t'=t}^{t+r(m(p^{\mathrm{s}}))} x^{\mathrm{s}}_{p^{\mathrm{s}}t'} = \sum_{p \in p^{\mathrm{s}}} \sum_{t'=t}^{t+r(m(p))} x_{pt'} \overset{(1e)}{\leq} \sum_{p \in p^{\mathrm{s}}} 1 = |p^{\mathrm{s}}|$$

for all $p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}$ and $0 \leq t \leq T - r\left(m\left(p^{\mathrm{s}}\right)\right)$. Regarding the equity, we set $\bar{y}^{\mathrm{s}}_m = \max_{p \in P(m)} \left\{\sum_{t \in \mathcal{T}} x_{pt}\right\}$ and $\underline{y}^{\mathrm{s}}_m = \min_{p \in P(m)} \left\{\sum_{t \in \mathcal{T}} x_{pt}\right\}$. Then we obtain

$$\bar{y}^{\mathrm{s}}_{m(p^{\mathrm{s}})} = \frac{|p^{\mathrm{s}}|}{|p^{\mathrm{s}}|} \max_{p \in P(m(p^{\mathrm{s}}))} \left\{\sum_{t \in \mathcal{T}} x_{pt}\right\} \geq \frac{1}{|p^{\mathrm{s}}|} \sum_{p \in p^{\mathrm{s}}, t \in \mathcal{T}} x_{pt} = \frac{1}{|p^{\mathrm{s}}|} \sum_{t \in \mathcal{T}} x^{\mathrm{s}}_{p^{\mathrm{s}}t}$$

and analogously $\underline{y}^{\mathrm{s}}_{m(p^{\mathrm{s}})} \leq \frac{1}{|p^{\mathrm{s}}|} \sum_{t \in \mathcal{T}} x^{\mathrm{s}}_{p^{\mathrm{s}}t}$ for all $p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}$. By construction and due to the constraints (1f), it holds $\bar{y}^{\mathrm{s}} \leq \bar{y}$ and $\underline{y}^{\mathrm{s}} \geq \underline{y}$. Therefore, since the inequalities (1g) and (1h) are satisfied, the constraints (2g) and (2h) are also satisfied. □

Note that for every fractional solution $\left(x^{\mathrm{s}}, \bar{y}^{\mathrm{s}}, \underline{y}^{\mathrm{s}}\right)$ to the LP relaxation of the superpharmacy MILP (2), there exists a fractional solution $\left(x, \bar{y}, \underline{y}\right)$ to the LP relaxation of the original MILP (1) given by $\bar{y} = \bar{y}^{\mathrm{s}}$ and $\underline{y} = \underline{y}^{\mathrm{s}}$ as well as $x_{pt} = \frac{1}{|p^{\mathrm{s}}|} x^{\mathrm{s}}_{p^{\mathrm{s}}t}$ for all $p \in \mathcal{P}$ and $t \in \mathcal{T}$, where $p^{\mathrm{s}} \in \mathcal{P}^{\mathrm{s}}$ is chosen such that $p \in p^{\mathrm{s}}$. Thus, the reduced size of the formulation does not come at the cost of a weaker LP relaxation. In fact, the relaxation of the superpharmacy MILP is even stronger because of the strengthened inequalities (2d).

Since the number of variables and constraints of the superpharmacy MILP (2) decreases for partitions $\mathcal{P}^{\mathrm{s}}$ of $\mathcal{P}$ with fewer superpharmacies, we are interested in superpharmacies that contain many pharmacies. We call a superpharmacy $p_1^{\mathrm{s}} \subseteq \mathcal{P}$ *maximal* if there exists no other superpharmacy $p_2^{\mathrm{s}} \subseteq \mathcal{P}$ with $p_1^{\mathrm{s}} \subsetneq p_2^{\mathrm{s}}$. Fortunately, there exists a unique partition $\mathcal{P}^{\mathrm{s}}$ of $\mathcal{P}$ into maximal superpharmacies that we obtain by greedily aggregating equivalent pharmacies. For three pharmacies $p, p', p'' \in \mathcal{P}$ with $p \sim p'$ and $p' \sim p''$, it follows $p \sim p''$. Hence, for two maximal superpharmacies $p_1^{\mathrm{s}} \neq p_2^{\mathrm{s}} \subseteq \mathcal{P}$, it must hold $p_1^{\mathrm{s}} \cap p_2^{\mathrm{s}} = \emptyset$, since otherwise $p_1^{\mathrm{s}} \cup p_2^{\mathrm{s}}$ would also be a superpharmacy, contradicting the maximality of p_1^{s} or p_2^{s}. It follows that every pharmacy is contained in exactly one maximal superpharmacy, which shows that it is sufficient to extend a superpharmacy greedily until there exists no further pharmacy to add.

3.3. Rolling Horizon Approach

For instances that consist of many pharmacies and a long time horizon, the MILP (1) given in Section 3.1 becomes very large, even if we use the superpharmacy approach of Section 3.2 to reduce its size. Therefore, we decompose the OHP into smaller, tractable pieces.

The decision variables $x_{pt}, x_{pt'}$ belonging to different days $t, t' \in \mathcal{T}$ are solely connected by the periods of rest, the equity, and minimum number of services constraints. Intuitively, the choice of $x_{pt} \in \{0,1\}$ becomes less important for the choice of $x_{pt'} \in \{0,1\}$ for days $|t - t'| > r\,(m\,(p))$ that are far apart from each other, because they are not connected directly by a period of rest constraint. This raises the idea that we may end up with an out-of-hours plan not too far from an optimal solution if we plan the first days of the time horizon without paying too much attention to later days. An approach that often leads to high quality solutions for planning problems with such characteristics is the use of a *rolling horizon algorithm.* The general idea of rolling horizon algorithms is to divide a problem with a large time horizon into a sequence of smaller subproblems in which one considers only a part of the time horizon. By fixing decisions arising from the solution of the subproblems and extending the considered part of the time horizon, one iteratively tries to compute a solution for the original problem.

For solving an instance of the OHP, we split the time horizon $\mathcal{T}$ into intervals $\mathcal{T} = \{1, \ldots, t_1\} \cup \{t_1 + 1, \ldots, t_2\} \cup \cdots \cup \{t_k + 1, \ldots, T\}$. We determine for each day $t \in \{1, \ldots, t_1\}$ a set of out-of-hours pharmacies $F\,(t) \subseteq \mathcal{P}$, fix these services and plan the next interval $\{t_1 + 1, \ldots, t_2\}$ while respecting the constraints arising from the assignments already made for $\{1, \ldots, t_1\}$.

Rolling horizon algorithms usually use some kind of look-ahead, such that the decisions made in the current iteration also consider subsequent iterations. This can be achieved, for example, by relaxing the original problem for the remainder of the time horizon through aggregation [21]. In our approach, we do not consider a relaxation for the remaining days but instead include an additional interval into our subproblem. We start with the computation of a partial solution on the days $\{1, \ldots, t_1\} \cup \{t_1 + 1, \ldots, t_2\}$ but then only fix services assigned in the first interval $\{1, \ldots, t_1\}$. By doing this, we pay attention to the direct connection induced by the periods of rest, which are arguably the most restrictive temporal constraints for our problem. We do not consider further days, as we want to keep the subproblems small and more days in a relaxed version do not give much additional information since the planning is somewhat repetitive.

For the computation of the partial plans on a subset of days $\mathcal{T}' = \{1, \ldots, T'\} \subseteq \mathcal{T}$, respecting services $S(p) \subseteq \mathcal{T}'$ already assigned to the pharmacies $p \in \mathcal{P}$, we use an MILP as follows

$$\min \quad \sum_{p \in \mathcal{P}} \sum_{t \in \mathcal{T}'} x_{pt} + \sum_{m \in \mathcal{M}} M v_m \tag{3a}$$

$$\text{s.t.} \quad \sum_{p \in C(m)} x_{pt} \geq 1 \qquad \forall m \in \mathcal{M}, t \in \mathcal{T}', \tag{3b}$$

$$\sum_{p \in P(m)} x_{pt} \geq d(m,t) \qquad \forall m \in \mathcal{M}, t \in \mathcal{T}', \tag{3c}$$

$$x_{pt} + x_{p't} \leq 1 \qquad \forall \{p, p'\} \in \mathcal{C}, t \in \mathcal{T}', \tag{3d}$$

$$\sum_{t'=t}^{t+r(m(p))} x_{pt'} \leq 1 \qquad \forall p \in \mathcal{P}, \; 1 \leq t \leq T' - r(m(p)), \tag{3e}$$

$$\underline{y}_{m(p)} \leq \sum_{t \in \mathcal{T}'} x_{pt} \leq \bar{y}_{m(p)} + v_{m(p)} \qquad \forall p \in \mathcal{P}, \tag{3f}$$

$$\bar{y}_m - \underline{y}_m \leq \left\lceil \frac{e(m)\, T'}{T} \right\rceil \qquad \forall m \in \mathcal{M}, \tag{3g}$$

$$\underline{y}_m \geq \left\lceil \frac{n^{\min} T'}{T} \right\rceil \qquad \forall m \in \mathcal{M}, \tag{3h}$$

$$x_{pt} = 1 \qquad \forall p \in \mathcal{P}, t \in S(p), \tag{3i}$$

$$x_{pt} \in \{0, 1\} \qquad \forall p \in \mathcal{P}, t \in \mathcal{T}' \backslash S(p), \tag{3j}$$

$$\bar{y}_m, \underline{y}_m \geq 0 \qquad \forall m \in \mathcal{M}, \tag{3k}$$

$$v_m \in \mathbb{Z}_{\geq 0} \qquad \forall m \in \mathcal{M}. \tag{3l}$$

In addition to the restriction of the time horizon to $\mathcal{T}'$, we fix the variables $x_{pt} = 1$ if we already assigned $p \in \mathcal{P}$ a service on $t \in \mathcal{T}$ in a previous iteration, i.e., $t \in S(p)$. Note that we do not force $x_{pt} = 0$ even if we already considered $t \in \mathcal{T}'$ in a previous iteration and did not assign $p \in \mathcal{P}$ a service on day t. This gives us more freedom when satisfying the constraints 3g and 3h. Regarding the equity, we introduce variables $v_m \in \mathbb{Z}_{\geq 0}$ for all $m \in \mathcal{M}$ that allow a violation of the constraints (1f) but are penalized by a big constant $M \in \mathbb{Z}_{\geq 0}$ in the objective (3a). By doing so, we guarantee that the MILP will not be infeasible for an interval $\{1, \ldots, T'\}$ due to the equity constraints. This is reasonable, since we may be able to balance the out-of-hours plan in the following iterations. We call the planning problem in which we allow, but punish, a violation of the equity constraints the *equity relaxed OHP*. Note that we scale the equity coefficients included in the constraints (3g) and the minimum number of services to assign in the inequalities (3h). The logic behind the scaling is that we do not want to assign a pharmacy many services within one iteration just in order to meet the minimum number of services constraint (1h). Instead, we want to encourage

Algorithm 1: Rolling horizon heuristic for the computation of out-of-hours plans.

Input: An OHP instance and a set of days $\{t_1, \ldots, t_k\} \subseteq \mathcal{T}$

Output: A solution $x \in \{0,1\}^{|\mathcal{P}||\mathcal{T}|}$ to the equity relaxed OHP or $\emptyset$

Initialize $S(p) = \emptyset$ for all $p \in \mathcal{P}$ and $t_{k+1} = T$;
for $i \in \{1, \ldots, k\}$ **do**
    Set $\mathcal{T}' = \{1, \ldots, t_{i+1}\}$;
    if *there exists a solution to the MILP (3)* **then**
        Compute a solution $x \in \{0,1\}^{|\mathcal{P}||\mathcal{T}'|}$;
        Update $S(p) = \{t \in \{1, \ldots, t_i\} \mid x_{pt} = 1\}$;
    end
end
if *x is a feasible out-of-hours plan* **then**
    return x
end
else
    return $\emptyset$
end

a useful assignment of these services, evenly distributed over the whole time horizon, which relieves other pharmacies with more services. For the equity, it would be unreasonable to allow a difference using unscaled equity coefficients $e(m)$ within the first time intervals, because this restricts us in the planning of the following time intervals.

Algorithm 1 states the rolling horizon algorithm, which uses the above MILP (3). Note that we only consider a so called forward rolling horizon approach, i.e., we always start at the beginning of the planning horizon, since in practice the planning is restricted by additional periods of rest constraints arising from previous planning periods.

3.4. Deletion of Equity Services

A potential problem of Algorithm 1 is that we fix services which are not necessary for the coverage solely to satisfy the equity and minimum number of services constraints within one iteration. These fixed *equity services* raise the total number of out-of-hours services and take away the possibility to satisfy the corresponding constraints in a later iteration by assigning more useful services. To resolve this problem, we extend Algorithm 1 with an additional step within each iteration of the for-loop in which we delete such equity services from $S(p)$ for $p \in \mathcal{P}$.

We want to delete as many services as possible while still satisfying all constraints but the ones for the equity and minimum number of services. Since we are restricted to services in $S(p)$, we do not have to consider the conflict constraints (3d) and periods of rest inequalities (3e), as they are always satisfied by the choice of $S(p)$. Therefore, we are only left with the covering constraints and can formulate the deletion of equity services as a covering problem in which we want to retain a minimum number of services. Covering problems are $\mathcal{NP}$-hard in general but in our practical case the problem of deleting equity services remains tractable. This is due to the fact that all days can be considered independently, as we have no more linking constraints, and the number of services per day is relatively small. In our case study, we will handle the deletion problem in iteration $i \in \{1, \ldots, k\}$ by directly solving the following MILP

$$\min \quad \sum_{p \in \mathcal{P}} \sum_{t \in \{1,\ldots,t_i\}} x_{pt} \tag{4a}$$

$$\text{s.t.} \quad \sum_{p \in C(m)} x_{pt} \geq 1 \qquad \forall m \in \mathcal{M}, t \in \{1, \ldots, t_i\}, \tag{4b}$$

$$\sum_{p \in P(m)} x_{pt} \geq d(m,t) \qquad \forall m \in \mathcal{M}, t \in \{1, \ldots, t_i\}, \tag{4c}$$

$$x_{pt} = 0 \qquad \forall p \in \mathcal{P}, t \notin S(p) \tag{4d}$$

$$x_{pt} \in \{0,1\} \qquad \forall p \in \mathcal{P}, t \in S(p). \tag{4e}$$

In each iteration except the last of the extended algorithm, after solving the MILP (3) and setting $S(p)$, we compute an optimal solution $x \in \{0,1\}^{|\mathcal{P}|t_i}$ to the MILP (4) and again update $S(p) = \{t \in \{1, \ldots, t_i\} \mid x_{pt} = 1\}$. The sets $S(p)$ contain no more equity services after the deletion step but the remaining services were planned with the equity and minimum number of services constraints in mind.

Although Algorithm 1 may return a solution that violates equity constraints or may terminate without finding a solution at all, we will see that, for the instances tested in our real-world case study and a proper size of intervals, it computes solutions with a small optimality gap and few violations of equity constraints. A benefit of the approach is that it tends to distribute the services assigned to a pharmacy evenly over the time horizon, since, even with the deletion of equity services, we encourage equity in each iteration.

Note that the superpharmacy approach of Section 3.2 can be easily adapted for the MILP (3) and combined with the rolling horizon heuristic.

4. Case Study

In this section, we study the performance of our approaches from Section 3 and the effect of the input parameters on the resulting plans of a real-world test instance. Finally, we will analyze the plans regarding their coverage properties and the distribution of the out-of-hours services. All tests are implemented in Java 8 on a Linux Ubuntu 14.04 distribution with kernel version 3.13 and run on an Intel® Core™i7-3770 CPU @ 3.4 GHZ with 32 GB RAM. We use CPLEX version 12.6.3 [26] to solve the MILPs.

4.1. Setup of the Computational Study

Our test instance is based on the planning of out-of-hours services for the year 2017 in the area of North Rhine in Germany. Accordingly, our time horizon is $\mathcal{T} = \{1, \ldots, 365\}$. The Chamber of Pharmacists North Rhine provided a set of $|\mathcal{P}| =$

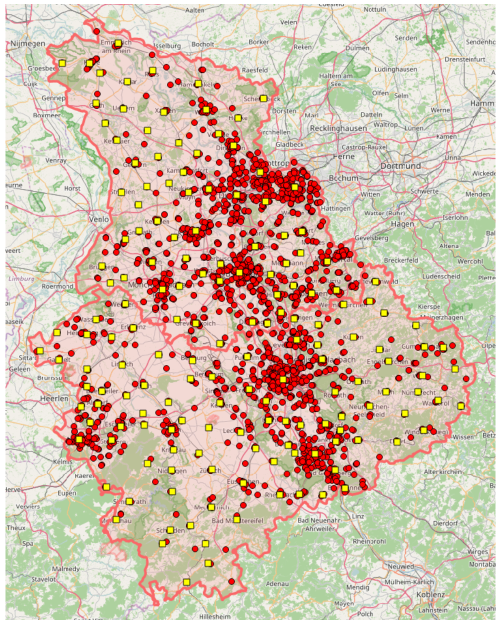


Figure 3: Geographical distribution of the pharmacies (red dots) and the municipalities (yellow squares).

2291 pharmacies that existed in October 2016 and $|\mathcal{M}| = 165$ municipalities, see Figure 3. The set of municipalities ranges from large cities such as Cologne in the mid-east of North Rhine, containing about one million residents and 244 pharmacies, to rural municipalities in the Eifel region in the south-west, containing a few thousand residents and often only one pharmacy. Data on the affiliation $m : \mathcal{P} \to \mathcal{M}$ of pharmacies and distances $\delta : (\mathcal{P} \cup \mathcal{M})^2 \to \mathbb{Q}_{\geq 0}$ on the road network is fixed by the geographical properties of the instance and provided by the Chamber of Pharmacists North Rhine.

The cover radii $\delta^{\text{cov}} : \mathcal{M} \to \mathbb{Q}_{\geq 0}$, minimum distances $\delta^{\text{con}} : \mathcal{M} \to \mathbb{Q}_{\geq 0}$ and periods of rest $r : \mathcal{M} \to \mathbb{Z}_{\geq 0}$ are defined via a classification of the municipalities into four categories: large cities, medium-sized towns, small towns and rural municipalities. For each category, the Chamber of Pharmacists North Rhine sets the three parameters based on legal restrictions or experience from prior planning periods. These parameters are then applied to the municipalities according to their category. Table 1 shows the parameters and the number of municipalities for each category as well as the number of pharmacies within the municipalities belonging to the corresponding category. The minimum distances, which determine the conflicts between out-of-hours pharmacies, range from 2 km for larger cities, where pharmacies are typically clustered, to 7 km for rural areas. Conversely, the periods of rest range from 15 days in larger cities to 5 days in rural areas, where few pharmacies usually have to perform more services. The cover radii range from 10 km for larger cities to 30 km for rural municipalities. For three municipalities in remote areas, we increase the cover radii up to 10%, since otherwise the number of covering pharmacies would be too

	total	large city	medium-sized town	small town	rural municipality
#municipalities	165	22	101	34	8
#pharmacies	2291	1408	799	70	14
cover radius δ^{cov}	-	10 km	15 km	20 km	30 km
period of rest r	-	15 days	7 days	7 days	5 days
minimum distance δ^{con}	-	2 km	4 km	4 km	7 km

Table 1: Numbers of municipalities and pharmacies as well as parameters by categories.

small for the period of rest. Together with the distances between municipalities and pharmacies, the cover radii give us sets of covering pharmacies $C(m)$ for all $m \in \mathcal{M}$ with sizes ranging from 10 to 237 pharmacies. From this, we know that there are pharmacies that perform at least 37 services over the course of the year.

In contrast to the parameters defined above, the demands $d : \mathcal{M} \times \mathcal{T} \to \mathbb{Z}_{\geq 0}$ are individually set by the chamber of pharmacists for each municipality. The demands are zero year-round for 117 of the 165 municipalities and go up to 9 services per day for the city of Cologne. Summing over the demands of all days $\sum_{t \in \mathcal{T}} d(m,t)$ for a municipality $m \in \mathcal{M}$ and dividing by the number of pharmacies $|P(m)|$ included gives us a lower bound on the average number of services per pharmacy going up to 20.28 for one of the municipalities.

For our first computational experiments, we choose $n^{\min} = 10$ as the minimum number of services per pharmacy. This value is based on existing plans of the Chamber of Pharmacists North Rhine. Since the Chamber of Pharmacists North Rhine aims to equally split the burden for all pharmacies within one municipality, we choose equity coefficients $e \equiv 1$, i.e., the difference in the number of services assigned to two pharmacies within the same municipality cannot exceed one.

As we already mentioned, the basic MILP (1) becomes very large for the given instance. In fact, CPLEX is not able to solve the LP relaxation in the root node of the branching tree within a time limit of one day. Nevertheless, CPLEX found a feasible solution that assigns 32,451 services, using primal heuristics in parallel while trying to solve the LP relaxation. In the following, we test combinations of the approaches from Section 3, namely the aggregation of pharmacies as well as the rolling horizon algorithm with its extension, and show that they are more reliable and lead to better results in less time.

The aggregation of pharmacies into superpharmacies, as proposed in Section 3.2, reduces the size of the MILP drastically. We can partition the set of $|\mathcal{P}| = 2291$ pharmacies into $|\mathcal{P}^{\text{s}}| = 1745$ superpharmacies, thus reducing the number of binary decision variables by almost a quarter. Here, 1414 pharmacies $p \in \mathcal{P}$ constitute their own superpharmacy $\{p\} \in \mathcal{P}^{\text{s}}$ and 877 pharmacies can be aggregated into 331 superpharmacies of size greater than or equal to two, with the largest superpharmacy containing 9 pharmacies. Table 2 shows the size of the constraint matrix for both MILPs after CPLEX performed its preprocessing and demonstrates the impact of the superpharmacies. As a result of the reduction, CPLEX is able to solve the LP

	#rows	#columns	#nonzero entries
original MILP (1)	$4,181,984$	$836,380$	$22,658,004$
superpharmacy MILP (2)	$1,060,723$	$637,036$	$14,630,106$

Table 2: Size of the constraint matrices after preprocessing of CPLEX.

relaxation of the superpharmacy MILP (2) within $42,955$ seconds. From the relaxed solution, we deduce that a feasible out-of-hours plan assigns at least $30,078$ services among the pharmacies, which is 7.3% below the number of services assigned by the plan computed with the original MILP (1) in 24 hours. Surprisingly, in contrast to the computation using the original MILP (1), we did not obtain a feasible integer solution within a time limit of one day. Although this may suggest that the aggregation of pharmacies is counterproductive for the computation of plans, we will see that the superpharmacies perform well together with the rolling horizon approach.

In order to test the rolling horizon approach, we need to define a penalty parameter M for the violation of the equity constraints and intervals in which we partition the time horizon. For the penalty parameter, we choose $M = 1000$, which seems to be large enough so that an optimal solution to the equity relaxed OHP, if possible, violates no equity constraint. For the partition of the time horizon, we set a *number of intervals* $2 \leq I \leq T$ and divide the time horizon $\mathcal{T}$ into intervals of lengths that are equal except for rounding, i.e., $\mathcal{T} = \left\{1, \ldots, \left\lfloor \frac{T}{I} \right\rfloor\right\} \cup \left\{\left\lfloor \frac{T}{I} \right\rfloor + 1, \ldots, \left\lfloor \frac{2T}{I} \right\rfloor\right\} \cup \cdots \cup \left\{\left\lfloor \frac{(I-1)T}{I} \right\rfloor + 1, \ldots, T\right\}$. Note that the choice of $I = 2$ corresponds to a single computation of the whole time horizon since we always consider two intervals in Algorithm 1. We test the values $I \in \{2, \ldots, 52\}$, thus starting with the whole time horizon for the sake of comparison and ending with an interval length of one week. For a comprehensive test of the performance of our approaches, we perform four computations for every value $I \in \{2, \ldots, 52\}$ using the following four algorithms arising of the possible combinations of the approaches proposed in Section 3:

Default — Algorithm Default makes no use of superpharmacies (cf. Section 3.2) and does not delete equity services (cf. Section 3.4).

DelEq — Algorithm DelEq also makes no use of superpharmacies but does delete equity services.

Sup — Algorithm Sup uses superpharmacies but does not delete equity services.

SupDelEq — Algorithm SupDelEq uses superpharmacies and also deletes equity services.

We set an aggregated time limit of 10,000 seconds for the computation of the MILPs (3). This reflects on the one hand, that we compute plans for a whole year and thus the construction of a plan does not have to be finished within few seconds. On the other hand, the computation should not take too long, since the algorithms are intended for a decision support. We expect that the decision makers have to perform many computations during the planning process, all with different input parameters and possibly new, tailored constraints for special cases among the pharmacies and municipalities.

Although the use of superpharmacies, deletion of equity services, and division into many intervals may cause some overhead in time, they do not dictate the complexity of the whole solution process. Therefore, we only consider the time taken to solve the MILPs (3). Let I' be the number of iterations left and T' the aggregated time it took to solve the previous MILPs. We set the time limit for the next MILP computation as $\frac{10000-T'}{I'}$. Accordingly, the time limit for each iteration is at least $\frac{10000}{I-1}$ and time saved in earlier iterations can be used for subsequent computations. Instead of using excessive time trying to prove optimality in each iteration, we terminate each MILP computation if we reach an optimality gap of 0.1%. By doing this, we save time for later, maybe more problematic iterations.

4.2. Performance of the Solution Approaches

In the following, we analyze the quality of the solutions obtained by the algorithms, i.e., the number of services assigned and the number of violated equity constraints. We pay particular attention to the reliability of our approaches and examine if the solution quality varies heavily for the choices of $I \in \{2, \dots, 52\}$ or if the performance is stable for a range of numbers of intervals. Furthermore, we compare the computational time of the algorithms. Exact values, corresponding to the plots within this section, are given in Table B.4 and Table B.5 in Appendix B.

Figure 4a shows for all solutions the number of services that we assign among the pharmacies as well as the lower bound of $30,078$ services, which equals the optimal solution value of the LP relaxation rounded up. A gap in the graph of an approach indicates that we did not obtain a solution for the corresponding number of intervals. Additionally, an unfilled circle indicates that we obtained a solution that violates the equity constraints, which is penalized in MILP (3). Figure 4b shows for all solutions the sum of violations $\sum_{m \in \mathcal{M}} v_m$, scaled logarithmically if greater than zero.

Apparently, the partial problems remain too hard to solve for a small number of large intervals. While the computations in which we use the superpharmacies consistently result in an out-of-hours plan, most of them violate the equity constraints for small I. If we do not use superpharmacies, then we sometimes obtain no plan at all after $10,000$ seconds. This indicates that the use of superpharmacies has a positive effect on the reliability of our algorithms. The same holds for the deletion of equity services. This additional step not only reduces the number of services assigned, but also results more frequently in plans that are not violated, compared to the algorithms in which we do not delete equity services. This is a logical consequence, since the absence of fixed equity services gives us more freedom to equalize an out-of-hours plan in later iterations. The algorithms in which we delete the equity services are also less sensitive to a variation in the number of intervals I, that is, we also obtain high quality solutions for large numbers I. This is in contrast to the algorithms in which we do not delete these services and therefore require equity in all $I-1$ iterations.

Starting from $I = 12$, SupDelEq returns consistently very good results that are also feasible, except for $I = 18$. The most efficient feasible plan that we obtain through this approach is computed for $I = 15$ and assigns only $30,174$ services, resulting

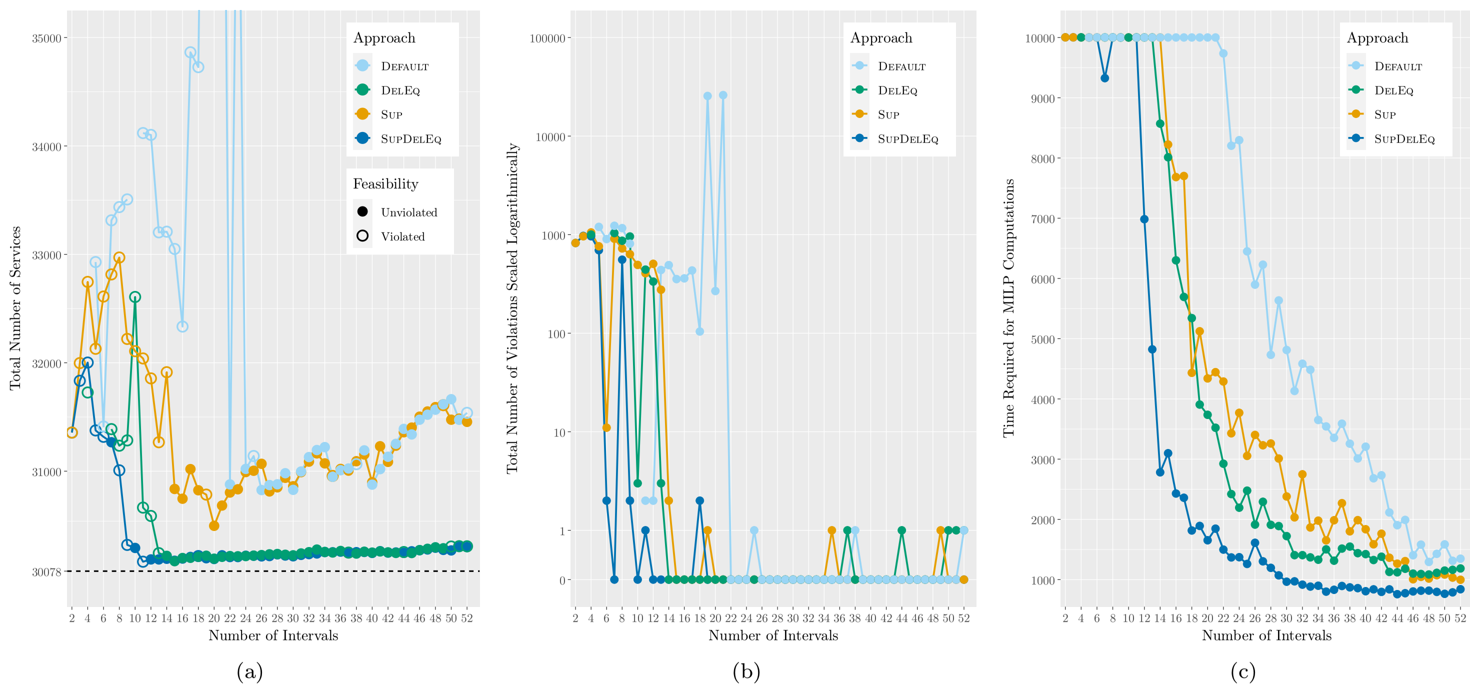


Figure 4: Total numbers of services assigned for all solutions as well as the lower bound 30078. Solutions that violate equity constraints are displayed as unfilled circles (a). Total number of violations of equity constraints for all solutions, represented with logarithmic scaling (b). Time required to solve the MILPs (c).

in a gap of 0.3192% compared to the lower bound. Even for $I = 52$, where we consider many small intervals and therefore tend to make more bad decisions during the planning, we obtain an out-of-hours plan that only assigns 30, 300 services. For comparison, DEFAULT computes a plan that assigns 31, 538 services for $I = 52$.

DELEQ leads to similar results for $I \geq 14$ and for $I = 15$ even to the best known feasible solution, which assigns 30, 170 services. Compared to the real plan used by the Chamber of Pharmacists North Rhine in 2017, which assigns 33, 574 services and does not satisfy all constraints, this is a reduction of 10.1%.

Figure 4c shows the total time required for the MILP computations. For $I \geq 14$, both approaches in which we delete equity services perform not only similar regarding the total number of services but also terminate before the time limit of 10, 000 seconds. If we consider $I = 23$ as an outlier then the same holds for both approaches in which we do not delete equity services for $I \geq 22$. This suggests that in general it makes no difference whether we use superpharmacies or not as long as the intervals are small enough to be solvable without superpharmacies. This is an expected result since the original MILP (1) and the superpharmacy MILP (2) are equivalent. However, the outcome could have been different if the formulations led CPLEX to construct solutions of different structure.

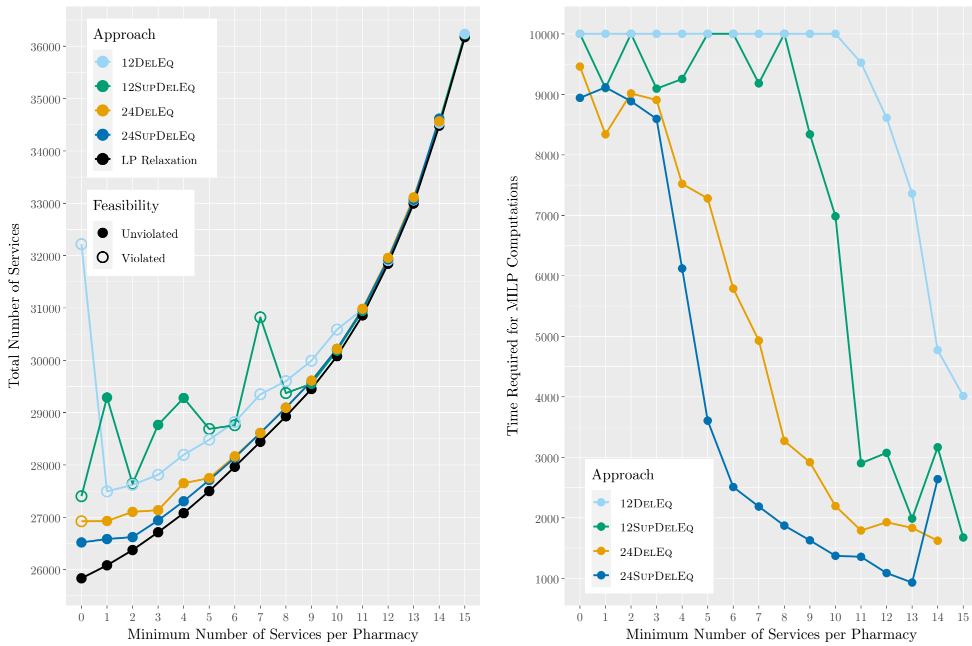
A result that was less expected is that the deletion of equity services speeds up the MILP computations, even though we fix fewer services from previous iterations. Even more surprisingly, the time saved by the deletion exceeds the savings achieved by using superpharmacies. We already mentioned that the deletion of equity services results in less violations of the equity constraints. Since the optimal solutions of the LP relaxations mostly do not violate any equity constraints, even for later iterations in which we fix many services, it is much harder and time-consuming to close the optimality gap if we cannot avoid the violation of an equity constraint. Therefore, it is easier to close the optimality gap if we delete equity services, which significantly reduces the computation time.

We summarize that, at least for the tested setting, using superpharmacies and deleting equity services speeds up the rolling horizon algorithm which enables us to find good solutions in less time within each iteration. Furthermore, the deletion of equity services makes the rolling horizon algorithm more reliable, since it reduces the total number of services assigned by fixing fewer bad decisions made in earlier intervals.

4.3. Influence of Input Parameters

In the following, we study the influence of the minimum number of services $n^{\min} \in \mathbb{Z}_{\geq 0}$ and equity coefficients $e : \mathcal{M} \to \mathbb{Q}_{\geq 0}$ on the resulting plans and also on the performance of our approaches. We choose these parameters for our analysis, as the Chamber of Pharmacists North Rhine requires the corresponding constraints but left a specification of these parameters open for discussion. Additionally, since both should ensure a balanced load on the pharmacies, it is particularly interesting to see how the plan changes if we adjust the level of balancing that we demand from a plan.

To test the influence of the minimum number of services, we consider the same setting as in the previous section except that we choose values $n^{\min} \in \{0, 1, \ldots, 15\}$. We solve the resulting problems by using the two approaches that performed best



(a) (b)

Figure 5: Total numbers of services assigned for all solutions. Solutions that violate equity constraints are displayed as unfilled circles (a). Time required to compute the solutions (b).

in the previous section, namely DelEq and SupDelEq. For each approach, we use a number of $I \in \{12, 24\}$ intervals, since we find it intuitive to consider intervals that have a length of one month, or half a month respectively. Additionally, both approaches showed a stable performance around $I = 24$ and the superpharmacy approach resulted in solutions of high quality for $I \geq 12$ and worse solutions for $I \leq 11$, which is why this breakpoint is of particular interest. We denote with **12DelEq** and **24DelEq** the use of approach DelEq together with a partition into 12 intervals, or 24 intervals respectively. Analogously, we denote with **12SupDelEq** and **24SupDelEq** the use of 12 or 24 intervals together with SupDelEq. Exact results are given in Table B.6 in Appendix B.

Figure 5a shows the total number of services assigned in the resulting plans. Additionally, it shows a lower bound on the number of services for the particular value of $n^{\min}$, which we obtain by solving the LP relaxation in the root node considering the whole time horizon. It becomes clear that the minimum number of services $n^{\min}$ has a large effect on the total number of services and the corresponding constraints (3h) are an actual restriction, even for low values of $n^{\min}$. Although we don't know the value of an optimal integer solution, we can deduce that an optimal solution for $n^{\min} \in \{0, 1, 2\}$ assigns not more than two services to at least one pharmacy since the best known integer solutions have lower values than the lower bound for $n^{\min} = 3$. The values of the relaxed solutions even suggest that there may be pharmacies to which an optimal plan would assign no services at all. Furthermore, we observe that the slope defined by the total number of services depending on the minimum number of services rises with increasing $n^{\min}$. This is due to the fact that the constraints (3h) are relevant for only a few pharmacies for low $n^{\min}$ and the number of pharmacies that are affected by these constraints rises with the value of $n^{\min}$.

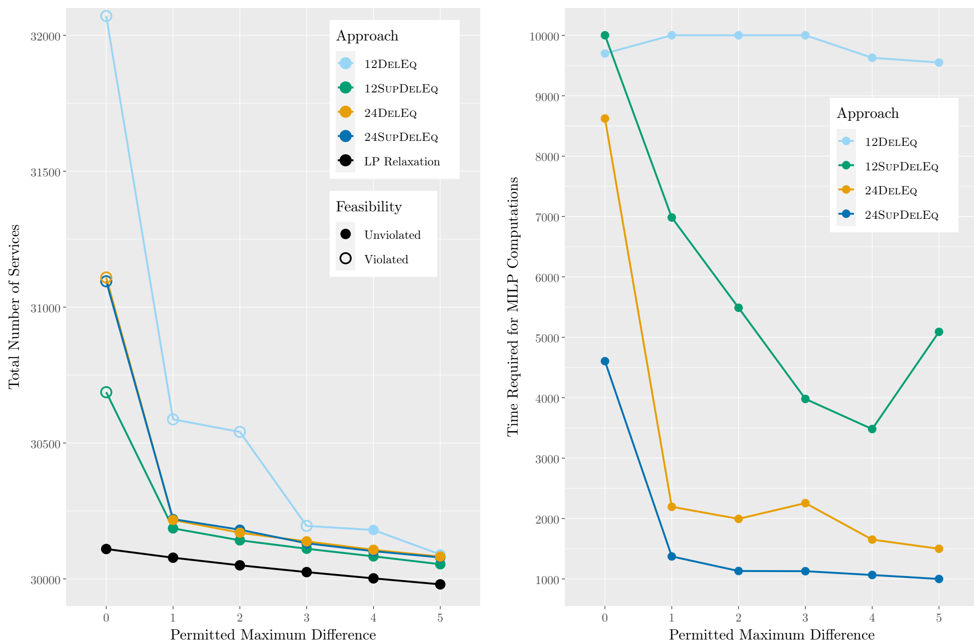
The minimum number of services $n^{\min}$ has not only an effect on the total number of services but also on the performance of our algorithms. For $n^{\min} = 15$, we find no out-of-hours plan within the time limit using $I = 24$. The small intervals often give us not enough freedom to meet the highly restricting minimum number constraints (3h), leading to some infeasible subproblems, for example here in the first iteration. The following subproblem becomes harder to solve, since we fix no services and additionally have a rather small time limit, as we distribute the remaining time on many intervals. These problems propagate through all iterations, which is why we obtain no plan after the last iteration.

Apart from this, the problems tend to be easier to solve and require less computation time for high values $n^{\min}$, as shown in Figure 4c. One intuitive reason for this is that a high minimum number of services raises the value of the LP relaxation more than it raises the value of an optimum integral solution, since the latter assigns more services anyway. Furthermore, we observe that the solutions for the LP relaxation tend to be less fractional for high $n^{\min}$. For example, the solution of the LP relaxation within the first iteration of the algorithm for $n^{\min} = 0$ using $I = 12$ and superpharmacies contains a total of 8863 integer infeasible variables, i.e., variables that should be integer but take fractional values. For $n^{\min} = 10$, the number of integer infeasible variables is only 1998. In practice, it is advantageous to have an LP relaxation solution that has few integer infeasible variables, since some primal heuristics in state-of-the-art MILP solvers perform better if the relaxed solution is almost integer. An example for this is the feasibility pump [27], which tries to convert the optimum solution of the LP relaxation to a feasible integer solution.

Once more, the difficult instances highlight the advantages of using superpharmacies, as these approaches perform better than the approaches in which we do not use them. We want to emphasize that the plans computed with 12SupDelEq often assign more services than the plans obtained by 12DelEq but they are much better regarding the violation of the equity constraints and thus have consistently better objective values. For $n^{\min} \in \{1, 3, 4\}$, the plans computed with 12SupDelEq violate no equity constraints and for $n^{\min} = 7$, we only have a violation of $\sum_{m \in \mathcal{M}} v_m = 2$, while the plans computed with 12DelEq each have a violation $\sum_{m \in \mathcal{M}} v_m \geq 334$ for $n^{\min} \leq 10$, i.e., on average two per municipality.

To test the equity coefficients, we set the minimum number of services back to $n^{\min} = 10$ and choose $e \equiv 0, \ldots, 5$ as a global parameter that is the same for all municipalities. We use the approaches from above to solve the resulting instances of the OHP. Exact results are given in Table B.7 in Appendix B.

We show the total number of services assigned in the computed plans as well as the corresponding lower bounds in Figure 6a. As expected, forcing all pharmacies within a municipality to perform an equal number of out-of-hours services leads to a significant increase in the total number of services assigned in the computed plans. Additionally, none of the plans is feasible regarding the equity constraints. The most



(a) (b)

Figure 6: Total numbers of services assigned for all solutions. Solutions that violate equity constraints are displayed as unfilled circles (a). Time required to compute the solutions (c).

efficient plan with respect to the number of services, computed by 12SupDelEq, has a violation of $\sum_{m \in \mathcal{M}} v_m = 476$, i.e., on average 2.88 per municipality, and is thus highly infeasible. The other three plans each have a cumulative violation of only 2 but are much less efficient. Since the optimality gap is quite large, it remains unclear if an optimal integer solution is equally inefficient or if the rolling horizon algorithms simply make too many bad decisions.

Surprisingly, compared to $e \equiv 1$, the number of services assigned does not decrease much with higher equity coefficients. This holds both for the lower bounds and the solutions computed by the rolling horizon algorithms. So, in contrast to the minimum number of services, where a rising value of $n^{\min}$ leads to significantly more services, the cost of stricter equity constraints is only marginal as long as we do not choose $e \equiv 0$. Regarding the time required by the algorithms, shown in Figure 6b, the problems tend to get easier to solve for higher e, where it is easier to satisfy the equity constraints.

Overall, the effect of the equity coefficients seems to be not as strong compared to the impact of the minimum number of services $n^{\min}$.

4.4. Discussion of the Out-of-Hours Plans

In the following, we analyze the plans computed in Section 4.3 regarding their coverage properties and the distribution of the services among the pharmacies, i.e.,

	$n^{\min} = 0$	$n^{\min} = 5$	$n^{\min} = 10$	$n^{\min} = 15$	Real Plan 2017
total number of services	26,520	27,712	30,186	36,224	33,574
mean distance to nearest pharmacy	8,705 m	8,456 m	8,030 m	7,559 m	7,204 m

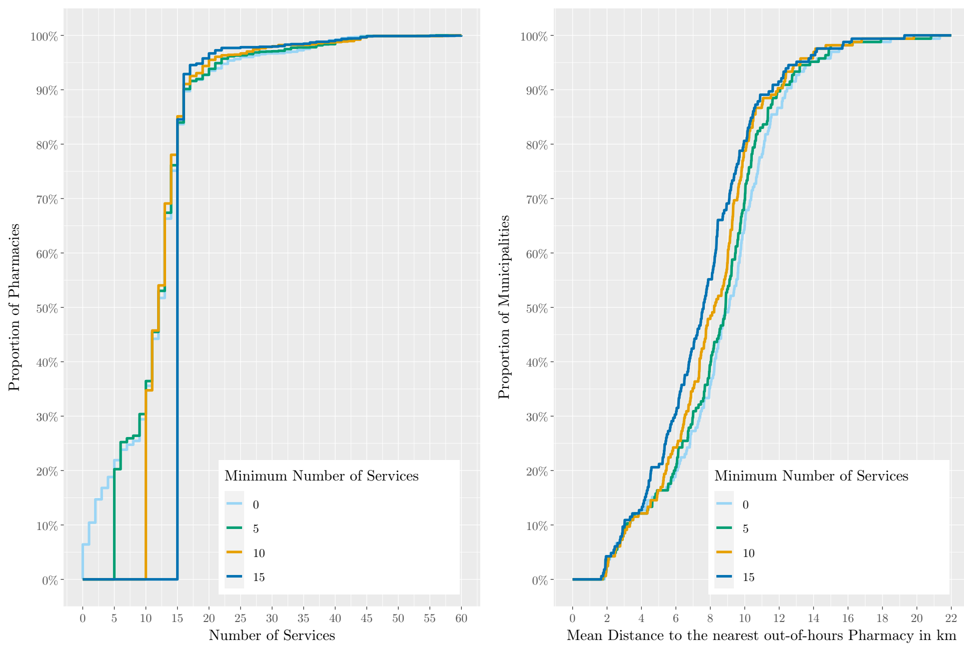
Table 3: Total number of out-of-hours services assigned and the mean distance from the center of the municipalities to the next out-of-hours pharmacies over the course of the year. For each value $n^{\min}$, we consider the best solution computed in the previous section.

their practicability. We start with a comparison of the plans on the global level. More precisely, we analyze how many pharmacies perform a specific number of services and how many municipalities are covered by a pharmacy that is on average not farther away than a specific number of kilometers. After that, we have a closer look at certain municipalities and groups of pharmacies.

We have already seen in Figure 6a that a variation of the equity coefficients $e \equiv 1, \ldots, 5$ has nearly no effect on the global level regarding the number of distributed services. Furthermore, choosing $e \equiv 0$ is not reasonable from a practical point of view, as the number of additional services compared to $e \equiv 1$ is disproportionate to the additional equity. Hence, for our evaluation on the global level, we only consider fixed equity coefficients $e \equiv 1$, together with a varying minimum number of services $n^{\min}$. For each value $n^{\min} \in \{0, 5, 10, 15\}$, we consider the plan computed in Section 4.3 that has the best objective value, i.e., the best plan computed by the algorithms 12DelEq, 12SupDelEq, 24DelEq, 24SupDelEq.

First, we analyze the distribution of out-of-hours services among the pharmacies. As a reminder, Table 3 shows for all four computed plans and the real plan of 2017 the total number of services assigned to all pharmacies. Figure 7a gives a more detailed view by showing for our plans the proportion of pharmacies for that the number of out-of-hours services over the course of the year is not greater than a specific value. The plan for $n^{\min} = 0$ assigns no services at all to 6.4% of the pharmacies and 18.8% of all pharmacies are assigned less than five services. In the last section, we already noted that the minimum number constraints (3h) are an actual restriction for the planning. From this, we conclude that it is not a coincidence that we assign so many pharmacies few or even no services. For our model, these pharmacies are in an unfavorable geographical position and assigning these pharmacies an out-of-hours service is not an optimal choice with respect to the total number of services. This means that we cannot distribute the services more evenly without loosing some efficiency. Note that we observe this effect, although the plan meets the equity constraints with $e \equiv 1$. It follows that the minimum number of services constraints cannot be omitted when applying the equity constraints.

Since an optimal plan assigns as few services as possible to pharmacies in an unfavorable location, it is not surprising that the introduction of a minimum number of services $n^{\min}$ results in many pharmacies that perform exactly $n^{\min}$ out-of-hours services. For $n^{\min} = 5$, we assign 20.3% of the pharmacies exactly $n^{\min}$ services. For $n^{\min} = 10$, this proportion increases to 34.7% and even to 84.5% for $n^{\min} = 15$. Of course, forcing such a high number of pharmacies to perform additional out-of-hours services reduces the efficiency of the plan but makes it also fairer. While 6.5% of the pharmacies perform more than 20 services for $n^{\min} = 0$, this proportion is



(a) (b)

Figure 7: The proportion of pharmacies that perform a specific number or fewer services (a). The proportion of municipalities for which the mean distance to the next out-of-hours pharmacy is less than or equal to the specific value (b).

nearly halved for $n^{\min} = 15$ to only 3.3% of all pharmacies. Unfortunately, this effect vanishes for pharmacies with a lot of services. For example, the number of pharmacies with 40 services or more is lower in the plan for $n^{\min} = 0$ compared to the one for $n^{\min} = 15$. Apparently, these highly occupied pharmacies are not in the vicinity of pharmacies with nearly no services that could take over some out-of-hours services.

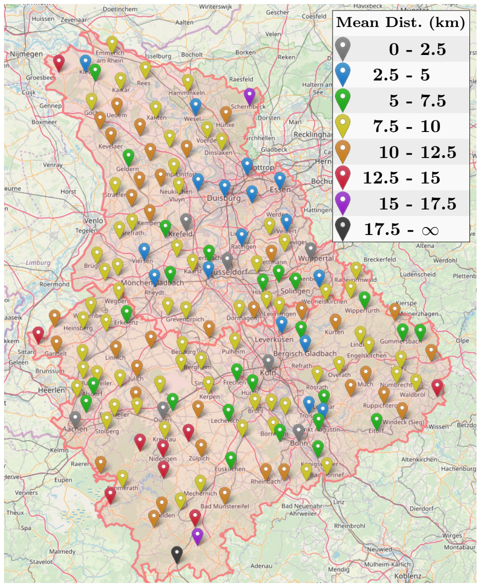
As a comparison, on the one hand, the real plan used by the Chamber of Pharmacists North Rhine in 2017 assigns more than 20 services to 10% of the pharmacies, which is more than twice as much as in the plan for $n^{\min} = 10$. On the other hand, no pharmacy has to perform more than 40 services (1.8% in the plan for $n^{\min} = 10$). In the real plan, the cover radii $\delta^{\text{cov}}(m)$ are locally adapted with respect to the number of pharmacies that are in the vicinity of a municipality $m \in \mathcal{M}$. This improves the coverage for municipalities with many pharmacies around and relieves highly occupied pharmacies in rural areas.

In addition to a higher fairness, the minimum number of services has also a positive effect on the coverage of the residents. Table 3 shows for all plans the mean distance that residents have to travel to get from the center of a municipality to the nearest out-of-hours pharmacy. We choose to only consider the mean distance, since the covering constraints (1b) already prohibit distances that are too long. Figure 7b gives us a better understanding which municipalities benefit the most from the minimum number of services. The plot shows for all plans the proportion of municipalities for which the mean distance to the next out-of-hours pharmacy over the course of the year is below a specific value. We see that the effect on the mean distance is primarily observable for municipalities with a mean distance of more than 4 km. This is due to the fact that municipalities with a very low mean distance are usually larger cities with a positive minimum demand $d > 0$ that already forces the pharmacies within the municipalities to perform several out-of-hours services. In our opinion, this is a positive effect, since the municipalities that previously had a bad coverage benefit the most from the minimum number of services. For example, for the two municipalities with the worst coverage, the mean distance to the nearest out-of-hours pharmacy is reduced by 1.5 km each in the plan for $n^{\min} = 10$ compared to the plan with $n^{\min} = 0$.

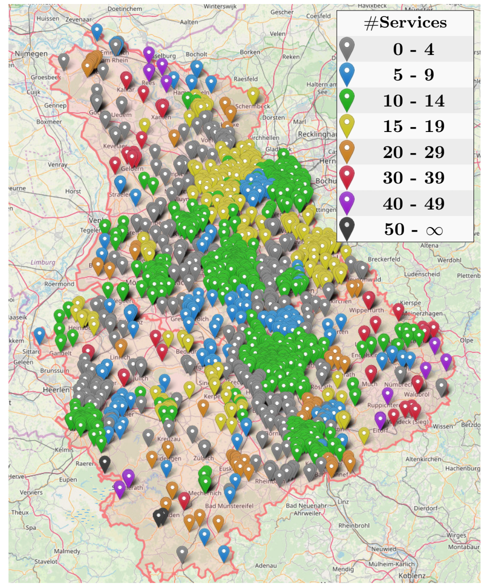
In Figure 8, we display the mean distances to the nearest out-of-hours pharmacies for individual municipalities as well as the number of services each pharmacy performs for values $n^{\min} = 0$ and $n^{\min} = 10$. In Figure 8a, we see that municipalities with a bad coverage are primarily located in rural areas with few close pharmacies and also often in border areas. Municipalities in border areas not only usually have fewer pharmacies in their vicinity, the nearby pharmacies also tend to perform less out-of-hours services. We can observe this in Figure 8b for the pharmacies around Dahlem, the southernmost municipality in North Rhine. This is due to the fact that pharmacies in border areas cover fewer municipalities and therefore are not so relevant within our model. While we cannot influence the number of nearby pharmacies, a minimum number of services guarantees that pharmacies in the border area also participate in the out-of-hours service and improve the coverage of the corresponding municipalities. For Dahlem, this results in a reduction of the mean distance from 21.3 km to 19.8 km.

Besides the pharmacies in border areas, pharmacies that perform nearly no services are mostly located in municipalities close to a larger city. The municipalities covered by these pharmacies often can also be covered by the pharmacies within the city. It is efficient to cover the surrounding municipalities by the pharmacies within the city, as they have to perform services anyway in order to meet the demand constraints (1c). Here, a minimum number of services leads to a better coverage of the municipalities near cities, as the residents do have to travel less often into the cities. The effect on the municipalities can be seen by comparing, for example, the coverage of the municipalities east of Düsseldorf, in the middle of North Rhine, or north of Aachen, in the west of North Rhine.

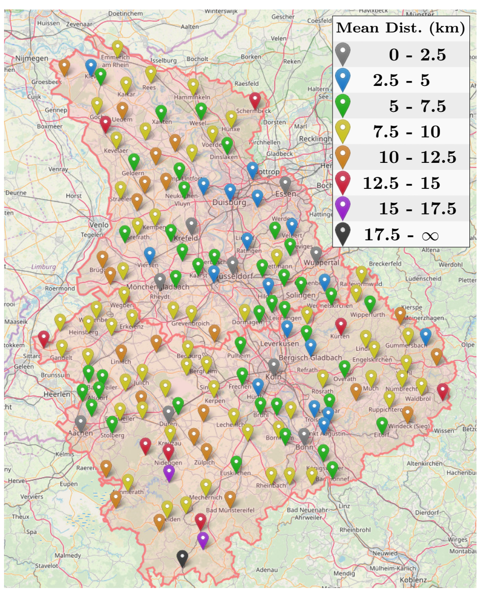
While a higher minimum number of services is beneficial for the coverage of the residents, the additional services are not always assigned in an efficient way. This can be seen when comparing the plan computed for $n^{\min} = 15$ with the real plan of 2017. Although our plan assigns more services, the mean distance from a municipality to the nearest pharmacy is 7.6 km, compared to 7.2 km in the real plan. This is due to the already mentioned adapted cover radii $\delta^{\text{cov}}(m)$, which are lowered for the real plan for municipalities $m \in \mathcal{M}$ that have many pharmacies in their vicinity. However, as the cover radii are increased for rural municipalities, the real plan has a worse coverage of these rural municipalities compared to our plans. For example, the mean distance to the next out-of-hours pharmacy from the municipality Monschau is 11.9 km in our plan for $n^{\min} = 10$ and 16.2 km in the real plan. In order to improve the mean



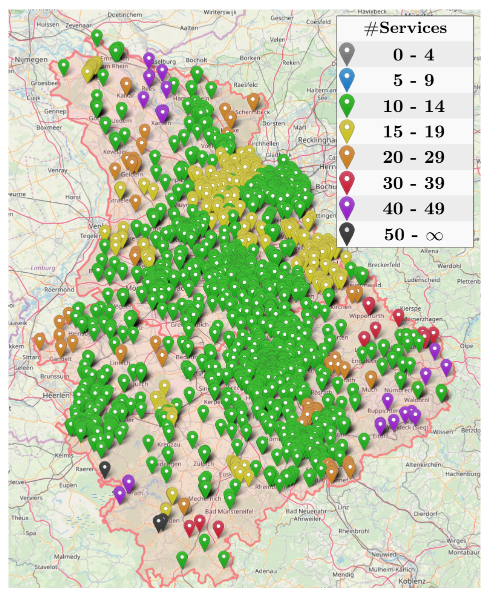
(a)



(b)



(c)



(d)

Figure 8: Properties for the plan computed for $n^{\min} = 0$ (a and b) as well as properties for the plan with $n^{\min} = 10$ (c and d). Each municipality is displayed and colored depending on the mean distance from the municipality center to the nearest out-of-hours pharmacy (a and c). Each pharmacy is colored depending on the number of services performed (b and d).

coverage of our plans, while preserving the superior coverage of rural municipalities, we could lower the cover radius $\delta^{\text{cov}}(m)$ for municipalities $m \in \mathcal{M}$ for which the covering pharmacies $p \in C(m)$ perform less than $n^{\min}$ services when the minimum number of services constraints are not enforced. This would lead to a more natural assignment of services that would improve the coverage and are not distributed for fairness reasons alone.

The effect of the minimum number of services on the fairness can be seen in the north of Aachen. In the plan for $n^{\min} = 0$, there are several pharmacies that perform more than 30 services, while many surrounding pharmacies perform less than five services. In the plan for $n^{\min} = 10$, these previously highly occupied pharmacies perform only ten to 14 services, just like the surrounding pharmacies.

Although the minimum number of services balances the number of services assigned to different pharmacies within the same area, the plan for $n^{\min} = 10$ still has some large local differences. In the rural Eifel region, in the south of North Rhine, there are two pharmacies (colored black in Figure 8d) within one municipality performing 58 and 59 services. In contrast to this, there are other pharmacies within a different municipality few kilometers to the north which cover nearly the same municipalities and are assigned only around 20 out-of-hours services. In practice, such a plan would not be approved by the pharmacists. Considering the most critical municipality covered by the highly occupied pharmacies, i.e., the municipality with the smallest number of covering pharmacies, we could reduce the burden via a redistribution to around 43 services. Here, the decision makers need to define some additional constraints such that the out-of-hours plan is not only fair within the municipalities but also beyond their borders.

The equity coefficients e, which we did not consider so far, since they have a low impact on the global properties of the resulting plans, are also a key parameter for the local fairness that decision makers have to consider. Figure 9 shows the pharmacies of Aachen and the Eifel region, which we analyzed above, for the plan using $n^{\min} = 10$ and $e \equiv 5$. Note that we change the coloring of the pharmacies compared to the one used in Figure 8 in order to highlight the difference in the numbers of services assigned to the pharmacies in Aachen. We see two pharmacies (colored yellow) in the southernmost area of Aachen which are assigned 16 services, while all other pharmacies in Aachen only perform eleven or twelve services. This is due to the fact that these two pharmacies can cover Monschau, the critical municipality we considered above. If we would not apply the equity constraints at all then the two pharmacies in Aachen could potentially perform even more services, resulting in an inequitable distribution within Aachen but reliving the highly occupied pharmacies in the Eifel region. We conclude that the equity constraints are necessary to guarantee that competing pharmacies within the same municipality have the same conditions. However, omitting the equity constraints or choosing a higher equity coefficient can be regarded as fairer in some locations, as we have the possibility to relieve highly occupied pharmacies within other municipalities.

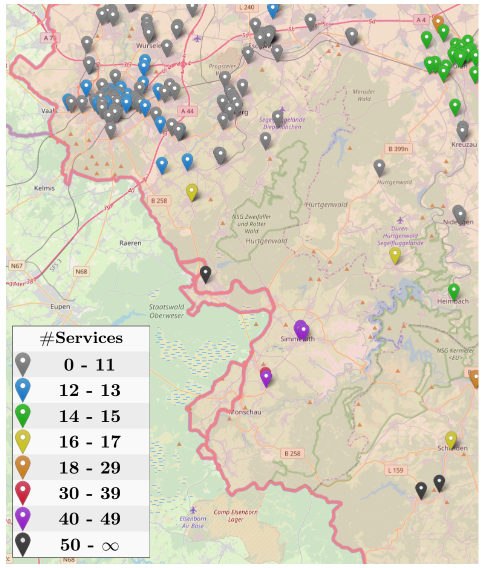


Figure 9: Pharmacies in Aachen and the Eifel region colored depending on the number of services performed.

5. Conclusions

In this paper, we proposed a mathematical model for the planning of out-of-hours services for pharmacies. To solve the problem, we presented an MILP formulation as well as problem specific solution approaches. In our aggregation approach, we identify pharmacies that are equivalent within our model and combine them into one artificial superpharmacy. This reduces not only the number of decision variables in our MILP but also avoids symmetric solutions. For our rolling horizon approach, we partition the planning period into a number of intervals. We obtain a plan by reducing the original problem to the intervals and sequentially computing plans for the subproblems. Furthermore, we proposed an extension to the rolling horizon algorithm in which we discard services that were assigned in the previous iterations solely to satisfy equity constraints.

Our approaches have proven to be very effective in our computational study. Here, we obtained in short computational time nearly optimal solutions in the tests conducted for a large-scale real-world instance. One reason for the effectiveness of our approaches is the strength of the MILP formulation. We have seen that its LP relaxation provides a lower bound on the total number of out-of-hours services that is extremely close to the number of services assigned in our solutions. For future work, it would be interesting to evaluate whether this property also applies for other practical instances or if our setting is special in this regard. For instances with a larger integrality gap, a problem-specific algorithm for computing lower bounds can be used to evaluate the potential of an optimal plan.

The computed plans are interesting on the one hand because they show the potential savings in the total number of out-of-hours services. For example, the best plan computed in Section 4.2 assigns 10.1% services less compared to the actual plan in North Rhine for 2017. On the other hand, the results show that it is possible to compute a plan that satisfies all constraints set by the Chamber of Pharmacists North Rhine, which was unclear before. Since we are able to reliably compute good solutions, the decision makers can test different parameters for the planning, analyze their effect on the plan and learn which restrictions are satisfiable. The out-of-hours services saved by the efficient planning may then also be redistributed to improve the coverage of the residents.

In our case study, we already tested different parameters for a minimum number of services each pharmacy has to perform. Introducing such a minimum number results in higher total number of services but in return also in a lower mean distance between the centers of the municipalities and the nearest out-of-hours pharmacy. Additionally, we observed that the distribution of the services on the pharmacies tends to be fairer for a higher minimum number. Nevertheless, there can still be large differences in the number of services assigned to pharmacies within the same region. These differences are mostly local problems, which the decision makers may resolve with tailored additional constraints. For future work, it would be interesting to extend the model with an integrated solution that guarantees equity globally and also considers fairness regarding weekends and holidays.

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Appendix A. Proof of Theorem2

The OHP is in $\mathcal{NP}$, since we only have to verify a polynomial number of constraints that are easy to evaluate in order to decide if a solution is feasible to the OHP. We proof the $\mathcal{NP}$-hardness by a reduction from the *k Disjoint Set Covers Problem* (k-DSC).

Proof. For the k-DSC we are given a finite basic set $\mathcal{I}$, a family of subsets $\mathcal{J} \subseteq 2^{\mathcal{I}}$ and a positive integer $k \in \mathbb{Z}_{\geq 2}$. The task is to determine whether there exists a partition of $\mathcal{J}$ into k disjoint subsets that cover $\mathcal{I}$, i.e., $\mathcal{J} = \mathcal{J}_1 \uplus \cdots \uplus \mathcal{J}_k$ with $\mathcal{I} = \bigcup_{J \in \mathcal{J}_i} J$ for all $i \in \{1, \ldots, k\}$. The k-DSC is strongly $\mathcal{NP}$-complete for all $k \in \mathbb{Z}_{\geq 2}$ [28].

Let $(\mathcal{I}, \mathcal{J}, k)$ be a given k-DSC instance. We construct an OHP instance by identifying the municipalities $\mathcal{M}$ with the basic elements $\mathcal{I}$, as both need to be covered, and the pharmacies $\mathcal{P}$ with the family of subsets $\mathcal{J}$, because they cover the pharmacies and the basic elements respectively. Accordingly, for each element $i \in \mathcal{I}$ we introduce one municipality m_i and for each set $J \in \mathcal{J}$ we introduce one pharmacy p_J. We define metric distances

$$\begin{aligned} \delta\left(p, p'\right) &= 1, \text{ for } p, p' \in \mathcal{P}, \\ \delta\left(m, m'\right) &= 1, \text{ for } m, m' \in \mathcal{P}, \\ \delta\left(p_J, m_i\right) = \delta\left(m_i, p_J\right) &= 1, \text{ for } p_J \in \mathcal{P}, m_i \in \mathcal{M} \text{ with } i \in J, \\ \delta\left(p_J, m_i\right) = \delta\left(m_i, p_J\right) &= 2, \text{ for } p_J \in \mathcal{P}, m_i \in \mathcal{M} \text{ with } i \notin J \end{aligned}$$

on the arcs and set the cover radii $\delta^{\text{cov}}(m) = 1$ for all $m \in \mathcal{M}$. Then we obtain $C(m_i) = \{p_J \in \mathcal{P} \mid i \in J\}$ for all $m_i \in \mathcal{M}$, i.e., a pharmacy $p_J \in \mathcal{P}$ can cover a municipality $m_i \in \mathcal{M}$ if the corresponding subset $J \in \mathcal{J}$ contains the element $i \in \mathcal{I}$. Furthermore, we define the time horizon $\mathcal{T} = \{1, \ldots, k\}$ and the periods of rest $r(m) = k - 1$ for all $m \in \mathcal{M}$. The remaining parameters are defined so that the corresponding constraints are not relevant for the reduction. We choose $m : \mathcal{P} \to \mathcal{M}$ arbitrary, $\delta^{\text{con}} \equiv 0$, $d \equiv 0$, $e : \mathcal{M} \to \mathbb{Q}_{\geq 0}$ arbitrary and $n^{\min} = 0$.

Now, let $\{\mathcal{J}_1, \cdots, \mathcal{J}_k\}$ be a feasible solution to the k-DSC instance. Then we define an out-of-hours plan for the constructed OHP instance by $F(t) = \{p_J \in \mathcal{P} \mid J \in \mathcal{J}_t\}$ for $t \in \mathcal{T} = \{1, \ldots, k\}$. Since each $\mathcal{J}_t$ is a cover for $\mathcal{I}$, by construction every municipality is covered on every day. Furthermore, each pharmacy is assigned exactly one out-of-hours service. Therefore, the plan is equitable and we violate no constraints regarding the periods of rest. Since no municipality has a positive demand, there are no conflicts between the pharmacies and we have no constraints on the minimum number of services, the plan is a feasible solution.

Let $F : \mathcal{T} \to 2^{\mathcal{P}}$ be a feasible solution to the constructed OHP instance. For $t \in \{1, \ldots, k-1\}$ we define $\mathcal{J}_t = \{J \in \mathcal{J} \mid p_J \in F(t)\}$ and $\mathcal{J}_k = \mathcal{J} \setminus \left(\bigcup_{t \in \{1,\ldots,k-1\}} \mathcal{J}_t\right)$. Due to the periods of rest, each pharmacy is assigned at most one out-of-hours service. Hence, we have $\mathcal{J} = \mathcal{J}_1 \uplus \cdots \uplus \mathcal{J}_k$. Additionally, since each municipality is covered on every day, by construction we have $\mathcal{I} = \bigcup_{J \in \mathcal{J}_t} J$ for all $t \in \{1, \ldots, k\}$. We conclude that $\{\mathcal{J}_1, \cdots, \mathcal{J}_k\}$ is a feasible solution to the k-DSC instance.

Note that, since the k-DSC is already $\mathcal{NP}$-hard for $k = 2$, we have shown the statement especially for $r \equiv 1$. □

Appendix B. Computational Results

The following tables list the exact results of the computations performed in Section 4.

	Approach											
	Default			DelEq			Sup			SupDelEq		
Number Intervals	Services	Violation	Time	Services	Violation	Time	Services	Violation	Time	Services	Violation	Time
2	-	-	-	-	-	-	31356	823	10003	31356	823	10003
3	-	-	-	-	-	-	31997	962	10003	31834	971	10003
4	-	-	-	31726	1001	10003	32746	1053	10002	32002	963	10002
5	32929	1199	10002	-	-	-	32129	762	10002	31376	695	10001
6	31412	902	10002	-	-	-	32611	11	10001	31317	2	10001
7	33314	1230	10002	31388	1038	10002	32815	908	10001	31267	0	9326
8	33436	1165	10002	31235	867	10002	32971	724	10001	31008	557	10001
9	33507	810	10001	31284	958	10001	32220	633	10001	30320	2	10001
10	-	-	-	32607	3	10001	32107	493	10001	30292	0	10001
11	34119	2	10001	30664	442	10002	32040	406	10001	30165	1	10001
12	34102	2	10001	30587	334	10001	31856	506	10001	30186	0	6984
13	33203	438	10001	30244	3	10001	31267	276	10001	30186	0	4823
14	33209	491	10001	30219	0	8570	31913	2	10001	30192	0	2781
15	33050	354	10001	30170	0	8012	30836	0	8224	30174	0	3098
16	32334	360	10001	30199	0	6301	30746	0	7683	30194	0	2429
17	34864	433	10001	30201	0	5693	31019	0	7701	30209	0	2359
18	34727	104	10001	30211	0	5342	30824	0	4433	30217	2	1815
19	40463	25524	10001	30220	0	3906	30784	1	5123	30197	0	1891
20	38819	268	10001	30193	0	3736	30496	0	4340	30190	0	1653
21	39750	26063	10001	30210	0	3521	30683	0	4444	30224	0	1846
22	30878	0	9736	30217	0	2924	30803	0	4289	30208	0	1498
23	37668	0	8204	30215	0	2420	30833	0	3429	30208	0	1368
24	31021	0	8296	30217	0	2194	30992	0	3769	30220	0	1372
25	31140	1	6450	30221	0	2477	31004	0	3054	30220	0	1260
26	30826	0	5897	30226	0	1913	31068	0	3402	30214	0	1611

Table B.4: Number of services assigned, number of violations of equity constrains and computational time required for each solution computed by the approaches Default, DelEq, Sup and SupDelEq with numbers of intervals $I \in \{2, \dots, 26\}$.

	Approach											
	Default			DelEq			Sup			SupDelEq		
Number Intervals	Services	Violation	Time	Services	Violation	Time	Services	Violation	Time	Services	Violation	Time
27	30872	0	6229	30233	0	2292	30813	0	3230	30217	0	1303
28	30879	0	4732	30236	0	1909	30851	0	3260	30233	0	1195
29	30981	0	5635	30229	0	1889	30939	0	3010	30222	0	1070
30	30828	0	4812	30224	0	1722	30861	0	2380	30215	0	965
31	30997	0	4132	30241	0	1406	30991	0	2033	30228	0	971
32	31130	0	4584	30253	0	1406	31088	0	2748	30235	0	916
33	31195	0	4484	30277	0	1369	31164	0	1865	30241	0	884
34	31222	0	3650	30255	0	1330	31072	0	1979	30254	0	898
35	30948	0	3543	30250	0	1502	30956	1	1651	30252	0	800
36	31009	0	3353	30264	0	1314	31021	0	1985	30258	0	831
37	31025	0	3589	30249	1	1515	31005	0	2269	30250	0	894
38	31065	1	3257	30240	0	1548	31092	0	1801	30253	0	871
39	31192	0	3011	30257	0	1439	31154	0	1987	30255	0	858
40	30876	0	3207	30249	0	1423	30898	0	1836	30246	0	807
41	31019	0	2682	30260	0	1325	31230	0	1586	30262	0	838
42	31133	0	2732	30251	0	1380	31086	0	1762	30249	0	796
43	31252	0	2115	30246	0	1127	31235	0	1363	30250	0	838
44	31390	0	1903	30254	1	1121	31359	0	1265	30247	0	758
45	31339	0	1991	30245	0	1180	31402	0	1307	30261	0	774
46	31475	0	1404	30272	0	1100	31503	0	1005	30271	0	803
47	31521	0	1581	30284	0	1093	31551	0	1045	30278	0	816
48	31567	0	1293	30298	0	1085	31591	0	1017	30294	0	817
49	31617	0	1428	30289	0	1112	31609	1	1072	30274	0	796
50	31664	0	1585	30305	1	1150	31475	0	1084	30269	0	764
51	31474	0	1312	30308	1	1162	31485	0	1033	30307	0	788
52	31538	1	1347	30307	1	1185	31455	0	997	30300	0	840

Table B.5: Number of services assigned, number of violations of equity constrains and computational time required for each solution computed by the approaches Default, DelEq, Sup and SupDelEq with numbers of intervals $I \in \{27, \ldots, 52\}$.

		Approach											
		12DelEq			12SupDelEq			24DelEq			24SupDelEq		
Min Number Services	LP Relaxation	Services	Violation	Time	Services	Violation	Time	Services	Violation	Time	Services	Violation	Time
0	25835	32218	36180	10001	27403	651	10001	26924	1	9460	26520	0	8942
1	26083	27494	911	10002	29289	0	9104	26930	0	8340	26585	0	9116
2	26375	27622	745	10001	27642	733	10001	27104	0	9016	26622	0	8887
3	26715	27812	1098	10001	28767	0	9095	27136	0	8907	26941	0	8595
4	27080	28194	923	10002	29280	0	9254	27652	0	7521	27307	0	6122
5	27501	28483	829	10002	28689	1	10001	27749	0	7280	27712	0	3606
6	27966	28821	624	10002	28759	517	10001	28165	0	5790	28139	0	2509
7	28443	29350	954	10001	30821	2	9182	28611	0	4929	28613	0	2186
8	28929	29602	621	10001	29373	328	10001	29093	0	3271	29094	0	1871
9	29451	29992	459	10001	29552	0	8339	29612	0	2918	29600	0	1627
10	30078	30587	334	10001	30186	0	6984	30217	0	2194	30220	0	1372
11	30858	30974	0	9524	30946	0	2902	30985	0	1790	30971	0	1355
12	31846	31900	0	8612	31902	0	3074	31957	0	1927	31929	0	1087
13	32996	33055	0	7359	33059	0	1987	33112	0	1833	33069	0	930
14	34483	34524	0	4772	34546	0	3166	34566	0	1621	34617	0	2639
15	36174	36236	0	4014	36224	0	1675	-	-	-	-	-	-

Table B.6: The number of services assigned, number of violations of equity constrains and computational time required for each solution computed by the approaches 12DelEq, 12SupDelEq, 24DelEq and 24SupDelEq, using different minimum numbers of services $n^{\min} \in \{0, \dots, 15\}$ ..

		Approach											
		12DelEq			12SupDelEq			24DelEq			24SupDelEq		
Equity Coeff.	LP Relaxation	Services	Violation	Time	Services	Violation	Time	Services	Violation	Time	Services	Violation	Time
0	30110	32071	2	9699	30687	476	10001	31109	2	8623	31095	2	4605
1	30078	30587	334	10001	30186	0	6984	30217	0	2194	30220	0	1372
2	30050	30541	290	10001	30142	0	5489	30170	0	1994	30181	0	1131
3	30025	30195	1	10001	30111	0	3980	30138	0	2255	30131	0	1128
4	30002	30180	0	9628	30083	0	3481	30107	0	1651	30102	0	1065
5	29980	30090	0	9550	30054	0	5091	30082	0	1499	30079	0	999

Table B.7: The number of services assigned, number of violations of equity constrains and computational time required for each solution computed by the approaches 12DelEq, 12SupDelEq, 24DelEq and 24SupDelEq, using different equity coefficients $e \equiv 0, \dots, 5$.