Lower Bounds for the Bandwidth Problem *

Franz Rendl[†] Renata Sotirov[‡] Christian Truden [§]

Abstract

The Bandwidth Problem asks for a simultaneous permutation of the rows and columns of the adjacency matrix of a graph such that all nonzero entries are as close as possible to the main diagonal. This work focuses on investigating novel approaches to obtain lower bounds for the bandwidth problem. In particular, we use vertex partitions to bound the bandwidth of a graph. Our approach contains prior approaches for bounding the bandwidth as special cases. By varying sizes of partitions, we have a trade-off between quality of bounds and efficiency of computing them.

To compute lower bounds, we derive several semidefinite programming relaxations. We evaluate the performance of our approach on a carefully selected set of benchmark instances.

Keywords: bandwidth problem, graph partition, semidefinite programming

1 Introduction

The Bandwidth Problem (BP) is the problem of labeling the vertices of a given undirected graph with distinct integers such that the maximum difference between the labels of adjacent vertices is minimal. Determining the bandwidth is NP-hard [26] and even approximating the bandwidth within a given factor is known to be NP-hard [31]. Moreover, the BP is known to be NP-hard even on trees with maximum degree three [13] and on caterpillars with hair length three [24]. On the other hand, the Bandwidth Problem has been solved for a few families of graphs having special properties. Among these are the path, the complete graph, the complete bipartite graph [5], the hypercube graph [17], the grid graph [6], the complete k-level t-ary tree [29], the triangular graph [21], and the triangulated triangle [19].

The Bandwidth Problem originated in the 1950s from sparse matrix computations, and received much attention since Harary's [15] description of the problem and Harper's paper [16] on the bandwidth of the n-cube. Berger-Wolf and Reingold [1] showed that the problem of designing a code to minimize distortion in multi-channel transmission can be formulated as the Bandwidth Problem for the generalized Hamming graphs. The Bandwidth Problem arises in many different engineering applications related to efficient storage and processing. It also plays a role in designing parallel computation networks, VLSI layout, constraint satisfaction problems, see e.g., [4, 11, 23] and the references therein.

Several lower and upper bounding approaches for bounding the bandwidth of a graph are considered in the literature. Well known heuristic for general graphs is Cuthill-McKee heuristic [7, 30]. For graphs with symmetry there exists an improved reverse Cuthill-McKee algorithm, see [32]. We list below results on lower bounding approaches. Fundamentals for obtaining lower bounds for the BP have been established by Juvan and Mohar in [22], where the authors derive eigenvalue bounds for the bandwidth of a graph. Helmberg et al. [18] derived a lower bound for the bandwidth of a graph by exploiting spectral properties of the graph. The same lower bound was derived by Haemers [14] by exploiting interlacing of Laplacian eigenvalues. The eigenvalue bound

*This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 764759 and the Austrian Science Fund (FWF): P 28008-N35.

†Alpen-Adria-Universität, Klagenfurt, Austria, franz.rendl@aau.at

‡Department of Econometrics and OR, Tilburg University, The Netherlands, r.sotirov@uvt.nl

§Alpen-Adria-Universität, Klagenfurt, Austria, christian.truden@aau.at

from [18, 14] is stronger than the bound from [22]. Povh and Rendl [27] showed that the eigenvalue bound from [18] can also be obtained by solving a Semidefinite Programming (SDP) relaxation for the Minimum Cut (MC) problem. They further tightened the SDP relaxation and consequently obtained a stronger lower bound for the Bandwidth Problem. Rendl and Sotirov [28] showed how to further tighten the SDP relaxation from [27]. Dunagan and Vempala [12] used an SDP relaxation for the BP derived by Blum et al. [2] to derive an $O(\log^3 n\sqrt{\log\log n})$ approximation algorithm for the BP, where n is the number of vertices. De Klerk et al. [8] proposed two lower bounds for the graph bandwidth based on SDP relaxations of the Quadratic Assignment Problem (QAP). The numerical results in [8] show that both their bounds dominate the bound of Blum et al. [2], and that in most of the cases their bounds are stronger than the bound by Povh and Rendl [27]. In [32], the authors derived an SDP relaxation of the minimum cut problem by strengthening the well known SDP relaxation for the QAP. They derive strong bounds for the bandwidth of highly symmetric graphs with up to 216 vertices by exploiting symmetry. For general graphs, their approach is rather restricted. The bounds mentioned above are either unsatisfyingly weak, or computationally very demanding already for small (general) graphs, that is for graphs of about 30 vertices. Here, we present new modeling approaches that are based on vertex partitions for obtaining strong lower bounds on the bandwidth.

Main results and outline. We formally introduce the Bandwidth Problem in Section 2. In Section 3, we present our modeling approach based on vertex partitions. For that purpose we introduce the Minimal Partition Problem. Two special cases of our modelling approach were previously used to obtain lower bounds for the BP. Both of them are extreme cases in our setting. In Section 4 we present several SDP models for the Minimal Partition Problem. Our smaller model has matrix variables of order n, while the larger model has a matrix variable of order $\mathcal{O}(nk)$. To solve our strongest relaxation efficiently, we implement the Alternating Direction Method of Multipliers (ADMM). Numerical results in Section 5 show the efficiency of our new bounding approach. We provide bounds for graphs with up to 128 vertices in a reasonable time frame.

Notation. The space of $n \times n$ symmetric matrices is denoted by $\mathcal{S}_n$ and the space of $n \times n$ symmetric positive semidefinite matrices by $\mathcal{S}_n^+$. For two matrices $X, Y \in \mathbb{R}^{n\times n}$, $X \geq Y$, means $x_{ij} \geq y_{ij}$, for all i, j. The set of $n \times n$ permutation matrices is denoted by Π_n.

We use I_n to denote the identity matrix of order n, and e_n^i to denote the i-th standard basis vector of length n. Similarly, J_n and e_n denote the $n \times n$ all-ones matrix and all-ones n-vector, respectively.

The **diag** operator maps an $n \times n$ matrix to the n-vector given by its diagonal, while the **vec** operator stacks the columns of a matrix in a vector. The adjoint operator of **diag** we denote by **Diag**. The trace operator is denoted by **trace**, and $\langle \cdot, \cdot \rangle$ denotes the trace inner product. The Hadamard product of two matrices A and B of the same size is denoted by $A \circ B$ and defined as $(A \circ B)_{ij} = a_{ij} \cdot b_{ij}$ for all i, j.

2 The Bandwidth Problem

In this section, we formally introduce the Bandwidth Problem (BP) and show its relation to the Quadratic Assignment Problem (QAP).

Let $G = (V, E)$ be an undirected simple graph with $|V| = n$ vertices and edge set E. A bijection $\phi : V \to \{1, \ldots, n\}$ is called a *labeling* of the vertices of G. The bandwidth of a graph G with respect to the labeling ϕ is defined as follows

$$\mathbf{bdw}(\phi, G) := \max_{[i,j]\in E} |\phi(i) - \phi(j)|.$$

The *bandwidth of a graph* G is defined as the minimum of $\mathbf{bdw}(\phi, G)$ over all labelings ϕ, i.e.,

$$\mathbf{bdw}(G) := \min \{\mathbf{bdw}(\phi, G) : \ \phi \text{ labeling of } G\}.$$

Equivalently, one can consider the adjacency matrix A of the graph G. The bandwidth of A amounts to a simultaneous permutation of the rows and columns of the adjacency matrix such that the largest distance of a nonzero entry from the main diagonal is as small as possible. Hence, the *bandwidth of an adjacency matrix* A is therefore defined as:

$$\mathbf{bdw}(A) := \mathbf{bdw}(G).$$

Therefore, from now on we assume that a graph G is given through its adjacency matrix A. Since in terms of matrices the BP asks for a simultaneous permutation of the rows and columns of A such that all nonzero entries are as close as possible to the main diagonal, a 'natural' problem formulation is as follows.

Let r be an integer such that $1 \leq r \leq n-2$, and $B_{r,n} = (b_{ij})$ be the symmetric matrix of order n defined as follows

$$b_{ij} := \begin{cases} 1, & \text{for } |i-j| > r, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the following holds:

$$\min_{Q \in \Pi_n} \langle Q^\mathsf{T} A Q, B_{r,n} \rangle = \begin{cases} 0, & \text{then } \mathbf{bdw}(A) \leq r, \\ > 0, & \text{then } \mathbf{bdw}(A) > r. \end{cases} \tag{1}$$

The minimization problem has the form of a QAP, which might be even harder to solve than actually computing $\mathbf{bdw}(A)$. The idea of formulating the Bandwidth Problem as a QAP was suggested by Helmberg et al. [18]. De Klerk et al. [8] considered two SDP-based bounds for the Bandwidth Problem that are obtained from the SDP relaxations for the QAP introduced in [35] and [9]. The results show that it is hard to obtain bounds for graphs with 32 vertices, even though the symmetry in the graphs under consideration was exploited.

Since it is very difficult to solve QAPs in practice for sizes larger than 30 vertices other approaches are needed for deriving bounds for the bandwidth of graphs.

3 Partition Approach

We show how to use vertex partitions in order to obtain lower bounds for the bandwidth of a graph. Our approach contains prior approaches for bounding the bandwidth of a graph as special cases.

For $3 \leq k \leq n$ let $m \in \mathbb{N}^k$ be given with $m_i \geq 1,\ \sum_{i=1}^k m_i = n$. We consider partitions of the vertex set V into k subsets $\{S_1, \ldots, S_k\}$ such that $|S_j| = m_j,\ j = 1, \ldots, k$. These are in one-to-one correspondence with $n \times k$ partition matrices:

$$\mathcal{P}_m := \{X \in \{0,1\}^{n \times k} : Xe_k = e_n,\ X^\mathsf{T} e_n = m\}, \tag{2}$$

where for the partition $(S_1, \ldots, S_k)$ we set $x_{ij} = 1$ whenever $i \in S_j$. Since any vertex $i \in V$ is assigned to precisely one of the blocks S_j we can define the map $p : V = \{1, \ldots, n\} \mapsto \{1, \ldots, k\}$ given by

$$p(i) = j \Leftrightarrow x_{ij} = 1 \Leftrightarrow i \in S_j,$$

which identifies the partition block containing vertex i. Thus, given the partition matrix $X \in \mathcal{P}_m$ we get $S_j = \{i \in V : p(i) = j\}$ for all $1 \leq j \leq k$. For $1 \leq r \leq k-2$ let $B_{r,k} = (b_{ij})$ be the 0–1 matrix of order k with

$$b_{ij} = \begin{cases} 1, & |i-j| > r, \\ 0, & |i-j| \leq r. \end{cases} \tag{3}$$

Suppose that $i \in S_u, j \in S_v$, i.e., $p(i) = u,\ p(j) = v$. Then for $X \in \mathcal{P}_m$ the following holds:

$$(XB_{r,k}X^\mathsf{T})_{ij} = e_k^{u\mathsf{T}} B_{r,k} e_k^v = \begin{cases} 1, & |u-v| > r, \\ 0, & |u-v| \leq r. \end{cases}$$

Therefore we can derive

$$\frac{1}{2}\langle A, XB_{r,k}X^\mathsf{T}\rangle = \sum_{\substack{i,j\in V,\\ i<j}} a_{ij}(XB_{r,k}X^\mathsf{T})_{ij} = \sum_{[i,j]\in E} (XB_{r,k}X^\mathsf{T})_{ij} = \sum_{\substack{[i,j]\in E,\\ |p(i)-p(j)|>r}} 1.$$

Hence, this term counts the number of edges with endpoints in partition subsets of distance greater than r. There exist several possibilities for these edges. In particular, for each $i = \{1, \dots, k-r-1\}$, we have at least one of the following cases:

$$\text{Case } (i)\dots \text{ the edge connects a vertex in } S_i \text{ to a vertex in } \{S_{r+1+i}, \dots, S_k\}. \tag{4}$$

For an illustration of this property, we use the following example.

Example 1. We consider a 15×15 matrix and the partitioning $m = (3, 3, 3, 3, 3)^\mathsf{T}$. Moreover, we choose $r = 2$. If $\langle A, XB_{r,k}X^\mathsf{T}\rangle > 0$ then there must be an edge with endpoints in blocks of distance larger than $r = 2$. Such edges are $S_1 \to S_4$, $S_1 \to S_5$, or $S_2 \to S_5$ which require to 'jump' over $\{S_2, S_3\}$ or $\{S_3, S_4\}$ at least.

Basic Partition. For a straightforward interpretation of the term $\frac{1}{2}\langle A, XB_{r,k}X^\mathsf{T}\rangle$ we consider a special partition matrix. We denote the assignment of vertices to subsets $\{S_1, \dots, S_k\}$ that is defined by the map $\overline{p}(i) := \min_{v\in 1,\dots,k}\{\sum_{\ell=1}^{v} m_\ell \geq i\}$, $i \in V$, as the *basic partition*, and its associated partition matrix by $\overline{X} \in \mathcal{P}_m$. Thus, $\overline{X}$ is characterized by consecutive bands of ones. Moreover, each entry (b_{uv}) of $B_{r,k}$ relates to a $|S_u| \times |S_v|$ block within $\overline{X}B_{r,k}\overline{X}^\mathsf{T}$ where all entries have the value b_{uv}. Thus, for a given $n \times n$ adjacency matrix A the term $\frac{1}{2}\langle A, \overline{X}B_{r,k}\overline{X}^\mathsf{T}\rangle$ counts the number of entries which overlap with the non-zero entries of the block matrix $\overline{X}B_{r,k}\overline{X}^\mathsf{T}$. Hence, it reflects the number of edges with start and end points in partitions of distance larger than r.

General Partition. In general, any matrix $X \in \mathcal{P}_m$ can be obtained from the basic partition matrix $\overline{X}$ by row-permutations that are defined by a permutation matrix $P \in \Pi_n$. Thus

$$\mathcal{P}_m = \{P\overline{X} \colon P \in \Pi_n\},$$

where $\overline{X}$ is the basic partition matrix. The following transformation is obtained by replacing X by $P\overline{X}$:

$$\frac{1}{2}\langle A, XB_{r,k}X^\mathsf{T}\rangle = \frac{1}{2}\mathbf{trace}\big(A^\mathsf{T}P\overline{X}B_{r,k}\overline{X}^\mathsf{T}P^\mathsf{T}\big) = \frac{1}{2}\langle P^\mathsf{T}AP, \overline{X}B_{r,k}\overline{X}^\mathsf{T}\rangle.$$

This shows that it is irrelevant if the permutation $P \in \Pi_n$ is applied to the adjacency matrix A or to the basic partition matrix $\overline{X}$.

We summarize the above observations in the following proposition.

Proposition 1. *Let A be a $n \times n$ adjacency matrix, and let $3 \leq k \leq n$ and $m \in \mathbb{N}^k$ be given with $\sum_{i=1}^{k} m_i = n$. Let $1 \leq r \leq k-2$. If*

$$\min_{X\in\mathcal{P}_m} \frac{1}{2}\langle A, XB_{r,k}X^\mathsf{T}\rangle > 0, \textit{ then}$$

$$\boldsymbol{bdw}(A) > \min\{m_2 + \ldots + m_{r+1},\ m_3 + \ldots + m_{r+2}, \ldots,\ m_{k-r} + \ldots + m_{k-1}\}.$$

Proof. From above observations we know that if the premise $\min_{X\in\mathcal{P}_m} \frac{1}{2}\langle A, XB_{r,k}X^\mathsf{T}\rangle > 0$ holds, at least one of the following $k-r-1$ cases does occur, see (4).

In case (1) there must be some edge joining S_1 and $\{S_{r+2}, \dots, S_k\}$. Therefore, regardless of which sets are connected, the respective edge must reach over the blocks $\{S_2, \dots, S_{r+1}\}$. This corresponds to a distance greater than $|\{S_2, \dots, S_{r+1}\}| = m_2 + \ldots + m_{r+1}$.

Similarly, in case (2) there must be some edge joining S_2 and $\{S_{r+3}, \dots, S_k\}$ which results in a distance greater than $m_3 + \dots + m_{r+2}$.

The distance is determined analogously for all other cases. Finally, in case $(k - r - 1)$ the resulting distance is $m_{k-r} + \dots + m_{k-1}$. Therefore $\mathbf{bdw}(A)$ must be larger than the smallest of these numbers. $\square$

Conversely, a proposition regarding the upper bound can be given.

Proposition 2. *Let A be a $n \times n$ adjacency matrix, and let $3 \le k \le n$ and $m \in \mathbb{N}^k$ be given with $\sum_{i=1}^{k} m_i = n$. Let $1 \le r \le k-2$. If*

$$\min_{X \in \mathcal{P}_m} \frac{1}{2} \langle A, X B_{r,k} X^\mathsf{T} \rangle = 0, \ \textit{then}$$

$\mathbf{bdw}(A) \le \max\{m_1 + m_2 + \dots + m_{r+1} - 1,\ m_2 + m_3 + \dots + m_{r+2} - 1, \dots,\ m_{k-r} + \dots + m_k - 1\}.$

In Figure 1 we illustrate the lower and upper bounds given by Propositions 1 and 2, respectively.

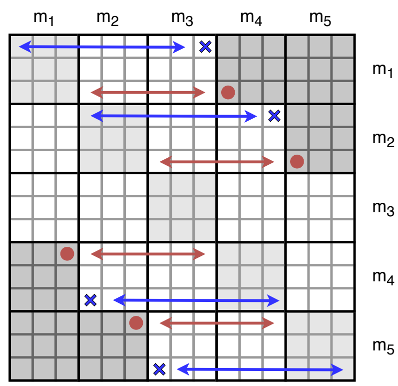

Figure 1: $A \in \mathcal{S}_{15}$, $m = (3, 3, 3, 3, 3)^\mathsf{T}$, and $r = 2$. The crosses (bullets) indicate possible possitions of the non-zero entries in terms of lower (upper) bounds.

We now formally introduce the *Minimal Partition Problem* (minPart):

$$\text{minPart}(m, r) := \min_{X \in \mathcal{P}_m} \frac{1}{2} \langle A, X B_{r,k} X^\mathsf{T} \rangle. \tag{5}$$

In other words, the minPart is the problem of finding a partition of the vertices that has minimal number of edges in the blocks, defined by the one-entries in matrix $B_{r,k}$. Note that there are many choices for k and $m \in \mathbb{N}^k$ such that $\sum_{i=1}^k m_i = n$.

Corollary 3. *Let A be a $n \times n$ adjacency matrix of G, and let $3 \le k \le n$ and $m \in \mathbb{N}^k$ be given with $\sum_{i=1}^k m_i = n$. Let $r = 1$. Further, suppose that $m_2 = \dots = m_{k-1}$.*

If there exists $X \in P_m$ such that $\langle A, X B_{r,k} X^\mathsf{T} \rangle > 0$ then $\mathbf{bdw}(A) > m_2$.

In a similar way we get good choices of m in case $r = 2$.

Corollary 4. *Let A be a $n \times n$ adjacency matrix of G, and let $r = 2$ and $m \in \mathbb{N}^k$ be given with $\sum_{i=1}^k m_i = n$. Further, suppose $m = (m_1, m_2, m_3, m_2, m_3, \dots, m_k)^\mathsf{T}$.*

If there exists $X \in P_m$ such that $\langle A, X B_{r,k} X^\mathsf{T} \rangle > 0$ then $\mathbf{bdw}(A) > m_2 + m_3$.

By cyclically repeating the sizes, we can insure that the minimum in Proposition 1 is attained in each term simultaneously as above.

Relation to Prior Work. We present below two important special cases of our new modelling approach and their relation to prior work.

The case $k = 3$**.** Given $k = 3$ the only allowable choice for r is $r = 1$ and therefore the only nonzero elements in $B_{1,3}$ are $b_{1,3} = b_{3,1} = 1$. Hence for $m = (m_1,\ m_2,\ m_3)^\top$ Proposition 1 states that if there exists $X \in P_m$ such that $\langle A, XB_{1,3}X^\top\rangle > 0$ then $\mathbf{bdw}(A) > m_2$. This observation is used in [18] to derive lower bounds on $\mathbf{bdw}(A)$, and is further refined in [27, 32].

The case $k = n$**.** Another notable case occurs for $k = n$, which implies that $m_1 = \ldots = m_n = 1$. Hence, for any $r \in \{1, \ldots, n-2\}$ it follows from Proposition 1 that $\mathbf{bdw}(A) > r$ if there exists a partition matrix $X \in P_m$ such that $\langle A, XB_{r,n}X^\top\rangle > 0$. However, in this case the basic partition matrix becomes the identity matrix of rank n, i.e., $\overline{X} = I_n$. Thus, X becomes a permutation matrix $Q \in \Pi_n$ and we recover the statement

$$\min_{Q\in\Pi_n} \langle Q^\top AQ, B\rangle > 0 \Rightarrow \mathbf{bdw}(A) > r,$$

from (1). This approach is used e.g., in [8] to derive lower bounds on $\mathbf{bdw}(A)$.

To summarize, we have shown that once the minPart problem has a positive value for given $B_{r,k}$ and m, we get a lower bound on the bandwidth from Proposition 1. The minPart problem is itself NP-complete, so our strategy is to consider tractable lower bounds for the minPart problem. If some lower bound turns out to be positive for given $B_{r,k}$ and m, then clearly minPart has a positive value, and our bounding argument can be applied. In the following section we consider relaxations of minPart, based on semidefinite optimization. This will result in matrix variables of order $nk + 1$ where the user has some flexibility in the choice of k (and m), as we will see below.

4 SDP models

In this section we derive several SDP relaxations for the Minimal Partition Problem. Our first two SDP relaxations are obtained by matrix lifting and therefore have matrix variables of order $\mathcal{O}(kn)$, while the third relaxation has k matrix variables of order n.

4.1 SDP model in $\mathcal{S}_{n\cdot k+1}$

In this section, we derive an SDP relaxation whose matrix variable is of order $nk + 1$.

Let $X \in \mathcal{P}_m$ be a partition matrix, see (2). Let $x_1, \ldots, x_k$ be the columns of X, i.e., $X = \begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix}$, and $x := \mathbf{vec}(X) \in \mathbb{R}^{n\cdot k}$. Now, the constraint $Y = xx^\top$ may be weakened to $Y - xx^\top \succeq 0$ which is well known to be equivalent to the following convex constraint

$$Z := \begin{bmatrix} Y & x \\ x^\top & 1 \end{bmatrix} \succeq 0.$$

In this section, we use the following block notation for $Z \in \mathcal{S}_{n\cdot k+1}$:

$$Z = \begin{bmatrix} X_1 & X_{12} & \ldots & X_{1k} & x_1 \\ X_{12}^\top & X_2 & \ldots & X_{2k} & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ X_{1k}^\top & X_{2k}^\top & \ldots & X_k & x_k \\ x_1^\top & x_2^\top & \ldots & x_k^\top & 1 \end{bmatrix},$$

where X_i corresponds to $x_i x_i^\top,\ i = 1, \ldots, k$, and X_{ij} to $x_i x_j^\top,\ i \neq j,\ i, j = 1, \ldots, k$.

For any $X_i,\ i = 1, \ldots, k$, we have $\mathbf{diag}(X_i) = \mathbf{diag}(x_i x_i^\top) = x_i$ and thus $\mathbf{trace}\,(X_i) = x_i^\top e_n = m_i$. For all $i = 1, \ldots, k$ we have:

$$\langle J_n, X_i\rangle = \mathbf{trace}\big(e_n e_n^\top x_i x_i^\top\big) = \mathbf{trace}\big((x_i^\top e_n)^2\big) = {m_i}^2.$$

Similarly, we have

$$\langle J_n, X_{ij} + X_{ij}^\mathsf{T} \rangle = \textbf{trace}\big(J_n X_{ij} + J_n X_{ij}^\mathsf{T}\big) = 2 \cdot \textbf{trace}\big(e_n e_n^\mathsf{T} x_i x_j^\mathsf{T}\big) = 2 m_i m_j, \quad \forall i, j.$$

From orthogonality of vectors x_i, $i = 1, \ldots, k$, it follows $\textbf{diag}\big(X_{ij}\big) = 0$.

Let us describe the matrix (3) as the sum of symmetric matrices having only one non-zero entry, i.e., $B_{r,k} = \sum\limits_{|u-v|>r} e_k^u e_k^{v\mathsf{T}} + e_k^v e_k^{u\mathsf{T}}$. Hence, we derive

$$X B_{r,k} X^\mathsf{T} = \sum_{|u-v|>r} X e_k^u e_k^{v\mathsf{T}} X^\mathsf{T} + X e_k^v e_k^{u\mathsf{T}} X^\mathsf{T} = \sum_{|u-v|>r} x_u x_v^\mathsf{T} + x_v x_u^\mathsf{T} = \sum_{|u-v|>r} X_{uv} + X_{vu}.$$

Therefore, we can rewrite the Minimal Partition Problem, see (5), as:

$$\min_{X \in \mathcal{P}_m} \frac{1}{2} \langle A, X B_{r,k} X^\mathsf{T} \rangle = \min \frac{1}{2} \langle A, \sum_{|u-v|>r} X_{uv} + X_{vu} \rangle = \min \sum_{|u-v|>r} \langle A, X_{uv} \rangle.$$

We collect now all above mentioned constraints and propose the following model for the Minimal Partition Problem based on the matrix lifting approach.

$$\min \quad \sum_{|u-v|>r} \langle A, X_{uv} \rangle, \tag{6a}$$

$$\text{s.t.} \quad \textbf{diag}\,(X_i) = x_i, \quad i = 1, \ldots, k, \tag{6b}$$

$$\textbf{diag}\,(X_{ij}) = 0, \quad i \neq j, \ i, j = 1, \ldots k, \tag{6c}$$

$$\textbf{trace}\,(X_i) = m_i, \quad i = 1, \ldots, k, \tag{6d}$$

$$\langle J_n, X_i \rangle = {m_i}^2, \quad i = 1, \ldots, k, \tag{6e}$$

$$\langle J_n, X_{ij} + X_{ij}^\mathsf{T} \rangle = 2 m_i m_j, \quad i \neq j, \ i, j = 1, \ldots, k, \tag{6f}$$

$$Z = \begin{bmatrix} X_1 & X_{12} & \ldots & X_{1k} & x_1 \\ X_{21} & X_2 & \ldots & X_{2k} & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ X_{k1} & X_{k2} & \ldots & X_k & x_k \\ x_1^\mathsf{T} & x_2^\mathsf{T} & \ldots & x_k^\mathsf{T} & 1 \end{bmatrix} \succeq 0. \tag{6g}$$

Here $Z \in \mathcal{S}_{kn+1}^+$. The feasible region of the above SDP relaxation equals the feasible region of the SDP relaxation for the graph partition problem derived by Wolkowicz and Zhao [33]. In order to further improve the relaxation, one can add nonnegativity constraints.

Below, we analyze the feasible region of the model (6).

Lemma 5. *Let Z satisfy* (6b)*,* (6c)*,* (6d)*,* (6e)*, and* (6g)*. Then*

$$\underbrace{\begin{pmatrix} e_n \\ 0_n \\ \vdots \\ 0_n \\ -m_1 \end{pmatrix}, \begin{pmatrix} 0_n \\ e_n \\ \vdots \\ 0_n \\ -m_2 \end{pmatrix}, \ldots, \begin{pmatrix} 0_n \\ 0_n \\ \vdots \\ e_n \\ -m_k \end{pmatrix}}_{k \ \text{vectors}}, \underbrace{\begin{pmatrix} I_n \\ I_n \\ \vdots \\ I_n \\ -e_n^\mathsf{T} \end{pmatrix}}_{n \ \text{vectors}}$$

forms the nullspace of Z.

For a proof we refer the reader to [28, Lemma 10 and Section 5.2] as well as to [33]. Note that the vectors from Lemma 5 correspond to a $(n \cdot k + 1) \times (n + k)$ matrix. As the sum of the first k columns is equal to the sum of the last n columns, the nullspace of Z has rank $n + k - 1$.

Lemma 6. *Let Z satisfies* (6b)*,* (6c)*,* (6d)*,* (6e)*, and* (6g)*. Then*

$$\begin{cases} X_1 & +X_{12} & +\ldots & +X_{1k} & = x_1 e_n^\top \\ \vdots & & & & \vdots \\ X_{k1} & +X_{k1} & +\ldots & +X_k & = x_k e_n^\top \\ x_1 & +x_2 & +\ldots & +x_k & = e_n \end{cases}.$$

Again, we refer the reader to [28, Section 5.2], and [33] for a formal proof. As a consequence of the previous lemma, the block $\begin{bmatrix} X_{k1} & X_{k2} & \ldots & X_{k,k-1} & X_k & x_k \end{bmatrix}$ is determined by $X_1, \ldots, X_{k-1}, X_{ij}$, $(i \neq j,\ i,j = 1, \ldots, k-1)$, and $x_1, \ldots, x_{k-1}$. Hence, matrix Z can be reduced by one block of rows and their corresponding columns without loss of information. This leads us to the reduced SDP model presented in the following section.

One can also derive the Slater feasible version of the SDP relaxation (6) by exploiting a basis of the orthogonal complement to the nullspace of Z given in Lemma 5. For details see e.g., [35, 28]. The Slater feasible version may be efficiently solved by using the Alternating Direction Method of Multipliers (ADMM) as described in [25]. The ADMM is a first-order method for convex problems that decomposes an optimization problem into subproblems that may be easier to solve.

4.2 Reduced SDP Model in $\mathcal{S}_{n \cdot (k-1)+1}$

In this section we provide an SDP relaxation that is equivalent to the one from the previous subsection, but contains less variables. In particular, based on Lemma 6 we propose the following SDP relaxation for the Minimal Partition Problem.

$$\min \quad \sum_{|u-v|>r} \langle A, X_{uv} \rangle, \tag{7a}$$

$$\text{s.t.} \quad \textbf{diag}\,(X_i) = x_i, \quad i = 1, \ldots, k-1, \tag{7b}$$

$$\textbf{diag}\,(X_{ij}) = 0, \quad i \neq j,\ i,j = 1, \ldots k-1, \tag{7c}$$

$$\textbf{trace}\,(X_i) = m_i, \quad i = 1, \ldots, k-1, \tag{7d}$$

$$\langle J_n, X_i \rangle = {m_i}^2, \quad i = 1, \ldots, k-1, \tag{7e}$$

$$\langle J_n, X_{ij} + X_{ij}^\top \rangle = 2 m_i m_j, \quad i \neq j,\ i,j = 1, \ldots, k-1, \tag{7f}$$

$$\widetilde{Z} = \begin{bmatrix} X_1 & X_{12} & \ldots & X_{1,k-1} & x_1 \\ X_{12}^\top & X_2 & \ldots & X_{2,k-1} & x_2 \\ \vdots & \vdots & & \vdots & \vdots \\ X_{1,k-1}^\top & X_{2,k-1}^\top & \ldots & X_{k-1} & x_{k-1} \\ x_1^\top & x_2^\top & \ldots & x_{k-1}^\top & 1 \end{bmatrix} \succeq 0. \tag{7g}$$

Here $\widetilde{Z} \in \mathcal{S}^+_{(k-1)n+1}$. Note that the nullspace of the reduced matrix $\widetilde{Z}$ has still rank $k-1$. We show below that the SDP relaxation (7) is equivalent to (6). The number of equations in this SDP is moderate i.e., $\mathcal{O}(nk)$. Additional sign constraints

$$X_{uv} \geq 0, \quad |u-v| > r \tag{8}$$

insure that the lower bound from this model is always nonnegative.

Lemma 7. *From* (7b) – (7g) *follows* (6b) – (6g).

Proof. **Step 1:** From Lemma 6 directly follows that, given $\widetilde{Z}$, the missing entries of Z can be expressed by:

$$
\begin{aligned}
x_k &= e_n - x_1 - \cdots - x_{k-1} \geq 0, \\
X_{ik} &= x_i e_n^\mathsf{T} - X_i - \sum_{\substack{j=1 \\ i \neq j}}^{k-1} X_{ij}, \quad i = 1, \ldots, k-1, \\
X_k &= x_k e_n^\mathsf{T} - \sum_{j=1}^{k-1} X_{kj}.
\end{aligned}
$$

Nonnegativity of x_k follows from (7c) and (7g).
Step 2:

Constraint (6g)**.** From [28, Section 5], we know that under (7b) – (7g) holds that

$$\widetilde{Z} \succeq 0 \quad \wedge \quad Z = \overline{W} U \overline{W}^\mathsf{T} \Rightarrow Z \succeq 0,$$

where

$$
\overline{W} := \begin{bmatrix} e_n & 0_n & \cdots & 0_n & I_n \\ 0_n & e_n & \cdots & 0_n & I_n \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0_n & 0_n & \cdots & e_n & I_n \\ -m_1 & -m_2 & \cdots & -m_k & -e_n \end{bmatrix}^\perp . \tag{9}
$$

Hence, holds (6g).

Constraint (6b)**.** In addition to (7b) $\mathbf{diag}(X_k) = x_k$ must hold. In particular, from **Step 1** it follows

$$\mathbf{diag}\big(X_k\big) = \mathbf{diag}\big(x_k e_n^\mathsf{T} - \sum_{j=1}^{k-1} X_{kj}\big) = x_k - \sum_{j=1}^{k-1} \mathbf{diag}(X_{kj}) = x_k.$$

Constraint (6c)**.** In addition to (7b) $\mathbf{diag}(X_{ik}) = 0$, $i = 1, \ldots, k-1$, must hold. Again, by using **Step 1** we have:

$$\mathbf{diag}\big(X_{ik}\big) = \mathbf{diag}\big(x_i e_n^\mathsf{T} - X_i - \sum_{\substack{j=1 \\ i \neq j}}^{k-1} X_{ij}\big) = 0.$$

Constraint (6d)**.** From (7b) and **Step 1** we have (6b). Thus, from $\mathbf{diag}(X_k) = x_k$ it follows $\mathbf{trace}\big(X_k\big) = m_k$.

Constraint (6e)**.** From $\langle J_n, x_k e_n^\mathsf{T} \rangle = m_k \cdot n$ and $\langle J_n, X_{kj} \rangle = \langle e_n e_n^\mathsf{T}, x_k x_j^\mathsf{T} \rangle = m_j \cdot m_k$, we have

$$\langle J_n, X_k \rangle = \langle J_n, x_k e_n^\mathsf{T} - \sum_{j=1}^{k-1} X_{kj} \rangle = m_k \big(n - \sum_{j=1}^{k-1} m_j\big) = m_k^2.$$

Constraint (6f)**.** In addition to (7f), $\langle J_n, X_{ik} + X_{ik}^\mathsf{T} \rangle = 2 m_i m_j$, $i = 1, \ldots, k-1$, must hold. Thus,

$$\langle J_n, X_{ik} + X_{ik}^\mathsf{T} \rangle = 2 \cdot \langle J_n, X_{ik} \rangle = \ 2 \cdot \big[\langle J_n, x_i e_n^\mathsf{T} \rangle - \langle J_n, X_i \rangle - \sum_{\substack{j=1 \\ i \neq j}}^{k-1} \langle J_n, X_{ij} \rangle\big] = \ 2 m_i m_k.$$

□

Note that the direction opposite to the one in the lemma follows directly. To make the SDP relaxation (7) with additional nonnegativity constraints equivalent to SDP relaxation (6) with additional nonnegativity constraints, we need to add nonnegativity constraints to the 'missing' blocks $[X_{k1}\ X_{k2}\ \dots\ X_{k,k-1}\ \ X_k\ \ \ x_k]$ in (7). In particular we have the following proposition.

Proposition 8. *The SDP relaxation* (6) *with additional constraints* $Z \geq 0$ *is equivalent to the SDP relaxation* (7) *with additional constraints* $\widetilde{Z} \geq 0$ *and*

$$1 - \sum_{r=1}^{k-1} (X_r)_{i,i} - \sum_{r=1}^{k-1} (X_r)_{j,j} + \sum_{r=1}^{k-1}\sum_{p=1}^{k-1} (X_{rp})_{i,j} \geq 0, \quad i > j,$$

$$(X_r)_{i,i} - \sum_{l=1}^{k-1} (X_{lr})_{i,j} \geq 0, \quad i \neq j, r \in \{1, \dots, k-1\}.$$

In Section 5 we demonstrate strength of our SDP relaxations.

4.3 SDP model in $\mathcal{S}_n$

To derive an SDP relaxation whose matrix variable is of order n, we exploit the spectral decomposition of the matrix $B_{r,k}$. Similar approach was exploited in [10, 34] to derive SDP relaxations for the quadratic assignment problem. It follows from the well known Spectral Decomposition theorem that $B_{r,k}$ can be written as

$$B_{r,k} = \sum_{i=1}^{k} \lambda_i u_i u_i^\mathsf{T},$$

where λ_i are the eigenvalues of $B_{r,k}$ and $u_i \in \mathbb{R}^k$ the corresponding normalized eigenvectors. Hence, the term $XB_{r,k}X^\mathsf{T}$ can be written as

$$XB_{r,k}X^\mathsf{T} = \sum_{i=1}^{k} \lambda_i (Xu_i)(Xu_i)^\mathsf{T}.$$

For each $i \in \{1 \dots, k\}$, we introduce a matrix variable $Q_i \in \mathcal{S}_n$ that corresponds to $(Xu_i)(Xu_i)^\mathsf{T}$, and derive constraints that relate Q_i and X.

By exploiting the fact that $X^\mathsf{T} e_n = m$, we obtain the following constraints:

$$Q_i e_n = (m^\mathsf{T} u_i) X u_i, \qquad i = 1, \dots, k. \tag{10}$$

From the definition of Q_i we derive:

$$\mathbf{diag}(Q_i) = \mathbf{diag}(Xu_i u_i^\mathsf{T} X^\mathsf{T}) = (Xu_i) \circ (Xu_i) = X(u_i \circ u_i). \tag{11}$$

The last equality above follows from $Xu_i = u_i^{(1)} \cdot \chi_1 + u_i^{(2)} \cdot \chi_2 + \dots + u_i^{(k)} \cdot \chi_k$, where χ_i is the indicator vector of ones in the ith column of X.

From

$$\sum_{i=1}^{k} Xu_i u_i^\mathsf{T} X^\mathsf{T} = XX^\mathsf{T} \succeq 0,$$

it follows:

$$\sum_{i=1}^{k} Q_i \succeq 0. \tag{12}$$

Now, we derive semidefinite programming constraints that correspond to matrices Q_i, $i = 1, \dots, k$. Namely, by relaxing $Q_i = (Xu_i)(Xu_i)^\mathsf{T}$ we obtain the following SDP constraint $Q_i -$

$(Xu_i)(Xu_i)^\mathsf{T} \succeq 0,\ i = 1,\dots k$. In the following lemma we show that the SDP constraint $Q_i - (Xu_i)(Xu_i)^\mathsf{T} \succeq 0$ is equivalent to

$$Q_i \succeq 0, \qquad i = 1,\dots k, \tag{13}$$

under additional conditions.

Lemma 9. *Let $Xe_k = e_n$ and $X^\mathsf{T} e_n = m$, and $Q_i,\ i = 1,\dots,k$, satisfy* (10)*. Then*

$$\begin{pmatrix} Q_i & Xu_i \\ (Xu_i)^\mathsf{T} & 1 \end{pmatrix} \succeq 0$$

if and only if $Q_i \succeq 0,\ i = 1,\dots k$.

Proof. Suppose that $Q_i \succeq 0,\ i = 1,\dots,k$. Let $\alpha \in \mathbb{R}^n$ be an arbitrary vector. Then, by using the Cauchy-Schwarz inequality we have:

$$(\alpha^\mathsf{T} Q_i \alpha)(e_n^\mathsf{T} Q_i e_n) \geq (\alpha^\mathsf{T} Q_i e_n)^2.$$

After substituting $Q_i e_n = (m^\mathsf{T} u_i) X u_i$ and $e_n^\mathsf{T} Q_i e_n = (m^\mathsf{T} u_i)^2$ into the above inequality, it reduces to

$$(\alpha^\mathsf{T} Q_i \alpha) \geq (\alpha^\mathsf{T} X u_i)^2.$$

Now, for any $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ we have

$$\begin{aligned} (\alpha^\mathsf{T}, \beta) \begin{pmatrix} Q_i & Xu_i \\ (Xu_i)^\mathsf{T} & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad &= \alpha^\mathsf{T} Q_i \alpha + 2\beta \alpha^\mathsf{T} (Xu_i) + \beta^2 \\ &\geq (\alpha^\mathsf{T} X u_i)^2 + 2\beta\alpha^\mathsf{T}(Xu_i) + \beta^2 \\ &= (\alpha^\mathsf{T} X u_i + \beta)^2 \geq 0. \end{aligned}$$

The other direction is trivial. □

Finally, we collect constraints (10)–(13), add several obvious ones, and arrive to the following SDP relaxation for the Minimal Partition Problem.

$$\begin{aligned} \min \quad & \tfrac{1}{2} \sum_{i=1}^{k} \langle \lambda_i A, Q_i \rangle, \\ \text{s.t.} \quad & Q_i e_n = (m^\mathsf{T} u_i) X u_i, \quad i = 1,\dots,k, \\ & \mathbf{diag}(Q_i) = X(u_i \circ u_i), \quad i = 1\dots,k, \\ & Xe_k = e_n,\ X^\mathsf{T} e_n = m, \\ & X \geq 0,\ \sum_{i=1}^{k} Q_i \geq 0,\ \sum_{i=1}^{k} \lambda_i Q_i \geq 0, \\ & Q_i \succeq 0, \quad i = 1,\dots,k. \end{aligned} \tag{14}$$

Note that $Q_i \in \mathcal{S}_n$ for all i. Thus, the order of SDP matrices does not depend on k.

It is not difficult to verify that for $Q_i,\ i = 1,\dots,k$, feasible for (14), the following is satisfied:

$$\mathbf{diag}\big(\sum_{i=1}^{k} Q_i\big) = e_n \quad \text{and} \quad \mathbf{diag}\big(\sum_{i=1}^{k} \lambda_i Q_i\big) = 0.$$

In the case of equipartition, the constraint $\frac{n}{k} I_n - \sum_{i=1}^{k} Q_i \succeq 0$ is also implied by the rest of the model constraints. We prove this in the following lemma.

Lemma 10. *Let $m_1 = m_2 = \ldots = m_k = \frac{n}{k}$ and $Q_i,\ i = 1,\dots,k$ be feasible for* (14)*. Then,*

$$\frac{n}{k} I_n - \sum_{i=1}^{k} Q_i \succeq 0. \tag{15}$$

Proof. As the columns of X are orthogonal and each column contains $\frac{n}{k}$ ones, it follows $X^\top X = \frac{n}{k} I_k$. Let $\widetilde{X} := \sqrt{\frac{k}{n}} X$ which contains k orthogonal columns. Consequently, holds $\widetilde{X}^\top \widetilde{X} = \frac{k}{n} X^\top X = I_n$. Due to the well known Basis Completion Theorem we can complete $\widetilde{X}$ to a basis of dimension n, i.e.,

$$\exists \widetilde{Y} \in \mathbb{R}^{n\times(n-k)}: \widetilde{Z} = \left[\widetilde{X},\ \widetilde{Y}\right] \text{ orthogonal.}$$

Hence, we have

$$I_n = \widetilde{Z}\widetilde{Z}^\top = \left[\widetilde{X}\widetilde{Y}\right] \begin{bmatrix} \widetilde{X}^\top \\ \widetilde{Y}^\top \end{bmatrix} = \widetilde{X}\widetilde{X}^\top + \widetilde{Y}\widetilde{Y}^\top,$$

from where it follows

$$I_n - \widetilde{X}\widetilde{X}^\top = \widetilde{Y}\widetilde{Y}^\top \succeq 0,$$

and thus $I_n - XX^\top \succeq 0$. Now, by using this and $\sum_{i=1}^{k} Q_i = XX^\top$, it follows (15). □

The constraint (15) does not hold for general partitions m, as instead of $X^\top X = m_0 I_k$ only the following weaker constraint $X^\top X = \mathbf{Diag}(m)$ holds. The SDP model (14) can be further strengthened by adding triangle inequalities.

Finally, one can derive an SDP relaxation by exploiting the spectral decomposition of the adjacency matrix of the graph A. This leads to an SDP relaxation with matrix variables of order k. It turns out that the resulted relaxation is too weak and therefore we do not present it here.

5 Computational Experiments

5.1 Solving the SDP relaxations

The partition-based lower bounds for the bandwidth problem lead to semidefinite programs with either k semidefinite matrices Q_i of order n, see (14), or one big matrix of $n\cdot(k-1)+1$, see (7). The resulting relaxations can be solved routinely with standard SDP packages such as SDPT3 only for limited values of n and k.

We now focus on the strongest model which has one matrix variable of order $n\cdot(k-1)+1$ and roughly nk^2 equality constraints. We also consider nonnegativity constraints which adds another $O(n^2k^2)$ potentially violated sign constraints to our relaxation. Interior point based methods for such a scenario turn out to be too slow, so we propose to use the ADMM method, which works well for SDP with simple sign constraints. To use the ADMM, we first derive the Slater feasible version of the SDP relaxation (6) by exploiting (9). The resulting SDP relaxation has a matrix variable of order $(k-1)\cdot(n-1)+1$, see e.g., [33]. Then, we proceed in the same manner as described in [25, 20].

5.2 Test problems

We investigate the practical performance of our lower bounds on the following classes of graphs.

5.2.1 Torus graphs

For given integer k the torus graph T_k has k^2 vertices which we label by (i,j) for $i,j \in \{1,\dots,k\}$. We introduce 'vertical' edges of the form $[(i,j),(i+1,j)]$ for $1 \le i \le k-1$ and $[(1,j),(k,j)]$. Altogether there are k^2 such edges. In a similar way we add 'horizontal' edges of the form $[(i,j),(i,j+1)]$ for $j < k$ together with $[(i,1),(i,k)]$. This graph therefore has $n := k^2$ vertices and $2n$ edges. Even though T_k is very sparse, its bandwidth is relatively large. We found labelings showing that $\mathbf{bdw}(T_k) \le 2k$, but we do not know the exact value of $\mathbf{bdw}(T_k)$.

5.2.2 Torus graphs plus Hamiltonian path

Here we start out with the torus graph T_k, choose a labeling of its vertices yielding a bandwidth of size $2k$, and add the Hamiltonian path from the first to the last vertex in this labeling. The resulting graph is denoted by TH_k. It is still sparse having roughly $3|V(TH_k)|$ edges and bandwidth again at most $2k$.

5.2.3 Dense embeddings of Torus graphs

We take the graph TH_k and randomly add edges within the bandwidth $2k$ where each edge appears with probability $\frac{1}{2}$. The resulting graph is denoted TD_k and has bandwidth at most $2k$.

5.2.4 Hamming graphs

The Hamming graph $H(d,q)$ is the Cartesian product of d copies of the complete graph K_q. The Hamming graph $H(d,2)$ is also known as the hypercube (graph) Q_d. Thus, the hypercube graph Q_d has 2^d vertices. The bandwidth of the hypercube graph was determined by Harper [17] and is given by the following expression:

$$\mathbf{bdw}(Q_d) = \sum_{i=0}^{d-1} \binom{i}{\lfloor \frac{i}{2} \rfloor}.$$

Hence, we use the hypercube graphs to test the quality of our bounds.

5.3 Computations

Torus graphs. In the tables to follow we always provide the following information. The first block of data contains the vector m of cardinalities for the partition blocks. We consider partitions into $k \in \{4,5,6\}$ blocks. We set $r = 1$ and ask that $m_2 = m_3 = \ldots = m_{k-1}$.

The sizes m_1 and m_k are chosen such that $\sum_{i=1}^{k} m_i = n$ and $|m_1 - m_k| \leq 1$. Next we provide upper and lower bounds for the Minimal Partition Problem. The upper bound (ub) is obtained by running a standard Simulated Annealing heuristic [3] to find a good partition. The lower bound (lb) is obtained from the SDP relaxation (6) with all nonnegativity constraints included. Our main interest lies in values of m, where the obtained lower bound is nontrivial, i.e., $lb > 0$. We give an illstrutation of the obtained solutions in Figure 2.

First, we consider Table 1, which contains computational results for the Torus graph T_7. Initially, we consider 4 blocks with $m_2 = m_3 = 8$ leading to a lower bound $lb > 1.23$. Hence, Corollary 3 allows us to conclude that $\mathbf{bdw}(T_7) > 8$. We next try $m_2 = m_3 = 9$ where we only obtain the trivial lower bound of 0. Therefore we get no further restriction on $\mathbf{bdw}(T_7)$ from 4-partitions. The 5-partition with $m_2 = m_3 = m_4 = 9$ however yields a positive lower bound and therefore $\mathbf{bdw}(T_7) > 9$. Also, 6-partitions, given in the last block of the Table 1, do not lead to a further tightening of $\mathbf{bdw}(T_7)$.

T_7 $(n = 49)$

m_1	m_2	m_3	m_4	m_5	m_6	ub	lb
16	8	8	17			6	1.23
15	9	9	16			5	-
11	9	9	9	11		6	0.68
9	10	10	10	10		5	-
6	9	9	9	9	7	6	0.56
4	10	10	10	10	5	4	-

Table 1: Torus graph T_7.

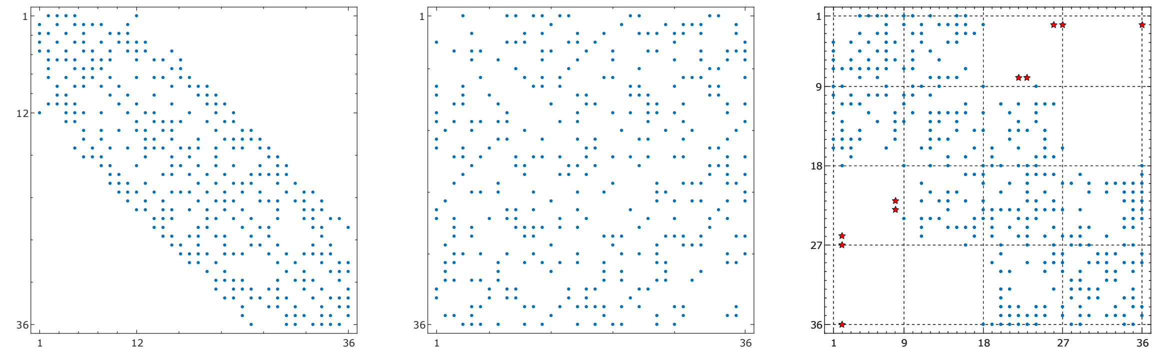

Figure 2: Illustration of the TD_6 graph. On the left, we show the unpermuted graph, in the center, the permuted graph is shown, on the right, the obtained solution of the minPart problem with $m = (9, 9, 9, 9)^\top$ is shown. The value of minPart is 5, the corresponding entries are indicated by stars.

The results for the Torus graphs T_8, T_9, and T_{10} are summarized in Table 2. We proceed as before and consider partitions with $k \in \{4, 5, 6\}$. We can prove a lower bound of 11 for $\mathbf{bdw}(T_8)$ and $\mathbf{bdw}(T_9)$. It turns out that proving positive lower bounds for our partition problems get increasingly difficult as either n or k increases. The use of 6-partition blocks for T_{10} allows us to prove a lower bound of 14 for T_{10}.

T_8 $(n = 64)$

m_1	m_2	m_3	m_4	m_5	m_6	ub	lb
23	9	9	23			7	1.01
22	10	10	22			6	-
17	10	10	10	17		7	0.84
15	11	11	11	16		7	-
12	10	10	10	10	12	8	0.99
10	11	11	11	11	10	6	-

T_9 $(n = 81)$

m_1	m_2	m_3	m_4	m_5	m_6	ub	lb
31	9	9	32			9	1.53
30	10	10	31			8	-
25	10	10	10	26		10	1.63
24	11	11	11	24		9	-
20	10	10	10	10	21	9	1.91
18	11	11	11	11	19	9	-

T_{10} $(n = 100)$

m_1	m_2	m_3	m_4	m_5	m_6	ub	lb
41	9	9	41			11	1.62
40	10	10	40			10	-
32	12	12	12	32		10	0.68
30	13	13	13	31		9	-
24	13	13	13	13	24	10	0.52
22	14	14	14	14	22	10	-

Table 2: Torus graphs $T_8,\ T_9,\ T_{10}$.

As a second experiment, we consider the graphs $TH_7, \ldots, TH_{10}$ consisting of the union of the

Torus graph and a Hamiltonian path such that $TH_k \leq 2k$ is insured. The results are summarized in Table 3. Compared to the Torus graphs we get slightly stronger lower bounds even though these graphs are still quite sparse, with $|E(TH_k)| < 3|V(TH_k)|$. Again, we see increasing gaps between lower and upper bounds as the number of nodes of the graph increases.

TH_7 $(n = 49)$

m_1	m_2	m_3	m_4	m_5	m_6	ub	lb
14	10	10	15			5	0.87
13	11	11	14			3	-
8	11	11	11	8		2	0.18
6	12	12	12	7		1	-
2	11	11	11	11	3	3	0.07

TH_8 $(n = 64)$

m_1	m_2	m_3	m_4	m_5	m_6	ub	lb
21	11	11	21			7	0.76
20	12	12	20			5	-
14	12	12	12	14		6	0.64
12	13	13	13	13		3	-
8	12	12	12	12	8	6	0.35
6	13	13	13	13	6	3	-

TH_9 $(n = 81)$

m_1	m_2	m_3	m_4	m_5	m_6	ub	lb
28	12	12	29			10	0.96
27	13	13	28			7	-
21	13	13	13	21		8	1.12
19	14	14	14	20		6	-
14	13	13	13	13	15	10	1.34
12	14	14	14	14	13	7	-

TH_{10} $(n = 100)$

m_1	m_2	m_3	m_4	m_5	m_6	ub	lb
37	13	13	37			11	0.64
36	14	14	36			9	-
29	14	14	14	29		11	1.20
27	15	15	15	28		9	-
22	14	14	14	14	22	11	1.64
20	15	15	15	15	20	9	-

Table 3: Torus graphs plus Hamiltonian paths TH_7, TH_8, TH_9, TH_{10}.

Next, we consider results for the dense embeddings of the Torus graphs, see Table 4. We partition into 4 blocks and note that in all cases we get tighter estimates for the bandwidth as compared to the previous tables. We find partitions of value 0 for values $m_2 \leq 2k-1$ such that our approach (with the number of blocks limited to 4) is not strong enough to close the gap between lower and upper bounds of the bandwidth for these graphs. In contrast to the previous results we always get lower bounds which rounded up give the optimal value of the partition problem.

To demonstrate the quality of the SDP relaxation (14) we compute the 4-partition with $m = (13, 11, 11, 14)^\mathsf{T}$ for TD_7. We obtain the optimal value 1.59. This result leads again to the conclusion that $\mathbf{bdw}(TD_7) \geq 12$. Similar experiment for TD_8 results with provable lower bound on the bandwidth of 14, while our strongest relaxation provides a better result, see Table 4.

We summarize the bandwidth information for all variations of the Torus graphs in Table 5.

TD_7		$(n = 49)$			
m_1	m_2	m_3	m_4	ub	lb
12	12	12	13	0	-
13	11	11	14	3	3.00
TD_8		$(n = 64)$			
m_1	m_2	m_3	m_4	ub	lb
17	15	15	17	0	-
18	14	14	18	1	0.97
19	13	13	19	5	4.71
TD_9		$(n = 81)$			
m_1	m_2	m_3	m_4	ub	lb
24	16	16	25	0	-
25	15	15	26	4	4.00
26	14	14	27	9	8.67
TD_{10}		$(n = 100)$			
m_1	m_2	m_3	m_4	ub	lb
31	19	19	31	0	-
32	18	18	32	1	0.58
33	17	17	33	5	4.37

Table 4: Dense embeddings of Torus graphs with bandwidth $\leq 2k$.

k	n	T_k	TH_k	TD_k	
			bdw $\geq$		**bdw** $\leq$
7	49	10	12	12	14
8	64	11	13	15	16
9	81	11	14	16	18
10	100	14	15	19	20

Table 5: Summary of bounds for the bandwidth.

Our partitioning approach provides nontrivial lower bounds on all instances. The bounds are particularly strong for the densest instances in this collection.

Now, let us provide some information on computation time. To compute 4-partitions for graphs with 49 vertices we need about 20 seconds, for 5-partitions about 30 seconds, and for 6-partitions about 90 seconds. On the other hand, to compute a 4-partition (resp. 6-partition) on a graph with 100 vertices, our ADMM code needs about 200 seconds (resp. 700 seconds). Clearly, computational times increase with respect to increasing partition sizes and vertices of graphs. However, we obtain bounds in reasonable time for all tested graphs.

Hamming graphs. Results for the Hamming graphs H_5, H_6, and H_7 are summarized in Table 6. The table reads similar to the previous tables. To show a lower bound of 10 for $\mathbf{bdw}(H_5)$, our ADMM needs only 4 seconds. For comparison purposes we computed a lower bound for H_5 and the case $k = 32$. Thus we solved the QAP relaxation for that instance, and obtained 11 for the lower bound of the BP.

For the Hamming graph H_6 the 4-partition with $m_2 = m_3 = 17$ and $r = 1$ yields a positive lower bound, and therefore $\mathbf{bdw}(H_6) \geq 18$. We also compute the 6-partition with $m = (15, 9, 8, 9, 8, 15)^\mathsf{T}$ and $r = 2$, and obtain a positive lower bound, which leads again to the conclusion that $\mathbf{bdw}(H_6) \geq 18$. Finally, we prove a lower bound of 33 for the Hamming graph H_7.

Hamming graph H_5 $n = 32$, $\mathbf{bdw} = 13$

m_1	m_2	m_3	m_4	-	-	ub	lb	$\mathbf{bdw} \geq$	
6	10	10	6			0	-		
7	9	9	7			4	0.99	10	

Hamming graph H_6 $n = 64$, $\mathbf{bdw} = 23$

m_1	m_2	m_3	m_4	m_5	m_6	ub	lb	$\mathbf{bdw} \geq$	r
15	17	17	15			10	1.18	18	1
15	9	8	9	8	15	14	1.18	18	2
14	9	9	9	9	14	9	-		

Hamming graph H_7 $n = 128$, $\mathbf{bdw} = 43$

m_1	m_2	m_3	m_4	m_5		ub	lb	$\mathbf{bdw} \geq$	
33	31	31	33			19	-		
34	30	30	34			31	3.11	31	
16	32	32	32	16		18	0.93	33	

Table 6: Hamming graphs.

5.4 Discussion

Based on our limited computational experiments we reach the following conclusion.
• The partitioning approach leads to acceptable lower bounds for the Bandwidth Problem. Our results indicate that the bounds get weaker as the number of nodes gets larger. This should come as no surprise in view of the hardness results known for the Bandwidth Problem.
• Our approach offers some flexibility in choosing the number k of partition blocks to estimate the bandwidth. A larger k would result in tighter bounds at higher computational cost.
• Further tightening of the semidefinite models are possible by adding additional constraints, e.g., triangle inequalities. This results in SDPs which require a refined computational setup.

6 Summary and Conclusion

We have shown that the partition approach provides a versatile tool to obtain lower bounds for the bandwidth of a graph. The choice of the model parameters k, m, and r are highly problem dependent. However, our experiments indicate that even with a small number of partition blocks ($k \ll n$) we are able to derive nontrivial lower bounds on the bandwidth, even for very sparse graphs. Further research is necessary to explore this approach for larger graphs.

References

[1] Tanya Y. Berger-Wolf and Edward M. Reingold. Index assignment for multichannel communication under failure. *IEEE Transactions on Information Theory*, 48(10):2656–2668, Oct 2002.

[2] Avrim Blum, Goran Konjevod, R. Ravi, and Santosh Vempala. Semi-definite relaxations for minimum bandwidth and other vertex-ordering problems. *Theoretical Computer Science*, 235(1):25 – 42, 2000.

[3] Rainer E. Burkard and Franz Rendl. A thermodynamically motivated simulation procedure for combinatorial optimization problems. *European Journal of Operational Research*, 17(2):169 – 174, 1984.

[4] P. Z. Chinn, J. Chvtalov, A. K. Dewdney, and N. E. Gibbs. The bandwidth problem for graphs and matricesa survey. *Journal of Graph Theory*, 6(3):223–254.

[5] Vclav Chvtal. A remark on a problem of Harary. *Czechoslovak Mathematical Journal*, 20(1):109–111, 1970.

[6] Jarmila Chvtalov. Optimal labelling of a product of two paths. *Discrete Mathematics*, 11(3):249 – 253, 1975.

[7] Elizabeth Cuthill and James McKee. Reducing the bandwidth of sparse symmetric matrices. In *Proceedings of the 1969 24th National Conference*, ACM '69, pages 157–172, New York, NY, USA, 1969. ACM.

[8] Etienne de Klerk, Marianna E.-Nagy, and Renata Sotirov. On semidefinite programming bounds for graph bandwidth. *Optimization Methods and Software*, 28(3):485–500, 2013.

[9] Etienne de Klerk and Renata Sotirov. Improved semidefinite programming bounds for quadratic assignment problems with suitable symmetry. *Mathematical Programming*, 133(1):75–91, Jun 2012.

[10] Etienne de Klerk, Renata Sotirov, and Uwe Truetsch. A new semidefinite programming relaxation for the quadratic assignment problem and its computational perspectives. *INFORMS Journal on Computing*, 27(2):378–391, 2015.

[11] Josep Díaz, Jordi Petit, and Maria Serna. A survey of graph layout problems. *ACM Comput. Surv.*, 34(3):313–356, September 2002.

[12] John Dunagan and Santosh Vempala. On euclidean embeddings and bandwidth minimization. In Michel Goemans, Klaus Jansen, José D. P. Rolim, and Luca Trevisan, editors, *Approximation, Randomization, and Combinatorial Optimization: Algorithms and Techniques*, pages 229–240, Berlin, Heidelberg, 2001. Springer Berlin Heidelberg.

[13] Michael R. Garey, R. Graham, David Johnson, and D. Knuth. Complexity results for bandwidth minimization. *SIAM Journal on Applied Mathematics*, 34(3):477–495, 1978.

[14] Willem H. Haemers. Interlacing eigenvalues and graphs. *Linear Algebra and its Applications*, 226–228:593 – 616, 1995. Honoring J.J.Seidel.

[15] Frank Harary. Problem 16. In Miroslav Fiedler, editor, *Theory of graphs and its applications*, page 191. Czechoslovak Academy of Science, Prague, 1967.

[16] Lawrence H. Harper. Optimal assignments of numbers to vertices. *Journal of the Society for Industrial and Applied Mathematics*, 12(1):131–135, 1964.

[17] Lawrence H. Harper. Optimal numberings and isoperimetric problems on graphs. *Journal of Combinatorial Theory*, 1(3):385 – 393, 1966.

[18] Christoph Helmberg, Franz Rendl, Bojan Mohar, and Svatopluk Poljak. A spectral approach to bandwidth and separator problems in graphs. *Linear and Multilinear Algebra*, 39(1–2):73–90, 1995.

[19] Robert Hochberg, Colin McDiarmid, and Michael Saks. On the bandwidth of triangulated triangles. *Discrete Mathematics*, 138(1):261 – 265, 1995. 14th British Combinatorial Conference.

[20] Hao Hu and Renata Sotirov. On solving the quadratic shortest path problem. *INFORMS Journal on Computing, to appear*, 2019.

[21] F.K. Hwang and J.C. Lagarias. Minimum range sequences of all k-subsets of a set. *Discrete Mathematics*, 19(3):257 – 264, 1977.

[22] Martin Juvan and Bojan Mohar. Laplace eigenvalues and bandwidth-type invariants of graphs. *J. Graph Theory*, 17(3):393–407, July 1993.

[23] Yung-Ling Lai and Kenneth Williams. A survey of solved problems and applications on bandwidth, edgesum, and profile of graphs. *J. Graph Theory*, 31(2):75–94, June 1999.

[24] Burkhard Monien. The bandwidth minimization problem for caterpillars with hair length 3 is NP-complete. *SIAM J. Algebraic Discrete Methods*, 7(4):505–512, October 1986.

[25] Danilo Elias Oliveira, Henry Wolkowicz, and Yangyang Xu. ADMM for the SDP relaxation of the QAP. *Mathematical Programming Computation*, 10(4):631–658, Dec 2018.

[26] Christos H. Papadimitriou. The NP-completeness of the bandwidth minimization problem. *Computing*, 16(3):263–270, Sep 1976.

[27] Janez Povh and Franz Rendl. A copositive programming approach to graph partitioning. *SIAM Journal on Optimization*, 18(1):223–241, 2007.

[28] Fanz Rendl and Renata Sotirov. The min-cut and vertex separator problem. *Computational Optimization and Applications*, 69(1):159—187, Jan 2018.

[29] Lawren Smithline. Bandwidth of the complete k-ary tree. *Discrete Mathematics*, 142(1):203 – 212, 1995.

[30] Jonathan S. Turner. On the probable performance of heuristics for bandwidth minimization. *SIAM Journal on Computing*, 15(2):561–580, 1986.

[31] Walter Unger. The complexity of the approximation of the bandwidth problem. In *Proceedings 39th Annual Symposium on Foundations of Computer Science (Cat. No.98CB36280)*, pages 82–91, Nov 1998.

[32] Edwin R. van Dam and Renata Sotirov. On bounding the bandwidth of graphs with symmetry. *INFORMS Journal on Computing*, 27(1):75–88, 2015.

[33] Henry Wolkowicz and Qing Zhao. Semidefinite programming relaxations for the graph partitioning problem. *Discrete Applied Mathematics*, 96(Supplement C):461–479, 1999.

[34] Yong Xia. Convex hull of the orthogonal similarity set with applications in quadratic assignment problems. *Journal of Industrial & Management Optimization*, 9(3):689–701, 2013.

[35] Qing Zhao, Stefan E. Karisch, Franz Rendl, and Henry Wolkowicz. Semidefinite programming relaxations for the quadratic assignment problem. *Journal of Combinatorial Optimization*, 2(1):71–109, Mar 1998.