Knapsack Polytopes
A Survey

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Abstract

The $0/1$ knapsack polytope is the convex hull of all $0/1$ vectors that satisfy a given single linear inequality with non-negative coefficients. This paper provides a comprehensive overview of knapsack polytopes. We discuss basic polyhedral properties, (lifted) cover and other valid inequalities, cases for which complete linear descriptions are known, geometric properties for small dimensions, and connections to independence systems. We also discuss the generalization to (mixed-)integer knapsack polytopes and variants. The results from the literature are complemented by some new results, examples, and open questions.

Keywords: knapsack polytope, cover inequality, lifting, separation problem, complete linear description, independence systems

Introduction

The *0/1 knapsack problem* consists of selecting items with given weights and profits such that the sum of the weights does not exceed a certain capacity and the sum of the profits is maximized. If the weights of the n items are given by $a = (a_1, \dots, a_n)^\top \in \mathbb{R}^n_+$, the capacity by $\beta \in \mathbb{R}_+$ and the profits by $c = (c_1, \dots, c_n)^\top \in \mathbb{R}^n_+$, it can be written as $\max\{c^\top x \,:\, x \in K^{a,\beta}\}$, where

$$K^{a,\beta} \coloneqq \{x \in \{0,1\}^n \,:\, a^\top x \leq \beta\}$$

is the *0/1 knapsack set* associated with weights a and capacity β.

The knapsack problem is of fundamental importance in discrete optimization and has several features that make it unique. First, it is weakly *NP*-hard, but can be solved by dynamic programming in pseudo-polynomial time $\mathcal{O}(n\,\beta)$ (if $a \in \mathbb{Z}_+^n$, $\beta \in \mathbb{Z}_+$) or $\mathcal{O}(n\,\bar{c})$ (if $c \in \mathbb{Z}_+^n$), where $\bar{c}$ is some upper bound on the optimal objective, e.g., $c_1 + \cdots + c_n$, see, e.g., Kellerer et al. [77, Lemma 2.3.1 and Lemma 2.3.2]. Thus, it serves as an intermediate problem with respect to its complexity status. Second, the problem is easily described, but has a rich structure depending on the weights and profits. Third, it appears as a subproblem in general binary programs $\max\{c^\top x \,:\, Ax \leq b,\; x \in \{0,1\}^n\}$. To derive cutting planes for such programs, one can, for example, take any row $a^\top x \leq \beta$ of the system $Ax \leq b$ and investigate the knapsack problem defined by $a^\top x \leq \beta$. Fourth, since it has a combinatorial structure (or is combined with additional combinatorial constraints) but has general coefficients, it serves as a bridge between combinatorial optimization and integer programming.

As a consequence, the literature on knapsack problems is huge, studying algorithmic, polyhedral, and computational questions. While the algorithmic side has been treated, for instance, in the book by Kellerer et al. [77] or Martello and Toth [91], we are not aware of an overview on polyhedral research. One motivation for this article was the appearance of special knapsack problems, whose polyhedral structure was studied [70] or even a complete linear description was found [69]. In this context, we (unsuccessfully) searched for an overview article on knapsack polytopes.

With this article we try to bridge this gap by providing an overview on the *0/1 knapsack polytope*

$$P^{a,\beta} := \operatorname{conv}\left(K^{a,\beta}\right),$$

i.e., the convex hull of all feasible solutions of the knapsack problem. Whenever the knapsack inequality is clear from the context, we write K and P instead of $K^{a,\beta}$ and $P^{a,\beta}$, respectively.

In the following, we concentrate on the most investigated $0/1$ case, but also cover the integer case or combinations of the knapsack problem and other constraints. In the integer case, the variables can attain any non-negative integer value (possibly with upper bounds). This case is of less practical importance and fewer polyhedral results exist for this case. However, it is of theoretical interest because the feasible region (possibly with slack variables) of every bounded integer program can be represented as the integer points fulfilling box constraints and a single linear equation, see Bradley [22]. Furthermore, a large variety of knapsack variants exists and covering all results seems hard; however, a selection of variants is covered in Section 10.

The selected topics of this paper and their presentation are naturally subjective. We try to be comprehensive for the important results, mentioning many other ones on the way. Sometimes, we will just refer to the literature for details.

The topics are structured in the following way: Section 1.1 starts with some basic results on the knapsack polytope. Afterwards, cover inequalities are introduced and we discus whether these inequalities define facets of the knapsack polytope in Section 1.2. Limits on the size of integer formulations are investigated in Section 2. In Section 3, the fundamentals on lifting are reviewed and illustrated for the case of cover inequalities. In practice, valid inequalities are generated on the fly, which is covered in Section 4. While complete linear descriptions of the knapsack polytope are treated in Section 5, Section 6 provides geometric properties of knapsack polytopes. Since the feasible set of a knapsack problem forms an independence system, there is a close connection between these two problems. In Section 7, we discuss the question of how to recognize whether an independence system arises from a knapsack or when a knapsack defines a matroid. In Section 8 and Section 9 we change the focus to the case where variables are no longer required to be binary but integer or mixed-integer valued, respectively. Variants of the knapsack problem are presented in Section 10. Finally, we close with some conclusions and open questions in Section 11.

Since this is an overview paper, the focus is on results from the literature. We provide proofs for some results where the required space is reasonably small. Moreover, we complement the literature results by some new material. For instance, most content in Section 2 and Section 6 as well as Lemma 20, Proposition 22 and Corollary 31 are new. Moreover, we provide running examples and present everything in a unified notation. Finally, we pose some open questions to stimulate further research.

Throughout this article, we assume basic knowledge of polyhedral theory and integer programming. There are numerous books that provide an introduction, for example, Schrijver [110], Nemhauser and Wolsey [98], or Korte and Vygen [79].

Notation. If not stated differently, n is a positive integer that denotes the dimension of the knapsack polytope's ambient space. We denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ the set of all positive integers, integers, and real numbers, respectively; $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of non-negative integers and real numbers, respectively. The set $\{1, \dots, k\}$ of the first k positive integers is denoted by $[k]$, where $[0] = \varnothing$. Moreover, we define $[k]_0 \coloneqq [k] \cup \{0\}$. The characteristic vector of a set $S \subseteq [n]$ is denoted by χ^S. In particular, we denote the all-ones vector $\chi^{[n]}$ by $\mathbb{1}$ and the i-th unit vector $\chi^{\{i\}}$ by e^i. Given a set $S \subseteq [n]$, we abbreviate the sum $\sum_{i \in S} x_i$ by $x(S)$. This in particular means $x(\varnothing) = 0$.

Contents

1 General Polyhedral Structure

In this section, we review basic polyhedral properties of arbitrary 0/1 knapsack polytopes. While Section 1.1 provides fundamental properties like dimension and facets that can be derived from trivial inequalities, Section 1.2 focuses on facets of knapsack polytopes that are based on cover inequalities. Most parts of the presented results were already published in the 1970s.

1.1 Basic Properties

Before stating polyhedral properties of knapsack polytopes, we discuss basic assumptions that can be made about the data $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ without loss of generality.

Assumption 1. *We have $0 < a_i \leq \beta$ for all $i \in [n]$ and $a_1 + \cdots + a_n > \beta$.*

Observation 2. Let $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$. Assumption 1 can be guaranteed by performing the following preprocessing steps in the specified order:

(a) If $a_i < 0$, we can complement variable x_i (i.e., replacing it by $1 - x_i$). Note that this step implies an increase of the value of of β by $|a_i|$.

(b) If $a_i = 0$, we can remove object i, since $P^{a,\beta}$ is (equivalent to) the Cartesian product of a knapsack polytope on variables $[n] \setminus \{i\}$ and the interval $[0, 1]$, i.e., variable i does not contribute to the relevant structure of $P^{a,\beta}$.

(c) If $a_i > \beta$, we can remove object i, since $x_i = 0$ in all solutions.

If $\beta < 0$ or $\beta = 0$, the polytope is empty or consists of the null vector only, respectively. Furthermore, if $a_1 + \cdots + a_n \leq \beta$, then $P^{a,\beta} = [0, 1]^n$.

Note that we have defined a knapsack polytope via a "less-than-or-equal"-constraint $a^\top x \leq \beta$. By complementing variables, it is also possible to consider $a^\top x \geq \beta$ knapsacks, i.e., the corresponding polytopes are affinely equivalent. Furthermore, for equality constrained knapsacks it holds that

$$\begin{aligned}&\operatorname{conv}\{x \in \{0,1\}^n \,:\, a^\top x = \beta\} = \\ &\quad \operatorname{conv}\{x \in \{0,1\}^n \,:\, a^\top x \leq \beta\} \cap \operatorname{conv}\{x \in \{0,1\}^n \,:\, a^\top x \geq \beta\},\end{aligned}$$

i.e., it suffices to understand the $\leq$- and $\geq$-polytopes. For this reason, we will only consider $\leq$-polytopes. For basic results on equality constrained knapsacks, e.g., concerning their dimensions and basic facet defining inequalities, we refer the reader to Lee [82].

To be able to characterize facet defining inequalities of knapsack polytopes, it is necessary to know their dimension. A characterization of their dimension as well as a characterization which trivial inequalities $x_i \geq 0$ and $x_i \leq 1$, $i \in [n]$, are facet defining is provided in the following lemma.

Lemma 3. *Let $a \in \mathbb{R}^n_+$ and $\beta \in \mathbb{R}_+$.*

(a) *The knapsack set* $K^{a,\beta}$ *is down-monotone, i.e., if* $x \in K^{a,\beta}$ *and* $y \leq x$ *with* $y \in \{0,1\}^n$*, then* $y \in K^{a,\beta}$ *as well.*

(b) *Let* $H \coloneqq \{i \in [n] : a_i > \beta\}$. *Then* $\dim(P^{a,\beta}) = n - |H|$.

(c) *For each* $i \in [n]$, $x_i \geq 0$ *defines a facet of* $P^{a,\beta}$ *if and only if* $i \notin H$.

(d) *For each* $i \in [n]$, $x_i \leq 1$ *defines a facet if and only if* $i \notin H$ *and* $a_i + a_j \leq \beta$ *for all* $j \in [n] \setminus (H \cup \{i\})$.

Proof. Throughout this proof, we use the notation $K = K^{a,\beta}$ and $P = P^{a,\beta}$. Although all results can be found in the literature, e.g., Part (b), (c), and (d) follow from Hammer et al. [62, Proposition 1–3] and Statement (d) also appears in Balas [10, Proposition 2], we provide proofs to make this survey self-contained and use the same notation.

(a) Let $x \in K$ and $y \in \{0,1\}^n$ fulfill $y \leq x$. Since $a \geq 0$, we have $a^\top x \geq a^\top y$. Thus, y is feasible as well.

(b) If $i \in H$, then $x_i = 0$ for every vertex of P. Thus, $\dim(P) \leq n - |H|$. Moreover, if $a_i \leq \beta$, $i \in [n]$, then $e^i \in P$. Together with 0 this forms $n - |H| + 1$ affinely independent points and thus $\dim(P) \geq n - |H|$.

(c) Because $x_i \geq 0$ is an implicit equation for every $i \in H$, it cannot define a facet of P. Conversely, if $i \notin H$, the points 0 and e^ℓ for $\ell \in [n] \setminus (H \cup \{i\})$ are $n - |H|$ affinely independent vectors contained in P. Since they fulfill $x_i \geq 0$ with equality and $\dim(P) = n - |H|$, the claim follows.

(d) Suppose $i \notin H$ and $a_i + a_j \leq \beta$ for all $j \in [n] \setminus (H \cup \{i\})$. Then the points e^i and $e^i + e^j$, $j \in [n] \setminus (H \cup \{i\})$ are $n - |H|$ affinely independent feasible points satisfying $x_i \leq 1$ with equality, i.e., $x_i \leq 1$ defines a facet. Conversely, if $x_i \leq 1$ defines a facet, then $i \notin H$ (otherwise $x_i = 0$ for every solution). If $a_i + a_j > \beta$ for $j \in [n] \setminus (H \cup \{i\})$, then $x_i \leq 1$ is dominated by the valid inequality $x_i + x_j \leq 1$, contradicting $x_i \leq 1$ defining a facet. □

Remark 4. *Note that Assumption 1 and Lemma 3(b) guarantee that* $P^{a,\beta}$ *is full-dimensional. Conversely, if* $a \geq 0$ *and* $P^{a,\beta}$ *is full-dimensional, then* $e^i \in P^{a,\beta}$ *and* $a_i \leq \beta$ *for all* $i \in [n]$.

Example 5. Consider the knapsack inequality

$$a^\top x = 4\,x_1 + 5\,x_2 + 6\,x_3 + 7\,x_4 + 9\,x_5 \leq 13 = \beta, \tag{1}$$

which will serve as a running example throughout this survey. Because $a_i \leq \beta$ for all $i \in [5]$, the set H defined in Lemma 3 is empty. Hence, the knapsack polytope $P^{a,\beta}$ is full-dimensional (Lemma 3(b)) and all non-negativity inequalities $x_i \geq 0$ define facets (Lemma 3(c)). Among all upper bound inequalities $x_i \leq 1$, only $x_1 \leq 1$ defines a facet (Lemma 3(d)). Finally, the only feasible 0/1 points satisfying the knapsack inequality $a^\top x \leq \beta$ with equality are $(1,0,0,0,1)^\top$, $(0,0,1,1,0)^\top$. Thus, $a^\top x \leq \beta$ defines a face of dimension 1.[1]

Remark 6. *Different knapsack constraints $a^\top x \leq \beta$ might lead to the same knapsack set/polytope, e.g., if $a \in \mathbb{Z}^n_+$ and $\beta \in \mathbb{Z}_+$ have a greatest common divisor at least 2. Sometimes it is also possible to increase components of a or decrease β without changing the knapsack set/polytope, e.g., $x_1 + 2\,x_2 \leq 2$ can be replaced by $2\,x_1 + 2\,x_2 \leq 2$ and then by $x_1 + x_2 \leq 1$.*

In the following, we investigate properties of general facet defining inequalities of $P^{a,\beta}$. An inequality $c^\top x \leq \gamma$ is called *homogeneous* if $\gamma = 0$ and *non-homogeneous* otherwise.

Lemma 7. *[62, Proposition 4] Let $a \in \mathbb{R}^n_+$ and $\beta \in \mathbb{R}_+$. If $P^{a,\beta}$ is full-dimensional, the only homogeneous facet defining inequalities $P^{a,\beta}$ are (positive multiples of) the trivial inequalities $-x_i \leq 0$, $i \in [n]$.*

Proof. Lemma 3(c) shows that $-x_i \leq 0$ defines a facet for all $i \in [n]$. Conversely, assume $c^\top x \leq 0$ defines a facet of $P = P^{a,\beta}$. Since P is full-dimensional and $a \geq 0$, $e^j \in P$ holds for all $j \in [n]$. Thus, $c_j \leq 0$ for all $j \in [n]$, and the claim follows if exactly one $c_j < 0$. Therefore assume there exist distinct $j, \ell \in [n]$ with $c_j < 0$ and $c_\ell < 0$. Then for any $\tilde{x} \in P$ with $c^\top \tilde{x} = 0$, it follows that $\tilde{x}_j = \tilde{x}_\ell = 0$, since otherwise $c^\top \tilde{x} < 0$. Thus, the facet defined by $c^\top x \leq 0$ is contained in the facets defined by $-x_j \leq 0$ and $-x_\ell \leq 0$, a contradiction. □

Lemma 8. *[62, Proposition 5] Let $a \in \mathbb{R}^n_+$ and $\beta \in \mathbb{R}_+$ such that $P^{a,\beta}$ is a full-dimensional knapsack polytope, and let $c^\top x \leq \gamma$ define a facet of $P^{a,\beta}$ with $\gamma \neq 0$. Then $\gamma > 0$ and $0 \leq c_i \leq \gamma$ for all $i \in [n]$.*

Proof. Since $0 \in P = P^{a,\beta}$, $\gamma \not< 0$. Thus, $\gamma > 0$.

For the sake of contradiction, assume $c_i < 0$ for some $i \in [n]$. Then there exists $\tilde{x} \in P \cap \{0,1\}^n$ such that $c^\top \tilde{x} = \gamma$ and $\tilde{x}_i = 1$ (otherwise the inequality would be dominated by $x_i \geq 0$). Due to Lemma 3(a), $\tilde{x} - e^i \in P$. However, $c^\top(\tilde{x} - e^i) = c^\top \tilde{x} - c_i > \gamma$, a contradiction to the validity of $c^\top x \leq \gamma$. Conversely, if $c_i > \gamma$ for some $i \in [n]$, then $x_i = 0$ for all feasible solutions. Hence, the face defined by $c^\top x \leq \gamma$ would be contained in the facet defined by $x_i \geq 0$, contradicting $c^\top x \leq \gamma$ being facet defining. □

In analogy to Lemma 3(a), Lemma 8 enables us to prove a monotonicity result for knapsack polytopes.

Corollary 9. *Let $a \in \mathbb{R}^n_+$ and $\beta \in \mathbb{R}_+$. The knapsack polytope $P^{a,\beta}$ is down-monotone, i.e., if $x \in P^{a,\beta}$ and $0 \leq y \leq x$, then $y \in P^{a,\beta}$.*

[1] It is easy to construct examples in which $a^\top x \leq \beta$ does not support $P^{a,\beta}$ (defines a face of dimension -1), e.g., $2\,x_1 + 2\,x_2 \leq 3$.

Proof. If $P = P^{a,\beta}$ is not full-dimensional, there exists $i \in [n]$ with $P \subseteq \{x \in \mathbb{R}^n_+ \,:\, x_i = 0\}$, which implies $0 = y_i = x_i$. Iteratively projecting on the remaining indices produces a full-dimensional polytope.

Let P be full-dimensional. It is sufficient to prove that y fulfills every facet defining inequality $c^\top x \leq \gamma$ of P. If $\gamma = 0$, Lemma 7 implies that the inequality is equivalent to a trivial inequality, which is fulfilled because $y \geq 0$. Otherwise, by Lemma 8, $c \geq 0$. Thus, $c^\top y \leq c^\top x \leq \gamma$ holds. □

Note that the results of Lemmas 3, 7, and 8 also follow from general statements for independence systems (or equivalently set covering problems), see, e.g., Conforti and Laurent [28], Balas and Ng [11], Laurent [80], and Sassano [109].

A general approach towards understanding general knapsack polytopes with integral coefficients is via so-called master knapsack polytopes. The *master 0/1 knapsack polytope* P_M is the knapsack polytope associated to the knapsack inequality

$$\sum_{i=1}^{\beta} \sum_{j=1}^{K_i} i\, x^i_j \leq \beta, \tag{2}$$

where $K_i \coloneqq 1 + \lfloor \frac{\beta}{i} \rfloor$, i.e., K_i is the smallest integer exceeding $\frac{\beta}{i}$ and the binary variable x^i_j corresponds to the jth variable with coefficient i. The following result is due to Hammer et al. [63] as stated by Hammer and Peled [64].[2]

Observe that any knapsack inequality $a^\top x \leq \beta$ with $a \in \mathbb{Z}^n_+$, $\beta \in \mathbb{Z}_+$ is represented in (2): For $i \in [\beta]$, let $A_i \coloneqq \{\ell \in [n] \,:\, a_\ell = i\}$ and define $I_{a,\beta} \coloneqq \{(i,j) \,:\, i \in [\beta],\ j = 1, \dots, |A_i|\}$. Then the knapsack polytope $P^{a,\beta}$ is isomorphic to the face of P_M that is defined by $x^i_j = 0$, $(i,j) \notin I_{a,\beta}$. Consequently, if we are given a complete linear description of P_M, we can directly deduce a complete linear description of arbitrary knapsack polytopes. Hammer and Peled [64] provide a complete description of P_M if $\beta \leq 7$. In general, however, a complete linear description is not available and one needs to take the knowledge on a concrete knapsack into account to find facet defining inequalities.

In contrast to the knapsack polytope P, the polytope corresponding to the standard LP relaxation

$$P_{\text{LP}} = \{x \in [0,1]^n \,:\, a^\top x \leq \beta\}$$

is well understood. If Assumption 1 holds, the knapsack inequality $a^\top x \leq \beta$ and the lower bound constraints $x_i \geq 0$, $i \in [n]$, are always facet defining. The upper bound constraint $x_i \leq 1$ is facet defining, provided $a_i < \beta$. Thus, P_{LP} has at least $n+1$ and at most $2n+1$ facets. Moreover, a complete characterization of the vertices of P_{LP} is available, which translates

[2] We could not access this reference online.

Dantzig's [31] characterization of optimal LP solutions into the language of polyhedra.

Proposition 10. *Let Assumption 1 hold. The vector* $\bar{x} \in [0,1]^n$ *is a vertex of* P_{LP} *if and only if there exist disjoint sets* I_0*,* $I_1 \subseteq [n]$ *with* $|I_0 \cup I_1| \geq n-1$ *such that* $a(I_1) \leq \beta$*,* $a_j > \beta - a(I_1)$ *for* $j \in [n] \setminus (I_0 \cup I_1)$*, and for every* $i \in [n]$ *we have*

$$\bar{x}_i = \begin{cases} 0, & \text{if } i \in I_0, \\ 1, & \text{if } i \in I_1, \\ \frac{\beta - a(I_1)}{a_j}, & \text{if } i \notin I_0 \cup I_1. \end{cases}$$

Proof. Every basis B of a vertex $\bar{x}$ of P_{LP} consists of n linearly independent constraints from the system $a^\top x \leq \beta$, $0 \leq x \leq \mathbb{1}$. Since $x_i \leq 1$ and $x_i \geq 0$ cannot both be contained in B, each basis contains at most n box constraints. Moreover, because there is exactly one non-box constraint in the LP relaxation, B contains at least $n-1$ box constraints. Hence, at least $n-1$ entries of $\bar{x}$ are integral.

Let I_0 and I_1 be the lower and upper bound constraints contained in B, respectively. If all entries of $\bar{x}$ are integral, then $a(I_1) \leq \beta$ since $\bar{x}$ is a vertex of P_{LP}, and also the remaining properties given in the statement of the proposition hold. Otherwise, there exists exactly one $j \in [n]$ with $\bar{x}_j \in (0,1)$. Inserting the fixings I_0 and I_1 and solving the knapsack equation (since it is contained in B in this case) for x_j yields $\bar{x}_j = \frac{\beta - a(I_1)}{a_j}$. Since $x_j \in (0,1)$, we conclude $a_j > \beta - a(I_1)$ and even $a(I_1) < \beta$.

To prove the reverse direction, one readily verifies that the box constraints corresponding to I_0 and I_1 as well as the knapsack constraint (if $\bar{x}_j \notin \mathbb{Z}$) form a basis for $\bar{x}$ provided $a_j > \beta - a(I_1)$ and $a(I_1) < \beta$ hold. □

Statements about certain inequalities, which will be introduced below, require the following ordering of coefficients.

Assumption 11. *The coefficients of the knapsack inequality are sorted non-decreasingly, i.e.,* $a_1 \leq a_2 \leq \cdots \leq a_n$*.*

At the end of this section, we pose some research questions; to the best of our knowledge these are currently open, but partial answers are discussed afterwards.

(Q1) How many different knapsack polytopes do (asymptotically) exist (w.r.t. affine or 0/1 equivalence)?

(Q2) In general it is *NP*-hard to decide whether the defining inequality $a^\top x \leq \beta$ defines a non-empty face of P, since one needs to determine whether the equality knapsack has a solution. Are there efficient sufficient conditions to guarantee binary feasibility of $a^\top x = \beta$ or that $a^\top x \leq \beta$ defines a facet?

(Q3) Is there a characterization of possible operations on the defining inequality such that the knapsack set/polytope does not change, cf. Remark 6? This would try to characterize the set

$$\big\{(\tilde{a}, \tilde{\beta}) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \,:\, K^{a,\beta} = K^{\tilde{a},\tilde{\beta}}\big\}.$$

Concerning Question (Q3), Bradley et al. [23] show that the solutions of the linear inequality system

$$\begin{aligned} &\tilde{a}(F) \le \tilde{\beta}, && F \subseteq \{0,1\}^n \text{ with } a(F) \le \beta,\\ &\tilde{a}(C) \ge \tilde{\beta} + 1, && C \subseteq \{0,1\}^n \text{ with } a(C) \ge \beta + 1,\\ &\quad 0 \le \tilde{a}_1 \le \tilde{a}_2 \le \cdots \le \tilde{a}_n \end{aligned}$$

are exactly the ones that define the same knapsack polytope as $a^\top x \le \beta$, provided Assumption 11 holds. Thus, for a concrete knapsack instance, (Q3) can be answered, whereas general operations/manipulations of the knapsack inequality that yield the same knapsack polytope are unknown.

1.2 Covers

One of the most important class of valid inequalities for knapsack polytopes is given by cover inequalities, which are discussed in this section.

Definition 12. Let $a \in \mathbb{R}^n_+$ and $\beta \in \mathbb{R}_+$. A set $C \subseteq [n]$ is called a *cover* of $P^{a,\beta}$ if $a(C) > \beta$. It is called *minimal* if it does not contain a proper subset that is a cover, i.e., $C \setminus \{i\}$ is not a cover for any $i \in C$. The corresponding *(minimal) cover inequality* is

$$x(C) \le |C| - 1.$$

Cover inequalities are clearly valid for $P^{a,\beta}$, see, e.g., Crowder et al. [29]. Moreover, the set of box constraints and all minimal cover inequalities provides an integer formulation for the knapsack set, since for a point $\tilde{x} \in \{0,1\}^n$ with $a^\top \tilde{x} > \beta$, the cover $\{i \in [n] \,:\, \tilde{x}_i = 1\}$ contains a minimal cover, whose cover inequality is violated by $\tilde{x}$.

Example 13 (Example 5 continued)**.** Consider again the knapsack inequality (1) given by $4\,x_1 + 5\,x_2 + 6\,x_3 + 7\,x_4 + 9\,x_5 \le 13$. The corresponding minimal covers are

$$\begin{aligned} &C_1 = \{1,2,3\}, && C_2 = \{1,2,4\}, && C_3 = \{1,3,4\}, && C_4 = \{2,3,4\},\\ &C_5 = \{2,5\}, && C_6 = \{3,5\}, && C_7 = \{4,5\}. \end{aligned}$$

The set $C = \{1,2,3,4\}$ is a cover, but not a minimal cover since $C_1 \subsetneq C$.

By analyzing minimal cover inequalities of knapsack polytopes, one is able to derive faces of the knapsack polytope. To this end, we restrict the knapsack polytope to the variables that are contained in the cover and obtain the following result, which has been implicitly shown by Padberg [99].

Proposition 14. *Let $a \in \mathbb{R}^n_+$ and $\beta \in \mathbb{R}_+$ be such that $P^{a,\beta}$ is full-dimensional, and let C be a minimal cover w.r.t. the knapsack inequality $a^\top x \leq \beta$. Then $x(C) \leq |C| - 1$ defines a facet of the knapsack polytope*

$$P_C \coloneqq \operatorname{conv}\Big\{x \in \{0,1\}^C \,:\, \sum_{i \in C} a_i\, x_i \leq \beta\Big\},$$

and it defines a face of $P^{a,\beta}$ of dimension at least $|C| - 1$.

Proof. Since $P^{a,\beta}$ is full-dimensional, the same holds for P_C. The $|C|$ affinely independent vectors $\mathbb{1} - e^i$, $i \in C$, fulfill the minimal cover inequality with equality. Thus, $x(C) \leq |C| - 1$ defines a facet of P_C and a face of $P^{a,\beta}$ of dimension at least $|C| - 1$. $\square$

In general, minimal cover inequalities need not define facets of P. In Section 3, however, we will see that so-called lifting can render these inequalities to be facet defining.

To strengthen the integer formulation of $P^{a,\beta}$ via minimal cover inequalities and to derive facet defining inequalities, we refine the concept of minimal covers.

Definition 15. Let $a \in \mathbb{R}^n_+$, $\beta \in \mathbb{R}_+$, and let Assumption 11 hold. Then a cover $C = \{i_1, \dots, i_k\}$, where $i_1 < i_2 < \dots < i_k$, is called *strong* if C is a minimal cover and

$$a(C) - a_{i_k} + a_j \leq \beta$$

for every $j \in [i_k - 1] \setminus C$. Moreover, the *extension* of a (not necessarily strong) cover $C = \{i_1, \dots, i_k\}$ is the set $E(C) \coloneqq C \cup \{i_k + 1, \dots, n\}$. Since the weights a_i are sorted non-decreasingly, the *extension inequality*

$$x\big(E(C)\big) = x(C) + x\big(\{i_k + 1, \dots, n\}\big) \leq |C| - 1 \tag{3}$$

is valid for P.

Example 16 (Example 5 continued)**.** Note that Assumption 11 holds in this example. The strong covers are C_1, C_5, C_6, and C_7. To see this for $C_5 = \{2, 5\}$, we check Definition 15:

$$a(C_5) - a_{i_k} + a_j \leq 14 - 9 + 7 = 12 \leq 13 = \beta,$$

for every $j \in \{1, 3, 4\}$ since $a_4 = 7 = \max\{a_\ell \,:\, \ell \in [i_k - 1] \setminus C_5\}$.

The minimal cover C_3, however, is not strong, because removing item 4 from and adding item 2 to the cover produces an infeasible collection of items.

Further details on (extension inequalities of) strong covers are provided by Balas [10]. Moreover, there exists a complete characterization of the cases in which extension inequalities define facets of $P^{a,\beta}$.

Theorem 17 (Balas [10])**.** *Let $a \in \mathbb{R}^n_+$, $\beta \in \mathbb{R}_+$, and let Assumption 11 hold. Let $E \subseteq [n]$ contain at least two elements and let $k \in [n]$. Then the inequality $x(E) \leq k-1$ defines a facet of $P^{a,\beta}$ if and only if it is the extension inequality of a strong cover $C = \{i_1, \dots, i_k\}$ with $i_1 < i_2 < \dots < i_k$ and*

$$a(C) - a_{i_k} - a_{i_{k-1}} + a_n \leq \beta. \tag{4}$$

Note that $x(E) \leq k-1$ in Theorem 17 is valid in both cases (either it defines a facet or it arises from a strong cover). A characterization equivalent to Theorem 17 in terms of independence systems is given by Laurent [80, Proposition 3.11 and 3.14].

Example 18 (Example 5 continued)**.** As seen in Example 16, the strong covers of the running example are $\{C_1, C_5, C_6, C_7\}$. Their extensions are

$$E(C_1) = \{1,2,3,4,5\}, \qquad E(C_5) = \{2,5\},$$
$$E(C_6) = \{3,5\}, \qquad E(C_7) = \{4,5\}.$$

Condition (4) of Theorem 17 yields:

$$\begin{aligned}
a(C_1) - a_3 - a_2 + a_5 &= 15 - 6 - 5 + 9 = 13 \leq 13,\\
a(C_5) - a_5 - a_2 + a_5 &= 14 - 9 - 5 + 9 = 9 \leq 13,\\
a(C_6) - a_5 - a_3 + a_5 &= 15 - 9 - 6 + 9 = 9 \leq 13,\\
a(C_7) - a_5 - a_4 + a_5 &= 16 - 9 - 7 + 9 = 9 \leq 13.
\end{aligned}$$

Thus, all inequalities $x(E(C_i)) \leq |C_i| - 1$ for $i \in \{1,5,6,7\}$ define facets and these are the only facet defining inequalities of the form $x(E) \leq k-1$.

(Q4) Can one obtain a characterization of knapsacks for which the cover inequalities provide a complete linear description of $P^{a,\beta}$?

(Q5) How strong is the formulation using (minimal) cover inequalities (e.g., in terms of LP-gap)?

2 Binary Formulations Based on Strong Covers

A set of linear inequalities $Ax \leq b$ separating $K = K^{a,\beta}$ and $\{0,1\}^n \setminus K$ is called a *binary formulation* of K, i.e., $K = \{x \in \{0,1\}^n : Ax \leq b\}$. Using the concept of (extensions of) strong covers, one can find a minimum size binary formulation of K all of whose inequalities have left-hand side coefficients that are either 0 or 1, so-called 0/1-*binary formulations*. Such formulations are interesting, since they are usually numerically more stable than the original inequality, if the latter contains large numbers. Moreover, these inequalities have a simple interpretation since they encode cardinality restrictions on subsets of variables. The following result is due to Glover [48] as stated by Wolsey [123].

Theorem 19 (Glover [48])**.** *Let $a \in \mathbb{R}^n_+$ and $\beta \in \mathbb{R}_+$. Let Assumption 11 hold and let $\mathfrak{S}$ be the set of all strong covers of $K^{a,\beta}$. Then*

$$x\big(E(C)\big) \leq |C| - 1 \quad \text{for all } C \in \mathfrak{S}$$

is a $0/1$-binary formulation of $K^{a,\beta}$ of minimum size. That is, there is no smaller set of inequalities with $0/1$-coefficients on the left-hand side that enforces that a binary vector is contained in $K^{a,\beta}$.

Note that Balas [10] cites Glover's result in a different way by restricting the strong covers in Theorem 19 to the set

$$\bar{\mathfrak{S}} = \{C \in \mathfrak{S} \,:\, E(C) \nsubseteq E(C') \text{ for all } C' \in \mathfrak{S} \setminus \{C\} \text{ with } |C| = |C'|\}.$$

Thus, if $\bar{\mathfrak{S}} \subsetneq \mathfrak{S}$, the size of Balas's version of the binary formulation in Theorem 19 is smaller than in Wolsey's version, which would be a contradiction to the minimality of the formulation. However, this discrepancy cannot occur.

Lemma 20. *Let C and $\bar{C}$ be distinct strong covers for the knapsack defined by $a^\top x \leq \beta$ with $|C| = |\bar{C}|$. Then neither $E(C) \subseteq E(\bar{C})$ nor $E(\bar{C}) \subseteq E(C)$ holds.*

Proof. Let C and $\bar{C}$ be two distinct strong covers with $k = |C| = |\bar{C}|$. Let $C = \{i_1, \ldots, i_k\}$ and $\bar{C} = \{j_1, \ldots, j_k\}$, where $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$. By symmetry, it suffices to show $E(C) \nsubseteq E(\bar{C})$ to prove the assertion.

For the sake of contradiction, suppose

$$\{i_1, \ldots, i_k\} \cup \{i_k + 1, \ldots, n\} = E(C) \subseteq E(\bar{C}) = \{j_1, \ldots, j_k\} \cup \{j_k + 1, \ldots, n\}.$$

Then $C \subseteq E(\bar{C}) \setminus \{i_k + 1, \ldots, n\}$. On the one hand, if $i_k \leq j_k$, this implies

$$\begin{aligned} C &\subseteq (\{j_1, \ldots, j_k\} \cup \{j_k + 1, \ldots, n\}) \setminus \{i_k + 1, \ldots, n\} \\ &= \{j_1, \ldots, j_k\} \setminus \{i_k + 1, \ldots, j_k\} \\ &\subseteq \bar{C}, \end{aligned}$$

contradicting either $|C| = |\bar{C}|$ or $C \neq \bar{C}$.

On the other hand, if $i_k > j_k$, then $C \subseteq \{j_1, \ldots, j_k\} \cup \{j_k + 1, \ldots, i_k\}$. Since $|C| = |\bar{C}|$ and $C \neq \bar{C}$, the index $j^\star \coloneqq \max\{j \in \bar{C} \setminus C\}$ exists and $a_{i_\ell} \geq a_{j_\ell}$ for every $\ell \in [k]$. Consequently, the knapsack weight of $C' \coloneqq (C \setminus \{i_k\}) \cup \{j^\star\}$ fulfills

$$\beta < a(\bar{C}) \leq a(C') \leq a(C),$$

which proves that C' is a cover, and thus, contradicts C being strong. By combining both cases in the case distinction, we obtain $E(C) \nsubseteq E(\bar{C})$. $\square$

Example 21 (Example 5 continued)**.** According to Example 16, the set of all strong covers is $\mathfrak{S} = \{C_1, C_5, C_6, C_7\}$ with extensions given in Example 18. Consequently, a minimum size 0/1-binary formulation of $K^{a,\beta}$ is

$$\begin{aligned}\big\{x \in \{0,1\}^5 \,:\, x_1 + x_2 + x_3 + x_4 + x_5 \le 2, &\qquad x_2 + x_5 \le 1,\\ x_3 + x_5 \le 1, &\qquad x_4 + x_5 \le 1\,\big\}.\end{aligned}$$

In general, the minimum size $|\mathfrak{S}|$ of a 0/1-binary formulation of K provided by Theorem 19 can be exponential in n and may be complicated to compute exactly. For this reason, we present a simple technique to produce lower bounds on $|\mathfrak{S}|$. Let $A \coloneqq \{a_i \,:\, i \in [n]\}$ be the set of different left-hand side coefficients of the knapsack inequality $a^\top x \le \beta$. For every $\alpha \in A$, we define $A_\alpha \coloneqq \{i \in [n] \,:\, a_i = \alpha\}$, i.e., A_α contains all indices of variables with coefficient α in the knapsack constraint. Moreover, define $\binom{A_\alpha}{n_\alpha}$ as the set containing all subsets of A_α with cardinality n_α.

Proposition 22. *Let Assumption 11 hold for $a \in \mathbb{R}^n_+$, let $\beta \in \mathbb{R}_+$, C be a strong cover of $K^{a,\beta}$, $n_\alpha \coloneqq |C \cap A_\alpha|$, $\alpha \in A$, and define $\bar{\alpha} \coloneqq \max\{\alpha \in A \,:\, n_\alpha > 0\}$. Then, for every choice of $C'_\alpha \in \binom{A_\alpha}{n_\alpha}$, $\alpha \in A \setminus \{\bar{\alpha}\}$, the set*

$$C' = \left(\bigcup_{\alpha \in A \setminus \{\bar{\alpha}\}} C'_\alpha\right) \cup (C \cap A_{\bar{\alpha}}),$$

is a strong cover of K.

Proof. Let $C = \{i_1, \dots, i_k\}$ be a strong cover and let $C' = \{i'_1, \dots, i'_k\}$ be defined as stated. Then $i'_k = i_k$ since elements in $A_{\bar{\alpha}}$ are not modified. Moreover, for every $\alpha \in A$ there exists a bijection $\phi_\alpha \colon A_\alpha \to A_\alpha$ that maps $C' \cap A_\alpha$ to $C \cap A_\alpha$, where $\phi_{\bar{\alpha}}$ is the identity on $A_{\bar{\alpha}}$. If C' is not strong, then $\beta < a(C') - a_{i'_k} + a_{j'}$ holds for the maximum element $j' \in [i'_k] \setminus C'$. In particular, since $C \cap A_{\bar{\alpha}} = C' \cap A_{\bar{\alpha}}$ and $\phi_\alpha(C' \cap A_\alpha) = C \cap A_\alpha$ for every $\alpha \in A \setminus \{\bar{\alpha}\}$ by definition, we have

$$j \coloneqq \phi_{a_{j'}}(j') \in [\phi_{\bar{\alpha}}(i'_k)] \setminus C = [i_k] \setminus C. \tag{5}$$

This implies

$$\begin{aligned}\beta &< \sum_{i' \in C'} a_{i'} - a_{i'_k} + a_{j'} = \sum_{\alpha \in A} \sum_{i' \in C' \cap A_\alpha} a_{i'} - a_{i'_k} + a_{j'}\\ &= \sum_{\alpha \in A} \sum_{i' \in C' \cap A_\alpha} a_{\phi_\alpha(i')} - a_{\phi_{\bar{\alpha}}(i'_k)} + a_{j'} = \sum_{i \in C} a_i - a_{i_k} + a_j,\end{aligned}$$

because j and j' have the same coefficient in the knapsack inequality by (5). Thus, since $j \in [i_k] \setminus C$, we conclude that C cannot be a strong cover, a contradiction. For this reason, every set C' that is generated as proposed above is a strong cover. □

Proposition 22 implies that we can generate $\prod_{\alpha \in A \setminus \{\bar{\alpha}\}} \binom{|A_\alpha|}{n_\alpha}$ many strong covers from one given strong cover. Thus, by guessing some strong covers, Proposition 22 and Theorem 19 yield a lower bound on the number of inequalities in a 0/1-binary formulation. In the following, we show two applications of this result to derive exponential lower bounds for particular knapsacks.

Example 23. Consider the knapsack inequality $\sum_{i=1}^{n} x_i + 2\sum_{i=n+1}^{2n} x_i \leq n$ written as $a^\top x \leq n$, where n is even. Let $I_k \coloneqq [k] \cup \{n+1, \dots, n+\frac{n-k-1}{2}\}$ for every odd $k \in [n]$. Then, $a^\top \chi^{I_k} = n-1$ implies that $\chi^{I_k} \in K$. Furthermore, adding any element with weight 2 to I_k increases the weight to $n+1$. Thus, the extended set is not in K. Consequently, the set $C_k \coloneqq I_k \cup \{n + \frac{n-k+1}{2}\}$ is a minimal cover, and it is strong because every element in $[n + \frac{n-k+1}{2}] \setminus C_k$ has weight 1. Hence, Proposition 22 shows that we can generate $\binom{n}{k}$ many strong covers from C_k and they are pairwise different. For this reason, any 0/1-binary formulation of K contains at least

$$\sum_{\substack{k \in [n]:\\ k \text{ odd}}} \binom{n}{k} = 2^{n-1}$$

inequalities. Thus, a restriction to inequalities with 0/1-coefficients leads to a drastic increase of the number of necessary constraints in comparison to a formulation with coefficients in $\{0, \pm 1, \pm 2\}$.

Example 24. Consider the knapsack inequality $\sum_{i=1}^{2n} 2^{\lceil i/2 \rceil - 1} x_i \leq 2^n - 1$ written as $a^\top x \leq \beta$, which is associated with the so-called orbisack, introduced by Kaibel and Loos [74]. Let $C = \{i \in [2n] : i \text{ odd or } i = 2\}$. Because $a^\top \chi^C = 2^n$, C is a minimal cover since all coefficients are greater than or equal to 1.

Moreover, C is strong: For $n \leq 2$, this follows by a case distinction. Otherwise, if $n > 2$, we have that $2n-2$ is the index with the largest coefficient not in C for which we estimate

$$a(C) - a_{2n-1} + a_{2n-2} = 1 + \sum_{i=1}^{n} 2^{i-1} - 2^{n-1} + 2^{n-2} = 2^n - 2^{n-1} + 2^{n-2} \leq 2^n - 1.$$

Thus, C is a strong cover.

Since C contains all elements of weight 1, one element of weight 2^{n-1} as well as one of the two items of weight 2^i for $i \in \{1, \dots, n-2\}$, Proposition 22 implies that we can generate 2^{n-2} strong covers from C. This shows that we need exponentially many inequalities in any $0/1$-binary formulation of K. In fact, one can show that the exact number of strong covers is 2^{n-1}. Thus, by guessing the single set C we can generate one half of all strong covers.

(Q6) Theorem 19 implicitly uses bounds by requiring $x \in \{0,1\}^n$. What is the minimal size of a formulation $Ax \le b$ with coefficients in $\{0, \pm 1\}$ such that $K = \{x \in \mathbb{Z}^n : Ax \le b\}$? An upper bound is $|\mathfrak{S}| + 2n$ by explicitly adding trivial inequalities.

(Q7) The lower bound on the number of strong covers is relying on the representation of K via the knapsack inequality $a^\top x \le \beta$. If all coefficients in a are different, however, the bounding technique is not applicable. Is there a simple mechanism to derive bounds on the number of strong covers also in this situation?

3 Lifting

In this section, we consider lifting, an important concept to strengthen valid inequalities for the knapsack polytope $P \coloneqq P^{a,\beta}$ with $a \in \mathbb{R}^n_+$, $\beta \in \mathbb{R}_+$. (Up-)lifting takes an inequality $\sum_{i \in S} c_i\, x_i \le \gamma$ with $S \subseteq [n]$ that is valid for the restricted knapsack set $K_S \coloneqq \{x \in K : x_i = 0,\ i \in [n] \setminus S\}$ and turns it into a valid inequality for $K \coloneqq K^{a,\beta}$ by lifting coefficients for the variables x_i, $i \notin S$, from 0 to appropriate lifting coefficients. The goal is to obtain a stronger or even facet defining inequality for P.

It is also possible to decrease coefficients of variables that are contained in the initial inequality, so-called *down-lifting*. In the following, we discuss up-lifting in detail and often write lifting instead of up-lifting. Down-Lifting will be discussed in Section 3.2.

We first note that "trivial lifting" is always possible, i.e., the inequality with 0 coefficients outside S is valid for K:

Lemma 25. *Let $a \in \mathbb{R}^n_+$ and $\beta \in \mathbb{R}_+$. If $\sum_{i \in S} c_i\, x_i \le \gamma$ is valid for K_S, then it is also valid for K.*

Proof. Consider any solution $\hat{x} \in K$ and define $\bar{x}$ by $\bar{x}_i = \hat{x}_i$ if $i \in S$ and $\bar{x}_i = 0$ otherwise. By monotonicity of K (Lemma 3(a)) and definition of K_S, we find $\bar{x} \in K_S$. Since the inequality is valid for K_S, we obtain

$$\sum_{i \in S} c_i\, \hat{x}_i = \sum_{i \in S} c_i\, \bar{x}_i \le \gamma,$$

which shows that the inequality is valid for K. $\square$

There are two ways to compute the lifting coefficients: either iteratively, one coefficient in each step, by so-called *sequential lifting* or all coefficients in one step, which is called *simultaneous lifting*.

3.1 Sequential Up-Lifting

Sequential lifting fixes an ordering of $[n] \setminus S$. By iteratively solving a sequence of optimization problems, the strongest possible lifting coefficients are computed such that the inequality remains valid for the corresponding restricted knapsack sets. In each iteration, all previously computed coefficients are taken into account.

This procedure will first be described for general valid inequalities. Then it is applied to minimal cover inequalities.

Lifting General Inequalities The sequential (up-)lifting procedure can be defined for an arbitrary valid inequality

$$\sum_{i \in S} \pi_i \, x_i \leq \pi_0 \tag{6}$$

for K_S with $\pi \geq 0$. Then for $k \in [n] \setminus S$ compute π_k so that

$$\pi_k \, x_k + \sum_{i \in S} \pi_i \, x_i \leq \pi_0 \tag{7}$$

is valid for $K_{S \cup \{k\}}$. Consider the lifting function $\Phi_S \colon \mathbb{R}_+ \to \mathbb{R}$ defined as

$$\Phi_S(u) \coloneqq \min_{x \in \{0,1\}^S} \Big\{ \pi_0 - \sum_{i \in S} \pi_i \, x_i \, : \, \sum_{i \in S} a_i \, x_i \leq \beta - u \Big\}. \tag{8}$$

Theorem 26 (Padberg [99])**.** *Let $a \in \mathbb{R}^n_+$, $\beta \in \mathbb{R}_+$, and let (6) be valid for K_S. Then Inequality (7) is valid for $K_{S \cup \{k\}}$ if $\pi_k \leq \Phi_S(a_k)$. Moreover, if $\dim(P) = n$, $\pi_k = \Phi_S(a_k)$, and (6) defines a face of dimension t of $\operatorname{conv}(K_S)$, then (7) defines a face of $\operatorname{conv}(K_{S \cup \{k\}})$ of dimension at least $t + 1$.*

After the lifting coefficient $\pi_k = \Phi_S(a_k)$ for $k \in [n] \setminus S$ has been computed, one can update S to be $S \cup \{k\}$ and compute the next coefficient for a new element of $[n] \setminus S$ (as long as $S \neq [n]$). By iterating this process, the other lifting coefficients can be computed one after the other such that the resulting inequality is valid for K. In general, the lifting function Φ_S can decrease as more variables are lifted in. Thus, the lifting coefficients depend on the order in which the variables are lifted. This lifting procedure is called *sequential lifting*.

Note that since $\pi \geq 0$, $\Phi_S(u) \leq \pi_0$ holds. Moreover, since (6) is valid for K_S and every feasible solution x considered in (8) is contained in K_S if $u \geq 0$, $\Phi_S(u) \geq 0$ follows. Thus, evaluating Φ_S in u amounts to solving a 0/1 knapsack problem whose objective value is contained in $[0, \pi_0]$. Applying Theorem 26 iteratively, the resulting inequalities are valid for the larger sets and therefore each intermediate objective value lies in the interval $[0, \pi_0]$. Moreover, if the original inequality (6) has integral coefficients, the computed lifting coefficients are integral as well.

Therefore, if π and π_0 are integral, $\Phi_S(u)$ can be computed by dynamic programming in $\mathcal{O}(n\,\pi_0)$ time. The overall running time of computing all lifting coefficients is thus in $\mathcal{O}(n^2\,\pi_0)$.

Note that if a and β are integral, $\Phi_S(u)$ can be computed in $\mathcal{O}(n\,\beta)$ time, since $u \geq 0$ and thus the right hand side is always at most β. However, usually $\pi_0 < \beta$, so the first way is faster; see also the next section.

Lifting Cover Inequalities One important class of valid inequalities is given by lifting minimal cover inequalities, i.e., the starting inequality (6) is given by $x(C) \leq |C| - 1$; in particular, $S = C$ and $\pi_0 = |C| - 1$. Consider some order $i_1, \dots, i_k$ of $[n] \setminus C$, and define

$$C_1 = C, \quad C_r = C_{r-1} \cup \{i_{r-1}\} \quad \text{for } r \in \{2, \dots, k\}.$$

Set $\alpha_i = 1$ for $i \in C$ and iteratively $\alpha_{i_r} = \Phi_{C_r}(a_{i_r})$ for $r \in [k]$. The resulting inequality

$$\sum_{i \in C} x_i + \sum_{i \in [n] \setminus C} \alpha_i\, x_i \leq |C| - 1 \tag{9}$$

is called a *simple lifted cover inequality (LCI)*; "simple" refers to the fact that we are only up-lifting here. Note that all coefficients in (9) are integral by construction.

Recall that by Proposition 14, $x(C) \leq |C| - 1$ defines a facet of $\operatorname{conv}(K_C)$, which is affinely equivalent to P_C. Thus, by Theorem 26, the resulting simple LCI is valid for K and defines a facet of $P = \operatorname{conv}(K)$. However, as the lifting coefficients depend on the ordering of $[n] \setminus C$, lifting one cover C can yield theoretically up to $(n - |C|)!$ (not necessarily different) facets of P.

All lifting coefficients can be computed in $\mathcal{O}(n\,|C|^2)$ time by the dynamic programming approach mentioned above. In fact, this can even be improved to $\mathcal{O}(n\,|C|)$ as shown by Zemel [131], i.e., computing all lifting coefficients for minimal cover inequalities has the same complexity as computing a single lifting coefficient. Moreover, Kaparis and Letchford [76] noted that by combining the ideas of Balas and Zemel [12] and Gu et al. [60], most of the lifting coefficients can be computed in time $\mathcal{O}(n + |C| \log|C|)$.

Example 27 (Example 5 continued)**.** Consider again the knapsack inequality $4\,x_1 + 5\,x_2 + 6\,x_3 + 7\,x_4 + 9\,x_5 \leq 13$. The minimal cover inequality associated to the minimal cover $C = \{2, 3, 4\}$ is given by $x_2 + x_3 + x_4 \leq 2$. Lifting this inequality, we have two choices in ordering $[n] \setminus C = \{1, 5\}$.

If we use the ordering $(i_1, i_2) = (1, 5)$, i.e., first lift the variable x_1 and then x_5 into the cover inequality, we have $C_1 = C = \{2, 3, 4\}$ and $C_2 = \{1, 2, 3, 4\}$.

In the first step, the lifting coefficient $\alpha_{i_1} = \Phi_{C_1}(a_{i_1})$ of x_1 is given by

$$\begin{aligned}
\Phi_{C_1}(a_{i_1}) &= \min_{x\in\{0,1\}^C} \Big\{ |C| - 1 - \sum_{i\in C} x_i \,:\, \sum_{i\in C} a_i\, x_i \le \beta - a_{i_1} \Big\} \\
&= \min_{x\in\{0,1\}^C} \{ 2 - x_2 - x_3 - x_4 \,:\, 5\,x_2 + 6\,x_3 + 7\,x_4 \le 13 - 4 \} \\
&= 2 - 1 = 1,
\end{aligned}$$

and we obtain the inequality $x_1 + x_2 + x_3 + x_4 \le 1$. The lifting coefficient $\alpha_{i_2} = \Phi_{C_2}(a_{i_2})$ of x_5 is then given by

$$\begin{aligned}
&\min_{x\in\{0,1\}^{C_2}} \Big\{ 2 - x_1 - x_2 - x_3 - x_4 \,:\, 4\,x_1 + 5\,x_2 + 6\,x_3 + 7\,x_4 \le 13 - 9 \Big\} \\
&= 2 - 1 = 1.
\end{aligned}$$

This yields the simple LCI $x_1 + x_2 + x_3 + x_4 + x_5 \le 2$.

Using the ordering $(i_1, i_2) = (5, 1)$ gives:

$$\begin{aligned}
\alpha_{i_1} &= \min_{x\in\{0,1\}^C} \{ 2 - x_2 - x_3 - x_4 \,:\, 5\,x_2 + 6\,x_3 + 7\,x_4 \le 13 - 9 \} \\
&= 2 - 0 = 2, \\
\alpha_{i_2} &= \min_{x\in\{0,1\}^{C_2}} \{ 2 - x_2 - x_3 - x_4 - 2x_5 \,:\, 5\,x_2 + 6\,x_3 + 7\,x_4 + 9\,x_5 \le 13 - 4 \} \\
&= 2 - 2 = 0.
\end{aligned}$$

This yields the different simple LCI $x_2 + x_3 + x_4 + 2x_5 \le 2$.

To speed up the computation of lifting coefficients, Balas [10] presents a weaker form of lifting that runs in linear time and which does not rely on solving an optimization problem. The trade-off is that Balas's simple LCIs do not necessarily define facets of P.

Theorem 28 (Balas [10])**.** *Let $a \in \mathbb{R}^n_+$, $\beta \in \mathbb{R}_+$, let Assumption 11 hold, and let C be a minimal cover for P, $E(C)$ the extension of C, and C_h be the set of the last h elements of C, $h \in \{1, \ldots, |C|\}$. Consider the partition $N_0, N_1, \ldots, N_q$, $q = |C| - 1$, of $[n]$, where*

$$\begin{aligned}
&N_0 = [n] \setminus E(C), \qquad N_1 = E(C) \setminus \bigcup_{h=2}^{q} N_h, \\
&N_h = \Big\{ j \in E(C) \,:\, \sum_{i\in C_h} a_i \le a_j < \sum_{i\in C_{h+1}} a_i \Big\}, \quad h \in \{2, \ldots, q\},
\end{aligned}$$

and define

$$\pi_i = h \quad \textit{for all } i \in N_h,\ h \in \{0, 1, \ldots, q\}.$$

Then, the inequality

$$\sum_{i\in C} x_i + \sum_{i\in[n]\setminus C} \pi_i\, x_i \le |C| - 1 \tag{10}$$

is valid for P. Furthermore, if

$$\sum_{i\in C\setminus C_{h+1}} a_i \le \beta - a_j \quad \textit{for all } j \in N_h,\ h \in \{0, 1, \dots, q\}, \tag{11}$$

then (10) defines a facet of P.

Recently, Letchford and Souli [83] introduced a modified lifting procedure that can also be applied if the underlying cover C is not minimal. If C is minimal, it produces the same inequality as the lifting procedure of Theorem 28. If C is not minimal, it generates an inequality that is not weaker than (10) and that can be stronger in some cases. In particular, Letchford's and Souli's method can modify coefficients of variables with indices contained in C.

Corollary 29 (Balas and Zemel [12])**.** *Let C be a minimal cover. If Condition (11) holds, every lifting sequence leads to the same sequentially lifted minimal cover inequality. This inequality is given by*

$$\sum_{i\in C} x_i + \sum_{i\in[n]\setminus C} \pi_i\, x_i \le |C| - 1,$$

where π_i is defined as in Theorem 28. Furthermore, it is the unique facet defining inequality of P having coefficients equal to 1 for all $i \in C$ and a right-hand side of $|C| - 1$.

Thus, provided Condition (11) is met, there is exactly one facet of P that can be found via sequentially lifting the minimal cover inequality for C. In particular, an inequality defining this facet can be found by Balas's procedure and one need not rely on the sequential lifting approach.

Example 30 (Example 5 continued)**.** Consider again the knapsack inequality $4\,x_1 + 5\,x_2 + 6\,x_3 + 7\,x_4 + 9\,x_5 \le 13$. For the minimal cover $C = \{1, 2, 3\}$ and its extension $E(C) = \{1, 2, 3, 4, 5\}$, the six sets C_1, C_2, C_3 and N_2, N_1, N_0 defined in Theorem 28 are given by

$$\begin{aligned}
&C_1 = \{3\}, \qquad C_2 = \{2, 3\}, \qquad C_3 = \{1, 2, 3\},\\
&N_2 = \{j \in E(C) \,:\, 5 + 6 \le a_j \le 5 + 6 + 4\} = \varnothing,\\
&N_1 = E(C) \setminus N_2 = \{1, 2, 3, 4, 5\},\\
&N_0 = [5] \setminus E(C) = \varnothing.
\end{aligned}$$

Thus, we obtain $\pi_4 = \pi_5 = 1$ and the valid inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 \le 2. \tag{12}$$

Since $N_0 = N_2 = \varnothing$, we need to check Condition (11) only for $h = 1$. Because $a_1 \leq 13 - a_j$ is true for all $j \in [5]$, the valid inequality (12) defines a facet of P. Furthermore, by Corollary 29, it is the unique facet of P which can be obtained from C by sequential lifting. In fact, (12) is the extension inequality for the cover $C = \{1, 2, 3\}$.

For the cover $C = \{2, 3, 4\}$ and its extension $E(C) = \{2, 3, 4, 5\}$ the six sets C_1, C_2, C_3 and N_2, N_1, N_0 are given by

$$\begin{aligned} &C_1 = \{4\}, \quad && C_2 = \{3, 4\}, \quad && C_3 = \{2, 3, 4\}, \\ &N_2 = \varnothing, && N_1 = \{2, 3, 4, 5\}, && N_0 = \{1\}. \end{aligned}$$

This results in the valid inequality

$$x_2 + x_3 + x_4 + x_5 \leq 2,$$

which is not facet defining for P, because it is dominated by the lifted inequality in Example 27. Hence, if Condition (11) does not hold (as in this example for $h = 0$ and $j = 1$), Theorem 28 does not necessarily yield sequentially lifted facets.

In Example 30, we have seen that sequential lifting may generate the extension inequality of a cover. The following corollary characterizes when this can occur. Recall that a strong cover is a minimal cover by definition.

Corollary 31. *Let Assumption 11 hold, and let $C = \{i_1, \dots, i_k\}$ be a cover of K, where $i_1 < i_2 < \cdots < i_k$ and $k \geq 2$. Then the extension inequality of C is a sequentially lifted cover inequality of P if and only if C is a strong cover and (4) holds.*

Proof. Since lifted cover inequalities are facet defining for P, necessity follows by Theorem 17. To see sufficiency, consider the quantity

$$a_j^\star = \max_{x \in \{0,1\}^C} \Big\{ x(C) \, : \, \sum_{i \in C} a_i \, x_i \leq \beta - a_j \Big\}.$$

If $j \in \{i_k + 1, \dots, n\}$, then $a_j^\star \geq |C| - 2$ by (4) since $a_j \leq a_n$ by Assumption 11. Thus, $a_j^\star = |C| - 2$ follows because $a_j \geq a_{i_k}$ and C is a minimal cover. Consequently, the lifting coefficient of x_j is 1. If $j \in [i_k] \setminus C$, then $a_j^\star = |C| - 1$ since C is a strong cover, and thus, the lifting coefficient is 0. Applying this argument iteratively (for any lifting sequence) yields the assertion. $\square$

(Q8) How strong (e.g., in terms of LP-gap) is the formulation using lifted cover inequalities?

(Q9) How many lifted cover inequalities are there?

(Q10) In Theorem 28, the additional assumption (11) not only depends on the knapsack polytope, but also on its representation via the knapsack inequality. Does there exist a similar characterization that is independent from the knapsack inequality?

3.2 Sequential Down-Lifting

The procedure described above is called up-lifting, because coefficients that were zero in the initial inequality are potentially increased. Recall that we assumed the initial inequality $\sum_{i\in M} \pi_i\, x_i \le \pi_0$ to be valid for K_S, i.e., the variables not in S are at their lower bound. In contrast to this, the down-lifting procedure assumes the initial inequality to be valid for the set $K^S \coloneqq \{x \in K \,:\, x_i = 1,\ i \in [n] \setminus S\}$, i.e., variables outside S are at their upper bound. While the initial inequality in the up-lifting process is valid for the whole knapsack set K, see Lemma 25, this inequality is typically not valid for K in the down-lifting case.

The aim of *down-lifting* is to turn an inequality valid for K^S into a valid inequality for K. To this end, the down-lifting procedure identifies coefficients π_i for $i \in [n] \setminus S$ such that

$$\sum_{i=1}^{n} \pi_i\, x_i \le \pi_0 + \sum_{i\in[n]\setminus S} \pi_i$$

is valid for K. To find such coefficients, analogously to the up-lifting case, a lifting sequence has to be fixed and the *up-lifting function* $\Phi^S \colon \mathbb{R}_+ \to \mathbb{R}$,

$$\begin{aligned}\Phi^S(u) &= \max_{x\in K^S} \Big\{ \sum_{i\in S} \pi_i\, x_i - \pi_0 \,:\, \sum_{i=1}^{n} a_i\, x_i \le \beta + u \Big\}\\ &= \max_{x\in\{0,1\}^S} \Big\{ \sum_{i\in S} \pi_i\, x_i - \pi_0 \,:\, \sum_{i\in S} a_i\, x_i + \sum_{i\in[n]\setminus S} a_i - u \le \beta \Big\},\end{aligned}$$

is considered. If x_k, $k \in [n] \setminus S$, is the first variable to be down-lifted, the lifting procedure removes k from $[n] \setminus S$ and determines the maximum violation $\pi_k = \Phi^S(a_k)$ of the initial inequality that can occur if x_k is fixed at its lower bound. Thus,

$$\pi_k\, x_k + \sum_{i\in S} \pi_i\, x_i \le \pi_0 + \pi_k$$

is valid for $K^{S\cup\{k\}}$. Note that this is in complete analogy to the up-lifting case since the up-lifting function Φ_S can be equivalently defined as

$$\Phi_S(u) = \min_{x\in K_S} \Big\{ \pi_0 - \sum_{i\in S} \pi_i\, x_i \,:\, \sum_{i=1}^{n} a_i\, x_i \le \beta - u \Big\}.$$

Example 32 (Example 5 continued)**.** Let $S = \{2, 3, 4, 5\}$. Then $x(S) \le 1$ is valid for K^S, however, it is invalid for K because it cuts off the feasible solution $(0, 1, 1, 0, 0)^\top$.

To make this inequality valid for K, we down-lift variable x_1 by computing

$$\pi_1 = \max_{x\in K^S} \Big\{ x(S) - 1 \,:\, 4\, x_1 + 5\, x_2 + 6\, x_3 + 7\, x_4 + 9\, x_5 \le 13 + 4 \Big\} = 1,$$

which results in the valid inequality $x_1 + x(S) \leq 1 + 1 = 2$.

Computing the lifting coefficient π_k exactly may be time-consuming, since it requires to solve a knapsack problem. Alternatively, it is possible to down-lift the initial inequality inexactly. Similar to the up-lifting case, we obtain an analogue to Theorem 26.

Theorem 33 (Nemhauser and Wolsey [98]). *Let $a \in \mathbb{R}^n_+$, $\beta \in \mathbb{R}_+$, and let (6) be valid for K^S. If $\{x \in K^{S\cup\{k\}} : x_k = 0\} \neq \varnothing$ and $\pi_k \geq \Phi^S(a_k)$, then*

$$\pi_k\, x_k + \sum_{i\in S} \pi_i\, x_i \leq \pi_0 + \pi_k \tag{13}$$

is valid for $K^{S\cup\{k\}}$. Moreover, if $\pi_k = \Phi^S(a_k)$ and (6) defines a face of dimension t of $\operatorname{conv}(K^S)$, then (13) defines a face of dimension at least $t+1$ of $\operatorname{conv}(K^{S\cup\{k\}})$.

Note that both the up- and down-lifting procedure can be applied jointly to one inequality. To illustrate this, consider up- and down-lifted minimal cover inequalities. Let C be a minimal cover and let (C_0, C_1) be a partition of C. The initial inequality for the lifting procedure is then $x(C_1) \leq |C_1| - 1$, which is valid for K^{C_1}, but not for K_{C_1} if $C_1 \neq C$. After fixing a lifting sequence, the variables in $[n] \setminus C$ are sequentially up- and the ones in C_0 down-lifted. This results in

$$\sum_{i\in[n]\setminus C} \pi_i\, x_i + \sum_{i\in C_0} \pi_i\, x_i + \sum_{i\in C_1} x_i \leq |C_1| - 1 + \sum_{i\in C_0} \pi_i, \tag{14}$$

which defines a facet of K, see Nemhauser and Wolsey [98]. In the literature, one refers to inequalities of this kind as *lifted cover inequalities (LCI)*, whereas inequalities with $C_0 = \varnothing$ are *simple LCIs* as before.

3.3 Sequence Independent Lifting and Superadditivity

The inequalities produced by sequential lifting may depend on the order, as we have seen in Example 27. Another approach is to compute *all* coefficients for variables with indices in $[n] \setminus S$ simultaneously, which may be faster. This concept is called *sequence independent or simultaneous lifting*. Since superadditivity of the lifting function plays an important role, this procedure is sometimes also called *superadditive lifting*. Note that Balas's lifting procedure in Theorem 28 can be considered as a simultaneous lifting approach, since it yields valid inequalities with coefficients that do not depend on any ordering of the indices $i \in [n] \setminus S$.

We first describe simultaneous lifting in a general setting for arbitrary valid inequalities, and afterwards, for cover inequalities more particularly.

Lifting General Inequalities Let $\pi \in \mathbb{R}_+^{[n]_0}$ and $\sum_{i \in S} \pi_i\, x_i \leq \pi_0$ be a valid inequality for $K_S \coloneqq \{x \in K \,:\, x_i = 0,\; i \in [n] \setminus S\}$ with $S \subseteq [n]$. Simultaneous lifting is performed via a function $\Psi\colon \mathbb{R}_+ \to \mathbb{R}$ such that

$$\sum_{i \in [n] \setminus S} \Psi(a_i)\, x_i + \sum_{i \in S} \pi_i\, x_i \leq \pi_0 \tag{15}$$

is valid for K. Note that taking $\Psi(u) = \Phi_S(u)$ does not lead to a valid inequality in general, since the set S grows as more variables are lifted in. Thus, if $i_1, \ldots, i_k$ is the lifting order of $[n] \setminus S$ chosen for the sequential lifting approach, the lifting function (8) satisfies

$$\Phi_S \geq \Phi_{S \cup \{i_1\}} \geq \Phi_{S \cup \{i_1, i_2\}} \geq \cdots \geq \Phi_{S \cup \{i_1, \ldots, i_k\}}.$$

To overcome this issue, one needs superadditive functions.

Definition 34. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *superadditive* on $D \subseteq \mathbb{R}$ if it satisfies $f(d_1) + f(d_2) \leq f(d_1 + d_2)$ for all $d_1,\, d_2 \in D$.

Theorem 35 (Gu et al. [60], Wolsey [125])**.** *Inequality* (15) *is valid for K if the function $\Psi\colon \mathbb{R} \to \mathbb{R}$ satisfies*

(a) $\Psi(u) \leq \Phi_S(u)$ for all $u \in [0, \pi_0]$ and

(b) Ψ is superadditive on $[0, \pi_0]$.

Examples for superadditive lifting functions can be found, among others, in Marchand et al. [87, Sec. 2.2.2], Gu et al. [60], or Letchford and Souli [83].

Lifting Cover Inequalities Let $C = \{i_1, \ldots, i_k\}$ with $i_1 < \cdots < i_k$ be a minimal cover and let Assumption 11 hold. Let A_j be the sum of the j largest elements of $\{a_{i_1}, \ldots, a_{i_k}\}$, i.e., $A_j = \sum_{i=1}^{j} a_{i_{k+1-i}}$, and define $A_0 \coloneqq 0$ as well as $\lambda \coloneqq \sum_{i \in C} a_i - \beta > 0$. Then, the function

$$\Psi(u) = \begin{cases} i, & \text{if } A_i \leq u < A_{i+1} - \lambda \quad \text{for } i \in \{0, \ldots, k-1\}, \\ i + \frac{1}{\lambda}(u - A_i), & \text{if } A_i - \lambda \leq u < A_i \quad\quad\;\; \text{for } i \in \{1, \ldots, k-1\}, \\ k + \frac{1}{\lambda}(u - A_k), & \text{if } A_k - \lambda \leq u, \end{cases}$$

for $u \geq 0$ is dominated by $\Phi_C(u)$ and is superadditive on $\mathbb{R}_+$, see Marchand et al. [87]. This leads to the valid inequality

$$\sum_{i \in [n] \setminus C} \Psi(a_i)\, x_i + \sum_{i \in C} x_i \leq |C| - 1.$$

Example 36 (Example 5 continued)**.** Consider again the knapsack inequality $4\, x_1 + 5\, x_2 + 6\, x_3 + 7\, x_4 + 9\, x_5 \leq 13$ and the minimal cover $C = \{2, 3, 4\}$.

Here, $\lambda = 5+6+7-13 = 5$, $A_0 = 0$, $A_1 = 7$, $A_2 = 13$, $A_3 = 18$, and the superadditive function $\Psi(u)$ is given by

$$\Psi(u) = \begin{cases} 0, & \text{if } 0 \leq u < 2, \\ 1 + \frac{1}{5}(u-7), & \text{if } 2 \leq u < 7, \\ 1, & \text{if } 7 \leq u < 8, \\ 2 + \frac{1}{5}(u-13), & \text{if } 8 \leq u < 13, \\ 3 + \frac{1}{5}(u-18), & \text{if } 13 \leq u. \end{cases}$$

This yields $\Psi(4) = \frac{2}{5}$ and $\Psi(9) = \frac{6}{5}$, and thus, the valid inequality

$$\tfrac{2}{5}\, x_1 + x_2 + x_3 + x_4 + \tfrac{6}{5}\, x_5 \leq 2, \tag{16}$$

which is different from the facet defining inequality that we obtained from C by sequential lifting in Example 27, and does not define a facet of P. However, this is not always the case. For the minimal cover $C = \{1, 2, 3\}$ the above method also yields the facet defining inequality (12).

Remark 37 (Remark for Examples 27, 30, 36)**.** *Note that for all minimal covers with two elements w.r.t.* $4x_1 + 5x_2 + 6x_3 + 7x_4 + 9x_5 \leq 13$*, i.e.,* $C = \{2, 5\}$*,* $C = \{3, 5\}$ *and* $C = \{4, 5\}$*, all three lifting procedures (sequential lifting in Theorem 26 and Theorem 28, as well as sequence independent lifting) yield a coefficient of* 0 *for the remaining three variables. This is due to the fact that all three covers are strong covers with* $E(C) = C$*, and it can easily be seen that the cover inequalities in all three cases already define facets of* P*.*

Easton and Hooker [35] present a procedure to simultaneously lift a set of variables into a cover inequality. If the cover and the set of variables to lift are sorted, this procedure only needs linear effort. Specifically, for a cover $C \subseteq [n]$ and a set $F \subseteq [n] \setminus C$, they define the (F, ρ)*-simultaneously lifted cover inequality (SLCI)*

$$x(C) + \rho\, x(F) \leq |C| - 1,$$

which is valid if $\rho \geq 0$ is sufficiently small. An exact characterization is given by Easton and Hooker [35, Theorem 2.1]. In general, SLCIs are not facet defining, but only increase the dimension of the face that is defined by the underlying cover inequality. However, Easton and Hooker [35, Theorem 2.3] state sufficient conditions under which SLCIs are indeed facet defining for $P_{C \cup F} = \operatorname{conv}(K_{C \cup F})$.

3.4 Complete Characterization of Facets from Simple LCIs

Balas and Zemel [12] give a complete characterization of the facet defining inequalities that can be obtained from a minimal cover by sequential or simultaneous lifting, which we describe in this section. A generalization of this result for arbitrary 0/1 polytopes is provided by Peled [102] and Zemel [130]. Wolsey [124] discusses lifting procedures for general integer programs. In order to state the result of Balas and Zemel, additional notation needs to be introduced.

Recall that Theorem 28 characterizes valid inequalities that are easy to compute and that even are sequentially lifted minimal cover inequalities if Condition (11) is satisfied. Let I be the set of indices $j \in [n] \setminus C$ that satisfy Condition (11) and let $J := [n] \setminus (C \cup I)$. Given the set J, define

$$\mathcal{M}(J) := \Big\{ M \subseteq J \, : \, \sum_{j \in M} a_j \leq \beta, \; M \neq \varnothing \Big\},$$

and for each $M \in \mathcal{M}(J)$ define β'_M as

$$\beta'_M := \min_{x \in \{0,1\}^C} \Big\{ |C| - 1 - x(C) \, : \, \sum_{i \in C} a_i \, x_i \leq \beta - a(M) \Big\}.$$

Theorem 38 (Balas and Zemel [12, Theorem 9])**.** *Let $a \in \mathbb{R}^n_+$, $\beta \in \mathbb{R}_+$, and let Assumption 11 hold, let C be a minimal cover for P and let π_i be defined as in Theorem 28. Then the simple LCI*

$$x(C) + \sum_{i \in [n] \setminus C} \alpha_i \, x_i \leq |C| - 1,$$

is facet defining for P if and only if

(a) for every $i \in J$ there exists $\delta_i \in [0,1]$ such that

$$\alpha_i = \begin{cases} \pi_i, & i \in I, \\ \pi_i + \delta_i, & i \in J, \end{cases}$$

and

(b) the vector $\delta \in \mathbb{R}^J$ is a vertex of the polyhedron

$$T = \Big\{ \delta \in \mathbb{R}^J \, : \, \sum_{j \in M} \delta_j \leq \beta'_M - \sum_{j \in M} \pi_j, \; M \in \mathcal{M}(J) \Big\}.$$

Corollary 39 (Balas and Zemel [12, Corollary 9.1])**.** *The simple LCI (9) of a minimal cover C with lifting coefficients α_i fulfilling (a) and (b) of Theorem 38 is a sequentially lifted minimal cover inequality if and only if $\delta \in \{0,1\}^J$.*

This means that all facet defining simple LCIs with integer coefficients can be generated by sequential lifting, whereas the simultaneously lifted minimal cover inequalities that cannot be found by sequential lifting have fractional coefficients. These can be obtained by computing the fractional vertices of the polyhedron T. Another implication is a characterization of the lifting coefficients of sequentially lifted facets.

Corollary 40 (Balas and Zemel [12, Theorem 3])**.** *Let* (9) *be a sequentially simple lifted cover inequality that defines a facet of P. Then the lifting coefficients α_i, $i \in [n] \setminus C$, satisfy*

$$\alpha_i = \begin{cases} \pi_i, & i \in I, \\ \pi_i \;\; \text{or} \;\; \pi_i + 1, & i \in J. \end{cases}$$

Moreover, Hartvigsen and Zemel [66] proved that one can decide in $\mathcal{O}(n^2)$ time whether the valid inequality $c^\top x \leq \gamma$ with $c \in \mathbb{Z}_+^n$ is a (facet defining) simple lifted minimal cover inequality. However, if $c \in \mathbb{R}^n$, deciding whether $c^\top x \leq \gamma$ defines a valid inequality obtained by lifting a minimal cover inequality is *coNP*-complete even if the underlying cover is known.

Example 41 (Example 5 continued)**.** Consider again the knapsack inequality $4\,x_1 + 5\,x_2 + 6\,x_3 + 7\,x_4 + 9\,x_5 \leq 13$ and the minimal cover $C = \{2, 3, 4\}$. As in Example 30,

$$C_1 = \{4\}, \quad C_2 = \{3, 4\}, \quad C_3 = \{2, 3, 4\},$$

and $\pi_1 = 0$, $\pi_5 = 1$. It follows that $I = \varnothing$ and $J = \{1, 5\}$. Moreover, $\mathcal{M}(J) = \{\{1\}, \{5\}, \{1, 5\}\}$. For $M = \{1\}$, the coefficient $\beta'_M = \beta'_{\{1\}}$ is given by

$$\begin{aligned} &\min_{x \in \{0,1\}^C} \left\{ |C| - 1 - x(C) \, : \, \sum_{i \in C} a_i\, x_i \leq \beta - a(M) \right\} \\ = &\min_{x \in \{0,1\}^C} \left\{ 2 - x_2 - x_3 - x_4 \, : \, 5\,x_2 + 6\,x_3 + 7\,x_4 \leq 13 - 4 \right\} = 1, \end{aligned}$$

which implies the inequality

$$\delta_1 \leq \beta'_M - \pi_1 = 1 - 0 = 1.$$

For $M = \{5\}$ and $M = \{1, 5\}$, the inequalities $\delta_5 \leq 1$ and $\delta_1 + \delta_5 \leq 1$ can be derived analogously. By Theorem 38, the inequality

$$\alpha_1\, x_1 + x_2 + x_3 + x_4 + \alpha_5\, x_5 \leq 2 \tag{17}$$

is facet defining for P if and only if

(*a*) $\alpha_1 = \pi_1 + \delta_1 = \delta_1$,
$\alpha_5 = \pi_5 + \delta_5 = 1 + \delta_5$,
with δ_1, $\delta_5 \in [0, 1]$, and

(*b*) $\delta = (\delta_1, \delta_5)^\top \in \mathbb{R}^2$ is a vertex of the polyhedron

$$T = \{(\delta_1, \delta_5)^\top \in \mathbb{R}^2 \, : \, \delta_1 \leq 1, \; \delta_5 \leq 1, \; \delta_1 + \delta_5 \leq 1\}.$$

Since T has the two vertices $(1,0)^\top$ and $(0,1)^\top$, the only two facet defining inequalities that can be obtained from the minimal cover $C = \{2,3,4\}$ by sequential or simultaneous lifting are

$$x_2 + x_3 + x_4 + 2\,x_5 \leq 2, \tag{18}$$
$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 2, \tag{19}$$

cf. Example 27. Here, Inequality (18) corresponds to the vertex $(0,1)^\top$ and Inequality (19) corresponds to the vertex $(1,0)^\top$. Since T has only integral vertices, Corollary 29 implies that both facet defining inequalities can be obtained by sequential lifting.

For the minimal cover $C = \{1,2,3\}$, it holds $I = \{4,5\}$ and $J = \varnothing$. Thus, the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$$

is the unique facet that can be obtained from sequential or simultaneous up-lifting from C. We have already concluded this uniqueness—at least for sequential lifting—in Example 30.

(Q11) Can one develop a similar theory for (sequentially or simultaneously) lifted inequalities different from covers? For example, can one characterize facet defining inequalities that arise from lifted pack inequalities (see Section 4)?

4 Valid Inequalities, Separation and Computations

The previous sections concentrated on strong inequalities for knapsack problems that are (liftings of) minimal cover inequalities. In practice, an important aspect is the computational effort needed to generate violated valid inequalities. In this context, there is often a trade-off between run time and the strength of inequalities. Moreover, strong inequalities (in the sense that they are facet defining) are not necessarily the best if applied in a branch-and-cut approach, for example, if they are dense.

Cutting planes generated from knapsacks are very important for a good performance of branch-and-cut based algorithms. This is highlighted by the fact that besides mixed-integer rounding and Gomory mixed-integer cuts, knapsack cuts are the most important class of cutting planes (the performance of CPLEX 12.5 degrades by 14 % (35 % on affected instances) if knapsack separation is turned off, see Achterberg and Wunderling [1]).

In this section, we present several families of valid inequalities and highlight several computational aspects in connection with the generation of these inequalities if we require that they cut off the current LP relaxation solution $x^\star$. This approach was pioneered by Crowder et al. [29] and further developed by many researchers as described below.

Cover Inequalities The basis for most generated inequalities in branch-and-cut codes are (minimal) cover inequalities (CIs). It turns out that the separation problem for cover inequalities is *NP*-hard, see Ferreira [39][3] and Klabjan et al. [78]. However, Crowder et al. [29] already observed that the problem can be written as the knapsack problem

$$\min_{y \in \{0,1\}^n} \Big\{ (1 - x^\star)^\top y \;:\; \sum_{j=1}^{n} a_j\, y_j \geq \beta + 1 \Big\}.$$

Thus, it can be solved in pseudo-polynomial time $\mathcal{O}(n\,\beta)$ if the weights a and β are integral. In practice, the cover is often found heuristically, e.g., by greedily adding items in non-decreasing order of $(1 - x_j^\star)/a_j$, see Crowder et al. [29].

Extended Cover Inequalities In practice, cover inequalities are almost always lifted. One easy possibility is to consider extended cover inequalities (ECI), see (3). Gabrel and Minoux [43] provided an exact separation algorithm for ECIs. Kaparis and Letchford [76] presented an $\mathcal{O}(n\,\beta)$ time algorithm for integral weights and several heuristics for the same problem.

Lifted Cover Inequalities The up-lifting procedure described in Section 3 yields simple lifted cover inequalities (9). Gu et al. [58] showed that the separation problem for simple LCIs is *NP*-hard. Hunsaker and Tovey [71] investigated the strength of these inequalities in practice. They showed that even if all simple LCIs are added to a knapsack problem, there exist instances that need exponentially many nodes in a branch-and-bound tree to solve the problem. Note that "this result is not suggested by the *NP*-hardness of binary knapsack problems, because cover inequality separation for these problems is *NP*-hard" [71, page 219].

As discussed in Section 3.2, up- and down-lifting can be applied to produce the possibly stronger inequalities (14). Gu et al. [57] adapted the algorithm by Zemel [131] to obtain down-lifting in $\mathcal{O}(|C|\,n^3)$ time.

If sequentially lifted cover inequalities are considered, a sequence of lifting variables has to be fixed. Hoffman and Padberg [68], for example, suggested to first up-lift variables with $x_j^\star > 0$, then down-lift variables in $D = \{j \in C \,:\, x_j^\star = 1\}$ and, finally, up-lift variables with $x_j^\star = 0$. Gu et al. [57] suggested an alternative heuristic; see also Martin [92]. Note that one can use different (minimal) covers to start with and that the corresponding cover inequality might not be violated before lifting.

Aspects of simultaneous lifting, in particular choices of lifting functions, are discussed, e.g., by Gu et al. [60].

[3] The thesis [39] is cited by [40], but we could not access it online.

$(1,k)$-Configurations A set $N \cup \{t\}$ with $N \subsetneq [n]$ and $t \in [n] \setminus N$ is called a $(1,k)$-*configuration* for $k \in \{2,\dots,|N|\}$, if $a(N) \le \beta$ and $Q \cup \{t\}$ is a minimal cover for every $Q \subseteq N$ with $|Q| = k$. Padberg [100] proved that for any $(1,k)$-configuration $N \cup \{t\}$, the $(1,k)$-*configuration inequality*

$$(|S| - k + 1)\, x_t + x(S) \le |S|, \qquad S \subseteq N,\ k \le |S|,$$

is valid for P and defines a facet of the knapsack polytope's restriction to $N \cup \{t\}$. Furthermore, Gottlieb and Rao [52] provide necessary and sufficient conditions on two disjoint $(1,k)$-configurations to define facets of P.

Example 42. Consider $K = \{x \in \{0,1\}^4 : 3\,x_1 + 5\,x_2 + 6\,x_3 + 7\,x_4 \le 14\}$. Then a $(1,2)$-configuration is given by $N = \{1,2,3\}$ and $t = 4$, since the union of $\{4\}$ and any two items of $\{1,2,3\}$ forms a minimal cover.

If $k = |N|$, a $(1,k)$-configuration is a minimal cover. Since the separation of minimal cover inequalities is *NP*-hard, Ferreira et al. [40] conjecture that the separation problem for $(1,k)$-configuration inequalities is *NP*-hard, too. For this reason, they present heuristics to separate these inequalities. A proof/refutation of this conjecture, however, is missing, and thus, the separation complexity is open.

Coefficient Increased Cover Inequalities Given a (not necessarily minimal) cover C, classical lifting procedures strengthen the cover inequality $x(C) \le |C| - 1$ by increasing 0-coefficients of variables that are not contained in C. Another way of inequality strengthening is described by Dietrich and Escudero [33]. They also allow to increase coefficients of cover variables; their strengthening procedure runs in $\mathcal{O}(n \log(n))$ time.

Lifted Pack Inequalities Based on an idea of Weismantel [122], Atamtürk [6] considered pack inequalities. A set $P' \subseteq [n]$ is a *pack* if $a(P') \le \beta$. Then the *pack inequality*

$$\sum_{j \in P'} a_j\, x_j \le a(P')$$

is trivially valid. It is dominated by the upper bounds $x_j \le 1$. However, *lifted pack inequalities* (LPIs) are not necessarily dominated. For details on the lifting procedure, we refer the reader to Atamtürk [6].

Example 43 (Example 5 continued)**.** Consider the running example with knapsack inequality $4\,x_1 + 5\,x_2 + 6\,x_3 + 7\,x_4 + 9\,x_5 \le 13$. A pack is given by $P' = \{1,5\}$ with inequality $4\,x_1 + 9\,x_5 \le 13$. Lifting x_2 yields

$$\Phi_{P'}(5) = \min\{13 - 4\,x_1 - 9\,x_5 \,:\, 4\,x_1 + 9\,x_5 \le 8,\ x_1, x_5 \in \{0,1\}\} = 9.$$

This yields the lifted pack inequality $4\,x_1 + 9\,x_2 + 9\,x_5 \leq 13$, which does not seem to be dominated by any lifted cover inequality.

Note that lifting x_3 into the pack $\{1,2\}$ yields $4\,x_1 + 5\,x_2 + 4\,x_3 \leq 9$, which is dominated by the minimal cover inequality $x_1 + x_2 + x_3 \leq 2$. Thus, not all lifted pack inequality are equally useful.

Weight Inequalities One particular example of LPIs was derived by Weismantel [122]. Let $r = \beta - a(P')$ be the residual capacity of the pack P'. Then the coefficients with $a_j > r$ are up-lifted to obtain the so-called *weight inequalities* (WIs):

$$\sum_{j \in P'} a_j\, x_j + \sum_{j \in [n] \setminus P'} \max\{a_j - r, 0\}\, x_j \leq a(P').$$

Weismantel also presented a variant of weight inequalities by reducing the coefficient of one variable in the pack and modifying the coefficient for variables in $[n] \setminus P'$. This larger class of inequalities contains weight inequalities and Weismantel proved that it can be separated in pseudo-polynomial time. His algorithm can also be used to separate weight inequalities itself. For integral weights, Kaparis and Letchford [76] improved the running time to $\mathcal{O}((n + \bar{a})\,\beta)$, where $\bar{a} = \max\{a_1, \ldots, a_n\}$, and designed an effective heuristic. The complexity of the separation problem for WIs seems to be open, although Martin [92] claims that a more general version is NP-hard.

Atamtürk [6] noted that stronger LPIs can be obtained by using superadditive functions.

Example 44. Taking the pack $\{1,5\}$ in the running example yields $r = 0$. Thus, its weight inequality is just the original knapsack inequality. However, the pack $\{1,2\}$ yields the residual capacity $r = 13 - 9 = 4$ and the weight inequality $4\,x_1 + 5\,x_2 + 2\,x_3 + 3\,x_4 + 5\,x_5 \leq 9$. Note that this is not equal to the corresponding lifted pack inequality.

Further inequalities are the *extended weight inequalities* of Weismantel [122], which are based on three mutually disjoint sets T, I, and $\{k\}$. We present the version with $I = \varnothing$ for brevity. In this case, $P' = T$ is a pack and $a(P') + a_k > \beta$. The inequality is then given by

$$\sum_{j \in P'} x_j + \alpha_k\, x_k \leq |P'|,$$

where

$$\alpha_k \coloneqq \min_{x \in \{0,1\}^{P'}} \Big\{ x(P') \,:\, \sum_{j \in P'} a_j\, x_j \geq a_k - r \Big\}.$$

Such inequalities can then be lifted to obtain lifted extended weight inequalities (LEWIs). For a given extended weight inequality, Weismantel [122] proved that, provided the knapsack data is integral, the lifting coefficients can be computed in polynomial time, see Martin [92] for computational results.

Example 45. Consider the knapsack inequality $2\,x_1 + 2\,x_2 + 2\,x_3 + 4\,x_4 \leq 7$. Taking the pack $P' = \{1, 2, 3\}$ with $r = 1$ and $k = 4$ yields

$$\alpha_4 := \min_{x \in \{0,1\}^{P'}} \left\{x_1 + x_2 + x_3 \,:\, 2\,x_1 + 2\,x_2 + 2\,x_3 \geq 4 - 1 = 3\right\} = 2$$

and the extended weight inequality $x_1 + x_2 + x_3 + 2\,x_4 \leq 3$, which is facet defining but not a lifted cover inequality.

Surrogate Knapsack Cuts Let $a^\top x \leq \beta$ be an n-dimensional knapsack inequality and let $J \subseteq [n]$. Glover et al. [50] suggest to use the Gomory cuts

$$\sum_{j \in J} \lfloor \lambda_0\, a_j + \lambda_j \rfloor\, x_j + \sum_{j \in [n] \setminus J} \lfloor \lambda_0\, a_j \rfloor\, x_j \leq \left\lfloor \lambda_0\, \beta + \sum_{j \in J} \lambda_j \right\rfloor,$$

where $\lambda \in \mathbb{R}_+^{J \times \{0\}}$, as cutting planes. Based on surrogate analysis, they show that these cutting planes can be separated in polynomial time. Furthermore, they show how to strengthen these cutting planes and they argue that these strengthenings cannot be found by classical lifting procedures.

Exact Separation Boyd [18, 19, 21] developed an algorithm to exactly separate inequalities for the knapsack polytope via the equivalence of separation and optimization. Another exact separation approach using the concept of polarity is described by Boccia [17] as stated by Kaparis and Letchford [76][4]. Yan and Boyd [129] also consider mixed-integer knapsack sets. This work has been refined and extended by Kaparis and Letchford [76] for the 0/1-case and Fukasawa and Goycoolea [42] for the mixed-integer case; see the Ph.D. theses [55] and [41] for more information. Fukasawa and Goycoolea also considered knapsacks resulting from tableau rows—note that all remaining approaches use original rows only. Vasil'ev (= Vasilyev) [119] introduced a different implementation with application to the generalized assignment problem, see also the extensive computational study in Avella et al. [7]. Avella et al. [8] consider MIPs in which the continuous variables are aggregated to a single one. Vasilyev et al. [120] further improve the implementation for exact knapsack separation.

Computations Wolter [127] performed a comparison with the framework SCIP and concluded that LEWIs (with $I = \varnothing$) have the strongest impact on the dual bound. Moreover, simultaneous lifting does not differ significantly from sequential LCIs.

Kaparis and Letchford [76] compare different exact and heuristic knapsack separation procedures to the exact separation procedure. On a subset of sparse MIPLIB instances, LCIs closed a significant amount of gap with small

[4] We could not access Boccia's article online.

computational effort if the heuristics are used. Moreover, they perform much better than ECIs or simple LCIs. On this test set, WIs and LPIs perform worse, and the performance of ECIs and simple LCIs is very similar. However, on a test set of dense multidimensional knapsack instances, LPIs perform significantly better than WIs and LCIs.

Complementing these experiments, Fukasawa and Goycoolea [42] compared the relative strength of the MIR-closure and the exact separation of knapsack cuts on MIPLIB instances. They observe that often a significant part of the gap is closed by MIR inequalities, making it hard to improve on them.

(Q12) What is the complexity of separating $(1,k)$-configuration, lifted pack, weight, and lifted extended weight inequalities?

(Q13) Can complementing variables help to strengthen inequalities similar to complemented mixed-integer rounding, see Marchand and Wolsey [89]?

5 Complete Linear Descriptions of Particular Knapsack Polytopes

In order to find complete linear descriptions of knapsack polytopes, we present two approaches: In Section 5.1 extended formulations are constructed and in Section 5.2 we consider special cases for which complete linear descriptions in the original space are known.

5.1 Extended Formulations

A large part of the previous exposition considers the description of the convex hull of the knapsack solutions via a system of linear inequalities, i.e., finding some matrix A and vector b with $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. One way of dealing with the difficulty of finding such a system and its possibly exponential size is to consider a higher dimensional representation, which can be projected down onto P, i.e., $P = \{x \in \mathbb{R}^n : \exists\, y \in \mathbb{R}^d\ Ax + By \leq b\}$. This higher dimensional polytope is called an *extended formulation* of the initial polytope. Surveys on extended formulations are given, e.g., by Conforti et al. [27] and Kaibel [73].

Unfortunately, the possibility to introduce additional variables does in general not allow to find exact polynomial sized formulations for knapsack polytopes, since there exist knapsack polytopes $P \subseteq \mathbb{R}^n$ that need $\Omega(2^{\sqrt{n}})$ many inequalities in any extended formulation, see Pokutta and Van Vyve [105]. Bienstock [16] was able to show, however, that for every $\varepsilon \in (0,1)$ there exists an ε-approximate extended formulation Q of P of size $\mathcal{O}(\varepsilon^{-1} n^{2+\lceil \varepsilon^{-1} \rceil})$. A formulation $Q = \{x \in \mathbb{R}^n : \exists\, y \in \mathbb{R}^d\ Ax + By \leq b\}$ is called *ε-approximate extended formulation* if $P \cap \{0,1\}^n = Q \cap \{0,1\}^n$ and

$$\max\{w^\top x : x \in P \cap \{0,1\}^n\} \geq (1-\varepsilon) \max\{w^\top x : x \in Q\}$$

holds for every $w \in \mathbb{R}^n$. Note that Bienstock's extended formulation is of polynomial size in n for any fixed ε, but grows exponentially in the inverse approximation quality $\frac{1}{\varepsilon}$. Moreover, while ε-approximate formulations exist in an extended space, Faenza and Sanità [38] proved that no ε-approximate formulation of size polynomial in n exists in the original space in general, showing the power of extended formulations.

One possibility to find extended formulations of knapsack polytopes is to exploit that knapsack problems can be solved via dynamic programming. For general problems that admit a dynamic programming scheme, Martin et al. [95] derived extended formulations for these problems' solution set. In the following, we present this extended formulation for knapsack problems, where we follow the presentation of Conforti et al. [27].

Consider the 0/1 knapsack problem $\max\{c^\top x : a^\top x \leq \beta,\ x \in \{0,1\}^n\}$ for integral weights $a \in \mathbb{Z}^n_+$ and $\beta \in \mathbb{Z}_+$. For each integral $0 \leq \beta' \leq \beta$ and $k \in [n]$, define the value function

$$f(k, \beta') \coloneqq \max\left\{c^\top x : a^\top x \leq \beta',\ x \in \{0,1\}^n,\ x_i = 0,\ i \in \{k+1, \dots, n\}\right\}.$$

Clearly, to solve the knapsack problem, one wants to compute $f(n, \beta)$. This can be done using the Bellman equation

$$f(k, \beta') = \max\{f(k-1, \beta'),\ f(k-1, \beta' - a_k) + c_k\},$$

with $f(0, \beta') = 0$ and $f(k, \beta') = -\infty$ if $\beta' < 0$.

This dynamic programming algorithm is equivalent to finding a minimum weight path from $v_{0,0}$ to $v_{n,\beta}$ in the graph $(V, A^0 \cup A)$ with nodes $v_{k,\beta'}$ for $k \in [n]_0$, $0 \leq \beta' \leq \beta$, and two types of arcs. The arc subset A^0 consists of the arcs $(v_{k-1,\beta'}, v_{k,\beta'})$ with weight 0, where $k \in [n]$ and $\beta' \in [\beta]_0$. The arc subset A contains the arcs $(v_{k-1,\beta'-a_k}, v_{k,\beta'})$ of weight c_k, where $k \in [n]$ and $\beta' \in \{a_k, \dots, \beta\}$.

Applying the results of Martin et al. [95] to this graph, yields the following theorem, where the y-variables model whether an arc is used in a minimum weight path or not.

Theorem 46. *Let $a \in \mathbb{Z}^n_+$ and $\beta \in \mathbb{Z}_+$. An extended formulation of $P^{a,\beta}$ of $\mathcal{O}(n\,\beta)$ size is given by*

$$\begin{aligned}
x_k - y(\{(v_{k-1,\beta'-a_k}, v_{k,\beta'}) : a_k \leq \beta' \leq \beta\}) &= 0, & k &\in [n],\\
y(\delta^-(v_{n,\beta})) &= 1, & &\\
y(\delta^-(v_{k,\beta'})) - y(\delta^+(v_{k,\beta'})) &= 0, & v_{k,\beta'} &\in V \setminus \{v_{0,0}, v_{n,\beta}\},\\
y_a &\geq 0, & a &\in A \cup A^0,
\end{aligned}$$

where $\delta^-(v)$ and $\delta^+(v)$ are the sets of in-coming and out-going arcs of a node v, respectively.

For particular knapsack polytopes, so-called orbisacks (described in more detail in Section 5.2), Loos [85] shows how the extended formulation based on a dynamic programming graph can be used to compute a complete linear description in the original space. In general, however, this extended formulation is too complicated to compute the projection to the original x-variables or too large to be useful in practice.

(Q14) The knapsack problem admits a fully polynomial time approximation scheme, but does the knapsack polytope admit also an ε-approximate extended formulation whose size grows polynomially in n and $\frac{1}{\varepsilon}$?[5]

5.2 Complete Linear Descriptions

We now turn to special cases for which complete linear descriptions can be derived by exploiting the particular combinatorial structure.

Weakly Super-Increasing Knapsack Polytopes. A knapsack is called *weakly super-increasing*, if $a([i-1]) \leq a_i$ for every $i \in [n]$, for example, if $a_i = 2^i$ for each $i \in [n]$. Such knapsacks were investigated by Laurent and Sassano [81] who showed that weakly super-increasing knapsack polytopes are completely described by box constraints and $\mathcal{O}(n)$ minimal cover inequalities. These covers can be constructed explicitly.

Sequential Knapsack Polytopes. A knapsack is called *sequential*, if a_i is a divisor of a_{i+1} for every $i \in [n-1]$. In addition to $a_i = 2^i$, another example for a sequential knapsack is $a_i = \prod_{j=1}^{i} j$. Pochet and Weismantel [103] provided a complete linear description of sequential knapsack polytopes. The description contains an inequality for every combination of a subset $W \subseteq [n]$, a partition $B = B_1 \cup \cdots \cup B_m$ of W, and a permutation of $[m]$, and thus, is very large. The separation complexity of this formulation is an open problem.

One Coefficient. Knapsacks whose inequality has exactly one weight, i.e., $a = (\lambda, \ldots, \lambda)^\top$, $\lambda \in \mathbb{N}$, are equivalent to a cardinality constraint $\mathbb{1}^\top x \leq \lfloor \beta/\lambda \rfloor$. The complete description is given by this single inequality and the trivial inequalities.

Two Coefficients. Knapsacks whose inequalities have two different coefficients, i.e., there exists $k \in [n]$ such that the knapsack inequality is given by

$$\lambda\, x([k]) + \mu\, x(\{k+1, \ldots, n\}) \leq \beta,$$

for λ, μ, and $\beta \in \mathbb{N}$, were investigated by Weismantel [121]; this includes the special case of one coefficient if $\lambda = \mu$. He developed a complete linear

[5] This question has already been posed by Van Vyve and Wolsey [118].

description of exponential size that consists of eight families of inequalities. The separation problem of this formulation is solvable in polynomial time. The special case $\lambda = 1$ was treated by Dahl and Foldnes [30], who proved that in this case a complete linear description is given by four families of inequalities via total dual integrality. Hartmann [65] showed that a complete linear description of the knapsack polytope can be separated in linear time in this case.

Small and Large Coefficients Bader et al. [9] considered knapsack inequalities $a^\top x \leq \beta$ whose coefficients are either relatively small or large in comparison to the right-hand side. Let $n, k \in \mathbb{N}$ with $3 \leq k \leq n$ and $s \in \{k, \dots, n\}$. Assume that the coefficients satisfy $\frac{\beta}{k+1} < a_i \leq \frac{\beta}{k}$ for every $i \in S \coloneqq [s]$ and $\frac{k-1}{k+1}\,\beta < a_i \leq \beta$ for every $i \in L \coloneqq \{s+1, \dots, n\}$. Then Bader et al. present the following complete linear description of the knapsack polytope induced by $a^\top x \leq \beta$ (without providing a proof):

$$\begin{aligned}
&& \sum_{j \in L} x_j &\leq 1, && \\
\sum_{j=i}^{s} x_j + \sum_{\substack{j \in L:\\ a_j > \beta - a_i}} x_j + (k-1) && \sum_{j \in L} x_j &\leq k, && i \in [s-k+1], \\
\sum_{j \in R} x_j + \sum_{\substack{j \in L:\\ a_j > \beta - a_{i(R)}}} x_j + (|R|-1) && \sum_{j \in L} x_j &\leq |R|, && R \subseteq L,\ |R| \in [k-1], \\
&& x_i &\geq 0, && i \in [n],
\end{aligned}$$

where $i(R) = \min R$ is the index of an item in R with smallest coefficient $a_{i(R)}$.

$(1,k)$-Configurations. Recall the definition of $(1,k)$-configurations from Section 4. Padberg [100] showed that if $N = [n-1]$ and $t = n$ form a $(1,k)$-configuration w.r.t. the knapsack inequality $a^\top x \leq \beta$, a complete linear description of K is given by the $(1,k)$-configuration inequalities

$$(|S| - k + 1)\, x_n + x(S) \leq |S|$$

for each subset $S \subseteq [n-1]$ with $k \leq |S|$ as well as box constraints and $a^\top x \leq \beta$. This result also extends to general packing or multidimensional knapsack problems, see Section 10.1.

Gaps in Coefficients. Weismantel [122] considered knapsacks given by

$$\sum_{j=1}^{\beta} \sum_{i \in N_j} j\, x_i \leq \beta,$$

where N_j is the set of all items with weight j for $j \in [\beta]$. He could derive complete linear descriptions for the two special cases

(a) $N_j = \varnothing$ for all $1 < j \leq \lfloor \frac{\beta}{2} \rfloor$,

(b) $N_j = \varnothing$ for all $1 < j \leq \lfloor \frac{\beta}{3} \rfloor$ and $N_j = \varnothing$ for all $j \geq \lfloor \frac{\beta}{2} \rfloor + 1$.

Examples for these special cases are

(a) $a = (1, 1, 5, 7, 7, 8)^\top$, $\beta = 9$,

(b) $a = (1, 1, 1, 4, 4, 5)^\top$, $\beta = 10$.

Matroids. If the knapsack defines a matroid, the knapsack polytope can be described completely using the results of Edmonds [37]. In this case, the set of all extensions of strong cover inequalities suffices to give a formulation, see Wolsey [123]. A characterization of the cases in which the knapsack defines a matroid is discussed in Section 7.

Graphic Knapsacks. Suppose Assumption 11 holds. Wolsey [123] called a knapsack *graphic* if there exists $t \in [n-1]$ such that its knapsack inequality satisfies $a_{t+1} + a_t > \beta$ and $a([t]) \leq \beta$. Again, the set of all strong cover inequalities provides a complete linear description, see Wolsey [123]. Note that $t = 1$ corresponds to the matroid case.

Further Cases. Gillmann and Kaibel [47] introduced *revlex-initial polytopes*, i.e., $\operatorname{conv}\{x \in \{0,1\}^n : x \prec_{\text{rlex}} v\}$ for some $v \in \{0,1\}^n$, where "$\prec_{\text{rlex}}$" is the strict reverse lexicographic order. If $\bar{a}_i = 2^i$, $i \in [n]$, then this corresponds to $\operatorname{conv}\{x \in \{0,1\}^n : \bar{a}^\top x \leq \bar{a}^\top v - 1\}$, i.e., a knapsack polytope. Gillmann and Kaibel provided a complete linear description with a polynomial number of inequalities in n.

Related objects are *orbisacks*, i.e., the convex hull of 0/1-matrices of size $m \times 2$ such that the first column is lexicographically not smaller than the second. Using the vector $\bar{a}$ as defined before, orbisacks can be seen to be special knapsack polytopes $\operatorname{conv}\{(x, y) \in \{0,1\}^{m \times 2} : \bar{a}^\top (y - x) \leq 0\}$ (after complementing variables). Kaibel and Loos [74] found a complete linear description, and Loos [84] presented an algorithm for optimization over orbisacks. The complete linear description has $\Theta(3^m)$ facets and there is a facet defining inequality such that the ratio of its largest and smallest coefficient is 2^{m-2}.

Finally, in [70] so-called *symresacks* $\operatorname{conv}\{x \in \{0,1\}^n : x \succeq_{\text{lex}} \gamma(x)\}$ for some coordinate permutation γ are introduced, where "$\succeq_{\text{lex}}$" refers to the lexicographic order. These polytopes are again knapsack polytopes (after complementing variables) and depending on γ, complete linear descriptions can sometimes be derived [69].

Example 47 (Example 5 continued)**.** A complete linear description of the knapsack polytope $\operatorname{conv}\{x \in \{0,1\}^5 : 4\,x_1 + 5\,x_2 + 6\,x_3 + 7\,x_4 + 9\,x_5 \leq 13\}$ is given by $x_i \geq 0$, $i \in [5]$, $x_1 \leq 1$, the four extension inequalities of strong covers (Example 21) as well as the lifted cover inequality $x_2 + x_3 + x_4 + 2\,x_5 \leq 2$ (Example 27). This result is not implied by any of the cases discussed above.

(Q15) Are there further classes of knapsack polytopes for which one can provide complete linear descriptions?

(Q16) Is there a common generalization of all the mentioned complete linear descriptions?

6 Geometric Properties of Knapsack Polytopes

The preceding sections dealt with algebraic properties and descriptions of knapsack polytopes. This section investigates their geometric properties.

Number of Vertices Since knapsack problems with integral weights can be solved by dynamic programming, the number of vertices $V(P)$ of P can in principle be computed by a modification of the dynamic programming algorithm with a zero objective. However, the running time of this procedure may be exponentially large in n, since $V(P)$ might be exponential.

To find approximations of $V(P)$, Dyer [34] presents a randomized algorithm to approximately count vertices of P. That is, for every $\varepsilon > 0$, the algorithm returns a number within the range $(1-\varepsilon)V(P)$ and $(1+\varepsilon)V(P)$ with high probability. Dyer's algorithm runs in $\mathcal{O}(n^{5/2}\sqrt{\log(n\varepsilon^{-1})} + \varepsilon^{-2}n^2)$ time and is based on dynamic programming.

For an integer knapsack without upper bounds, i.e, $K' = \{x \in \mathbb{Z}^n_+ : a^\top x \leq \beta\}$, Hayes and Larman [67] show that the number of vertices of $\operatorname{conv}(K')$ is bounded from above by $\log_2^n(\frac{4\beta}{a_1})$. For the binary case, however, we are not aware of such an upper bound.

Adjacency Motivated by optimization algorithms that are based on local improvements of sub-optimal solutions, Geist and Rodin [45] investigated adjacency of vertices of P. They show that two distinct vertices u, v of P are adjacent if and only if there do not exist further distinct vertices w^1, w^2 of P such that $\frac{u+v}{2} = \frac{w^1+w^2}{2}$. They also show that deciding whether two vertices are adjacent is NP-complete.

Distribution of Classes of Facets As mentioned in the previous sections, the most important classes of inequalities for $P^{a,\beta}$ are given by trivial inequalities $0 \leq x \leq \mathbb{1}$ and simple lifted minimal cover inequalities (LCI). As we have seen in Section 4, simple LCIs play an important role in solving knapsack problems via branch-and-cut. However, it is unclear which amount of non-trivial facets are defined by simple LCIs. To get an estimate for this value, we enumerated all full-dimensional knapsack polytopes in small

Table 1: Number of full-dimensional 0/1 polytopes and knapsack polytopes in small dimensions.

	dimension				
type of 0/1 polytope	1	2	3	4	5
general	1	2	12	347	1 226 525
knapsack	1	2	5	17	92

dimensions, computed its facets, and checked which facets are defined by simple LCIs. In the remainder of this section, we describe our investigation in more detail.

In the following, we consider all possible 0/1 knapsack polytopes up to 0/1 equivalence, i.e., up to coordinate permutations and complementing variables. Thus, if we refer to a knapsack polytope in the following, we in fact refer to all members of its 0/1 equivalence class. To enumerate all equivalence classes of full-dimensional knapsack polytopes in dimensions $n \in [5]$, we used a tool by Aichholzer [2] that allows to access all equivalence classes of 0/1 polytopes in dimensions up to 5, which is available through his web page[6]. Since the web tool allows to access at most 1000 polytopes per query, we used an offline version that Aichholzer made accessible to us.

Using Aichholzer's tool, we first generated a list of all (equivalence classes of) 0/1 polytopes and removed all polytopes that are not full-dimensional. To filter the knapsack polytopes from this list, we checked for each polytope P whether there exists a linear inequality $a^\top x \leq \beta$ that separates the vertices of P from the remaining binary points by solving an LP, cf. Bradley et al. [23]. Note that we cannot restrict to inequalities $a^\top x \leq \beta$ with non-negative coefficients and $a_i \leq \beta$, $i \in [n]$, because we deal with equivalence classes of polytopes. That is, we cannot guarantee that Assumption 1 holds. Moreover, note that we classify $[0,1]^n$ as a knapsack polytope.

Table 1 shows that the vast majority of 0/1 polytopes are not knapsacks if $n \geq 4$. In particular, only 92 out of 1 226 617 equivalence classes of 0/1 polytopes consist of knapsacks if $n = 5$. For smaller dimensions, all ($n \in [2]$) or about half of all polytopes ($n = 3$) are knapsacks.

Let $\{P_1, \dots, P_N\}$ be the set of representatives for each equivalence class of full-dimensional knapsack polytopes. For each representative P_i, we computed a facet description of P_i using POLYMAKE [44]. By iterating over all facet defining inequalities of P_i, we grouped the facets into two classes: trivial facets defined by box constraints and non-trivial facets. For each non-trivial facet, we checked whether it is the extension inequality of a strong cover (and thus, a simple lifted cover inequality). For the remaining non-trivial facets, we checked by hand whether there exists a minimal cover such that

[6] http://www.ist.tugraz.at/staff/aichholzer/research/rp/rcs/info01poly/

Table 2: Statistical measures on distribution of simple lifted minimal cover inequalities in set of non-trivial facet-defining inequalities in dimensions 2–5.

statistical measure	dimension 2	3	4	5
minimum	100.00 %	100.00 %	75.00 %	54.54 %
maximum	100.00 %	100.00 %	100.00 %	100.00 %
median	100.00 %	100.00 %	100.00 %	100.00 %
arithmetic mean	100.00 %	100.00 %	98.44 %	92.99 %

the corresponding facet defining inequality is a sequentially simple LCI.

To evaluate the distribution of simple LCIs among the non-trivial facet defining inequalities of P_i, $i \in [N]$, let f_i, t_i, and c_i be the number of all facets, the number of trivial facets, the number of facets associated with simple LCIs, and the number of remaining facets, respectively. The fraction $\rho_i = \frac{c_i}{f_i - t_i}$ describes the portion of non-trivial facets that are defined by simple LCIs. For describing the distribution of non-trivial facets based on simple LCIs, we used the statistical measures

- ▷ maximum: $\max\{\rho_i : i \in [N]\}$,
- ▷ minimum: $\min\{\rho_i : i \in [N]\}$,
- ▷ median: $\mathrm{median}\{\rho_i : i \in [N]\}$, and
- ▷ arithmetic mean: $\frac{1}{N} \sum_{i \in [N]} \rho_i$.

Table 2 shows that in dimensions 2 and 3, every non-trivial facet defining inequality is a simple LCI. In dimensions 4 and 5, the majority of all non-trivial facets are defined by simple LCIs since the arithmetic mean is larger than 90.0 % and the median value is even 100.0 % in both cases. However, there are also equivalence classes in which only relatively few facets are defined by simple LCIs, e.g., there exists a knapsack polytope in dimension 5 such that only about 54.5 % of all facet defining inequalities are simple LCIs.

Our experiments show that simple LCIs are important for describing knapsack polytopes in small dimensions. These findings have also implications for general binary programs. To strengthen a binary program, we can consider the knapsack polytopes P defined by the inequalities of its constraint matrix and separate valid inequalities for P. If the constraint matrix is sparse, Table 2 indicates that separating simple LCIs is important to find strong cutting planes for P, and thus, the binary program. This hypothesis is also supported by the numerical experiments of Kaparis and Letchford [76] on MIPLIB instances mentioned in Section 4. If the constraint matrix gets denser, however, simple LCIs might not suffice to give tight formulations of knapsack polytopes, cf. minimum percentage value in Table 2 for dimension 4 and 5.

(Q17) Does there exist a strong lower/upper bound on the number of vertices $V(P)$ of binary knapsack polytopes?

(Q18) Is there a routine computing $V(P)$ that is faster than enumerating all vertices of P?

(Q19) Is there a (recursive) scheme to construct all knapsack polytopes in fixed dimension?

(Q20) What is the minimum percentage of non-trivial facets of a knapsack polytope that can be described by simple LCIs? Does this tend to 0 for $n \to \infty$?

7 Independence Systems and Matroids

7.1 Independence Systems

An *independence system* is a tuple $(F, \mathcal{I})$, where F is a finite set and $\mathcal{I} \subseteq 2^F$ such that (i) $\varnothing \in \mathcal{I}$ and (ii) if $X \in \mathcal{I}$, then $Y \in \mathcal{I}$ for every $Y \subseteq X$. The sets contained in $\mathcal{I}$ are called *independent*, whereas sets in $2^F \setminus \mathcal{I}$ are called *dependent*. Let $a \in \mathbb{R}^n_+$, $\beta \in \mathbb{R}_+$, and consider the system $\mathcal{I}_K \coloneqq \{I \subseteq [n] : a(I) \leq \beta\}$ containing the index sets of feasible solutions of the knapsack $K = K^{a,\beta}$. Due to Lemma 3, $\mathcal{M} \coloneqq ([n], \mathcal{I}_K)$ is an independence system, which motivates the following natural question:

(Q21) Which independence systems can be represented as knapsack problems?

While this problem is open in general, a complete characterization is available for the class of stable sets as we outline in the following. Given an undirected graph $G = (V, E)$, a *stable set* in G is a set $S \subseteq V$ such that the nodes in S are pairwise non-adjacent in G. Denote the set of all stable sets in G by $\mathcal{I}(G)$, i.e., $\mathcal{I}(G) = \{S \subseteq V : S \text{ is stable}\}$. Clearly, $\mathcal{S}(G) \coloneqq (V, \mathcal{I}(G))$ is an independence system. We need the following concept.

Definition 48. A graph $G = (V, E)$ is a *threshold graph* if and only if one (and thus all) of the following equivalent properties holds:

(a) The graph G can be constructed from a one-vertex graph by repeated application of the following two operations:
  - (i) Addition of a single isolated vertex to the graph.
  - (ii) Addition of a single vertex to the graph that is connected with all other vertices.

(b) There exist a real number S as well as real vertex weights $w(v)$ such that for every pair of distinct vertices u, $v \in V$ the pair $\{u, v\}$ is an edge of G if and only if $w(u) + w(v) > S$.

(c) There exist a real number β and real vertex weights $a(v)$ such that for every $S \subseteq V$ we have $\sum_{v \in S} a(v) \leq \beta$ if and only if S is a stable set.

In particular, the property of being a threshold graph is closed under taking induced subgraphs. Thus, once the construction sequence from (a) is

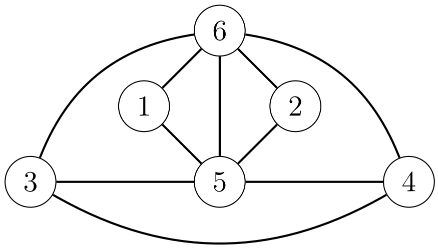

Figure 1: An example for a threshold graph.

known, one can easily construct a knapsack inequality characterizing $\mathcal{S}(G)$: Assume that the nodes of G are labeled $1, \ldots, |V|$ in the order in which they are added in the construction sequence. Let $\sum_{i=1}^{j} a_i^j\, x_i \leq \beta_j$ be the inequality characterizing the graph constructed in step j, and let $x_1 \leq 1$ be the inequality for the one-vertex graph. If we apply construction rule *(i)* to obtain the $(j+1)$-st graph, we define the $(j+1)$-st inequality to be

$$2\sum_{i=1}^{j} a_i^j\, x_i + x_{j+1} \leq 2\,\beta_j + 1;$$

if we apply construction rule *(ii)*, we define the inequality

$$\sum_{i=1}^{j} a_i^j\, x_i + \beta_j\, x_{j+1} \leq \beta_j.$$

Theorem 49 (Chvátal and Hammer [26])**.** *$\mathcal{S}(G)$ can be represented as a knapsack problem if and only if G is a threshold graph.*

Example 50. Figure 1 shows an example for a threshold graph. Using the construction rules *(i)* and *(ii)*, this graph can be constructed in the following order, which also yields the knapsack characterization for the corresponding subgraphs.

- ▷ Starting with node 3. $\quad x_3 \leq 1$
- ▷ Adding node 4 by *(ii)*. $\quad x_3 + x_4 \leq 1$
- ▷ Adding node 2 by *(i)*. $\quad x_2 + 2x_3 + 2x_4 \leq 3$
- ▷ Adding node 1 by *(i)*. $\quad x_1 + 2x_2 + 4x_3 + 4x_4 \leq 7$
- ▷ Adding node 5 by *(ii)*. $\quad x_1 + 2x_2 + 4x_3 + 4x_4 + 7x_5 \leq 7$
- ▷ Adding node 6 by *(ii)*. $\quad x_1 + 2x_2 + 4x_3 + 4x_4 + 7x_5 + 7x_6 \leq 7$

Then the stable sets of the graph given in Figure 1 coincide with the feasible solutions of the final knapsack inequality.

Generalizing the concept of being threshold to general independence systems seems to be complicated. To the best of our knowledge, it is only known for matroids, see Theorem 55 in the next subsection.

7.2 Knapsacks and Matroids

In general, solving linear optimization problems over independence systems is *NP*-hard. However, for the special case of matroids it can be solved very efficiently by the greedy algorithm. Recall that an independence system $(F, \mathcal{I})$ is a *matroid* if for every $S \subseteq F$, the maximal independent sets contained in S have the same size.

Since optimization over matroids is easy, a natural question concerning knapsacks K is how to characterize whether the associated independence system $\mathcal{M} = ([n], \mathcal{I}_K)$, where $\mathcal{I}_K \coloneqq \{I \subseteq [n] : a(I) \leq \beta\}$, is a matroid.

Definition 51. Suppose Assumption 11 holds. A maximal independent set $C \subseteq [n]$ is a *ceiling* of $\mathcal{M}$ if for all $i \in C$ with $i+1 \in [n] \setminus C$ it follows that $(C \setminus \{i\}) \cup \{i+1\} \notin \mathcal{I}_K$.

Using the concept of ceilings, Wolsey [123] provides a full characterization of knapsacks that are representable via a matroid. Furthermore, a characterization via strong covers exists due to Cerdeira and Barcia [24].

Theorem 52 (Cerdeira and Barcia [24], Wolsey [123])**.** *The following statements are equivalent:*

(a) *$\mathcal{M} = ([n], \mathcal{I}_K)$ is a matroid.*

(b) *The number of strong covers of K is at most 2.*

(c) *$\mathcal{M}$ has a unique ceiling.*

Example 53 (Example 5 continued)**.** The independence system corresponding to the knapsack defined via $4\,x_1 + 5\,x_2 + 6\,x_3 + 7\,x_4 + 9\,x_5 \leq 13$ has two ceilings, namely $\{3, 4\}$ and $\{1, 5\}$. Therefore, by Theorem 52(c), the corresponding knapsack is not a matroid. Alternatively, this can be shown by Theorem 52(b), since we have seen before that $C_1 = \{1, 2, 3\}$, $C_5 = \{2, 5\}$, $C_6 = \{3, 5\}$ and $C_7 = \{4, 5\}$ are strong covers for the knapsack.

Besides the two aforementioned results, Amado and Barcia [3] consider a certain family $\mathcal{F}$ of matroids and they characterize whether a knapsack K is contained in $\mathcal{F}$. These findings are of particular interest if the coefficients in the knapsack inequality are almost the same because such knapsacks are typically hard to solve, see Martello and Toth [91, Section 2.10]. For such knapsacks, Amado and Barcia used their findings to strengthen the standard LP relaxation of knapsack problems.

While the previous results used the knapsack inequality to characterize if K is a matroid, we can also pose the reverse question: which matroids are knapsack representable? Therefore, we need to adapt the threshold property that we defined for graphs in the last section to matroids. An independence system $(F, \mathcal{I})$ is called *threshold* if there exists $a \in \mathbb{R}^F_+$ and $\beta \in \mathbb{R}_+$, such that $\mathcal{I} = \{I \subseteq F : a(I) \leq \beta\}$, i.e., $\mathcal{I}$ is knapsack representable. To characterize whether a matroid is threshold, we need a monotonicity property.

Definition 54. Let $(F, \mathcal{I})$ be an independence system and let R, $S \subseteq F$. We say that R *is greater than or equal to* S, in formula $R \geq S$, if for every $T \subseteq F \setminus (R \cup S)$

$$(R \cup T) \in \mathcal{I} \implies (S \cup T) \in \mathcal{I}.$$

The system $(F, \mathcal{I})$ is said to be k*-monotone* if for each pair (R, S) of subsets of F with $|R \cup S| < k$ either $R \geq S$ or $R \leq S$.

Theorem 55 (Giles and Kannan [46]). *A matroid is threshold if and only if it is* 3*-monotone.*

Theorem 55 extends Theorem 52 by a third characterization of knapsacks having the matroid property. A generalization to arbitrary independence systems, and thus an answer to Question (Q21), is open.

8 Integer Knapsacks

For the integer knapsack problem, the variables are bounded and integer valued:

$$\max \{c^\top x \,:\, a^\top x \leq \beta,\ 0 \leq x \leq u,\ x \in \mathbb{Z}^n_+\}.$$

Some authors, like Kellerer et al. [77], distinguish between bounded knapsack problems with $u_i < \infty$ and unbounded knapsack problems with $u_i = \infty$. Observe that each unbounded knapsack problem can be made bounded by setting $u_i \coloneqq \lfloor \beta / a_i \rfloor$, provided $a_i > 0$. Thus, complementing variables x_i by $u_i - x_i$ is valid which generalizes Observation 2 to the integer case. For this reason, we assume that Assumption 1 holds for the remainder of this section.

Note that we can always reduce a (bounded) integer knapsack problem to a corresponding binary one by replacing each integer item i by u_i binary copies of it, but some of the structure may be lost in this transition and the formulation may become large. Moreover, while every binary $\leq$-knapsack can be transformed into a binary $\geq$-knapsack, see Section 1.1, such a transformation does not exists if we consider integer knapsacks. The reason for this is that integer $\leq$-knapsacks are bounded, whereas integer $\geq$-knapsacks are unbounded. Thus, different polyhedral properties arise, as, for example, investigated by Yaman [128].

Analogously to the binary case, complete linear descriptions of the integer knapsack polytope $\operatorname{conv}\{x \in \mathbb{Z}^n_+ \,:\, a^\top x \leq \beta\}$ for $a \in \mathbb{Z}^n_+$ can be found by investigating faces of the *master knapsack polytope* $\operatorname{conv}\{x \in \mathbb{Z}^n_+ : \sum_{i=1}^n i\, x_i \leq n\}$, see, e.g., Aráoz et al. [4], Shim et al. [113], or Tyber and Johnson [116]. Although some facet defining inequalities are known, we are far from fully understanding this polytope. Thus, as for binary knapsack polytopes, investigations of particular knapsack polytopes are necessary to find strong cutting planes.

In contrast to the classical binary knapsack problem, there are only few polyhedral investigations of the integer knapsack problem, see, e.g., Atamtürk [5, 6], Ceria et al. [25], Pochet and Weismantel [103], or Pochet and Wolsey [104]. Among others, Ceria et al. [25] discuss a generalization of minimal cover inequalities to the general integer case. Following their definition, an *(integer) cover* is a set $C \subseteq [n]$ with the property $\sum_{i \in C} a_i u_i > \beta$. A cover C is called *minimal* if no proper subset of C is a cover. Analogously to the binary case, every minimal integer cover C gives rise to the *(minimal) (integer) cover inequality*

$$x(C) \leq u(C) - 1,$$

which states that not all variables of a (minimal) cover can simultaneously be at their upper bound. In contrast to the binary setting, however, these inequalities do not define an integer programming formulation of the integer knapsack set in general.

Besides the straightforward generalization of minimal cover inequalities above, Atamtürk [6] discusses a richer class of inequalities derived from covers $C \subseteq [n]$, showing that for any $\rho > 0$, the inequality

$$\sum_{i \in C} \left\lceil \frac{\min\{a_i, \lambda\}}{\rho} \right\rceil (u_i - x_i) \geq \left\lceil \frac{\lambda}{\rho} \right\rceil, \tag{20}$$

where $\lambda = \sum_{i \in C} u_i - \beta$, is valid for the integer knapsack polytope. For the special case where $\rho = a_\ell$, a complete characterization when (20) is facet defining is available, using

$$\kappa_{i\ell} \coloneqq \left\lceil \frac{\min\{a_i, \lambda\}}{a_\ell} \right\rceil.$$

Theorem 56 (Atamtürk [6])**.** *Let $C \subseteq [n]$ be a cover and let $\ell \in C$ be such that $\mu = u_\ell a_\ell - \lambda \geq 0$. Inequality (20) with $\rho = a_\ell$ is facet defining for the restriction of the knapsack polytope to variables in C if and only if $a_i \geq \min\{\lambda, \kappa_{i\ell} a_\ell - r\}$ for all $i \in C \setminus \{\ell\}$, where $r = \mu - \lfloor \mu / a_\ell \rfloor a_\ell$.*

Moreover, there exist some families of integer knapsack polytopes for which complete linear descriptions are available. These families generalize weakly super-increasing and sequential binary knapsack polytopes to the general case of bounded knapsack polytopes (see Section 5.2).

A bounded knapsack with defining inequality $a^\top x \leq \beta$ and upper bound vector u is called *weakly super-increasing* if $\sum_{j=1}^{i-1} a_j u_j \leq a_i$ for every $i \in [n]$. Using this definition, the classical result for the binary case (Section 5.2) can be generalized to the case of arbitrary integer variables.

Theorem 57 (Gupte [61])**.** *A weakly super-increasing (not necessarily binary) knapsack polytope is completely described by box constraints and $\mathcal{O}(n)$ inequalities. An explicit construction scheme of the facet defining inequalities is available.*

Furthermore, the complete linear description of sequential binary knapsack polytopes, i.e., binary knapsacks with $a_i \mid a_{i+1}$ for every $i \in [n-1]$, can be generalized to arbitrary integer knapsack polytopes fulfilling the divisibility property, see Pochet and Weismantel [103]. In general, this description is very complicated. For the special case where no explicit upper bounds on the variables are present, i.e., $x_i \leq \lfloor \frac{\beta}{a_i} \rfloor$, Marcotte [90] derived a complete linear description of the knapsack polytope of linear size.

Theorem 58 (Marcotte [90]). *Let $a \in \mathbb{N}^n$ such that $a_i \mid a_{i+1}$ holds for every $i \in [n-1]$. Then $\operatorname{conv}\{x \in \mathbb{Z}^n_+ : a^\top x \leq \beta\}$ is completely described by non-negativity constraints and*

$$x_i \leq \left\lfloor \frac{\beta}{a_i} \right\rfloor - \sum_{j=i+1}^{n} \frac{a_j}{a_i} x_j, \quad i \in [n].$$

(Q22) Most knapsack examples in practice are of binary form. Are there interesting practical general integer examples?

(Q23) How can the integer knapsack case be best handled algorithmically?

9 Mixed-Integer Knapsacks

In this section, we discuss two variants of mixed-integer knapsacks: a general model for mixed-integer knapsacks and a variant describing a generalized flow model containing variable lower and upper bound constraints.

9.1 General Mixed-Integer Knapsacks

A general mixed-integer knapsack is the straightforward generalization of integer knapsacks to the mixed-integer case, i.e.,

$$X_{\text{Mix}} = \Big\{(x, y) \in \mathbb{Z}^m \times \mathbb{R}^n : \sum_{i=1}^{m} a_i\, x_i + \sum_{j=1}^{n} b_j\, y_j \leq \delta,\ 0 \leq x \leq u,\ 0 \leq y \leq \tilde{u}\Big\}$$

for non-negative vectors a, b, u, and $\tilde{u}$ of appropriate dimensions. If $u = \mathbb{1}$, X_{Mix} is called a *mixed-binary knapsack*, while it is called a *mixed-integer knapsack* otherwise.

For mixed-integer knapsacks, Martin and Weismantel [93] derive the family of *weight inequalities*

$$\sum_{i \in I} a_i\, x_i + \sum_{j \in J} b_j\, y_j + \sum_{i \in [m] \setminus I} \max\{0, a_i - r(I, J)\}\, x_i \leq \delta - r(I, J),$$

where $I \subseteq [m]$, $J \subseteq [n]$, and $r(I, J)$ is the residual capacity of the knapsack if all variables in I and J are fixed at their upper bound, i.e., $r(I, J) =$

$\delta - \sum_{i \in I} a_i\, u_i - \sum_{j \in J} b_j\, \tilde{u}_j$. Weight inequalities are valid for the mixed-integer knapsack polytope $P_{\text{Mix}} = \text{conv}(X_{\text{Mix}})$ and Martin and Weismantel describe cases in which they define facets of P_{Mix}.

The case of mixed-binary knapsacks is investigated by Richard et al. [106]. Besides providing basic polyhedral properties of P_{Mix}, they investigate the lifting problem of continuous variables and present a pseudo-polynomial time sequential lifting algorithm. In [107], the same authors adapt the concept of simultaneous lifting via superadditive functions for classical knapsacks to superlinear lifting functions for continuous variables in mixed-binary knapsacks. This allows them to derive several facet defining inequalities for P_{Mix}.

Marchand and Wolsey [88] consider the special case of mixed-binary knapsacks with exactly one unbounded continuous variable whose coefficient is -1, i.e., $X'_{\text{Mix}} = \big\{(x, y) \in \{0,1\}^m \times \mathbb{R}_+ \,:\, \sum_{i=1}^m a_i\, x_i \le \delta + y\big\}$. Note that X'_{Mix} is not a classical knapsack set, since the continuous variable has a negative coefficient. This model can be interpreted as an extension of the classical binary knapsack, where the capacity bound can be violated using the excess variable y.

In contrast to classical knapsack problems, the knapsack inequality of X'_{Mix} always defines a facet of $\text{conv}(X'_{\text{Mix}})$, see Marchand and Wolsey [88]. Moreover, they derive further valid inequalities. In the case that also some of the coefficients of binary variables are allowed to be negative, Marchand and Wolsey derive several facet defining inequalities and investigate their lifting problem.

A variant of the latter is $\big\{(x, y) \in \{0,1\}^m \times \mathbb{Z}^n_+ \,:\, \sum_{i=1}^m x_i \le \sum_{j=1}^n b_j\, y_j\big\}$, the so-called *integer capacity set*. Mazur and Hall [96] provide basic polyhedral properties of the corresponding polytope and they show how coefficients in its formulation can be reduced, which is of particular interest in preprocessing. Furthermore, they present facet defining inequalities that are based on certain knapsack covers.

9.2 Generalized Flow Models

The most general form of a mixed-integer knapsack with variable bound constraints is specified as follows. Let N_1 and N_2 be (not necessarily disjoint) sets, let $a \in \mathbb{R}^{N_1}$ and $b \in \mathbb{R}^{N_2}$ as well as ℓ', $u' \in \mathbb{R}^{N_1 \cup N_2}$ and $\delta \in \mathbb{R}$. A *mixed-integer knapsack with variable upper/lower bounds* is a set

$$K'_F = \Big\{(f', y') \in \mathbb{R}^{N_1} \times \{0,1\}^{N_2} \,:\, \begin{array}{ll} \sum_{j \in N_1} a_j\, f'_j + \sum_{j \in N_2} b_j\, y'_j \le \delta, & \\ \ell'_j \le f'_j \le u'_j, & j \in N_1 \setminus N_2, \\ \ell'_j\, y'_j \le f'_j \le u'_j\, y'_j, & j \in N_1 \cap N_2 \Big\}. \end{array}$$

The last families of inequalities in this definition are called *variable lower and upper bound constraints*, because depending on the value of y'_j they are active ($y'_j = 1$) or inactive ($y'_j = 0$). Note that we use ℓ' and u' instead of ℓ and u for the bounds, since we will adapt this notation below. Moreover, since we will interpret the continuous variables as flow values, we refer to the continuous variables as f-variables.

To simplify the analysis of such mixed-integer knapsacks, one typically transforms K'_F into a set with an easier integer formulation, which we present next and whose presentation follows Van Roy and Wolsey [108]. By appropriate substitutions of variables and bounds, they showed that K'_F can be represented as

$$\big\{(f,y) \in \mathbb{R}^N \times \{0,1\}^N \;:\; f(M_1) - f(M_2) \leq \delta,\; \ell_j\, y_j \leq f_j \leq u_j\, y_j,\; j \in N\big\},$$

where $N \coloneqq N_1 \cup N_2$ and (M_1, M_2) is a partition of N. This model is the so-called *standard form* of a mixed-integer knapsack, denoted by K_F. In particular, one can assume w.l.o.g. that $0 \leq \ell \leq u$.

The mixed-integer knapsack in standard form can be interpreted as a model of a flow network that consists of a single node v as well as a set of arcs M_1 that point into v and a set of arcs M_2 that leave v. The variable f_j models the flow value on arc j and variable y_j encodes via the generalized lower and upper bound constraints whether arc j can be used. Finally, the first inequality in K_F, to which we refer to as *flow knapsack inequality*, expresses that the net outflow of node v is at most δ.

To strengthen the formulation K_F of $P_F \coloneqq \operatorname{conv}(K_F)$, Van Roy and Wolsey [108] introduced generalized flow cover cuts. A pair (C_1, C_2), with $C_1 \subseteq M_1$ and $C_2 \subseteq M_2$, is called a *generalized cover* if

$$\lambda \coloneqq u(C_1) - \ell(C_2) - \delta > 0,$$

i.e., fixing variables in C_1 at their upper and variables in C_2 at their lower bound violates the flow knapsack inequality. Let

$$\begin{aligned} \bar{u} &\geq \max\big\{\lambda, \max_{j \in C_1}\{u_j\}\big\}, & \bar{u}_j &= \max\{\bar{u}, u_j\},\\ \bar{\ell}_j &= \max\{\bar{u}, \ell_j\}, & \underline{\ell}_j &= \min\{\bar{u}, \ell_j\}, \end{aligned}$$

and $\alpha^+ = \max\{0, \alpha\}$ for a real value α.

Given a generalized flow cover (C_1, C_2), let $L_i \subseteq M_i \setminus C_i$, $i \in \{1,2\}$. Following the presentation of Gu et al. [59], the *simple generalized flow cover* *inequality* for (C_1, L_1, C_2, L_2) is

$$\delta \geq \sum_{j \in C_1} \big(f_j + (u_j - \lambda)^+(1 - y_j)\big) + \sum_{j \in L_1} (f_j - (\bar{u}_j - \lambda) y_j)$$
$$- \sum_{j \in C_2} (f_j + \min\{\ell_j, \bar{\ell}_j - \lambda\})(1 - y_j) - \sum_{j \in L_2} (f_j - (\underline{\ell}_j - \lambda)^+ y_j)$$
$$- \sum_{j \in M_2 \setminus (C_2 \cup L_2)} f_j,$$

which is a valid inequality for P_F. Moreover, a generalization to so-called *extended generalized flow cover inequality* exists, see, e.g., Gu et al. [59].

Van Roy and Wolsey [108] investigate cases in which generalized flow cover inequalities define facets of P_F and they investigate their separation problems. Stallaert [114] derives a class of inequalities, so-called μ-inequalities, that complements flow cover inequalities. He further analyzes their separation problem and experimentally finds that using both generalized flow cover inequalities and μ-inequalities closes the integrality gap by 75 % while using either of these classes reduces the gap by 65 % only. An extensive study of the lifting problem of generalized flow cover inequalities as well as numerical experiments are provided by Gu et al. [59].

10 Variants of the Knapsack Problem

In this section, we briefly discuss polyhedral aspects of variants of the knapsack problem. For algorithmic aspects of the variants discussed in Sections 10.1–10.4, we again refer the reader to the book of Kellerer et al. [77].

10.1 Multidimensional Knapsack Problem

A multidimensional knapsack set K_{mult} is defined by m knapsack constraints instead of a single knapsack constraint, i.e.,

$$K_{\text{mult}} = \{x \in \{0,1\}^n \,:\, Ax \leq b\},$$

where $A \in \mathbb{Z}_+^{m \times n}$ and $b \in \mathbb{Z}_+^m$. That is, a multidimensional knapsack K_{mult} is the intersection of m knapsacks K_i and thus an independence system. Conversely, every independence system has a representation as a multidimensional knapsack via the cuts $x(C) \leq |C| - 1$ for every minimal dependent set C. Therefore, every result on general polytopes associated with independence systems holds also for multidimensional knapsack polytopes P_{M}, e.g., characterization of facets by maximal cliques in conflict hypergraphs, see Easton et al. [36].

The concept of cover inequalities for classical knapsacks directly transfers to the multidimensional case by considering each constraint in $Ax \leq b$

separately. Bektas and Oğuz [14] suggested an IP model to separate violated cover inequalities for P_{M}. Lifted cover inequalities have been investigated by Kaparis and Letchford [75]. By a straightforward generalization of minimal covers of a single constraint in K_{mult} to minimal covers of the complete system $Ax \leq b$, Balas and Zemel [13] show that every non-trivial facet defining inequality of P_{M} can be found by complementing variables x_i, $i \in S$, for an appropriately chosen set $S \subseteq [n]$, detecting a suitable minimal cover of the complemented knapsack, and (sequentially or simultaneously) lifting the corresponding minimal cover inequality. Thus, a theoretical mechanism to find a complete linear description of P_{M} is known.

Martin and Weismantel [94] considered a more general version of the multidimensional knapsack problem, where the upper bounds on variables are arbitrary positive integers. That is, they investigated the case of intersecting several *integer* knapsacks instead of *binary* ones, but of course, their results are applicable in the binary case. They derived so-called feasible set inequalities that are valid for $\bigcap_{i=1}^{m} K_i$ but not necessarily for every K_i, and they derived bounds on the lifting coefficients of these inequalities. If the knapsacks K_i, $i \in [m]$, can be labeled such that $K_i \cap K_k = \varnothing$ whenever $|i - k| \geq 2$, Martin and Weismantel completely characterized whether a facet defining inequality for some $\mathrm{conv}(K_i)$ also defines a facet of P_{M}.

10.2 Cardinality Constrained Knapsack Problem

The cardinality constrained knapsack set $K_{\leq}$ consists of an ordinary knapsack inequality together with the additional constraint that at most k items may be selected, i.e.,

$$K_{\leq} = \{x \in \{0,1\}^n \,:\, a^\top x \leq \beta,\ x([n]) \leq k\}.$$

A cardinality constrained knapsack is a special case of a multidimensional knapsack with two knapsack constraints, i.e., the intersection of two classical knapsacks. Polyhedral aspects of the cardinality constraint knapsack have been investigated by Louveaux and Weismantel [86] who developed so-called incomplete set inequalities to exploit structure of the intersection that is not already present in the separate knapsacks. Moreover, all of the results mentioned in Section 10.1 also apply to the cardinality constrained case.

Glover and Sherali [49, 111] derive special cover inequalities that exploit the cardinality constraint. For integer programs containing a cardinality constraint, Bienstock [15] introduced mixed-integer rounding inequalities, disjunctive cuts, and critical set inequalities. Zeng and Richard [132, 133] generalize the above problem by introducing several cardinality constraints on different variables and give a lifting scheme. If all variables are continuous, De Farias and Nemhauser [32] investigate a variant of $K_{\leq}$ in which the cardinality constraint is replaced by the condition that at most k variables attain positive values.

10.3 Generalized Upper Bound Constraints

Given a set $J \subseteq [n]$ of indices of variables $x \in \{0,1\}^n$, a constraint $x(J) \leq 1$ is called a *generalized upper bound (GUB)*. In many applications such constraints are combined with knapsack constraints. To this end, consider a partition of $[n]$ into k sets $Q_1, \ldots, Q_k$, i.e., $Q_1 \cup \cdots \cup Q_k = [n]$ and $Q_i \cap Q_j = \varnothing$ for all distinct $i, j \in [k]$. Then the *knapsack set with (non-overlapping) GUB constraints* is

$$K_{\text{GUB}} \coloneqq \Big\{x \in \{0,1\}^n \,:\, a^\top x \leq \beta,\ \sum_{i \in Q_j} x_i \leq 1,\ j \in [k]\Big\}.$$

Note that by adding Q_i's of cardinality 1, the assumption that the Q_i cover the set $[n]$ is without loss of generality.

Similarly to the basic case, a set $C \subseteq [n]$ is a (minimal) *GUB cover* if C is a (minimal) cover for the knapsack constraint and no two elements of C belong to the same Q_i, i.e., $a(C) > \beta$ and $|C \cap Q_i| \leq 1$ for all $i \in [k]$. The corresponding *GUB cover inequality* $x(C) \leq |C| - 1$ is valid for $P_{\text{GUB}} \coloneqq \text{conv}(K_{\text{GUB}})$. Note that ordinary cover inequalities are also valid, but are either GUB covers or are redundant for integer solutions.

Wolsey [126] considers knapsacks with GUB constraints in which the knapsack contains positive as well as negative coefficients and introduces GUB covers. He also investigates special cases in which P_{GUB} can be described completely, extensions of GUB cover inequalities, and the GUB analogue of flow covers. Note that negative coefficients can be eliminated by complementing to reach the above form, see Johnson and Padberg [72] for more details[7].

Sherali and Lee [112] investigate polyhedral properties of P_{GUB}. In particular, they characterize when minimal GUB cover inequalities define facets and investigate when sequential and simultaneous lifting of these inequalities defines facets. Moreover, they derive bounds on the lifting coefficients, which can be computed in time $\mathcal{O}(n\,k)$. These results provide a generalization of the classical case discussed in Section 3.4. In particular, for the case of a knapsack set without GUB constraints, i.e., $|Q_i| = 1$ for all $i \in [k]$, the characterization provided by Sherali and Lee [112, Proposition 4.2] reduces to Theorem 38.

Independently, Nemhauser and Vance [97] characterize the cases in which the sequentially lifted minimal GUB cover inequality

$$x(C) + \sum_{j \in [n] \setminus C} \alpha_j\, x_j \leq |C| - 1$$

defines a facet of P_{GUB}. Here, $\alpha_j = \pi_j + 1$ or $\alpha_j = \pi_j$ (see Theorem 28 and Theorem 38). This result both improves Corollary 40 for knapsack problems

[7] Johnson and Padberg also treat the related case of K_{GUB} with continuous variables.

without GUB constraints and generalizes Corollary 40 in the presence of GUB constraints.

Gu et al. [59] discuss algorithmic and practical aspects of lifting GUB cover inequalities. For instance, one can up- and down-lift these inequalities. The lifting problem, however, becomes more involved, see Gu [56].

Gokce and Wilhelm [51] consider the $\geq$-form, i.e., $a^\top x \geq \beta$, of K_{GUB} and define (strong) α-covers C_α.[8] They show that the corresponding inequalities $\sum_{j \in C_\alpha} x_j \geq \alpha$ are valid and characterize when they are facet defining. Furthermore, they describe sequential and sequence independent lifting for α-covers.

(Q24) Can one transfer some of the mentioned results to the case in which the sets Q_i are overlapping? This would correspond to a combination of a stable set and a knapsack problem.

10.4 Precedence Constrained Knapsack Problem

This variant of the knapsack problem adds precedence constraints $x_i \geq x_j$ to the ordinary binary knapsack problem, enforcing that item j may only be added to the knapsack if item i is also taken. This leads to the knapsack set

$$K_{\text{prec}} = \{x \in \{0,1\}^n \,:\, a^\top x \leq \beta,\ x_i \geq x_j\ (i,j) \in \mathcal{A}\},$$

where $\mathcal{A}$ is a set consisting of ordered pairs from $[n]$. Note that the set $\mathcal{A}$ can be interpreted as the arcs of a directed graph with node set $[n]$.

Boyd [20] investigates basic polyhedral properties of $P_{\text{prec}} \coloneqq \operatorname{conv}(K_{\text{prec}})$ and generalizes the concept of cover inequalities. Moreover, based on the graph defined by the precedence constraints, Boyd derives classes of facet defining inequalities. Park and Park [101] discuss lifted cover inequalities for the precedence constrained knapsack problem and they characterize in which cases these inequalities are facet defining for P_{prec}. They also discuss properties of the LP relaxation of K_{prec}. Van de Leensel et al. [117] investigate the complexity of lifting several classes of valid inequalities, e.g., $(1,k)$-configurations or (generalizations of) cover inequalities.

10.5 Generalized Assignment Problem

The generalized assignment problem is a variant of the multiple knapsack problem that has an application in scheduling. There are m knapsacks (machines) and each of the n items has to be assigned to exactly one machine, leading to the knapsack set

$$K_{\text{GAP}} = \Big\{x \in \{0,1\}^n \,:\, \sum_{j=1}^{n} a_{ij} x_{ij} \leq \beta_j,\ i \in [m],\ \text{and}\ \sum_{i=1}^{m} x_{ij} = 1,\ j \in [n]\Big\}.$$

[8] Note that complementing variables does not necessarily yield a set K_{GUB} with the same Q_i.

A polyhedral investigation of $P_{\text{GAP}} \coloneqq \text{conv}(K_{\text{GAP}})$ is provided by Gottlieb and Rao [54]. They also investigate properties of the LP relaxation of K_{GAP}. In [53], the same authors focus on facet defining properties of $(1, k)$-configurations. Trick [115] also investigates the LP relaxation of K_{GAP} and uses his findings to develop efficient primal heuristics.

11 Conclusions

This overview article demonstrates the high level of research on knapsack polytopes in the literature. Nevertheless, there are still some open questions, some of which we stated in the course of this article. From a theoretical side, one would expect more special cases in which a complete linear description can be determined. On the computational side, the handling of dense knapsack constraints would be an interesting topic.

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