A Framework for Solving Chance-Constrained Linear Matrix Inequality Programs

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Abstract

We propose a novel partial sample average approximation (PSAA) framework to solve the two main types of chance-constrained linear matrix inequality (CCLMI) problems: CCLMI with random technology matrix, and CCLMI with random right-hand side. We propose a series of computationally tractable PSAA-based approximations for CCLMI problems, analyze their properties, and derive sufficient conditions ensuring convexity. We derive several semidefinite programming PSAA-reformulations efficiently solved by off-the-shelf solvers and design a sequential convex approximation method for the PSAA formulations containing bilinear matrix inequalities. We carry out a comprehensive numerical study on three practical CCLMI problems: robust truss topology design, calibration, and robust control. The tests attest the superiority of the PSAA reformulation and algorithmic framework over the scenario and sample average approximation methods.
Keywords: stochastic programming, chance-constrained programming, linear matrix inequalities, sampling-based approximation, semidefinite programming

1 Introduction

In this study, we investigate chance-constrained programming problems with linear matrix inequalities (CCLMI). The generic mathematical formulation of the CCLMI program is:

$$\begin{aligned} \min \quad & c^T \boldsymbol{x} && \text{(1a)} \\ s.t. \quad & \mathbb{P}\{G(\boldsymbol{x}, \boldsymbol{\xi}) \succeq 0\} \geq p && \text{(1b)} \\ & \boldsymbol{x} \in X, && \text{(1c)} \end{aligned}$$

where $\boldsymbol{x} \in \mathbb{R}^m$ is a continuous decision vector, $X \subset R^m$ is a convex set representing a set of deterministic constraints, $c \in \mathbb{R}^m$ is a vector of fixed parameters, and $\boldsymbol{\xi}$ is an n-dimensional random vector with a multivariate probability distribution F supported on a set $\Xi \subset \mathbb{R}^n$. Moreover, $G : \mathbb{R}^m \times \Xi \to \mathcal{S}^d$ is a random matrix-valued function, taking values in the space $\mathcal{S}^d$ of $d \times d$ real symmetric matrices, of the form

$$G(\boldsymbol{x}, \boldsymbol{\xi}) = A_0(\boldsymbol{x}) + \sum_{i=1}^{n} A_i(\boldsymbol{x}) \xi_i \ , \tag{2}$$

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where $A_i(\boldsymbol{x}) : \mathbb{R}^m \rightarrow \mathcal{S}^d, i = 0, \ldots, n$ are affine functions. In addition, $G \succeq 0$ means that G is a symmetric positive semidefinite (PSD) matrix. The chance constraint (1b) ensures that the random matrix $G(\boldsymbol{x}, \boldsymbol{\xi})$ is PSD with a probability at least equal to p.

We study problem (1) associated with continuously distributed random variables. More precisely, we consider two types of chance-constrained problems: the first type is the so-called CCLMI with random technology matrix (CCLMI-RTM), and the second type is CCLMI with random right-hand side (CCLMI-RHS). The explicit formulations of CCLMI-RTM and CCLMI-RHS are:

$$\text{CCLMI-RTM} \quad \min \quad c^T \boldsymbol{x}$$
$$s.t. \quad \mathbb{P}\{A_0(\boldsymbol{x}) + \sum_{i=1}^{n} A_i(\boldsymbol{x})\xi_i \succeq 0\} \geq p \tag{3a}$$
$$(1c),$$

$$\text{CCLMI-RHS} \quad \min \quad c^T \boldsymbol{x}$$
$$s.t. \quad \mathbb{P}\{A_0(\boldsymbol{x}) + \sum_{i=1}^{n} A_i\xi_i \succeq 0\} \geq p \tag{4a}$$
$$(1c),$$

where each $A_i, i = 1, \ldots, n$ is a general constant symmetric matrix of size $d \times d$. Note that CCLMI-RHS is a special case of CCLMI-RTM with $A_i(\boldsymbol{x}) = A_i$. The next sections in this manuscript will focus on the normal and uniform distributions. We illustrate in Appendix the above formulations and the reformulations presented in this study with small-size numerical examples. We refer the reader to Appendix A for CCLMI-RHS, Appendix B for CCLMI-RTM, and Appendix C for a special case of CCLMI-RTM, in which we have $A_1(\boldsymbol{x}) = \hat{a}_1(\boldsymbol{x})A_1$ and $\hat{a}_1(\boldsymbol{x})$ is an affine function of $\boldsymbol{x}$.

While chance-constrained programming provides a natural way to model optimization problems with uncertainty, finding exact solutions to these problems is notoriously challenging for two main reasons. First, the feasible set of the chance-constrained programs is generally nonconvex even if the set X is convex and the function G is affine in $\boldsymbol{x}$. Next, computing $\mathbb{P}\{G(\boldsymbol{x}, \boldsymbol{\xi}) \succeq 0\}$ when $\boldsymbol{\xi}$ is continuously distributed is complicated since it involves the calculation of multi-dimensional integrals which is computationally expensive [29, 33]. The derivation of computationally efficient approximations for the chance-constrained programs is therefore crucial.

1.1 Related work

This paper describes chance-constrained programs with LMI constraints in the presence of uncertainty. As such, there are essentially two bodies of literature: robust optimization with LMI constraints and chance-constrained nonlinear programs.

Robust optimization seeks solutions that are immunized against uncertainty. Although robust optimization has been studied in a wide range of areas, the robust optimization literature for problems with LMI constraints is relatively scant. El Ghaoui et al. [20] studied robust LMI problems in which the underlying probability distribution is uniform. They formulated sufficient conditions for the existence of a robust solution as semidefinite programs. They also identified special cases for which the conditions are necessary and sufficient and the robust solution is unique. Ben-Tal and Nemirovski [7] focused on uncertain LMI problems with interval uncertainty and provided verifiable sufficient conditions for the validity of specific semi-infinite systems of LMIs. Later, Ben-Tal et al. [9] extended their previous work from the case of scalar perturbations onto more general perturbation models. For the case where the uncertainty set is finitely generated, Scherer [30] developed a sequence of asymptotically exact relaxations for the general classes of LMIs. Recently, Jeyakumar and Li [22] studied linear problems with uncertain LMI constraints and established strong duality between the robust counterpart of the uncertain LMI problem and the optimistic counterpart of its uncertain dual. The result holds without any constraint qualification. Lee and Lee [24] considered the uncertain LMI problems and derived approximate optimality and duality conditions for ϵ-solutions under the robust characteristic cone constraint qualification. In particular, the authors formulated a (Wolfe-type) dual problem for the robust counterpart of the semidefinite programming (SDP) problem and then provided approximate weak and strong duality theorems. For more information about robust optimization with LMI constraints, the reader is referred to [3, 4].

An alternative to robust optimization is stochastic programming (SP), in which the information about the underlying probability distribution is fully exploited. An important variant of SP is chance-constrained programming which requires the random constraints to be satisfied with high probability. In contrast to chance-constrained linear programs, the literature on chance-constrained nonlinear programming is sparse, among which the two main types (continuous and discrete) of probability distributions are considered. The case with a continuous underlying probability distribution typically requires the calculation of multi-dimensional integrals and poses severe computational challenges. It is then vital to develop approximation approaches, which can be classified into two main categories: convex (or tractable) approximations and sampling-based approaches which use representations of the empirical distribution obtained from Monte Carlo samples. To solve joint chance-constrained problems, Hong and Yang [21] proposed a sequential convex approximation approach, which is based on a sequence of conservative nonsmooth difference of convex (DC) approximations of the step function. They showed that the solutions of the sequential approximations converge to a KKT point of the original chance-constrained problem. Building on the latter work, Shan et al. [31, 32] developed a sequence of conservative smooth DC approximations of the step function to approximate joint chance-constrained problems and provided similar convergence results (as in [21]) but under weaker assumptions. Cao and Zavala [14] proposed a sigmoidal conservative approximation to tackle chance-constrained nonlinear problems by outer-approximating the step function. With regard to the sampling-based approaches, Calafiore et al. [10, 12] proposed a versatile approximation approach, named scenario approximation (SA), and developed upper bounds on the number of samples required to guarantee that the solution of the SA approach is feasible to the corresponding chance-constrained problem with high probability. Luedtke and Ahmed [27] considered a variant of the sample average approximation (SAA) approach, which takes a finite sample of random variables extracted from the underlying distribution and substitutes the chance constraint with a set of sampled constraints. Curtis et al. [19] attempted to directly solve the SAA formulation using nonlinear programming techniques. They employed an exact penalty function for the SAA, which is minimized by solving quadratic subproblems with linear cardinality constraints.

Alternatively, for the case of discrete probability distributions, most work resort to Boolean and integer programming solution techniques to reformulate and solve the problem. Lejeune and Margot [26] studied a class of nonlinear chance-constrained programs that involve a joint chance constraint with random technology matrix and stochastic quadratic inequalities. They derived two mixed-integer nonlinear programming (MINLP) reformulations such that the number of binary variables is not an increasing function of the number of scenarios used to represent uncertainty. They also designed two nonlinear branch-and-bound algorithms for solving the corresponding MINLP problems. Following their previous study, Lejeune et al. [25] considered a class of nonlinear chance-constrained programs with decision-dependent uncertainty. Additionally, the authors proposed deterministic MINLP reformulations using the Boolean modeling approach as well as exact MINLP reformulations for the chance-constrained problems with decision-dependent service uncertainty. They also proposed an algorithmic framework which includes derivation of tight lower and upper bounds and a nonlinear branch-and-bound algorithm to solve those MINLP problems. Adam and Branda [1] studied chance-constrained nonlinear programs with differentiable nonlinear random functions and a discrete distribution. They reformulated the chance-constrained program as an MINLP problem.

In addition to the above-mentioned literature, there exist some investigations focusing on gradient-based approximations for solving chance-constrained nonlinear programs. For instance, Kannan and Luedtke [23] proposed an algorithm for approximating the frontier of optimal objective value versus risk level for the chance-constrained nonlinear programs. They used a projected stochastic subgradient algorithm and solved a sequence of partially smoothened stochastic programs using a scenario-based approach. Van Ackooij and Henrion [34, 35] provided approximations for values and gradients of probability functions by using a spherical-radial decomposition of Gaussian random vectors. They utilized an internal sampling method for numerically evaluating the (sub)gradients.

CCLMI problems belong to the family of chance-constrained nonlinear programs and have received some attention in the past decade. For instance, Nemirovski [28] derived a tractable approximation for solving the CCLMI programs under the assumption that the random vector $\boldsymbol{\xi}$ has a Gaussian distribution and $A_0 \succeq 0$. Ben-Tal and Nemirovski [8] proposed a safe approximation for CCLMI by assuming the primitive perturbations to be independent with light-tail distributions. Cheung et al. [18] extended this latter work and developed a safe tractable approximation for CCLMI by establishing an upper bound on the violation probability of the chance constraint without having the independence assumption for the random variables.

Recently, Cheng et al. [16, 17] proposed the partial sample average approximation (PSAA) idea for solving chance-constrained programs. Such an idea has been successfully applied to solve chance-constrained linear programs. In this paper, we extend the PSAA idea to CCLMI and propose a general efficient framework to solve CCLMI problems. The contributions of this paper can be summarized as follows.

1. We develop the PSAA approach to approximate CCLMI problems and derive deterministic PSAA formulations of CCLMI-RHS and CCLMI-RTM respectively. We also provide sufficient conditions under which the PSAA formulations are convex optimization problems.

2. We develop SDP approximation and bilinear matrix inequality (BMI) approximation of the PSAA formulations for CCLMI-RHS and CCLMI-RTM respectively, both of which are inner approximations. Additionally, we derive an equivalent SDP reformulation of the PSAA formulation of CCLMI-RTM under special conditions.

3. For the general BMI approximation of CCLMI-RTM, we propose a sequential convex approximation method for solving the BMI problems, which consists of solving a set of SDP problems.

4. We demonstrate the efficacy of the proposed PSAA-based approximations through a comprehensive numerical study on three different industrial applications: robust truss topology design, calibration, and robust control. The numerical results demonstrate the strengths of our proposed approximations, in comparison with two basic approaches, i.e., SAA and SA.

5. We use Schur's complement to derive a new CCLMI-RHS reformulation of the truss topology design which originally takes the form of a CCLMI-RTM problem. The PSAA-based method permits to derive an approximation taking the form of an SDP problem instead of one with BMI constraints.

The remainder of this paper is organized as follows. Section 2 introduces two popular approximation methods for approximating CCLMI problems: the SA approach and the SAA method. Section 3 contains an overview of the main idea behind the PSAA method, presents new deterministic PSAA reformulations for the CCLMI-RHS and CCLMI-RTM problems, and exposes convexity results about the PSAA reformulations. Section 4 includes efficient solution methods for solving different PSAA formulations of problems CCLMI-RHS and CCLMI-RTM. Section 5 presents a comprehensive numerical study, based on three application problems, of the proposed PSAA framework. Finally, Section 6 summarizes our contributions and discusses possible future research.

2 Basic approximation approaches for CCLMI

In this section, we present two basic methods that can be used to approximate CCLMI problems. These two methods will be used as benchmarks against the main PSAA solution framework developed in this paper (see Section 3). We present below the scenario approximation (SA) approach and the sample average approximation (SAA) method for CCLMI-RTM and CCLMI-RHS problems, respectively.

2.1 Scenario approximation

Consider N independent and identically distributed (i.i.d) samples $\hat{\boldsymbol{\xi}}^1, \ldots, \hat{\boldsymbol{\xi}}^N$ extracted from the distribution F. The SA formulation of the CCLMI-RTM problem is given as

$$\begin{aligned} \text{SA-RTM} \quad \min \quad & c^T \boldsymbol{x} \\ s.t. \quad & A_0(\boldsymbol{x}) + \sum_{i=1}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq 0, \quad \forall t = 1, \ldots, N \qquad (5a) \\ & (1c), \end{aligned}$$

while the SA formulation of the CCLMI-RHS problem is

$$\begin{aligned} \text{SA-RHS} \quad \min \quad & c^T \boldsymbol{x} \\ s.t. \quad & A_0(\boldsymbol{x}) + \sum_{i=1}^{n} A_i\hat{\xi}_i^t \succeq 0, \quad \forall t = 1, \ldots, N \qquad (6a) \\ & (1c). \end{aligned}$$

We provide numerical illustrations of the above formulations in Appendix. See Appendix A.1 for SA-RHS, and Appendix B.1 and Appendix C.1 for SA-RTM.

Campi et al. [13] (see Theorem 1) provided an estimation on the number of scenarios to be included in the SA formulation. We denote by $\beta \in (0, 1)$ a confidence parameter, and we use the expression $\lceil \cdot \rceil$ to refer to the smallest integer greater than or equal to the argument $(\cdot)$.

Theorem 1. *The SA problem is either infeasible, and hence, the CCLMI problem is infeasible, or is feasible and attains a unique optimal solution $\boldsymbol{x}_N^*(\hat{\boldsymbol{\xi}})$. Given β, if the number of scenarios N satisfies the relation*

$$N \geq N^* := \left\lceil \frac{2}{1-p}(\log \frac{1}{\beta} + m) \right\rceil, \qquad (7)$$

then $\boldsymbol{x}_N^*(\hat{\boldsymbol{\xi}})$ satisfies the chance constraint (1b) *with confidence* $(1 - \beta)$.

An advantage of this approach is that when X is a convex set and $G(\boldsymbol{x}, \boldsymbol{\xi})$ is concave in $\boldsymbol{x}$ for each $\boldsymbol{\xi}$, the scenario approximation is a convex optimization problem. However, it has been shown in [27] that this approach can be very conservative, since it requires the satisfaction of the conditions imposed by all scenarios.

2.2 Sample average approximation

An alternative approach to approximate the CCLMI problems is to employ the sample average approximation (SAA) method. The fundamental idea of SAA is to approximate the expectation of random variable by its sample mean. The general probability constraint can be reformulated as an expectation

$$\mathbb{P}\{G(\boldsymbol{x},\boldsymbol{\xi}) \succeq 0\} = \mathbb{E}[\mathbb{I}(G(\boldsymbol{x},\boldsymbol{\xi}) \succeq 0)] \ , \tag{8}$$

where $\mathbb{I}(\cdot)$ is an indicator function which takes value 1 if $(\cdot)$ is true, and zero otherwise. The expectation can be approximated by the sample mean: $\frac{1}{N}\sum_{t=1}^{N}\mathbb{I}(G(\boldsymbol{x},\hat{\boldsymbol{\xi}}^t) \succeq 0)$, where scenarios $\hat{\boldsymbol{\xi}}^1,\ldots,\hat{\boldsymbol{\xi}}^N$ are independent samples extracted from the distribution of $\boldsymbol{\xi}$. The SAA formulation of problem (1) is:

$$\begin{aligned} \min \quad & c^T\boldsymbol{x} \\ s.t. \quad & \frac{1}{N}\sum_{t=1}^{N}\mathbb{I}(G(\boldsymbol{x},\hat{\boldsymbol{\xi}}^t) \succeq 0) \geq p && \text{(9a)} \\ & \text{(1c)}. \end{aligned}$$

Using the "big-M" method and introducing the binary variables $\eta_t, t = 1,\ldots,N$, problem (9) can be reformulated as a mixed-integer semidefinite programming (MISDP) problem

$$\begin{aligned} \min \quad & c^T\boldsymbol{x} \\ s.t. \quad & G(\boldsymbol{x},\hat{\boldsymbol{\xi}}^t) + M_t\eta_t I \succeq 0, \quad \forall t = 1,\ldots,N && \text{(10a)} \\ & \frac{\sum_{t=1}^{N}\eta_t}{N} \leq 1-p && \text{(10b)} \\ & \eta_t \in \{0,1\}, \quad \forall t = 1,\ldots,N && \text{(10c)} \\ & \text{(1c)}, \end{aligned}$$

where I is the identity matrix, and M_t is a large constant, such that $G(\boldsymbol{x},\hat{\boldsymbol{\xi}}^t)+M_tI \succeq 0$ for any $\boldsymbol{x} \in X$. More specifically, the best possible value of M_t is the smallest value such that the constraint (10a) holds for any $\boldsymbol{x} \in X$ when $\eta_t = 1$. If $\eta_t = 0$, we have $G(\boldsymbol{x},\hat{\boldsymbol{\xi}}^t) \succeq 0$ while $\eta_t = 1$ implies $G(\boldsymbol{x},\hat{\boldsymbol{\xi}}^t) \not\succeq 0$. Constraints (10a)–(10c) are the equivalent big-M reformulation of (9a). The constraint (10b) implies that at most $\lceil N \times (1-p)\rceil$ number of scenarios do not satisfy the LMI $G(\boldsymbol{x},\hat{\boldsymbol{\xi}}^t) \succeq 0$. For additional details on SAA approaches for solving chance-constrained optimization problems, we refer to [27].
Accordingly, the SAA formulation of the CCLMI-RTM problem is given as

$$\begin{aligned} \text{SAA-RTM} \quad \min \quad & c^T\boldsymbol{x} \\ s.t. \quad & A_0(\boldsymbol{x}) + \sum_{i=1}^{n}A_i(\boldsymbol{x})\hat{\xi}_i^t + M_t\eta_t I \succeq 0, \quad \forall t = 1,\ldots,N && \text{(11a)} \\ & \text{(1c)}; \text{(10b)}; \text{(10c)}, \end{aligned}$$

and the SAA formulation of the CCLMI-RHS problem is

$$\begin{aligned} \text{SAA-RHS} \quad \min \quad & c^T\boldsymbol{x} \\ s.t. \quad & A_0(\boldsymbol{x}) + \sum_{i=1}^{n} A_i\hat{\xi}_i^t + M_t\eta_t I \succeq 0, \quad \forall t = 1, \ldots, N \qquad (12a) \\ & (1c); (10b); (10c). \end{aligned}$$

Note that, in contrast to the SA approach which requires the satisfaction of the constraints imposed by all scenarios, the SAA approach requires to satisfy the conditions imposed by at least $\lceil N \times p \rceil$ scenarios. See Appendix A.2 for an illustration of SAA-RHS, and Appendix B.2 and Appendix C.2 for illustrations of SAA-RTM.

3 PSAA formulations of CCLMI

Section 3.1 presents an overview of the PSAA approach introduced in [16, 17] to tackle chance constraints with random linear inequalities. In Section 3.2, we derive deterministic reformulations of the PSAA-based approximations for the CCLMI-RTM and CCLMI-RHS problems. Convexity results for PSAA formulations are given in Section 3.3.

3.1 PSAA overview

Reconsider the general chance-constrained optimization model

$$\begin{aligned} \min \quad & c^T\boldsymbol{x} \\ s.t. \quad & \mathbb{P}\{G(\boldsymbol{x}, \boldsymbol{\xi}) \succeq 0\} \geq p \qquad (13a) \\ & (1c), \end{aligned}$$

where the random vector $\boldsymbol{\xi}$ is divided into two sub-vectors ξ_1 and $\boldsymbol{\xi}_2$ (i.e., $\boldsymbol{\xi} = (\xi_1, \boldsymbol{\xi}_2)$, where $\boldsymbol{\xi}_2 = (\xi_2, \ldots, \xi_n)$).

Under the assumption that ξ_1 is independent of $\boldsymbol{\xi}_2$, we can replace constraint (13a) by

$$\begin{aligned} \mathbb{P}\{G(\boldsymbol{x}, \boldsymbol{\xi}) \succeq 0\} & = \mathbb{E}[\mathbb{I}(G(\boldsymbol{x}, \boldsymbol{\xi}) \succeq 0)] \\ & = \mathbb{E}_{\xi_1, \boldsymbol{\xi}_2}[\mathbb{I}(G(\boldsymbol{x}, \xi_1, \boldsymbol{\xi}_2) \succeq 0)] \\ & = \mathbb{E}_{\xi_1}[\mathbb{E}_{\boldsymbol{\xi}_2}[\mathbb{I}(G(\boldsymbol{x}, \xi_1, \boldsymbol{\xi}_2) \succeq 0)]] \geq p. \end{aligned}$$

Moreover, applying the same idea as SAA, namely the substitution of the inner expectation by its sample mean, we obtain:

$$\begin{aligned} \mathbb{E}_{\xi_1}[\mathbb{E}_{\boldsymbol{\xi}_2}[\mathbb{I}(G(\boldsymbol{x}, \xi_1, \boldsymbol{\xi}_2) \succeq 0)]] & \approx \frac{1}{N}\sum_{t=1}^{N} \mathbb{E}_{\xi_1}[\mathbb{I}(G(\boldsymbol{x}, \xi_1, \hat{\boldsymbol{\xi}}_2^t) \succeq 0)] \qquad (14) \\ & = \frac{1}{N}\sum_{t=1}^{N} \mathbb{P}\{G(\boldsymbol{x}, \xi_1, \hat{\boldsymbol{\xi}}_2^t) \succeq 0\} \geq p, \qquad (15) \end{aligned}$$

where $\hat{\boldsymbol{\xi}}_2^1, \ldots, \hat{\boldsymbol{\xi}}_2^N$ are N independent Monte Carlo samples of $\boldsymbol{\xi}_2$. Thus, the PSAA formulation of problem (13) is:

$$
\begin{aligned}
\text{PSAA-G} \quad \min \quad & c^T \boldsymbol{x}, && \text{(16a)}\\
s.t. \quad & \mathbb{P}\{G(\boldsymbol{x}, \xi_1, \hat{\boldsymbol{\xi}}_2^t) \succeq 0\} \geq y_t, \quad \forall t = 1, \ldots, N && \text{(16b)}\\
& \frac{\sum_{t=1}^N y_t}{N} \geq p && \text{(16c)}\\
& y_t \geq 0, \quad \forall t = 1, \ldots, N && \text{(16d)}\\
& \text{(1c)},
\end{aligned}
$$

in which y_t is the lower bound on the probability that the random inequality $G(\boldsymbol{x}, \xi_1, \hat{\boldsymbol{\xi}}_2^t) \succeq 0$ holds. Constraints (16b) and (16c) are equivalent to (15).

Inspired by previous studies [16, 17] which apply the PSAA approach to solve chance-constrained linear programs, we extend the PSAA approach to tackle the CCLMI problems and derive deterministic reformulations.

3.2 Deterministic Reformulations of PSAA Formulations

Throughout the paper, we work under the following assumption on the random variables:

Assumption 1. *At least one of the random variables among $\xi_1, \ldots, \xi_n$ is independent of the others. Without loss of generality, we assume ξ_1 is independent of the others. Moreover, we assume that ξ_1 is either (a) uniform or (b) normal.*

Note that via a simple linear transformation (normalization), we can define the range of ξ_1 as the interval $[0, 1]$ in (a) and enforce ξ_1 to be standard in (b). Therefore, we hereafter assume that either $\xi_1 \sim U(0, 1)$ or $\xi_1 \sim \mathcal{N}(0, 1)$.

Remark 1. *The assumption that a random variable is independent of the others is non-restrictive for $\boldsymbol{\xi} \sim \mathcal{N}(\mu, \Sigma)$. Indeed, using the eigen-decomposition $\Sigma = U\Lambda U^T$, we can represent $\boldsymbol{\xi}$ by a linear combination of a standard normal vector $\boldsymbol{\xi}'$:*

$$
\boldsymbol{\xi} \sim \mathcal{N}(\mu, \Sigma) \Leftrightarrow \boldsymbol{\xi} = U\Lambda^{1/2}\boldsymbol{\xi}' + \mu = V\boldsymbol{\xi}' + \mu, \boldsymbol{\xi}' \sim \mathcal{N}(0, I) \ , \tag{17}
$$

where the columns of U are unit eigenvectors, Λ is a diagonal matrix of the eigenvalues, and $V = U\Lambda^{1/2}$. We can then rewrite:

$$
A_i'(\boldsymbol{x}) = \sum_{j=1}^n V_{ji} A_j(\boldsymbol{x}) \quad \textit{and} \quad A_0'(\boldsymbol{x}) = A_0(\boldsymbol{x}) + \sum_{i=1}^n \mu_i A_i(\boldsymbol{x}) \ .
$$

Let $\hat{\xi}_i^1, \ldots, \hat{\xi}_i^N$ ($\forall i = 2, \ldots n$) be independent Monte Carlo samples of ξ_i. Then, applying the PSAA idea (see PSAA-G formulation) to the CCLMI-RTM problem, we have the following PSAA formulation:

$$
\begin{aligned}
\min \quad & c^T \boldsymbol{x}\\
s.t. \quad & \mathbb{P}\{A_0(\boldsymbol{x}) + \sum_{i=2}^n A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq -A_1(\boldsymbol{x})\xi_1\} \geq y_t, \quad \forall t = 1, \ldots, N && \text{(18)}\\
& \text{(1c); (16c); (16d).}
\end{aligned}
$$

In the following theorem, we derive an alternative equivalent reformulation of the above PSAA formulation.

Theorem 2. *Problem* (18) *is equivalent to*

$$
\begin{aligned}
\textit{PSAA-RTM} \quad \min \quad & c^T \boldsymbol{x} \\
s.t. \quad & A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq -A_1(\boldsymbol{x}) z_1^t, \quad \forall t = 1, \dots, N && \text{(19a)} \\
& A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq -A_1(\boldsymbol{x}) z_2^t, \quad \forall t = 1, \dots, N && \text{(19b)} \\
& \Phi(z_2^t) - \Phi(z_1^t) \geq y_t, , \quad \forall t = 1, \dots, N && \text{(19c)} \\
& -\infty \leq z_1^t \leq z_2^t \leq +\infty, \quad \forall t = 1, \dots, N && \text{(19d)} \\
& \text{(1c)}; \text{(16c)}; \text{(16d)},
\end{aligned}
$$

where z_1^t and z_2^t are continuous auxiliary variables associated with each scenario t and $\Phi(\cdot)$ is the cumulative distribution function (CDF) of ξ_1.

Proof. Suppose that $(\boldsymbol{x}, y_t, z_1^t, z_2^t)$ is feasible for problem (19). Since z_1^t and z_2^t are feasible points for (19a) and (19b), any point between z_1^t and z_2^t is also feasible for (19a) and (19b) due to the convexity of the left sides in those constraints. Thus, by constructing the set

$$
S(t) := \{\xi_1 : A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq -A_1(\boldsymbol{x}) z_1^t,\ A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq -A_1(\boldsymbol{x}) z_2^t,\ z_1^t \leq \xi_1 \leq z_2^t\},
$$

we can rewrite (19a) and (19b) as

$$
A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq -A_1(\boldsymbol{x})\xi_1, \tag{20}
$$

and construct another related set:

$$
S'(t) := \{\xi_1 : A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq -A_1(\boldsymbol{x})\xi_1\}.
$$

It is easy to see that $S(t) \subset S'(t)$, and hence

$$
\mathbb{P}\{\xi_1 \in S'(t)\} \geq \mathbb{P}\{\xi_1 \in S(t)\}. \tag{21}
$$

Considering (19c), along with $\mathbb{P}\{z_1^t \leq \xi_1 \leq z_2^t\} = \Phi(z_2^t) - \Phi(z_1^t)$, we have

$$
\mathbb{P}\{A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i \succeq -A_1(\boldsymbol{x})\xi_1\} \geq \mathbb{P}\{\xi_1 \in S(t)\} \geq y_t,
$$

which shows that $(\boldsymbol{x}, y_t)$ is also feasible to (18).

Conversely, suppose that $(\boldsymbol{x}, y_t)$ is feasible to problem (18). With $S'(t) = \{\xi_1 : A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq -A_1(\boldsymbol{x})\xi_1\}$, the constraint (18) becomes equivalent to $\mathbb{P}\{\xi_1 \in S'(t)\} \geq y_t$. Then we define the auxiliary variables z_1^t and z_2^t as

$$z_2^t = \sup\{\xi_1 : \xi_1 \in S'(t)\} \quad \text{and} \quad z_1^t = \inf\{\xi_1 : \xi_1 \in S'(t)\}. \tag{22}$$

Since $S'(t)$ is a closed set, its infimum and supremum belong to $S'(t)$ which means z_1^t and z_2^t are feasible for constraints (19a) and (19b) respectively. Moreover, we have $S'(t) \subset [z_1^t, z_2^t]$ and thus $\Phi(z_2^t) - \Phi(z_1^t) = \mathbb{P}\{z_1^t \leq \xi_1 \leq z_2^t\} \geq \mathbb{P}\{\xi_1 \in S'(t)\} \geq y_t$. Therefore, $(\boldsymbol{x}, y_t, z_1^t, z_2^t)$ is also feasible to (19), which completes the proof. □

It is worth mentioning that the PSAA and SAA approaches share the same idea that consists in replacing the true probability distribution by its empirical distribution obtained from Monte Carlo samples. Accordingly, the PSAA formulation PSAA-RTM (i.e., problem (19)) has the same convergence properties as the SAA formulation (11) [17]. This means that, under some regularity conditions, the optimal value of PSAA-RTM converges to the optimal value of the original problem as the number N of scenarios tends to infinity. The main difference between the SAA and the PSAA approaches lies in the combinatorial structure of the reformulated probelms. While the SAA formulation requires the introduction of N additional binary variables, thereby giving rise to an MISDP reformulation, the proposed PSAA formulation contains only continuous variables. Hereafter, we focus on investigating the convexity properties of problem PSAA-RTM.

3.3 Convexity Results

In general, the feasible set of problem PSAA-RTM is nonconvex due to the bilinear terms in constraints (19a) and (19b) and to the nonconvexity of constraint (19c). However, under some specific assumptions presented next, the reformulation PSAA-RTM is convex.

Theorem 3. *If (a) $\xi_1 \sim \mathcal{N}(0,1)$ or (b) $\xi_1 \sim U(0,1)$, then the set*

$$Z_0 := \{(z_1^1, \ldots, z_1^N, z_2^1, \ldots, z_2^N, y_1, \ldots, y_N) : \Phi(z_2^t) - \Phi(z_1^t) \geq y_t, \ \frac{\sum_{t=1}^{N} y_t}{N} \geq p, \ z_1^t \leq z_2^t\}$$

is convex if (a) $p > \frac{N-\frac{1}{2}}{N}$ or (b) $p > \frac{N-1}{N}$. Moreover, the reformulation PSAA-RTM is convex if $A_1(\boldsymbol{x}) = A_1$.

Proof. We first provide the proof for $\xi_1 \sim \mathcal{N}(0,1)$. Since $0 \leq \Phi(\cdot) \leq 1$, we have $y_t \leq 1$. If $p > \frac{N-\frac{1}{2}}{N}$, then we must have $y_t > \frac{1}{2}, \ t = 1, \ldots, N$. Indeed, if we assume the opposite and suppose that there exists $y_t, \ t = 1, \ldots, N$, such that $y_t \leq \frac{1}{2}$, we have $p \leq \frac{N-\frac{1}{2}}{N}$, which contradicts our initial assumption.

As a result, taking into account that $y_t > \frac{1}{2}$, and $0 \leq \Phi(\cdot) \leq 1$, we have:

$$\Phi(z_2^t) > \frac{1}{2} \Longrightarrow z_2^t > 0, \text{ and } \Phi(z_1^t) < \frac{1}{2} \Longrightarrow z_1^t < 0.$$

Furthermore, it is well known that for $z_2^t \geq 0$ and $z_1^t \leq 0$, the cumulative distribution functions $\Phi(z_2^t)$ and $\Phi(z_1^t)$ are respectively concave and convex. Therefore, $\Phi(z_2^t) - \Phi(z_1^t)$ is concave, and hence Z_0 is a convex set, which concludes the proof.

The proof can be derived in a similar fashion for $\xi_1 \sim U(0,1)$. □

Note that problem CCLMI-RHS satisfies the condition that $A_1(\boldsymbol{x}) = A_1$, thus its PSAA formulation is a convex optimization problem when $p > \frac{N-\frac{1}{2}}{N}$ in the case of $\xi_1 \sim \mathcal{N}(0,1)$ or $p > \frac{N-1}{N}$ in the case of $\xi_1 \sim U(0,1)$.

4 Solution Methods

This section is devoted to the development of efficient solution methods for solving the proposed PSAA-RTM/RHS problems. A piecewise linear approximation is employed in Section 4.1 to handle the non-convex constraint (19c). In addition, approximate deterministic formulations are derived for problems PSAA-RTM and PSAA-RHS, and take the form of BMI and SDP problems, respectively. In Section 4.2, an SDP reformulation is developed for problem PSAA-RTM in a special case. Section 4.3 presents a sequential convex optimization algorithm to solve the general PSAA-RTM problem whose reformulation takes the form of a problem with BMIs.

4.1 Convex Approximations of Constraint (19c)

The problem PSAA-RTM (i.e., problem (19)) is in general non-convex due to the nonconvexity of the feasible area defined by the constraint (19c) and by the BMI constraints (19a) and (19b). In this section, we develop a set of linear inequalities to approximate (19c).

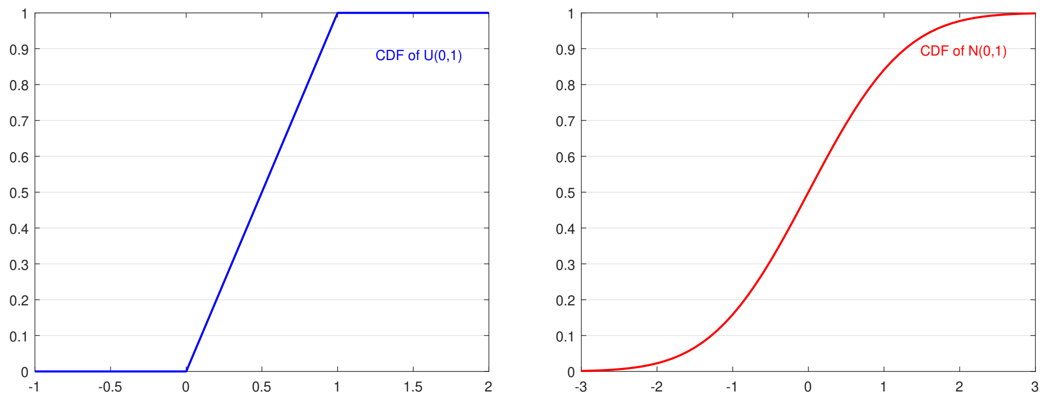
Note that for both the normal and uniform distributions, the CDF $\Phi(\cdot)$ is concave when $\Phi(\cdot) \geq 0.5$ and is convex when $\Phi(\cdot) \leq 0.5$ (See Figure 1 (a) and (b)). Considering the constraint (19c) along with (16c), one can infer that when the probability level p is close to one (e.g., 0.95), $\Phi(z_2^t)$ is usually greater than 0.5 and thus concave, while $\Phi(z_1^t)$ is usually less than 0.5 and thus convex. As a result, function $\Phi(z_2^t)$ can be approximated with a piecewise linear concave function

$$\Phi(z_2^t) \approx \min_{k \in \{1,\ldots,K\}} a_k z_2^t + b_k, \tag{23}$$

while function $\Phi(z_1^t)$ can be approximated by a piecewise linear convex function

$$\Phi(z_1^t) \approx \max_{s \in \{1,\ldots,S\}} a_s z_1^t + b_s, \tag{24}$$

where $s \in \{1,\ldots,S\}$ and $k \in \{1,\ldots,K\}$ are the indices of S and K linear segments for $\Phi(z_1^t)$ and $\Phi(z_2^t)$, respectively.



(a) Uniform distribution (b) Normal distribution

Figure 1: Cumulative distribution functions

In what follows, convex approximations of (19c) are provided for $\xi_1 \sim U(0,1)$, and $\xi_1 \sim \mathcal{N}(0,1)$, respectively.

4.1.1 PSAA-RTM with Uniform Distribution

Using the following piecewise linear approximation of (19c), we derive an inner approximation of PSAA-RTM and PSAA-RHS for the uniform distribution case. More precisely, we use a concave piecewise linear approximation and a convex piecewise linear approximation to approximate $\Phi(z_2^t)$ and $\Phi(z_1^t)$, respectively. Indeed, when $\xi_1 \sim U(0,1)$, the CDF $\Phi(\cdot)$ consists of three linear segments as shown in Figure 2. We use two upper linear segments (i.e., q_{21} and q_{22} in Figure 2) to approximate $\Phi(z_2^t)$ and two lower linear segments (i.e., q_{11} and q_{12} in Figure 2) to approximate $\Phi(z_1^t)$. Accordingly, the pair of slopes and intercepts $(a_k,\, b_k)$ (or $(a_s,\, b_s)$) of the concave (or convex) piecewise linear approximation of $\Phi(z_2^t)$ (or $\Phi(z_1^t)$) are given as:

$$(a_k, b_k) \in \{(1,\ 0), (0,\ 1)\} \tag{25}$$
$$(a_s, b_s) \in \{(1,\ 0), (0,\ 0)\}. \tag{26}$$

Subsequently, using (23) and (24), (19c) can be approximated by

$$\min_{k\in\{1,2\}} a_k z_2^t + b_k - \max_{s\in\{1,2\}} a_s z_1^t + b_s \geq y_t. \tag{27}$$

By introducing the auxiliary variables q_2^t and q_1^t to represent $\min_{k\in\{1,2\}} a_k z_2^t + b_k$ and $\max_{s\in\{1,2\}} a_s z_1^t + b_s$, respectively, (27) can be equivalently rewritten with these inequalities:

$$q_2^t - q_1^t \geq y_t, \qquad \forall t = 1, \ldots, N \tag{28a}$$
$$q_2^t \leq a_k z_2^t + b_k, \qquad \forall t = 1, \ldots, N, \quad \forall k = 1, 2 \tag{28b}$$
$$q_1^t \geq a_s z_1^t + b_s, \qquad \forall t = 1, \ldots, N, \quad \forall s = 1, 2 \tag{28c}$$

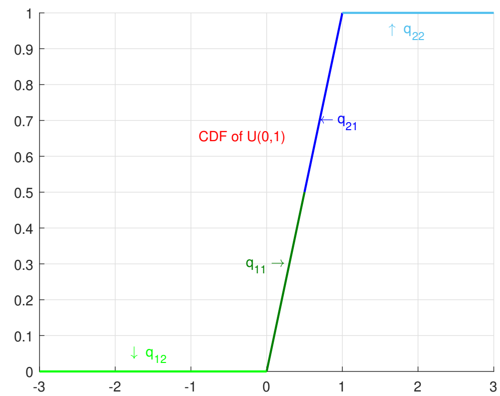


Figure 2: Piecewise linear approximation of the CDF of a uniform distribution

Consequently, using the above piecewise linear approximation of (19c), we obtain a BMI approximation of PSAA-RTM:

$$\begin{aligned} \text{PSAA-RTM-U} \quad \min \quad & c^T \boldsymbol{x} \\ s.t. \quad & A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq -A_1(\boldsymbol{x})z_1^t, \quad \forall t = 1, \ldots, N \quad \text{(29a)} \\ & A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq -A_1(\boldsymbol{x})z_2^t, \quad \forall t = 1, \ldots, N \quad \text{(29b)} \\ & (1c); (16c); (16d); (28a) - (28c). \end{aligned}$$

Theorem 4. *Problem PSAA-RTM-U, i.e., problem* (29)*, is an inner approximation of problem PSAA-RTM when* $\xi_1 \sim U(0,1)$*.*

Proof. To prove that (29) is an inner approximation of (19), it suffices to show that the feasible set defined by (28a)-(28c) is an inner approximation of the feasible area defined by (19c). Indeed, $\Phi(z_2^t)$ is lower-bounded by the piecewise linear concave function

$$\Phi(z_2^t) \geq q_2^t = \min_{k \in \{1,\ldots,K\}} a_k z_2^t + b_k,$$

while $\Phi(z_1^t)$ is upper-bounded by the piecewise linear convex function (see Figure 2)

$$\Phi(z_1^t) \leq q_1^t = \max_{s \in \{1,\ldots,S\}} a_s z_1^t + b_s.$$

Thus,

$$\Phi(z_2^t) - \Phi(z_1^t) \geq q_2^t - q_1^t,$$

which completes the proof. □

It is worth pointing out that when $A_1(\boldsymbol{x})$ is independent of the vector of decision variables $\boldsymbol{x}$, i.e., $A_1(\boldsymbol{x}) = A_1$, PSAA-RTM-U is an SDP problem, which can be efficiently and directly solved by off-the-shelf solvers, such as MOSEK. Note that problem CCLMI-RHS satisfies the condition that $A_1(\boldsymbol{x}) = A_1$, which implies that the corresponding PSAA-RTM-U problem is an SDP one.

4.1.2 PSAA-RTM with Normal Distribution

Using similar piecewise linear approximation as the previous section, we derive an inner approximation of the PSAA-RTM problem for the case of normal distribution. Indeed, when $\xi_1 \sim \mathcal{N}(0,1)$, we use the following slopes and intercepts [16]:

$$a_k = \frac{\Phi(z_2(k+1)) - \Phi(z_2(k))}{z_2(k+1) - z_2(k)},\ b_k = \frac{z_2(k+1)\Phi(z_2(k)) - z_2(k)\Phi(z_2(k+1))}{z_2(k+1) - z_2(k)}, k = 1, \ldots, K-1$$

$$a_s = \frac{\Phi(z_1(s+1)) - \Phi(z_1(s))}{z_1(s+1) - z_1(s)},\ \ b_s = \frac{z_1(s+1)\Phi(z_1(s)) - z_1(s)\Phi(z_1(s+1))}{z_1(s+1) - z_1(s)}, s = 1, \ldots, S-1$$

$$a_K = 0,\ b_K = \Phi(z_2(K)), \quad a_S = 0,\ b_S = \Phi(z_1(S)),$$

where $z_1(S) < z_1(S-1) < \ldots < z_1(1)$ and $z_2(1) < z_2(2) < \ldots < z_2(K)$ are given breakpoints.

Figure 3 uses three green segments to approximate $\Phi(z_1^t)$ and three blue segments to approximate $\Phi(z_2^t)$. It is evident that the larger the number of segments we employ, the more accurate the approximation is and the larger the size of the problem becomes. In our numerical study, we shall use more (than 3) segments to get better approximations. Figure 3 also illustrates how $\Phi(z_1^t)$ can be upper-bounded by a piecewise linear convex function and $\Phi(z_2^t)$ can be lower-bounded by a piecewise linear concave function at a finite number of points when $z_1^t \leq 0$ and $z_2^t \geq 0$.

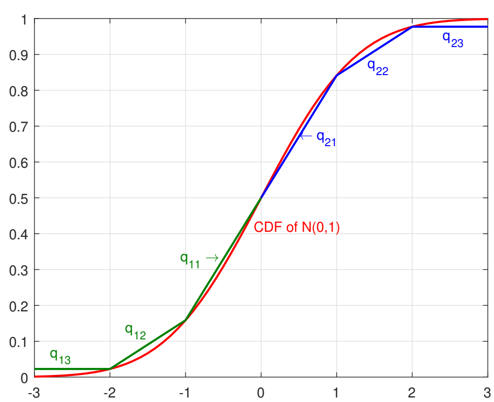
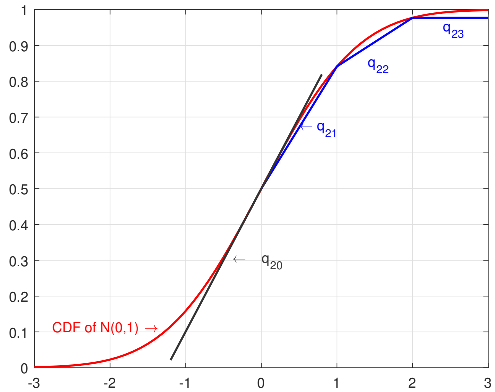
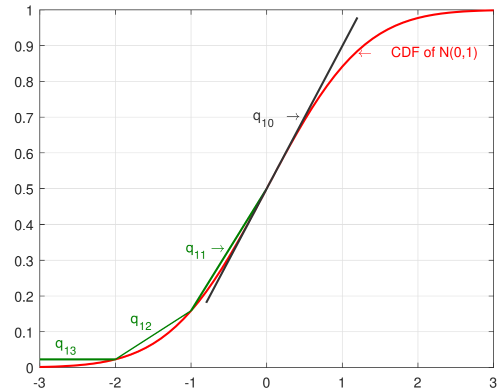


Figure 3: Piecewise linear approximation of the CDF of a standard normal distribution

Note that, if $z_1^t > 0$ (or $z_2^t < 0$), the piecewise linear approximations of $\Phi(z_1^t)$ (or $\Phi(z_2^t)$) will not provide an upper (or lower) bound of the corresponding function. Therefore, we can add two more segments to the previous ones to get better approximations of $\Phi(z_1^t)$



(a) approximation of $\Phi(z_2^t)$



(b) approximation of $\Phi(z_1^t)$

Figure 4: Modified Piecewise linear approximation

and $\Phi(z_2^t)$ when $z_1^t > 0$ and $z_2^t < 0$. To the end of approximating $\Phi(z_2^t)$, we use a piecewise tangent function with one segment for the interval $[-\infty, 0]$ in addition to the K piecewise linear segments placed in the interval $[0, \infty]$. Likewise, in order to approximate $\Phi(z_1^t)$, we use a piecewise tangent function with one segment for the interval $[0, \infty]$. Figure 4 illustrates this approach. The slopes and the intercepts of these extra two segments are:

$$a_{K+1} = \phi(0),\ b_{K+1} = \Phi(0), \qquad a_{S+1} = \phi(0),\ b_{S+1} = \Phi(0),$$

where ϕ is the PDF of the standard normal distribution.

As for the uniform distribution, we introduce the auxiliary variables q_1^t and q_2^t to represent the approximations of $\Phi(z_1^t)$ and $\Phi(z_2^t)$, respectively, and we approximate constraint (19c) with the following inequalities:

$$q_2^t \le a_k z_2^t + b_k, \qquad \forall t = 1, \dots, N, \quad \forall k = 1, \dots, K+1 \tag{30a}$$
$$q_1^t \ge a_s z_1^t + b_s, \qquad \forall t = 1, \dots, N, \quad \forall s = 1, \dots, S+1 \tag{30b}$$
$$(28a).$$

The benefit of adding those extra segments is that the set of constraints (28a);(30a)-(30b) defines an inner convex approximation of (19c) when $\xi_1 \sim \mathcal{N}(0,1)$.

Accordingly, with the above piecewise linear approximation of constraint (19c), we have a BMI approximation of problem PSAA-RTM:

$$\text{PSAA-RTM-N} \quad \min \quad c^T \boldsymbol{x}$$
$$s.t. \quad A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq -A_1(\boldsymbol{x}) z_1^t, \quad \forall t = 1, \dots, N \tag{31a}$$
$$A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq -A_1(\boldsymbol{x}) z_2^t, \quad \forall t = 1, \dots, N \tag{31b}$$
$$(1c); (16c); (16d); (28a); (30a) - (30b).$$

Theorem 5. *Problem PSAA-RTM-N* (31) *is an inner approximation of PSAA-RTM when* $\xi_1 \sim \mathcal{N}(0,1)$.

The proof is analogous to the proof of Thereom 4. Detailed numerical illustrations of the PSAA-RTM-N approximations of CCLMI-RHS and CCLMI-RTM are provided in Appendix A.3 and Appendix B.3, respectively.

It is worth indicating that, when $A_1(\boldsymbol{x})$ is independent of $\boldsymbol{x}$, i.e., $A_1(\boldsymbol{x}) = A_1$, the PSAA-RTM-N formulation is an SDP problem and can be efficiently solved by off-the-shelf solvers. Note also that problem CCLMI-RHS satisfies the condition $A_1(\boldsymbol{x}) = A_1$, and its corresponding PSAA-RTM-N formulation is thus an SDP problem as well. However, if $A_1(\boldsymbol{x})$ depends on $\boldsymbol{x}$, both PSAA-RTM-U (problem (29)) and PSAA-RTM-N (problem (31)) are nonconvex BMI problems and harder to solve. This is what motivates Sections 4.2 and 4.3 in which we respectively: 1) derive SDP reformulations of PSAA-RTM-U and PSAA-RTM-N under the assumption that $A_1(\boldsymbol{x})$ can be represented as a product of an affine function of decision variables by a constant matrix, and 2) design a sequential convex optimization algorithm to solve the BMI problems for the general case of $A_1(\boldsymbol{x})$.

4.2 Convex Reformulations of PSAA-RTM

Consider a special case for CCLMI-RTM in which we have $A_1(\boldsymbol{x}) = \hat{a}_1(\boldsymbol{x})A_1$, with $\hat{a}_1(\boldsymbol{x})$ an affine function of $\boldsymbol{x}$ and A_1 a given symmetric matrix of size $d \times d$:

$$\begin{aligned} \min \quad & c^T\boldsymbol{x} \\ s.t. \quad & \mathbb{P}\left\{A_0(\boldsymbol{x}) + \hat{a}_1(\boldsymbol{x})A_1\xi_1 + \sum_{i=2}^{n} A_i(\boldsymbol{x})\xi_i \succeq 0\right\} \geq p \qquad (32a) \\ & (1c). \end{aligned}$$

The corresponding PSAA-RTM-N (or PSAA-RTM-U) problem is:

$$\begin{aligned} \min \quad & c^T\boldsymbol{x} \\ s.t. \quad & A_0(\boldsymbol{x}) + \hat{a}_1(\boldsymbol{x})A_1 z_1^t + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq 0, \quad \forall t = 1,\ldots,N \qquad (33a) \\ & A_0(\boldsymbol{x}) + \hat{a}_1(\boldsymbol{x})A_1 z_2^t + \sum_{i=2}^{n} A_i(\boldsymbol{x})\hat{\xi}_i^t \succeq 0, \quad \forall t = 1,\ldots,N \qquad (33b) \\ & q_2^t - q_1^t \geq y_t, \quad \forall t = 1,\ldots,N \qquad (33c) \\ & q_2^t \leq a_k z_2^t + b_k, \quad \forall t = 1,\ldots,N, \quad \forall k = 1,\ldots,K+1 \qquad (33d) \\ & q_1^t \geq a_s z_1^t + b_s, \quad \forall t = 1,\ldots,N, \quad \forall s = 1,\ldots,S+1 \qquad (33e) \\ & -\infty \leq z_1^t \leq z_2^t \leq +\infty, \quad \forall t = 1,\ldots,N \qquad (33f) \\ & \frac{\sum_{t=1}^{N} y_t}{N} \geq p, \quad y_t \geq 0, \quad \forall t = 1,\ldots,N \qquad (33g) \\ & (1c), \end{aligned}$$

where we explicitly write down the constraints in problem (31) (i.e., (31a);(31b); (16c); (16d);(28a);(30a)-(30b)) to facilitate the latter reformulation. Theorem 6 proposes an equivalent convex reformulation of problem (33) which does not include any bilinear term.

Theorem 6. *When $\hat{a}_1(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \in X$, problem (33) has the same optimal value as the following SDP problem*

$$
\begin{aligned}
\textit{PSAA-RTM-S} \quad \min \quad & c^T \boldsymbol{x} \\
s.t. \quad & A_0(\boldsymbol{x}) + w_1^t A_1 + \sum_{i=2}^{n} A_i(\boldsymbol{x}) \hat{\xi}_i^t \succeq 0, \ \forall t = 1, \ldots, N && \text{(34a)} \\
& A_0(\boldsymbol{x}) + w_2^t A_1 + \sum_{i=2}^{n} A_i(\boldsymbol{x}) \hat{\xi}_i^t \succeq 0, \ \forall t = 1, \ldots, N && \text{(34b)} \\
& \bar{q}_2^t - \bar{q}_1^t \geq \bar{y}_t, \quad \forall t = 1, \ldots, N && \text{(34c)} \\
& \bar{q}_2^t \leq a_k w_2^t + b_k \hat{a}_1(\boldsymbol{x}), \quad \forall t = 1, \ldots, N, \quad \forall k = 1, \ldots, K+1 && \text{(34d)} \\
& \bar{q}_1^t \geq a_s w_1^t + b_s \hat{a}_1(\boldsymbol{x}), \quad \forall t = 1, \ldots, N, \quad \forall s = 1, \ldots, S+1 && \text{(34e)} \\
& \frac{\sum_{t=1}^{N} \bar{y}_t}{N} \geq \hat{a}_1(\boldsymbol{x}) p, \quad \bar{y}_t \geq 0, \quad \forall t = 1, \ldots, N && \text{(34f)} \\
& -\infty \leq w_1^t \leq w_2^t \leq +\infty, \quad \forall t = 1, \ldots, N && \text{(34g)} \\
& \text{(1c)},
\end{aligned}
$$

where w_1^t, w_2^t, $\bar{q}_1^t$, $\bar{q}_2^t$ and $\bar{y}_t$ are continuous auxiliary variables.

Proof. We first denote the optimal values of problems (33) and (34) by f_1^* and f_2^* respectively. It is easy to see that the optimal value of (34) is a lower bound for the original problem (33). Indeed, if we define the variables w_1^t, w_2^t, $\bar{q}_1^t$, $\bar{q}_2^t$ and $\bar{y}_t$ as

$$
w_1^t = \hat{a}_1(\boldsymbol{x}) z_1^t, \ w_2^t = \hat{a}_1(\boldsymbol{x}) z_2^t, \ \bar{q}_1^t = \hat{a}_1(\boldsymbol{x}) q_1^t, \bar{q}_2^t = \hat{a}_1(\boldsymbol{x}) q_2^t, \bar{y}_t = \hat{a}_1(\boldsymbol{x}) y_t \ , \tag{35}
$$

problem (34) becomes equivalent to problem (33). So without having constraint (35), problem (34) is a relaxation of (33) and thus $f_2^* \leq f_1^*$.

Conversely, we denote an optimal solution of problem (34) by $(\boldsymbol{x}^*, \bar{y}_t^*, {w_1^t}^*, {w_2^t}^*, \bar{q}_1^{t*}, \bar{q}_2^{t*})$. Then, we define the following variables

$$
z_1^t = \frac{{w_1^t}^*}{\hat{a}_1(\boldsymbol{x}^*)}, \ z_2^t = \frac{{w_2^t}^*}{\hat{a}_1(\boldsymbol{x}^*)}, \ y_t = \frac{\bar{y}_t^*}{\hat{a}_1(\boldsymbol{x}^*)} \geq 0, q_1^t = \frac{\bar{q}_1^{t*}}{\hat{a}_1(\boldsymbol{x}^*)}, \ q_2^t = \frac{\bar{q}_2^{t*}}{\hat{a}_1(\boldsymbol{x}^*)}, \Longrightarrow
$$

$$
{w_1^t}^* = z_1^t \hat{a}_1(\boldsymbol{x}^*), \ {w_2^t}^* = z_2^t \hat{a}_1(\boldsymbol{x}^*), \ \bar{y}_t^* = y_t \hat{a}_1(\boldsymbol{x}^*), \ \bar{q}_1^{t*} = q_1^t \hat{a}_1(\boldsymbol{x}^*), \bar{q}_2^{t*} = q_2^t \hat{a}_1(\boldsymbol{x}^*).
$$

By replacing $(\bar{y}_t^*, {w_1^t}^*, {w_2^t}^*, \bar{q}_1^{t*}, \bar{q}_2^{t*})$ with the variables defined in constraints (34a)-(34g) and then dividing $\hat{a}_1(\boldsymbol{x}^*)$ in constraints (34c)-(34g), we can conclude that $(\boldsymbol{x}^*, z_1^t, z_2^t, q_1^t, q_2^t, y_t)$ is feasible for problem (33). Thus, we have $f_2^* \geq f_1^*$, which concludes the proof. □

The above approach is illustrated with a numerical example in Appendix C.3.

4.3 Sequential Convex Optimization Algorithm for PSAA-RTM

In general, both PSAA-RTM-U (29) and PSAA-RTM-N (31) are nonconvex BMI problems. However, when the values of z_1^t and z_2^t are fixed, such as $z_1^t = z_1^{t*}$ and $z_2^t = z_2^{t*}$, they become convex and tractable as follows:

$$\begin{aligned}
\min \quad & c^T \boldsymbol{x} \\
s.t. \quad & A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\xi_i^t + A_1(\boldsymbol{x}) z_1^{t*} \succeq 0, \quad \forall t = 1, \dots, N && \text{(36a)} \\
& A_0(\boldsymbol{x}) + \sum_{i=2}^{n} A_i(\boldsymbol{x})\xi_i^t + A_1(\boldsymbol{x}) z_2^{t*} \succeq 0, \quad \forall t = 1, \dots, N && \text{(36b)} \\
& \text{(1c)}.
\end{aligned}$$

In the remainder of this section, we present an algorithm, inspired by [15, 36], which repeatedly solves problem (36) while systematically improving the variables z_1^t and z_2^t. The main idea of this approach, which is called *Sequential Convex Optimization Method*, is to enlarge the feasible region of the constraints (19a) and (19b) over z_1^t and z_2^t and to improve the objective value of problem (36), until a termination criterion is met. To this end, we introduce an SDP problem which depends parametrically on $\boldsymbol{x} = \boldsymbol{x}^*$.

$$\begin{aligned}
\max \quad & \lambda && \text{(37a)} \\
s.t. \quad & A_0(\boldsymbol{x}^*) + \sum_{i=2}^{n} A_i(\boldsymbol{x}^*)\xi_i^t + A_1(\boldsymbol{x}^*) z_1^t \succeq \lambda I, \quad \forall t = 1, \dots, N && \text{(37b)} \\
& A_0(\boldsymbol{x}^*) + \sum_{i=2}^{n} A_i(\boldsymbol{x}^*)\xi_i^t + A_1(\boldsymbol{x}^*) z_2^t \succeq \lambda I, \quad \forall t = 1, \dots, N && \text{(37c)} \\
& \text{(1c); (16c); (16d); (33c)} - \text{(33g)}.
\end{aligned}$$

Hereafter, we refer to problem (36) as the "master problem" and problem (37) as its "subproblem". The termination criterion is set as:

$$|f^*(s') - f^*(s'-1)| < \delta,$$

where s' denotes the number of iteration, $f^*(s')$ denotes the optimal value of the master problem at iteration s', and δ is a tolerance parameter set by the user. A numerical illustration of this method is given in Appendix D.

5 Numerical results

This section is decomposed into three subsections. Section 5.1 considers CCLMI-RHS and the application of choice is a truss topology problem. Section 5.2 is devoted to CCLMI-RTM and a study of a calibration problem with normally distributed random variables. Section 5.3 considers CCLMI-RTM and a control theory problem with uniformly distributed random variables.

The numerical tests assess the computational efficiency and exactness of the proposed PSAA-based reformulation and solution framework described in Section 4 for the aforementioned CCLMI formulations. We consider the SA and the SAA approaches presented in Section 2 as the benchmarks. The following three criteria are used to evaluate and benchmark the proposed framework: the optimal objective value, the CPU time, and the empirical probability of attaining the prescribed probability level. In order to do so, we generate 100,000 new, testing (i.e., unused in the solution of the optimization problems) samples. The empirical probability is defined as the proportion of testing samples for which the optimal solution x^* calculated with one of the tested formulations is feasible. All problem formulations are modeled with the YALMIP 3 Matlab package. The optimization problems are solved with MOSEK solver on a machine with Intel(R) Core(TM) $i7 - 4600U$ @ 3.40 GHz and 16.0 GB RAM.

Note that in order to carry out the numerical experiments, we develop a new chance-constrained formulation for the truss topology problem which, using Schur's complement converts the original CCLMI-RTM problem into a CCLMI-RHS problem.

5.1 CCLMI with RHS - truss topology problem

In this section, we consider a truss topology design (TTD) problem which has been analyzed in several earlier SDP studies [5, 6, 8]. In particular, we derive a new formulation of the TTD problem which transforms the original CCLMI-RTM problem into a CCLMI-RHS problem by using the Schur's complement concept. Consequently, its corresponding PSAA-based formulation will be an SDP problem rather than a BMI problem.

A *truss* is a mechanical structure consisting of thin elastic *bars* linked to each other at *nodes*. In a simple TTD problem, a finite 2D or 3D nodal set, a list of allowed pair connections of nodes by bars and an external load are given. The purpose is to assign the tentative bars weights in order to get a truss that is most rigid against the loads. The mathematical model of a TTD problem can be represented by the following SDP problem:

$$\min_{\tau,t} \left\{ \tau : \left[\begin{array}{c|c} 2\tau & h^T \\ \hline h & \sum_{i=1}^m t_i b_i b_i^T \end{array} \right] \succeq 0, \quad t \geq 0, \quad \sum_{i=1}^m t_i = 1 \right\}, \tag{38}$$

where τ is (an upper bound on) the *compliance* which is a measure of truss' rigidity (the less the compliance, the more rigid the truss will be), n is the total degrees of freedom of the nodes, and m is the number of tentative bars. Each variable $t_i \in \mathbb{R}$ represents the weight of bar i, $h \in \mathbb{R}^n$ is a vector of given external loads (i.e., the forces applied at the nodes), and the vector $b_i \in \mathbb{R}^n$ is a parameter given by the geometry of the nodal set.

5.1.1 CCLMI formulation with RTM

In reality, the truss is subjected not only to the primary load h but also to occasional relatively small secondary loads, affecting the nodes used by the construction. The designed truss must tolerate these loads as well; otherwise, it can be crushed by a very small occasional load. A natural way to design a more robust truss is to reformulate the TTD problem such that it can carry occasional random loads as well. Specifically, we assume that the occasional loads follow a normal distribution $\mathcal{N}(0, \bar{\rho}^2\Sigma)$, where $\bar{\rho}$ is an uncertainty level and Σ is an $n \times n$ covariance matrix. Accordingly, we have the following CCLMI formulation:

$$\max_{\bar{\rho},t} \left\{ \bar{\rho} : \begin{array}{c} \overbrace{\left[\begin{array}{c|c} 2\hat{\tau} & h^T \\ \hline h & \sum_{i=1}^m t_i b_i b_i^T \end{array}\right]}^{A(t)} \succeq 0, \quad t \geq 0, \quad \sum_{i=1}^m t_i = 1 \\ \mathbb{P}_{\boldsymbol{\zeta} \sim \mathcal{N}(0,\Sigma)} \left\{ \underbrace{\left[\begin{array}{c|c} 2\hat{\tau} & \bar{\rho}\boldsymbol{\zeta}^T \\ \hline \bar{\rho}\boldsymbol{\zeta} & \sum_{i=1}^m t_i b_i b_i^T \end{array}\right]}_{\mathscr{A}'_0[t] + \bar{\rho}\sum_{l=1}^n \zeta_l \mathscr{A}'_l[t]} \succeq 0 \right\} \geq p \end{array} \right\}, \tag{39}$$

where $\hat{\tau} > 0$ is a given parameter which is slightly greater than the optimal value of the simple TTD problem (38), denoted by τ_* (In the latter numerical test, we choose $\hat{\tau} = 1.025\tau_*$). Therefore, we are looking for a truss which is capable of tolerating equally well "nearly all" (up to probability p) random occasional loads in the form of $\bar{\rho}\boldsymbol{\zeta}$, where the n-dimensional random variable $\boldsymbol{\zeta}$ follows a normal distribution $\boldsymbol{\zeta} \sim \mathcal{N}(0, \Sigma)$. In other words, we intend to maximize $\bar{\rho}$ which is a measure of the rigidity of the truss with respect to occasional loads. Problem (39) is a CCLMI problem with random technology matrix.

5.1.2 CCLMI formulation with RHS

In the following theorem, we use the concept of Schur complement to derive an equivalent reformulation of problem (39). The reformulation takes the form of a CCLMI with random right-hand side, while the CCLMI problem (39) has a random technology matrix.

Theorem 7. *The CCLMI problem with random technology matrix* (39) *is equivalent to the CCLMI problem with random right-hand side*

$$\min_{\rho,t} \left\{ \rho : \begin{array}{c} \overbrace{\left[\begin{array}{c|c} 2\hat{\tau} & h^T \\ \hline h & \sum_{i=1}^m t_i b_i b_i^T \end{array}\right]}^{A(t)} \succeq 0, \quad t \geq 0, \quad \sum_i t_i = 1 \\ \mathbb{P}_{\boldsymbol{\zeta} \sim \mathcal{N}(0,\Sigma)} \left\{ \underbrace{\left[\begin{array}{c|c} 2\hat{\tau}\rho & \boldsymbol{\zeta}^T \\ \hline \boldsymbol{\zeta} & \sum_{i=1}^m t_i b_i b_i^T \end{array}\right]}_{\mathscr{A}_0[t,\rho] + \sum_{l=1}^n \zeta_l \mathscr{A}_l[t]} \succeq 0 \right\} \geq p \end{array} \right\}, \tag{40}$$

where ρ is an auxiliary variable defined as $\rho = \frac{1}{\bar{\rho}^2}$ and the objective is to minimize ρ instead of maximizing $\bar{\rho}$.

Proof. Consider the following positive semidefinite matrix inside the chance constraint (39).

$$\left[\begin{array}{c|c} 2\hat{\tau} & \bar{\rho}\boldsymbol{\zeta}^T \\ \hline \bar{\rho}\boldsymbol{\zeta} & \sum_{i=1}^m t_i b_i b_i^T \end{array}\right] \succeq 0, \tag{41}$$

Using Schur complement, we have

$$\sum_{i=1}^m t_i b_i b_i^T \succeq \frac{\bar{\rho}^2 \boldsymbol{\zeta}\boldsymbol{\zeta}^T}{2\hat{\tau}} = \frac{\boldsymbol{\zeta}\boldsymbol{\zeta}^T}{\frac{2\hat{\tau}}{\bar{\rho}^2}} \tag{42}$$

which is equivalent to

$$\left[\begin{array}{c|c} \frac{2\hat{\tau}}{\bar{\rho}^2} & \boldsymbol{\zeta}^T \\ \hline \boldsymbol{\zeta} & \sum_{i=1}^m t_i b_i b_i^T \end{array}\right] \succeq 0. \tag{43}$$

Then, introducing an auxiliary variable ρ and substituting $\frac{1}{\bar{\rho}^2}$ in (43) with ρ, we have

$$\left[\begin{array}{c|c} 2\hat{\tau}\rho & \boldsymbol{\zeta}^T \\ \hline \boldsymbol{\zeta} & \sum_{i=1}^m t_i b_i b_i^T \end{array}\right] \succeq 0. \tag{44}$$

Therefore, the maximization problem (39) which is a CCLMI-RTM problem is equivalent to the minimization problem (40) which is a CCLMI-RHS problem. □

5.1.3 Experimental setup and numerical results

Ben-Tal and Nemirovski [8] employed a standard normal distribution $\mathcal{N}(0, I)$ for the random vector $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_n)$ in the TTD problem. We consider a setting in which $\boldsymbol{\zeta}$ follows a more general normal distribution $\mathcal{N}(0, \Sigma)$. The dimension of $\boldsymbol{\zeta}$ is considered to be $n = 20$ and the parameter m is set to be $m = 54$. The standard deviation of $\boldsymbol{\zeta}$ is randomly generated on the interval $[0, 2]$ while the correlation matrix is generated randomly using the MATLAB function "*gallery('randcorr',n)*". Additionally, we also consider specific structured covariance matrices in our tests. With the randomly generated covariance matrix Σ, we replace its i-th largest eigenvalue by three different types of generating functions: The first one is constant and equal to 1 for $i = 1, \ldots, n$; the second one is linear and equal to $1 - 0.5\frac{i-1}{n-1}$ for $i = 1, \ldots, n$; the third one is exponential as $1 - {(e^{\frac{-i+n+1}{n}\gamma})}/{(e^{\gamma}-1)}$ for $i = 1, \ldots, n$, with slope γ. Herein, we consider four different slopes $\gamma \in \{0.1, 1, 5, 15\}$ for the exponential generating function. In other words, we consider three cases for eigenvalue decay of Σ: no decay, linear decay and exponential decay (with four different rates γ). The six different eigenvalue generation functions are displayed in

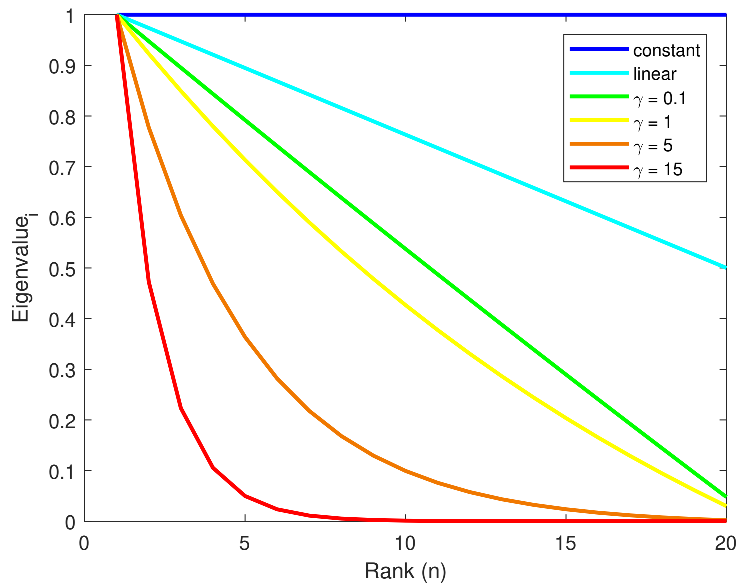


Figure 5: Different eigenvalue generating functions used in the experiments

Figure 5. One can observe that for the exponential generation functions, the eigenvalues decay rapidly by increasing γ.

Tables 1-6 compare the performances of the PSAA-RHS approximation with two different approximation approaches existing in the literature i.e., SA-RHS, and SAA-RHS. In our experiments, we consider various numbers of scenarios N, including $N^* = 2253$ obtained from Theorem 1 using $\beta = 0.05$. Moreover, we choose ten segments for the piecewise linear approximations in the interval $[-4, 4]$. The reliability parameter is set to $p = 95\%$ in all of the experiments, and we use $100,000$ testing samples to calculate the empirical probability. We use ten randomly generated datasets for the experiments.

We report in Tables 1-6 the averages (over the 10 datasets) of the metrics for each compared method or formulation. Each table reports the statistics of the solution corresponding to different covariance matrices generated using different eigenvalue generation functions. The sign "*" in the tables indicates that the corresponding method could not provide any feasible solution in five hours. We denote by "obj" the average optimal objective value, by "time(sec)" the average CPU time, by "Pr(%)" the average empirical probability, by "gap" the difference between the average empirical probability and the prescribed probability level (i.e., gap(%) $= \text{Pr} - p$), by "frac" the proportion of instances in which the prescribed probability level p is not achieved, and by "imp(%)" the average relative improvement of the optimal values of SAA-RHS and PSAA-RHS in comparison to SA-RHS. In other words, we define imp $= \frac{\text{obj}_0 - \text{obj}_j}{\text{obj}_0} \times 100\%, \quad \forall j = 1, 2$, where obj_0, obj_1, and obj_2 are the optimal objective values of the SA-RHS, SAA-RHS, and PSAA-RHS problems, respectively. The quality of the optimal solution can be measured by obj (the smaller, the better) or imp(%) (the larger, the better), and the tightness of the approximation can be evaluated by the defined gap (%) and frac.

Table 1: *Performance of SA-RHS, SAA-RHS, and PSAA-RHS approximations, when the eigenvalue generation function is exponential with rate* $\gamma = 15$.

$p = 95\%$	SA-RHS					SAA-RHS						PSAA-RHS					
N	obj	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac
500	3.36	40.94	99.06	4.06	0	2.17	35.49	12268.59	92.89	-2.11	1	2.11	37.26	100.65	95.26	0.26	0.2
1000	3.70	85.16	99.39	4.39	0	*	*	*	*	*	*	2.20	40.46	228.47	95.36	0.36	0
2000	4.11	194.07	99.73	4.73	0	*	*	*	*	*	*	2.37	42.43	586.10	95.50	0.50	0
2253	4.74	281.99	99.88	4.88	0	*	*	*	*	*	*	2.38	49.79	768.98	95.62	0.62	0

Table 1 reports the average value of the considered metrics (i.e., obj, imp(%), time(sec), Pr(%), gap(%) and frac) for the exponential generation function with $\gamma = 15$ when the number N of scenarios ranges from 500 to 2253. Table 1 highlights the superiority of the PSAA-RHS formulation as it not only results in better optimal solutions for each value of N, but also provides the tightest approximations (compared to the SA-RHS and SAA-RHS formulations) for all cases. Consider for example $N = 2253$. Compared to SA-RHS, the PSAA-RHS formulation improves the optimal value by almost 50%. Table 1 also emphasizes the remarkable tightness of the PSAA-RHS approximation for all sizes N of the scenario set, with an average gap of at most 0.62%. Moreover, across the ten generated instances, we observe that SAA-RHS never reaches the prescribed probability level 95%, while, in contrast, it is only for a small portion of the instances that the 95% reliability level is not achieved with the PSAA-RHS formulation. For example, for $N = 500$, the prescribed probability level is not attained in only 2 of the 10 instances. Moreover, in these two cases, the empirical probabilities are very close, i.e., equal to 94.63% and 94.82%, to the targeted 95% level. It can also be seen that, as the number of samples N grows (i.e., $N \to N^*$), the proportion of problem instances in which the reliability level is not attained decreases (i.e., frac $\to 0$) with the PSAA-RHS formulation. More specifically, by increasing N, the solutions obtained with PSAA-RHS are not only above the desired probability level p for all instances, but they are also very close to p. As an illustration, for $N = 2253$, the average empirical probability with PSAA-RHS is 95.36%. However, the solution obtained with SA-RHS is such that the empirical probability amounts on average to 99.88% and is overly conservative.

One can also observe that the SA-RHS method is the fastest approach but also the most conservative one as indicated by the average optimal value and the average gap. For instance, for $N = 500$, the optimal solution of SA-RHS is about 35% more expensive than those obtained with the PSAA-RHS and SAA-RHS formulations.

The SAA-RHS approach is the slowest method and unable to provide any feasible solution for moderate to large number of scenarios. For instance, for $N = 500$, more than 3 hours are needed to solve the SAA-RHS problem. The computational time of the SAA-RHS increases exponentially as the number of scenario increases, which explains why we do not have any solution within 5 hours for large numbers of scenarios. The root cause for this well-known issue is that the SAA-RHS approach requires the solution of a large-scale MISDP problem with N binary variables and whose continuous relaxation is typically loose. Even more problematic, the obtained solution is not feasible for the true problem (40), since its empirical probability is 92.89% which is below the prescribed 95% level. By contrast, the PSAA-RHS formulation solves problems with 500 scenarios in 101 seconds on average. As shown in Table 1, the computational time with the PSAA-RHS formulation increases linearly with the number of samples. Finally, note that despite being under the prescribed probability level (i.e, thus failing to meet the minimal requirements), the objective value of SAA-RHS is larger (lower quality) than that of PSAA-RHS.

Table 2: *Performance of SA-RHS, SAA-RHS, and PSAA-RHS approximations, when the eigenvalue generation function is exponential with rate* $\gamma = 5$.

$p = 95\%$	SA-RHS					SAA-RHS						PSAA-RHS					
N	obj	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac
500	6.76	37.87	98.38	3.38	0	5.43	19.69	15641.01	93.45	-1.55	1	5.40	20.05	102.64	95.26	0.26	0.4
1000	7.21	74.92	99.08	4.08	0	*	*	*	*	*	*	5.57	22.68	231.43	95.31	0.31	0.3
2000	8.43	189.16	99.62	4.62	0	*	*	*	*	*	*	6.10	27.63	635.23	95.58	0.58	0.1
2253	8.66	207.26	99.70	4.70	0	*	*	*	*	*	*	6.15	29.98	794.35	95.63	0.63	0.1

Table 3: *Performance of SA-RHS, SAA-RHS, and PSAA-RHS approximations, when the eigenvalue generation function is exponential with rate* $\gamma = 1$.

$p = 95\%$	SA-RHS					SAA-RHS						PSAA-RHS					
N	obj	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac
500	12.93	35.37	97.84	2.84	0	11.58	10.43	16084.07	92.47	-2.53	1	11.14	13.86	97.79	95.26	0.26	0.2
1000	13.68	76.93	98.84	3.84	0	*	*	*	*	*	*	11.50	11.04	220.79	95.52	0.52	0.2
2000	15.42	172.23	99.45	4.45	0	*	*	*	*	*	*	10.49	18.86	590.15	95.79	0.79	0
2253	15.99	192.08	99.46	4.46	0	*	*	*	*	*	*	14.13	11.63	604.88	96.18	1.18	0

Table 4: *Performance of SA-RHS, SAA-RHS, and PSAA-RHS approximations, when the eigenvalue generation function is exponential with rate* $\gamma = 0.1$.

$p = 95\%$	SA-RHS					SAA-RHS						PSAA-RHS					
N	obj	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac
500	14.02	34.41	97.88	2.88	0	11.44	18.37	15807.10	93.03	-1.97	1	12.78	8.84	98.81	95.27	0.27	0.4
1000	15.17	70.90	98.80	3.80	0	*	*	*	*	*	*	13.62	10.22	232.28	95.67	0.67	0.2
2000	16.62	175.95	99.36	4.36	0	*	*	*	*	*	*	14.27	14.15	532.04	96.58	1.58	0
2253	16.78	194.71	99.51	4.51	0	*	*	*	*	*	*	13.57	19.13	600.54	96.61	1.61	0

The results observed in Table 1 with the rate $\gamma = 15$ generalize to the other rates ($\gamma = 5, 1, 0.1$) as shown in Tables 2-4. The only minor difference is in Table 4 in which the optimal value obtained with SAA-RHS is lower than that obtained with PSAA-RHS. This is not really surprising given that the SAA-RHS solution does not reach the prescribed reliability level 95% and is therefore not feasible for the original CCLMI-RHS problem, whereas the PSAA-RHS formulation permits to attain the desired reliability level. Again, PSAA-RHS clearly outperforms SA-RHS and SAA-RHS.

Table 5: *Performance of SA-RHS, SAA-RHS, and PSAA-RHS approximations, when the eigenvalue generation function is linear.*

$p = 95\%$	SA-RHS					SAA-RHS						PSAA-RHS					
N	obj	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac
500	19.75	32.50	97.20	2.20	0	13.78	30.21	11278.69	92.01	-2.99	1	17.58	10.99	82.83	94.52	-0.48	0.9
1000	21.16	59.41	98.52	3.52	0	*	*	*	*	*	*	19.02	10.10	201.91	95.60	0.60	0.3
2000	23.53	154.23	99.22	4.22	0	*	*	*	*	*	*	20.85	11.38	568.47	96.79	1.79	0.1
2253	23.89	198.46	99.41	4.41	0	*	*	*	*	*	*	21.14	11.16	594.07	97.02	2.02	0

Table 6: *Performance of SA-RHS, SAA-RHS, and PSAA-RHS approximations, when the eigenvalue generation function is constant.*

$p = 95\%$	SA-RHS					SAA-RHS						PSAA-RHS					
N	obj	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac
500	25.07	35.08	97.44	2.44	0	20.62	17.75	14889.43	92.55	-2.45	1	22.72	9.37	103.51	95.08	0.08	0.6
1000	26.56	73.36	98.54	3.54	0	*	*	*	*	*	*	24.60	7.39	259.59	95.98	0.98	0.2
2000	30.03	196.64	99.39	4.39	0	*	*	*	*	*	*	26.96	10.23	630.87	97.08	2.08	0
2253	31.46	204.49	99.44	4.44	0	*	*	*	*	*	*	28.34	9.92	686.49	97.59	2.59	0

The results in Tables 5-6 for linear and constant eigenvalue generation functions provide a further confirmation of the results presented above. One can observe that the performance of the PSAA-RHS formulation is better for the exponential generating functions than for the linear and constant generating functions. For instance, Table 5 shows that for $N = 500$ the average empirical probability obtained with PSAA-RHS is slightly lower than 95% (i.e., gap =-0.48 %) but still superior to the one obtained with the SAA-RHS approach for which the gap is -2.99%. In addition, the proportion of instances that does not achieve the prescribed probability level is smaller for PSAA-RHS with exponential functions than with linear and constant functions. This can be explained by the structure of the n-dimensional random variable $\boldsymbol{\zeta} = U\Lambda^{1/2}\boldsymbol{\xi} = U\Lambda^{1/2}(\xi_1, \ldots, \xi_n)$, where the columns of U are unit eigenvectors of Σ and Λ is a diagonal matrix of the eigenvalues. Indeed, PSAA-RHS approximates the CDF function of ξ_1, and it samples the remaining random variables $\xi_2, \ldots, \xi_n$. Compared with the exponential function, a larger proportion of variance is allocated to $\xi_2, \ldots, \xi_n$, which requires more samples to achieve the prescribed probability level. Therefore, for the same number of samples, one can expect better solutions for the exponential functions than for linear and constant functions.

5.2 CCLMI with RTM - calibration problem

We analyze the benefits of the convex PSAA reformulations proposed in Section 4.2 for the CCLMI-RTM problem. The numerical tests are conducted on a new variant of a calibration problem (Ben-Tal and Nemirovski [8]), taking the following form

$$\rho^* = \max\left\{\sum_{l=1}^n \rho_l : \mathbb{P}\left\{-A_0 \preceq \sum_{l=1}^n \rho_l \xi_l A_l \preceq A_0\right\} \geq p\right\}, \tag{45}$$

where $\boldsymbol{\rho} \in \mathbb{R}^n$ is a decision vector, $\boldsymbol{\xi}$ is an n-dimensional random variable, and $A_0, \ldots, A_n$ are given symmetric $d \times d$ matrices such that for a given $\vartheta^* > 0$,

$$\text{Arrow}(\vartheta^* A_0, A_1, \ldots, A_n) \succeq 0,$$

where

$$\text{Arrow}(\vartheta^* A_0, A_1, \ldots, A_n) = \begin{bmatrix} \vartheta^* A_0 & A_1 & \ldots & A_n \\ A_1 & \vartheta^* A_0 & & \\ \vdots & & \ddots & \\ A_n & & & \vartheta^* A_0 \end{bmatrix}.$$

5.2.1 Experimental setup and numerical results

In the following numerical experiments, we consider three different sizes for the matrices $A_0, A_1, \ldots, A_n$ by setting $d = n \in \{10, 20, 30\}$. Similarly to Ben-Tal and Nemirovski [8], we set $A_0 = I_d$ wherein I_d is the identity matrix of size $d \times d$ and we randomly generate symmetric matrices $A_1, \ldots, A_n$. We then scale the generated matrices to ensure that $\text{Arrow}(\vartheta I_d, A_1, \ldots, A_n) \succeq 0$ if and only if $\vartheta \geq 1$; therefore, the input to the calibration procedure is the collection $A_0 = I_d, A_1, \ldots, A_n,\ \vartheta^* = 1$. We assume that the random vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ follows a standard normal distribution $\mathcal{N}(0, I)$. The rest of the notations and criteria are the same as those in Section (5.1.3). In our experiments, we consider eight segments for the piecewise linear approximations in the interval $[-4, 4]$. We impose a 2-hour time limit. The sign "*" in the tables indicates the corresponding method did not provide any feasible solution in two hours. The notation "$**$" indicates that the corresponding method reaches YALMIP's default stopping criterion (300 iterations), in which case we report the best solution found up to that stage.

Tables 7-9 compare the performances of the PSAA-RTM formulation with the two benchmarks, i.e., SA-RTM, and SAA-RTM, for different problem sizes (i.e., d, n) and numbers of scenarios N. The reported results are averaged over ten instances for each size.

Table 7: *Performance of SA-RTM, SAA-RTM, and PSAA-RTM approximations:* $d = n = 10$.

$p = 95\%$	SA-RHS					SAA-RHS						PSAA-RHS					
N	obj	time(*sec*)	Pr(%)	gap(%)	frac	obj	imp(%)	time(*sec*)	Pr(%)	gap(%)	frac	obj	imp(%)	time(*sec*)	Pr(%)	gap(%)	frac
520	5.23	0.95	98.71	3.71	0	6.19	18.27	1470.53	95.30**	0.30	0.4	6.53	24.94	6.17	94.96	-0.04	0.5
1000	4.78	1.71	99.36	4.36	0	5.73	19.82	2230.43	97.20**	2.20	0	6.39	33.62	11.06	95.39	0.39	0.2
2000	4.47	4.70	99.71	4.71	0	5.33	19.27	5264.72	98.36**	3.36	0	6.16	37.87	25.83	95.58	0.58	0

"**" incumbent solutions

Table 7 reports the results for the SA-RTM, SAA-RTM, and PSAA-RTM approximations with $d = n = 10$ and $N = 520$ (value obtained for N^* in Theorem 1 with $\beta = 0.05$), 1000, and 2000. The results in Table 7 demonstrate that the PSAA-RTM approach outperforms the other two approximations in terms of both solution quality and computational time. First, PSAA-RTM provides less conservative solutions than SA-RTM across all problem instances. In other words, the optimal value of PSAA-RTM is better than the one of SA-RTM. For instance, when $N = 1000$, PSAA-RTM improves the objective value of SA-RTM by more than 33% on average. Furthermore, the empirical probability of the PSAA-RTM solution is very close to the prescribed probability level $p = 95\%$ for all instances, whereas the empirical probability obtained with SA-RTM is exceedingly high and close to 99%, which confirms the conservatism of this method. Second, the PSAA-RTM approximation problem can be solved significantly faster than the SAA-RTM approximation problem. For instance, the CPU time with PSAA-RTM is on average more than 100 times smaller than that with SAA-RTM.

Table 8 reports the results for $d = n = 20$ and N=920 (value obtained from Theorem 1), 1000, and 2000 while Table 9 provides the same results for $d = n = 30$ and N=1000, 1320 (value obtained from Theorem 1), and 2000.

Table 8: *Performance of SA-RTM, SAA-RTM, and PSAA-RTM approximations: $d = n = 20$.*

$p = 95\%$	SA-RHS					SAA-RHS						PSAA-RHS					
N	obj	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac
920	8.58	8.89	98.09	3.09	0	*	*	*	*	*	*	10.80	25.93	54.46	94.95	-0.05	0.5
1000	8.46	10.83	98.17	3.17	0	*	*	*	*	*	*	10.72	26.73	78.84	95.05	0.05	0
2000	7.96	22.14	99.08	4.08	0	*	*	*	*	*	*	10.53	32.26	114.97	95.17	0.17	0

Table 9: *Performance of SA-RTM, SAA-RTM, and PSAA-RTM approximations: $d = n = 30$.*

$p = 95\%$	SA-RHS					SAA-RHS						PSAA-RHS					
N	obj	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac
1000	13.59	32.27	96.92	1.92	0	*	*	*	*	*	*	15.68	15.41	184.50	94.24	-0.76	1
1320	13.30	44.61	97.49	2.49	0	*	*	*	*	*	*	15.59	17.19	265.50	94.37	-0.63	1
2000	12.58	81.05	98.41	3.41	0	*	*	*	*	*	*	15.34	21.92	403.62	94.95	-0.05	0.6

The comments on the results displayed in Table 7 extend to Tables 8-9 and attest the superiority of PSAA-RTM over the SA-RTM and SAA-RTM approximation methods. It is also worth noting that SAA-RTM fails to solve any of the instances in two hours, whilst the PSAA-RTM formulation can solve the corresponding problem of largest size, i.e., $N = 2000$, and $d, n = 30$, in 404 seconds.

5.3 CCLMI with RTM - control problem

In this section, we consider a state-feedback stabilization problem in control theory. As an alternative to robust control [2,11], we focus on a probabilistically robust control which requires that a system be stable with high probability. This application is used to assess the proposed PSAA-RTM approach for solving the general CCLMI-RTM problems. More precisely, we aim at evaluating the PSAA-RTM-U formulation derived in Section 4.1.1 and taking the form of a BMI problem. We solve it by using the sequential convex optimization algorithm presented in Section 4.3.

5.3.1 State-feedback stabilization

Consider the class of uncertain linear systems described by

$$\dot{\mathbf{x}} = A(\boldsymbol{\xi})\mathbf{x} + B(\boldsymbol{\xi})\mathbf{u}, \tag{46}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state variable, A and B are $n \times n$ and $n \times m$ matrix functions of $\boldsymbol{\xi} \in \Xi$, respectively. $\boldsymbol{\xi} \in \mathbb{R}^r$ is an uncertain parameter, and the control input $\mathbf{u} \in \mathbb{R}^m$ is chosen using the state feedback control law

$$\mathbf{u}(t) = K\mathbf{x}, \tag{47}$$

where $K \in \mathbb{R}^{m \times n}$ is the feedback gain matrix. The uncertain closed-loop system (46) is robustly stable if and only if $A_{cl}(\boldsymbol{\xi}) = A(\boldsymbol{\xi}) + B(\boldsymbol{\xi})K$ is Hurwitz (i.e., all eigenvalues of A_{cl} have strictly negative real parts) for all the admissible $\boldsymbol{\xi} \in \Xi$ [11].

Using the enhanced LMI characterization proposed in [2], one can conclude that the uncertain linear system (46) is robustly stable if there exists a Lyapunov symmetric matrix function $P(\boldsymbol{\xi}) \in \mathbb{R}^{n\times n}$, a matrix $V \in \mathbb{R}^{n\times n}$, and a matrix $R \in \mathbb{R}^{m\times n}$, such that

$$\Pi(\boldsymbol{\xi}) := \begin{bmatrix} -(V+V^T) & \mathcal{M}(\boldsymbol{\xi}) & V^T \\ \mathcal{M}^T(\boldsymbol{\xi}) & -P(\boldsymbol{\xi}) & 0 \\ V & 0 & -P(\boldsymbol{\xi}) \end{bmatrix} \prec 0 \tag{48}$$

for all $\boldsymbol{\xi} \in \Xi$ and with

$$\mathcal{M}(\boldsymbol{\xi}) = V^T A^T(\boldsymbol{\xi}) + R^T B^T(\boldsymbol{\xi}) + P(\boldsymbol{\xi}).$$

Therefore, the feedback gain matrix is recovered as $K = RV^{-1}$, if a feasible solution to (48) is found. Subsequently, finding a feasible solution to (48), which is a sufficient condition for robust stability, can be achieved by solving the following robust convex optimization problem [11].

$$\begin{aligned} \min \quad & \alpha \\ s.t. \quad & -I \preceq \Pi(\boldsymbol{\xi}) \preceq \alpha I \ , \quad \forall \boldsymbol{\xi} \in \Xi. \end{aligned} \tag{49}$$

Accordingly, the corresponding probabilistic-robust control for the linear system, can be expressed as:

$$\begin{aligned} \min \quad & \alpha \\ s.t. \quad & \mathbb{P}\{-I \preceq \Pi(\boldsymbol{\xi}) \preceq \alpha I\} \geq p. \end{aligned} \tag{50}$$

5.3.2 Experimental setup and numerical results

Considering the same robust state-feedback stabilization problem as in [11] together with similar parameters and assumptions, we define the uncertain system (46) in the form

$$\dot{\mathbf{x}} = A(\boldsymbol{\xi})\mathbf{x} + B\mathbf{u}, \tag{51}$$

where

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}^T,$$

$$A(\boldsymbol{\xi}) = \begin{bmatrix} 2(1+\xi_3)\Omega & -2-2\xi_1-2(1+\xi_3)\Omega & 2(1+\xi_3)\sin(0.785+\xi_4) & 2+2\xi_1+2(1+\xi_3)\Omega \\ 0 & -2-2\xi_1 & 0 & 4+2\xi_1+2\xi_2 \\ 4(1+\xi_3)\cos(0.785+\xi_4) & -4(1+\xi_3)\cos(0.785+\xi_4) & -2(1+\xi_3)\Omega & 4(1+\xi_3)\Omega \\ 0 & 0 & 0 & 2+2\xi_2 \end{bmatrix},$$

$\Omega = \cos(0.785+\xi_4) - \sin(0.785+\xi_4)$, $\boldsymbol{\xi} = [\xi_1, \xi_2, \xi_3, \xi_4]^T$, and

$$\Xi := \{\boldsymbol{\xi} : |\xi_1| \leq 0.2, |\xi_2| \leq 0.2, |\xi_3| \leq 0.2, |\xi_4| \leq 0.2\}.$$

The uncertain parameter $\boldsymbol{\xi}$ follows a uniform distribution and we assume that ξ_1 is independent of ξ_2, ξ_3, and ξ_4. We implement the sequential convex optimization method presented in Section 4.3 to solve the CCLMI-RTM problem. The initial solution used for the sequential convex optimization algorithm is obtained from the SA-RTM approach (5), and the tolerance parameter δ is set to 10^{-5}. The rest of the notations and criteria are similar to those in Section 5.1.3. Table 10 reports the performances of SA-RTM, SSA-RTM, and PSAA-RTM for various N. The reported results are averaged over ten instances.

Table 10: *Performance of SA-RTM, SAA-RTM, and PSAA-RTM approximations.*

$p = 95\%$	SA-RHS					SAA-RHS						PSAA-RHS					
N	obj	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac	obj	imp(%)	time(sec)	Pr(%)	gap(%)	frac
1000	-0.0716	14.22	99.04	4.04	0	-0.0861	6.80	7135.20	94.08	-0.92	1	-0.0802	4.16	1071.79	95.04	0.04	0.3
1780	-0.0715	22.26	99.41	4.41	0	*	*	*	*	*	*	-0.0796	3.38	3432.75	95.50	0.50	0
2500	-0.0703	37.65	99.56	4.56	0	*	*	*	*	*	*	-0.0794	4.31	5507.78	95.36	0.36	0

As showcased in the above table, the results are consistent with the previously observed results in Tables 1-9 and underline the better performance of the PSAA-RTM approach (as compared to SA-RTM and SAA-RTM). Although the SAA-RTM objective value is smaller than that of PSAA-RTM for $N = 1000$, it is, as before, because the SAA-RTM solution does not satisfy the conditions imposed by the 95% reliability requirement.

6 Conclusion

In this paper, we develop a general and efficient framework based on the PSAA method to solve the two main subclasses – CCLMI-RHS and CCLMI-RTM – of chance-constrained LMI programs. We first derive deterministic PSAA formulations of CCLMI-RHS and CCLMI-RTM, and provide sufficient conditions under which the corresponding formulations are convex problems. Second, we develop SDP and BMI inner approximations of the PSAA formulations for CCLMI-RHS and CCLMI-RTM, respectively. Third, we derive an equivalent SDP reformulation of the PSAA formulation for CCLMI-RTM under special conditions. Fourth, we present a sequential convex approximation method to solve the general BMI approximation of CCLMI-RTM. Finally, we conduct a comprehensive numerical study on three applications to evaluate and benchmark our approach. The numerical results show that the PSAA approach dominates the SAA and SA approaches, and provides the tightest approximations (close to the prescribed probability level) and the highest quality solutions. Regarding the computational time, the SA approach is clearly the best performer but delivers solutions that are overly conservative. The SAA approach is always the slowest, more than 100 times slower than PSAA, and fails to provide any feasible solution within five hours when moderate numbers of scenarios are considered. In the future, we plan to investigate extensions of these results to more complex/general CCLMI problems in which the matrix-valued function $G(\boldsymbol{x}, \boldsymbol{\xi})$ takes the following form:

$$G(\boldsymbol{x}, \boldsymbol{\xi}) = A_0(\boldsymbol{x}) + \sum_{i=1}^{n} A_i(\boldsymbol{x})\xi_i + \sum_{1 \leq j \leq k \leq n} B_{jk}(\boldsymbol{x})\xi_j\xi_k.$$

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Appendix

In the following appendices, we deliberately use small-size models to illustrate the different approaches that can be used to solve CCLMI-RHS/RTM problems.

Appendix A Example of CCLMI-RHS

Consider the following data and parameters:

$$A_0(x) = \begin{bmatrix} 3x_1 & 4x_2 \\ 4x_2 & -2x_2 \end{bmatrix}, A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},$$
$$m = 2,\ n = 2,\ d = 2,\ p = 0.9,\ \beta = 0.01,\ \boldsymbol{\xi} = (\xi_1, \xi_2) \sim N(0, I)\ .$$

The corresponding CCLMI-RHS problem (52) reads:

$$\begin{aligned} \min \quad & -x_1 + 4x_2 && \text{(52a)} \\ s.t. \quad & \mathbb{P}\left\{ \begin{bmatrix} 3x_1 & 4x_2 \\ 4x_2 & -2x_2 \end{bmatrix} + \xi_1 \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + \xi_2 \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \succeq 0 \right\} \geq 0.9 && \text{(52b)} \\ & 2x_1 + 5x_2 \geq 10. && \text{(52c)} \end{aligned}$$

Appendix A.1 SA-RHS

According to Theorem 1, the number of samples needed in the SA approach is

$$N \geq N^* = \lceil \frac{2}{0.1} (\log \frac{1}{0.01} + 2) \rceil = 132.$$

The SA formulation of CCLMI-RHS (52) is:

$$\begin{aligned} \min \quad & -x_1 + 4x_2 \\ s.t. \quad & \begin{bmatrix} 3x_1 & 4x_2 \\ 4x_2 & -2x_2 \end{bmatrix} + \hat{\xi}_1^t \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \succeq 0, \quad \forall t = 1, \ldots, 132 \\ & \text{(52c)}, \end{aligned}$$

where $\hat{\xi}_1^1, \ldots, \hat{\xi}_1^{132}$ and $\hat{\xi}_2^1, \ldots, \hat{\xi}_2^{132}$ are iid samples extracted from the distribution $\mathcal{N}(0, 1)$.

Appendix A.2 SAA-RHS

The SAA formulation of CCLMI-RHS (52) is given as

$$\begin{aligned} \min \quad & -x_1 + 4x_2 \\ s.t. \quad & \begin{bmatrix} 3x_1 & 4x_2 \\ 4x_2 & -2x_2 \end{bmatrix} + \hat{\xi}_1^t \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + M_t \eta_t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \succeq 0, \forall t = 1, \ldots, 132 && \text{(54a)} \\ & \frac{\sum_{t=1}^{132} \eta_t}{132} \leq 0.1 && \text{(54b)} \\ & \eta_t \in \{0, 1\}, \quad \forall t = 1, \ldots, 132 && \text{(54c)} \\ & \text{(52c)}, \end{aligned}$$

where M_t is the smallest possible value such that the inequality $A_0(x) + \hat{\xi}_1^t A_1 + \hat{\xi}_2^t A_2 + M_t I \succeq 0$ always holds true for any $(x_1, x_2) \in X := \{(x_1, x_2) : 2x_1 + 5x_2 \geq 10\}$. More specifically, the optimal value of M_t can be found by solving the optimization problem :

$$\begin{aligned} \min \quad & M_t \qquad (55) \\ s.t. \quad & \begin{bmatrix} 3x_1 & 4x_2 \\ 4x_2 & -2x_2 \end{bmatrix} + \hat{\xi}_1^t \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + M_t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \succeq 0, \ \forall (x_1, x_2) \in X. \end{aligned}$$

Appendix A.3 PSAA-RTM-N

We illustrate here the piecewise linear approximation method explained in Section 4.1.2. We use $S+1$ segments for $\Phi(z_1^t)$ and $K+1$ segments for $\Phi(z_2^t)$ with parameters:

$$S = 3, \ K = 3, \ z_1(3) = -3, \ z_1(2) = -1.5, \ z_1(1) = 0, \ z_2(1) = 0, \ z_2(2) = 1.5, \ z_2(3) = 3.$$

The corresponding PSAA-RTM-N approximation problem of CCLMI-RHS (52) reads:

$$\begin{aligned}
\min \quad & -x_1 + 4x_2 \\
s.t. \quad & \begin{bmatrix} 3x_1 & 4x_2 \\ 4x_2 & -2x_2 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + z_1^t \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \succeq 0, \quad \forall t = 1, \dots, 132 && \text{(56a)} \\
& \begin{bmatrix} 3x_1 & 4x_2 \\ 4x_2 & -2x_2 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + z_2^t \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \succeq 0, \quad \forall t = 1, \dots, 132 && \text{(56b)} \\
& \frac{\sum_{t=1}^{132} y_t}{132} \geq 0.9 && \text{(56c)} \\
& y_t \geq 0, \quad \forall t = 1, \dots, 132 && \text{(56d)} \\
& q_2^t - q_1^t \geq y_t, \quad \forall t = 1, \dots, 132 && \text{(56e)} \\
& q_2^t \leq 0.3989 z_2^t + 0.5, \quad \forall t = 1, \dots, 132 && \text{(56f)} \\
& q_2^t \leq 0.2888 z_2^t + 0.5, \quad \forall t = 1, \dots, 132 && \text{(56g)} \\
& q_2^t \leq 0.0436 z_2^t + 0.8677, \quad \forall t = 1, \dots, 132 && \text{(56h)} \\
& q_2^t \leq 0.9987, \quad \forall t = 1, \dots, 132 && \text{(56i)} \\
& q_1^t \geq 0.3989 z_1^t + 0.5, \quad \forall t = 1, \dots, 132 && \text{(56j)} \\
& q_1^t \geq 0.2888 z_1^t + 0.5, \quad \forall t = 1, \dots, 132 && \text{(56k)} \\
& q_1^t \geq 0.0436 z_1^t + 0.1323, \quad \forall t = 1, \dots, 132 && \text{(56l)} \\
& q_1^t \geq 0.0013, \quad \forall t = 1, \dots, 132 && \text{(56m)} \\
& z_2^t \geq z_1^t, \quad \forall t = 1, \dots, 132 && \text{(56n)} \\
& 0 \leq q_2^t \leq 1, \quad \forall t = 1, \dots, 132 && \text{(56o)} \\
& 0 \leq q_1^t \leq 1, \quad \forall t = 1, \dots, 132 && \text{(56p)} \\
& \text{(52c)}.
\end{aligned}$$

Appendix B Example of CCLMI-RTM

Consider the following data and parameters:

$$A_0(x) = \begin{bmatrix} 5x_1 & 2x_2 \\ 2x_2 & 1 \end{bmatrix}, A_1(x) = \begin{bmatrix} 2x_1 & x_2 \\ x_2 & 3x_1 \end{bmatrix}, A_2(x) = \begin{bmatrix} x_2 & -1 \\ -1 & 2x_1 \end{bmatrix},$$

$$m = 2,\ n = 2,\ d = 2,\ p = 0.9,\ \beta = 0.01,\ \boldsymbol{\xi} = (\xi_1, \xi_2) \sim \mathcal{N}(0, I)\ .$$

The corresponding CCLMI-RTM problem is:

$$\begin{aligned} \min \quad & 2x_1 - 3x_2 && \text{(57a)} \\ s.t. \quad & \mathbb{P}\{\begin{bmatrix} 5x_1 & 2x_2 \\ 2x_2 & 1 \end{bmatrix} + \xi_1 \begin{bmatrix} 2x_1 & x_2 \\ x_2 & 3x_1 \end{bmatrix} + \xi_2 \begin{bmatrix} x_2 & -1 \\ -1 & 2x_1 \end{bmatrix} \succeq 0\} \geq 0.9 && \text{(57b)} \\ & x_1 - 2x_2 \geq 10. && \text{(57c)} \end{aligned}$$

Appendix B.1 SA-RTM

The SA formulation of CCLMI-RTM (57) is:

$$\begin{aligned} \min \quad & 2x_1 - 3x_2 && \text{(58)} \\ s.t. \quad & \begin{bmatrix} 5x_1 & 2x_2 \\ 2x_2 & 1 \end{bmatrix} + \hat{\xi}_1^t \begin{bmatrix} 2x_1 & x_2 \\ x_2 & 3x_1 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} x_2 & -1 \\ -1 & 2x_1 \end{bmatrix} \succeq 0, \quad \forall t = 1, \ldots, 132 \\ & \text{(57c)}. \end{aligned}$$

Appendix B.2 SAA-RTM

The SAA formulation of problem CCLMI-RTM (57) is

$$\begin{aligned} \min \quad & 2x_1 - 3x_2 && \text{(59)} \\ s.t. \quad & \begin{bmatrix} 5x_1 & 2x_2 \\ 2x_2 & 1 \end{bmatrix} + \hat{\xi}_1^t \begin{bmatrix} 2x_1 & x_2 \\ x_2 & 3x_1 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} x_2 & -1 \\ -1 & 2x_1 \end{bmatrix} + M_t \eta_t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \succeq 0,\ \forall t = 1, \ldots, 132 \\ & \text{(54b); (54c); (57c)}, \end{aligned}$$

where the optimal value of M_t is derived by solving the optimization problem

$$\begin{aligned} \min \quad & M_t && \text{(60)} \\ s.t. \quad & \begin{bmatrix} 5x_1 & 2x_2 \\ 2x_2 & 1 \end{bmatrix} + \hat{\xi}_1^t \begin{bmatrix} 2x_1 & x_2 \\ x_2 & 3x_1 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} x_2 & -1 \\ -1 & 2x_1 \end{bmatrix} + M_t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \succeq 0,\ \forall (x_1, x_2) \in X, \end{aligned}$$

where $X := \{(x_1, x_2) : x_1 - 2x_2 \geq 10\}$.

Appendix B.3 PSAA-RTM-N

Applying the PSAA-RTM-N reformulation (31), we obtain the following BMI formulation:

$$
\begin{aligned}
\min \quad & 2x_1 - 3x_2 \\
s.t. \quad & \begin{bmatrix} 5x_1 & 2x_2 \\ 2x_2 & 1 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} x_2 & -1 \\ -1 & 2x_1 \end{bmatrix} + z_1^t \begin{bmatrix} 2x_1 & x_2 \\ x_2 & 3x_1 \end{bmatrix} \succeq 0, \quad \forall t = 1, \ldots, 132 \quad (61a) \\
& \begin{bmatrix} 5x_1 & 2x_2 \\ 2x_2 & 1 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} x_2 & -1 \\ -1 & 2x_1 \end{bmatrix} + z_2^t \begin{bmatrix} 2x_1 & x_2 \\ x_2 & 3x_1 \end{bmatrix} \succeq 0, \quad \forall t = 1, \ldots, 132 \quad (61b) \\
& (56c) - (56p); (57c).
\end{aligned}
$$

Appendix C Example of a special case of CCLMI-RTM with $A_1(\boldsymbol{x}) = \hat{a}_1(\boldsymbol{x})A_1$

We consider now the case of a CCLMI-RTM problem with $A_1(\boldsymbol{x}) = \hat{a}_1(\boldsymbol{x})A_1$, $\hat{a}_1(\boldsymbol{x})$ is an affine function of $\boldsymbol{x}$, and the following data:

$$A_0(x) = \begin{bmatrix} 3x_1 & 2x_2 \\ 2x_2 & x_1 \end{bmatrix}, A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix},$$

$$\hat{a_1} = x_1 + x_2, \ \hat{a_2} = 2x_2 - x_1$$

$$m = 2, \ n = 2, \ d = 2, \ p = 0.9, \ \beta = 0.01$$

$$\boldsymbol{\xi} = (\xi_1, \xi_2) \sim \mathcal{N}(0, I) \ .$$

The corresponding CCLMI-RTM formulation is:

$$
\begin{aligned}
\min \quad & x_1 + 2x_2 && (62a) \\
s.t. \quad & \mathbb{P}\{\begin{bmatrix} 3x_1 & 2x_2 \\ 2x_2 & x_1 \end{bmatrix} + (x_1 + x_2)\xi_1 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + (2x_2 - x_1)\xi_2 \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \succeq 0\} \geq 0.9 && (62b) \\
& 2x_1 - 5x_2 \geq 10 && (62c) \\
& x_1, x_2 \geq 0. && (62d)
\end{aligned}
$$

Appendix C.1 SA-RTM

The SA formulation of CCLMI-RTM (62) is:

$$
\begin{aligned}
\min \quad & x_1 + 2x_2 && (63) \\
s.t. \quad & \begin{bmatrix} 3x_1 & 2x_2 \\ 2x_2 & x_1 \end{bmatrix} + (x_1 + x_2)\hat{\xi}_1^t \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + (2x_2 - x_1)\hat{\xi}_2^t \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \succeq 0, \quad \forall t = 1, \ldots, 132 \\
& (62c); (62d).
\end{aligned}
$$

Appendix C.2 SAA-RTM

The SAA formulation of CCLMI-RTM (62) is:

$$\begin{aligned}
\min \quad & x_1 + 2x_2 && (64)\\
s.t. \quad & \begin{bmatrix} 3x_1 & 2x_2 \\ 2x_2 & x_1 \end{bmatrix} + (x_1+x_2)\hat{\xi}_1^t \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + (2x_2 - x_1)\hat{\xi}_2^t \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} + M_t\eta_t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \succeq 0, \quad \forall t = 1,\ldots,132\\
& (54b); (54c); (62c); (62d),
\end{aligned}$$

where the optimal value of each M_t is derived by solving the optimization problem:

$$\begin{aligned}
\min \quad & M_t && (65)\\
s.t. \quad & \begin{bmatrix} 3x_1 & 2x_2 \\ 2x_2 & x_1 \end{bmatrix} + (x_1+x_2)\hat{\xi}_1^t \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + (2x_2 - x_1)\hat{\xi}_2^t \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} + M_t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \succeq 0, \ \forall (x_1, x_2) \in X,
\end{aligned}$$

where $X := \{(x_1, x_2) : 2x_1 - 5x_2 \geq 10, \ x_1 \geq 0, \ x_2 \geq 0\}$.

Appendix C.3 PSAA-RTM-S

We have $\hat{a}_1(\boldsymbol{x}) = x_1 + x_2 \geq 0$ because $x_1, x_2 \geq 0$. Applying Theorem 6, we derive the following convex SDP approximation problem of CCLMI-RTM as follows:

$$\begin{aligned}
\min \quad & x_1 + 2x_2 \\
s.t. \quad & \begin{bmatrix} 3x_1 & 2x_2 \\ 2x_2 & x_1 \end{bmatrix} + w_1^t \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + (2x_2 - x_1)\hat{\xi}_2^t \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \succeq 0, \ \forall t = 1,\ldots,132 && (66a)\\
& \begin{bmatrix} 3x_1 & 2x_2 \\ 2x_2 & x_1 \end{bmatrix} + w_2^t \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + (2x_2 - x_1)\hat{\xi}_2^t \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \succeq 0, \ \forall t = 1,\ldots,132 && (66b)\\
& \bar{q}_2^t \leq 0.3989 w_2^t + 0.5(x_1+x_2), \quad \forall t = 1,\ldots,132 && (66c)\\
& \bar{q}_2^t \leq 0.2888 w_2^t + 0.5(x_1+x_2), \quad \forall t = 1,\ldots,132 && (66d)\\
& \bar{q}_2^t \leq 0.0436 w_2^t + 0.8677(x_1+x_2), \quad \forall t = 1,\ldots,132 && (66e)\\
& \bar{q}_2^t \leq 0.9987(x_1+x_2), \quad \forall t = 1,\ldots,132 && (66f)\\
& \bar{q}_1^t \geq 0.3989 w_1^t + 0.5(x_1+x_2), \quad \forall t = 1,\ldots,132 && (66g)\\
& \bar{q}_1^t \geq 0.2888 w_1^t + 0.5(x_1+x_2), \quad \forall t = 1,\ldots,132 && (66h)\\
& \bar{q}_1^t \geq 0.0436 w_1^t + 0.1323(x_1+x_2), \quad \forall t = 1,\ldots,132 && (66i)\\
& \bar{q}_1^t \geq 0.0013(x_1+x_2), \quad \forall t = 1,\ldots,132 && (66j)\\
& \bar{q}_2^t - \bar{q}_1^t \geq \bar{y}_t, \quad \forall t = 1,\ldots,132 && (66k)\\
& \frac{\sum_{t=1}^{132} \bar{y}_t}{N} \geq 0.9(x_1+x_2) && (66l)\\
& \bar{y}_t \geq 0, \quad \forall t = 1,\ldots,132 && (66m)\\
& w_1^t \leq w_2^t, \quad \forall t = 1,\ldots,132 && (66n)\\
& 0 \leq \bar{q}_2^t \leq 1, \quad \forall t = 1,\ldots,132 && (66o)\\
& 0 \leq \bar{q}_1^t \leq 1, \quad \forall t = 1,\ldots,132 && (66p)\\
& (62c); (62d).
\end{aligned}$$

Appendix D Sequential Convex Optimization Algorithm

One can observe that constraints (61a) and (61b) include the bilinear terms $x_1 z_1^t$, $x_2 z_1^t$, $x_1 z_2^t$, and $x_2 z_2^t$. To handle these bilinear terms, one can use the sequential convex optimization algorithm presented in Section 4.3. The master problem is

$$
\begin{aligned}
\min \quad & 2x_1 - 3x_2 \\
s.t. \quad & \begin{bmatrix} 5x_1 & 2x_2 \\ 2x_2 & 1 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} x_2 & -1 \\ -1 & 2x_1 \end{bmatrix} + z_1^{t*} \begin{bmatrix} 2x_1 & x_2 \\ x_2 & 3x_1 \end{bmatrix} \succeq 0, \quad \forall t = 1, \ldots, 132 \quad (67a) \\
& \begin{bmatrix} 5x_1 & 2x_2 \\ 2x_2 & 1 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} x_2 & -1 \\ -1 & 2x_1 \end{bmatrix} + z_2^{t*} \begin{bmatrix} 2x_1 & x_2 \\ x_2 & 3x_1 \end{bmatrix} \succeq 0, \quad \forall t = 1, \ldots, 132 \quad (67b) \\
& (57c),
\end{aligned}
$$

and the corresponding subproblem is:

$$
\begin{aligned}
\max \quad & \lambda \quad (68a) \\
s.t. \quad & \begin{bmatrix} 5x_1^* & 2x_2^* \\ 2x_2^* & 1 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} x_2^* & -1 \\ -1 & 2x_1^* \end{bmatrix} + z_1^t \begin{bmatrix} 2x_1^* & x_2^* \\ x_2^* & 3x_1^* \end{bmatrix} \succeq \lambda I, \quad \forall t = 1, \ldots, 132 \quad (68b) \\
& \begin{bmatrix} 5x_1^* & 2x_2^* \\ 2x_2^* & 1 \end{bmatrix} + \hat{\xi}_2^t \begin{bmatrix} x_2^* & -1 \\ -1 & 2x_1^* \end{bmatrix} + z_2^t \begin{bmatrix} 2x_1^* & x_2^* \\ x_2^* & 3x_1^* \end{bmatrix} \succeq \lambda I, \quad \forall t = 1, \ldots, 132 \quad (68c) \\
& (56c) - (56p).
\end{aligned}
$$

Note that, unlike problem (61) which contains bilinear terms, the master problem (67) and subproblem (68) are all SDP problems which can be solved directly and efficiently by off-the-shelf solvers, such as MOSEK.