

# Risk-Sensitive Variational Bayes: Formulations and Bounds

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## Abstract

We study data-driven decision-making problems in a parametrized Bayesian framework. We adopt a risk-sensitive approach to modeling the interplay between statistical estimation of parameters and optimization, by computing a risk measure over a loss/disutility function with respect to the posterior distribution over the parameters. While this forms the standard Bayesian decision-theoretic approach, we focus on problems where calculating the posterior distribution is intractable, a typical situation in modern applications with large datasets, heterogeneity due to observed covariates and latent group structure. The key methodological innovation we introduce in this paper is to leverage a dual representation of the risk measure to introduce an optimization-based framework for approximately computing the posterior risk-sensitive objective, as opposed to using standard sampling based methods such as Markov Chain Monte Carlo. Our analytical contributions include rigorously proving finite sample bounds on the ‘optimality gap’ of optimizers obtained using the computational methods in this paper, from the ‘true’ optimizers of a given decision-making problem. We illustrate our results by comparing the theoretical bounds with simulations of a newsvendor problem on two methods extracted from our computational framework.

## 1 Introduction

In this paper we consider optimization problems of the form

$$\min_{a \in \mathcal{A}} \varrho_{\gamma}^R(a) := -\frac{1}{\gamma} \log \mathbb{E}_{\pi_n} [\exp(-\gamma R(a, \theta))] = -\frac{1}{\gamma} \log \mathbb{E}_{\pi_n} [\exp(-\gamma F(G(a, \theta)))], \quad (\text{SO})$$

where  $\mathcal{A} \subset \mathbb{R}^s$  ( $s \geq 1$ ) is the set of decision variables,  $\theta$  is a random variable distributed according to a sequence of distributions  $\{\pi_n\}$ , and  $G(a, \theta)$  is a problem-specific loss function. The scalar  $\gamma \in \mathbb{R}$ , and the function  $F : \mathbb{R} \rightarrow \mathbb{R}$  are user-specified, with  $R(a, \theta) = F(G(a, \theta))$  characterizing the risk-sensitivity of the decision-maker to the distribution  $\pi_n$ . This optimization problem represents a canonical model of operational decision-making and/or system design under uncertainty, including, for instance, classic operational decision-making problems like inventory management, wherein a manager must decide when and how much to re-stock an inventory system in the face of stochastic demand and supply constraints. As another example, staffing a call center or hospital ward to maximize throughput or quality of service as a function of stochastic inputs is a complicated optimal system design problem that is also a special case of this class of problems. Similarly, achieving a prescribed service-level performance agreement in a large scale cloud computing system

typically requires the careful design of optimal job replication (or ‘redundancy’) levels as a function of stochasticity in the job sizes.

Our interest lies in data-driven settings where  $\theta$  is an unknown parameter, and the probability distribution  $\pi_n(\theta) := \pi(\theta|\tilde{X}_n)$  depends on a set of  $n$  observations of covariates  $\tilde{X}_n = \{X_1, \dots, X_n\}$  through a Bayesian model, resulting in a *risk-sensitive Bayes-predictive stochastic program* for (SO). Such an approach starts with a prior  $\pi(\theta)$  over the unknown parameter  $\theta$ , quantifying domain or expert knowledge about how ‘reasonable’ different parameter values are. Also present is a likelihood  $p(\tilde{X}_n|\theta)$ , a parametric stochastic model over  $\tilde{X}_n$  that summarizes structural knowledge about the scientific or engineering problem at hand. Examples include queueing models, supply-chain models or models of financial data. This prior, together with the likelihood define the posterior distribution  $\pi(\theta|\tilde{X}_n)$ , summarizing all information over  $\theta$ .

In practical settings, the posterior  $\pi(\theta|\tilde{X}_n)$  typically cannot be easily computed, and DM’s are often led to restrictive modeling choices such as assuming the likelihood function has a conjugate prior. Indeed, one might argue that this is a predominant reason Bayesian methods are not widely used in operations research and engineering. Nonetheless, incorporating non-conjugate priors and complicated likelihood functions is critical for realizing the full utility of decision-theoretic Bayesian methods - however this entails the use of computational approximations.

*A goal of this paper is to introduce a framework from which can be extracted computational methods for approximately computing and optimizing posterior decision risk.*

Our approach exploits the dual representation of the log-exponential risk measure in (SO), which is convex (or extended coherent) (Rockafellar, 2007; Föllmer and Knispel, 2011). From the Donsker-Varadhan variational free energy principle (Donsker and Varadhan, 1975b,a, 1976, 1983) we observe that,

$$\varrho_\gamma^R(a) = \begin{cases} \min_{q \in \mathcal{M}} \left\{ \mathbb{E}_q[R(a, \theta)] + \frac{1}{\gamma} \text{KL}(q \parallel \pi_n) \right\} & \gamma > 0, \\ \max_{q \in \mathcal{M}} \left\{ \mathbb{E}_q[R(a, \theta)] + \frac{1}{\gamma} \text{KL}(q \parallel \pi_n) \right\} & \gamma < 0, \end{cases} \quad (\text{DV})$$

where  $\mathcal{M}$  is the set of all distribution functions that are absolutely continuous with respect to the posterior distribution  $\pi_n$  and ‘KL’ is the Kullback-Leibler divergence. Notice that this dual formulation exposes the reason we choose to use the log-exponential risk – the right hand side provides a combined assessment of the risk associated with model estimation (computed by the KL divergence  $\text{KL}(q \parallel \pi_n)$ ) and the decision risk under the estimated posterior  $q$  (computed by  $\mathbb{E}_q[R(a, \theta)]$ ). It is entirely possible to use other convex risk measures that yield ‘penalty’ functions other than the KL divergence present here.

As stated above, the reformulation presented in (DV) offers no computational gains. However, restricting ourselves to an appropriately chosen subset  $\mathcal{Q} \subset \mathcal{M}$ , that consists of distributions where the integral  $\mathbb{E}_q[R(a, \theta)]$  can be tractably computed, we immediately obtain a risk-sensitive variational Bayesian formulation of (DV):

$$-\frac{|\gamma|}{\gamma} \varrho_\gamma^R(a) \geq \max_{q \in \mathcal{Q}} \left\{ -\frac{|\gamma|}{\gamma} \mathbb{E}_q[R(a, \theta)] - \frac{|\gamma|}{\gamma} \frac{1}{\gamma} \text{KL}(q \parallel \pi_n) \right\} \quad \gamma \in \mathbb{R}. \quad (\text{RVB})$$

(RVB) is our framework for data-driven decision-making: the left hand side is the log-exponential risk of taking the decision  $a$  with disutility  $R$  under the (posterior) measure  $\pi_n$ , and the parameter  $\gamma$  measures the ‘risk sensitivity’ of the DM; see Section 2.1.1 below for a more detailed discussion. The choice of the family  $\mathcal{Q}$ , disutility  $R$ , and parameter  $\gamma$  encodes specific algorithms. Our analysis

in Section 4 below reveals general guidelines on how to choose  $\mathcal{Q}$  that ensures a small optimality gap (defined below) with high probability.

In machine learning, where the focus is solely on computing probability densities, computational approximations that minimizing the KL divergence  $\text{KL}(\cdot \parallel \pi)$  to the posterior are called *variational Bayes* (VB). We adopt this terminology. Indeed, our RVB framework includes two existing VB-based algorithms for approximately computing and optimizing Bayes-predictive stochastic programs. First, the *Naive VB* (NVB) algorithm, summarized in Algorithm 1 in Section 3.1.1, is a *separated estimation and optimization* (SEO) method wherein we first compute a ‘standard’ VB approximation to the posterior distribution by minimizing the KL divergence from the posterior over the family  $\mathcal{Q}$  and then using this estimated posterior as a plug-in to (SO). An approximate optimal action is chosen by optimizing this estimator of (SO). This amounts to minimizing (RVB) with  $R(a, \theta)$  equal to a constant. Intuitively, one anticipates that such a separation can result in a sub-optimal decision rule  $\mathbf{a}_{\text{NV}}^*$ : observe that minimizing the KL divergence alone (over  $\mathcal{Q}$ ) fits the most dominant mode of the posterior distribution, which may be less important for optimizing (SO).

A second, more ‘loss aware’ approach proposed in the literature is the so-called *Loss-calibrated variational Bayes* (LCVB) algorithm (Lacoste-Julien et al., 2011), wherein a minimax optimization problem jointly estimates and optimizes the posterior and decisions (respectively). The LCVB objective is obtained from (RVB) by setting  $\gamma = -1$  and choosing  $R(a, \theta) = \log G(a, \theta)$ . See Algorithm 2 in Section 3.1.2 below for a detailed description of this method.

In this paper, we consider the broad family of methods that fall under (RVB), and focus on the question

*What is the impact of the computational approximations on the statistical performance of inferred decision rules?*

We address this question by proving finite sample probabilistic bounds on the optimality gap between the approximate and true decision-making problems. In particular, we prove such bounds for an ‘oracle regret’ function that compares the expected Bayes risk of using the approximate optimal decision-rules (obtained by optimizing (RVB)) to the ‘true’ optimal expected Bayes risk. To summarize, our main analytical contributions include:

1. Bounding the optimality gap of the decision-rules inferred by (RVB), including the NVB and LCVB algorithms. In particular, we identify conditions on the prior density function (which need not be conjugate), the tractable family of distributions  $\mathcal{Q}$  and the Bayes risk function under which an oracle regret function is bounded.
2. Identifying gradient conditions on the aforementioned Bayes risk function under which the distance between the approximate optimal decision-rule and the ‘true’ decision rule is bounded away from zero with low probability, as a function of the number of samples and the Kullback-Leibler distance between the variational approximator and the true posterior.
3. The sample complexity analysis (and concomitant sufficiency conditions) yield guidelines on how to choose the set  $\mathcal{Q}$  for a given decision-making problem.

Here’s a brief roadmap for the rest of the paper. In the next section we provide a literature survey of relevant results from machine learning, theoretical statistics and operations research, placing our results in appropriate context. In Section 3, we describe the Bayes-predictive stochastic programming model we study, as well as the RVB framework and specific VB algorithms. We develop our theoretical results in Sections 4. We then illustrate the bounds obtained in Section 4

by specializing the results to the Newsvendor problem in Section 5 and also present some numerical results. We end with concluding remarks in Section 6.

## 2 Existing literature and our work

Our paper fits in with a growing body of work in operations research that lies at the intersection of decision-making under uncertainty and statistical estimation. Our results are also aligned with recent developments of a rigorous theoretical understanding of computational Bayesian methods in statistics and machine learning.

### 2.1 Operations research literature

The primary goal in data-driven decision-making is to learn empirical decision-rules (or *predictive prescriptions* as Bertsimas and Kallus (2014) term them)  $a^*(\tilde{X}_n)$  that prescribes a decision, given an observation of the covariates  $\tilde{X}_n$ . Early work in this direction, including classic work by Herbert Scarf on Bayesian solutions to the newsvendor problem (Scarf, 1960a), focused on two-stage solutions - estimation followed by optimization. Our setting is most related to recent work on *Bayesian risk optimization* (BRO) in Wu et al. (2018); Zhou and Wu (2017). In BRO, the authors consider optimal decision-making using various coherent risk measures computed under the posterior distribution. The authors establish several important results, including that the optimal values and decisions are asymptotically consistent as the sample size tends to infinity, and central limit theorems for these quantities. However, there are substantial differences with our paper. First, all of the analysis in Wu et al. (2018) presumes that the posterior risk measures are actually computable. The authors do not address the critical computational questions surrounding Bayesian methods or the impact of (inevitable) computational approximations on BRO – indeed, this is not their focus. Second, extended coherent risk measures are not considered (in particular, the log-exponential risk measure used here), and it is unclear if the asymptotic results continue hold otherwise. Third, while we use a risk measure to derive the computational framework in (RVB), the focus in Wu et al. (2018) is purely on the analytical properties of optimal decisions.

More recently, there has been significant interest in methods that use empirical risk minimization (ERM) or sample average approximation (SAA) for directly estimating decision-rules that optimize Monte Carlo or empirical approximations (Bertsimas and Kallus, 2014; Bertsimas et al., 2016; Ban and Rudin, 2018; Bertsimas and McCord, 2018; Deng et al., 2018; Elmachtoub and Grigas, 2017; Wilder et al., 2018). The survey by Homem-de Mello and Bayraksan (2014) consolidates recent results on Monte Carlo methods for stochastic optimization. It is important to note that this recent surge of work in data-driven decision-making has largely focused on explicit black-box models. On the other hand, there are many situations where optimal decisions must be made in the presence of a well-defined parametrized stochastic model. Bayesian methods are a natural means for estimating distributions over the parameters of a stochastic model; though, as noted before, the computational complexity of Bayesian algorithms can be high. The interplay between optimization and estimation, in the sense of discovering predictive prescriptions for Bayesian models has largely been ignored. Furthermore, as Liyanage and Shanthikumar (2005) show in the newsvendor context, SEO methods can be suboptimal in terms of expected regret and long-term average losses. Liyanage and Shanthikumar (2005) introduced *operational statistics* (OS) as an alternative to SEO (see Chu et al. (2008); Lu et al. (2015) as well), whereby the optimal empirical order quantity is determined

as a function of an optimization parameter that can be determined for each sample size. OS has demonstrably better performance, especially on single parameter newsvendor problems (though there is much less known about its statistical properties).

### 2.1.1 Distributionally robust optimization literature

There is an oblique connection between our computational methodology and the growing body of research on distributionally robust stochastic optimization (DRSO); see, for instance, Delage and Ye (2010); Goh and Sim (2010); Hu and Hong (2013); Wiesemann et al. (2014); Bayraksan and Love (2015); Wang et al. (2016); Watson et al. (2016); Blanchet et al. (2018). In particular, consider a DRSO problem with uncertainty set determined by a Kullback-Leibler divergence ball:

$$\min_{a \in \mathcal{A}} \max_{q \in \tilde{\mathcal{Q}}_c} \mathbb{E}_q[R(a, \theta)], \quad (\text{DR})$$

where  $\tilde{\mathcal{Q}}_c := \{q \in \mathcal{M} : \text{KL}(q \parallel \pi) \leq c\}$ , where  $c > 0$ . Following (Föllmer and Knispel, 2011, Prop. 3.1), it can be shown that, for fixed  $\gamma < 0$ , (DV) satisfies

$$\varrho_\gamma^R(a) = \max_{c > 0} \left\{ \frac{c}{\gamma} + \rho_c^R(a) \right\},$$

where  $\rho_c^R(a) = \max_{q \in \tilde{\mathcal{Q}}_c} \mathbb{E}_q[R(a, \theta)]$ . Notice that this equivalence holds for each  $a$ , and therefore, there exists  $c \equiv c(\gamma, a)$  at which the optimization over (DV) satisfies

$$\min_{a \in \mathcal{A}} \varrho_\gamma^R(a) = \min_{a \in \mathcal{A}} \left\{ \frac{c(\gamma, a)}{\gamma} + \rho_{c(\gamma, a)}^R(a) \right\}.$$

This connection between the log-exponential risk measure and robustness has been recognized in the optimal control literature as well (Whittle, 2002; Dupuis et al., 2000). However, it should be clear that our methodology is not an explicitly robust method - notice that we ‘trust’ the posterior distribution  $\pi$ , but we cannot compute  $\varrho_\gamma^R$  explicitly. In other words, (RVB) aims to find the best possible approximation to the ‘true’ posterior distribution in the set  $\mathcal{Q}$  calibrated by the loss  $\mathbb{E}_q[R(a, \theta)]$ . Nonetheless, there is much to be explored in relation to the robust properties of (DV); see the conclusion section for further discussion.

## 2.2 Statistics and machine learning literature

Lacoste-Julien et al. (2011) observe that calibrating a Gaussian process classification algorithm to a fixed loss function can improve classification performance over a loss-insensitive algorithm – indeed, this is the first documented presentation of the LCVB algorithm. Similarly, surrogate loss functions (Bartlett et al., 2006; Taskar et al., 2005) that are regularized upper bounds that depend on the cost function, also implicitly loss-calibrate frequentist classification algorithms. While standard VB methods for posterior estimation have been extensively used in machine learning (Blei et al., 2017), it is only recently that the theoretical questions surrounding VB have been addressed. In particular, we note Wang and Blei (2018) who prove asymptotic consistency of VB in the large sample limit, Zhang and Gao (2017) on the other hand establish bounds on the rate of convergence of the VB posterior to the ‘true’ posterior providing a more refined analysis, and Jaiswal et al. (2019b) where asymptotic consistency of  $\alpha$ -Rényi VB was demonstrated. Our analysis in this paper, extends these results to establish sample complexity bounds for computational Bayesian decision-making. These bounds, in turn, are complementary to large sample analyses in Jaiswal et al. (2019a).

### 3 Problem Setup and Methodology

In the subsequent sections, we introduce the Bayes-predictive stochastic programming model and the RVB computational framework. First, we present notations and important definitions used throughout the paper.

Let  $\otimes_n A$  denote the  $n$ -fold product of the set  $A$ ,  $\lfloor \cdot \rfloor$  the greatest integer function, and  $\mathbb{1}_{\{\cdot\}}$  the indicator function.  $\xi \in \mathcal{Y} \subseteq \mathbb{R}^m$  represents an  $\mathbb{R}^m$ -valued random variable that is conditionally independent of an  $\mathbb{R}^k$ -valued covariate  $X \in \mathcal{X} \subseteq \mathbb{R}^k$  given a (random) parameter  $\theta \in \mathbb{R}^d$ :

$$p(\xi \in dy, X \in dx|\theta) = z(\xi \in dy|\theta)p(X \in dx|\theta);$$

that is,  $\xi$  has the *likelihood*  $z(\cdot|\theta)$  with parameter  $\theta \in \Theta \subseteq \mathbb{R}^d$ ; typically  $z$  and  $p$  are the same distribution (though they need not be in our model). Let  $\tilde{X}_n := \{X_1, \dots, X_n\}$  represents a set of  $n$  samples of the covariate *likelihood*  $p(\cdot|\theta)$  associated with the distribution  $P_\theta$  with parameter  $\theta \in \Theta \subseteq \mathbb{R}^d$ . We denote the ‘true model parameter’ by  $\theta_0$  and its corresponding covariate-generating distribution as  $P_0$ . We denote the sigma-algebra generated by  $\otimes_n \mathcal{X}$ . We represent  $\pi(\theta|\tilde{X}_n) \propto p(\tilde{X}_n \in d\mathbf{x}|\theta)\pi(\theta)$  as the *posterior distribution* and  $\pi(\theta)$  as the *prior density function* that captures the DM’s *a priori* belief about the parameters of the stochastic model. We denote the general *model risk* as  $G(a, \theta)$ , representing the risk of taking action ‘ $a$ ’ when the model parameter are posited to be  $\theta$ .

Next, the *optimality gap* for any  $\mathbf{a} \in \mathcal{A}$  with value  $V$  is defined as,

**Definition 3.1** (Optimality Gap). Let  $V_0^* := \min_{a \in \mathcal{A}} G(a, \theta_0)$  and  $a_0^* := \operatorname{argmin}_{a \in \mathcal{A}} G(a, \theta_0)$  be the optimal value and decision respectively under the ‘true’ likelihood  $z(\cdot|\theta_0)$ . Then, the optimality gap in the value is the difference  $|V_0^* - V|$ , and the optimality gap in decision variables is  $\|a_0^* - \mathbf{a}\|$ , where  $\|\cdot\|$  is the Euclidean norm.

Observe that the optimality gap is a random variable, conditional on the data  $\tilde{X}_n$ .

The model risk can be obtained from other problem primitives. To see this, let  $(a, y) \mapsto \ell(a, y)$  be such that it is lower-semicontinuous in  $a \in \mathcal{A} \subset \mathbb{R}^s$  and continuous in  $y \in \mathcal{Y}$ . Consider the *predictive-posterior* risk defined as

$$\mathbb{E}_{p(\xi|\tilde{X}_n)}[\ell(a, \xi)] = \int_{\mathcal{Y}} \ell(a, y) p(\xi \in dy|\tilde{X}_n) \quad (1)$$

where  $p(\xi \in dy|\tilde{X}_n) = \int_{\Theta} z(\xi \in dy|\theta) \pi(\theta|\tilde{X}_n) d\theta$ . Observe that the objective above can also be expressed as

$$\begin{aligned} \mathbb{E}_{p(\xi|\tilde{X}_n)}[\ell(a, \xi)] &= \int_{\Theta} \left( \int_{\mathcal{Y}} \ell(a, y) z(\xi \in dy|\theta) \right) \pi(\theta|\tilde{X}_n) d\theta \\ &= \int_{\Theta} G(a, \theta) \pi(\theta|\tilde{X}_n) d\theta = \mathbb{E}_{\pi(\theta|\tilde{X}_n)}[G(a, \theta)]; \end{aligned} \quad (\text{BP})$$

observe that  $G(a, \theta) = \int_{\mathcal{Y}} \ell(a, y) z(\xi \in dy|\theta)$  is completely defined by the loss function  $\ell(a, y)$  and the likelihood function.

As noted in the introduction, we optimize the posterior log-exponential risk measure  $\varrho_\gamma^R$  of  $R(a, \theta) = (F \circ G)(a, \theta)$ :

$$\min_{a \in \mathcal{A}} \varrho_\gamma^R(a) = -\frac{1}{\gamma} \log \mathbb{E}_{\pi_n}[\exp(-\gamma R(a, \theta))],$$

where  $\gamma \in \mathbb{R}$  and  $F : \mathbb{R} \rightarrow \mathbb{R}$  is a well-defined continuous function. We place no further restrictions on  $F$ . Computing the posterior risk, however, is a formidable task. In much of the operations research and engineering literature, the focus has been on the choice of conjugate priors and likelihoods that lead to easy integration. However, very few models actually conform to these conditions in practice, and one must resort to some computational approximation to the desired integral. This leads us to the development of a computational framework for Bayes-predictive stochastic programs.

### 3.1 Computational framework for Bayes-Predictive Stochastic Programs

Recall, (RVB) in the introduction and consider the expression for  $\gamma < 0$ . Multiplying by  $-\gamma$  on either side of the equation, we obtain

$$\log \mathbb{E}_\pi [\exp(-\gamma R(a, \theta))] \geq \max_{q \in \mathcal{Q}} \{-\gamma E_q[R(a, \theta)] - \text{KL}(q \| \pi_n)\} =: \mathcal{F}(a; q(\cdot), \tilde{X}_n, \gamma), \quad (\text{RSVB})$$

where  $\mathcal{Q}$  is a family of distributions with respect to which the expectation on the RHS is tractable. Note, we focus on the  $\gamma < 0$  case above, and the same theoretical insights will hold when  $\gamma > 0$ . With an appropriate choice of  $\mathcal{Q}$ , the optimization on the right hand side (RHS) yields a good approximation to the log-exponential risk measurement on the left hand side (LHS).

For brevity, for a given  $a \in \mathcal{A}$  and  $\bar{\gamma}' = -\gamma > 0$ , we define the RSVB approximation to the true posterior  $\pi(\theta | \tilde{X}_n)$  as

$$q_{a, \bar{\gamma}'}^*(\theta | \tilde{X}_n) := \operatorname{argmax}\{q \in \mathcal{Q} : \mathcal{F}(a; q(\cdot), \tilde{X}_n, -\bar{\gamma}')\}$$

and the RSVB optimal decision as

$$\mathbf{a}_{\text{RS}}^* := \operatorname{argmin}_{a \in \mathcal{A}} \mathbb{E}_{q_{a, \bar{\gamma}'}^*(\theta | \tilde{X}_n)}[f(G(a, \theta))],$$

where  $G(a, \theta)$  is the model risk and  $f(\cdot) : \mathbb{R} \mapsto \mathbb{R}$  is a monotonic mapping.

**Example 3.1.** If  $\mathcal{Q} = \mathcal{M}$ ,  $R(\cdot, \cdot) = \text{constant}$ , and  $f(x) = x$ , then it is easy to observe that  $\mathbf{a}_{\text{RS}}^*$  is the Bayes posterior risk minimizer, denoted as  $\mathbf{a}_{\text{OS}}^*$ .

**Example 3.2.** If  $\mathcal{Q} \subset \mathcal{M}$ ,  $R(\cdot, \cdot) = \log G(\cdot, \cdot)$  (assuming positivity of  $G$ ),  $f(x) = \log x$  and  $\bar{\gamma}' = 1$ , we recover the loss-calibrated VB method (see below).

Examples of  $\mathcal{Q}$  include the family of Gaussian distributions, delta functions, or the family of factorized ‘mean-field’ distributions that discard correlations between components of  $\theta$ . The choice of  $\mathcal{Q}$  is decisive in determining the performance of the algorithm. In general, however the requirements on  $\mathcal{Q}$  are minimal, and part of the analysis in this paper is to articulate sufficient conditions on  $\mathcal{Q}$  that ensure small optimality gap for the optimal decision,  $\mathbf{a}_{\text{RS}}^*$ ; this establishes the “goodness” of the procedure.

Note that the RSVB algorithm described above is idealized - clearly the objective  $\mathcal{F}(a; q(\cdot), \tilde{X}_n, -\bar{\gamma}')$  cannot be computed since it requires the calculation of the posterior distribution - the very object we are approximating! Note, however that optimizing  $\mathcal{F}(a; q(\cdot), \tilde{X}_n, -\bar{\gamma}')$  is equivalent to optimizing  $\{\bar{\gamma}' \mathbb{E}_q[R(a, \theta)] - \text{KL}(q(\theta) \| p(\theta, \tilde{X}_n))\}$ , where  $p(\theta, \tilde{X}_n)$  is known, and for which the optimizers are the same. Since our focus is on bounding the optimality gap, in the remainder of the paper any reference to the RSVB algorithm is an allusion to the idealized objective  $\mathcal{F}(a; q(\cdot), \tilde{X}_n, -\bar{\gamma}')$ .

Next, we show that the naive and loss-calibrated variational Bayes algorithm are the special cases of RSVB.

### 3.1.1 The naive variational Bayes algorithm

The naive variational Bayes algorithm (hereafter referred to as ‘naive VB’) is a *separated estimation and optimization* (SEO) approach wherein we first compute an approximation  $q^* := q^*(\theta|\tilde{X}_n)$  to the posterior distribution  $\pi(\theta|\tilde{X}_n)$  using VB, compute the approximate posterior risk  $\mathbb{E}_{q^*}[G(a, \theta)]$ , followed by an optimization over the decisions. The naive VB algorithm is summarized in Algorithm 1.

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#### Algorithm 1: Naive VB

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**Input** :  $G(\cdot, \cdot)$ ,  $\tilde{X}_n$ ,  $\mathcal{Q}$

**Output**:  $\mathbf{a}_{\text{NV}}^*$

Step 1. Compute approximate posterior:  $q_n^* := \arg \min_{q \in \mathcal{Q}} \text{KL}(q(\cdot) \parallel \pi(\cdot|\tilde{X}_n))$ ;

Step 2. Compute:  $\mathbf{a}_{\text{NV}}^* := \arg \min_{a \in \mathcal{A}} \int_{\theta} G(a, \theta) q_n^*(d\theta)$

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Observe that , for  $R(\cdot, \cdot) = \text{constant}$ ,  $q_{a, \tilde{\gamma}'}^*(\theta|\tilde{X}_n)$  is same as  $q^*(\theta|\tilde{X}_n)$ , that is

$$q^*(\theta|\tilde{X}_n) = \arg \min_{\tilde{q} \in \mathcal{Q}} \text{KL}(\tilde{q}(\theta) \parallel \pi(\theta|\tilde{X}_n)) \quad (2)$$

and for  $f(x) = x$ , the optimal decision  $\mathbf{a}_{\text{RS}}^*$  is same as  $\mathbf{a}_{\text{NV}}^* \in \arg \min_{a \in \mathcal{A}} \mathbb{E}_{q^*}[G(a, \theta)]$ . We again note that the naive VB algorithm described above is idealized and the objective can be lower bounded by  $\text{KL}(\tilde{q}(\theta) \parallel p(\theta, \tilde{X}_n))$ . This is called the *evidence lower bound* (ELBO) in the VB literature, and calculating this objective only involves known terms. From an optimization perspective, minimizing the ELBO is equivalent to minimizing (2). In the remainder of the paper any reference to the naive VB algorithm implies the ‘idealized’ objective in (2).

### 3.1.2 Loss-calibrated variational Bayes algorithm

The so-called Loss-calibrated variational Bayes (LCVB) predictive method can be recovered from RSVB, as noted in Example 2, by applying  $R(a, \theta) = \log G(a, \theta)$ , where  $G(\cdot, \cdot) > 0$  and with  $\tilde{\gamma}' = 1$ , so that

$$\log \mathbb{E}_{\pi(\theta|\tilde{X}_n)}[e^{\log G(a, \theta)}] \geq \max_{q \in \mathcal{Q}} \{ \mathbb{E}_q[\log G(a, \theta)] - \text{KL}(q(\theta) \parallel \pi(\theta|\tilde{X}_n)) \}.$$

Now, using the monotonicity of the minimum functional, we obtain

$$\begin{aligned} \min_{a \in \mathcal{A}} \log \mathbb{E}_{\pi(\theta|\tilde{X}_n)}[G(a, \theta)] &\geq \min_{a \in \mathcal{A}} \max_{q \in \mathcal{Q}} \{ -\text{KL}(q(\theta) \parallel \pi(\theta|\tilde{X}_n)) + \mathbb{E}_q[\log G(a, \theta)] \} \\ &=: \min_{a \in \mathcal{A}} \max_{q \in \mathcal{Q}} \mathcal{F}_{\text{LC}}(a; q(\theta), \tilde{X}_n). \end{aligned} \quad (\text{LCVB})$$

We call  $\mathcal{F}_{\text{LC}}(\cdot; \cdot, \cdot)$  the *loss-calibrated VB* loss function. For brevity, for a given  $a \in \mathcal{A}$ , we define the loss-calibrated VB approximation to the true posterior  $\pi(\theta|\tilde{X}_n)$  as

$$q_a^*(\theta|\tilde{X}_n) := \arg \max_{q \in \mathcal{Q}} \mathcal{F}_{\text{LC}}(a; q(\cdot), \tilde{X}_n)$$

and the loss-calibrated optimal decision as

$$\mathbf{a}_{\text{LC}}^* := \arg \min_{a \in \mathcal{A}} \max_{q \in \mathcal{Q}} \mathcal{F}(a; q(\theta), \tilde{X}_n).$$



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**Algorithm 2:** Loss-calibrated VB

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**Input** :  $G(\cdot, \cdot)$ ,  $\tilde{X}_n$ ,  $\mathcal{Q}$ **Output:**  $\mathbf{a}_{\text{LC}}^*$ Step 1. Compute:  $\mathbf{a}_{\text{LC}}^* := \arg \min_{a \in \mathcal{A}} \max_{q \in \mathcal{Q}} -\text{KL}(q(\cdot) \| \pi(\cdot | \tilde{X}_n)) + \int \log G(a, \theta) q(\theta) d\theta$ .

---

LCVB is summarized in Algorithm 2. We observe that Lacoste-Julien et al. (2011) provide an alternative derivation of LCVB (presented in the appendix). However, our derivation demonstrates that (LCVB) is but a special case of a larger family of variational algorithms defined using (RVB). Again, we note that the objective in (LCVB) is idealized, and the KL divergence term can be bounded below by  $\text{KL}(q(\theta) \| p(\theta, \tilde{X}_n))$ , and any reference to the loss-calibrated VB method implies the idealized objective.

Ignoring the decision optimization, observe that (RSVB) shows that the maximization in the lower bound computes a regularized approximate posterior. Regularized Bayesian inference Zhu et al. (2014) views posterior computation as a variational inference problem with constraints on the posterior space represented as bounds on certain expectations with respect to the approximate posterior. Thus, the inner maximization can also be viewed as a regularized Bayesian inference procedure, where regularization constraints are imposed through  $\mathbb{E}_q[f(\cdot)]$ . However, note that the full optimization problem in our setting also involves a minimization over the decisions as well (which does not exist in the regularized Bayesian inference procedure).

**A special case.** As an interesting aside, fix  $a \in \mathcal{A}$  and let  $q_a^{**} := \arg \max_{q \in \mathcal{Q}} \mathcal{F}(a; q(\theta), \tilde{X}_n)$ , and assume that  $\mathcal{Q} = \mathcal{M}$ . In this case, it is clear that  $q_a^{**}(\cdot) = p(\cdot | \tilde{X}_n) G(a, \cdot) P(a, \tilde{X}_n)^{-1}$ , where  $P(a, \tilde{X}_n) = \int \pi(\theta | \tilde{X}_n) G(a, \theta) d\theta$ . Let  $P^2(a, \tilde{X}_n) := \int p(\theta | \tilde{X}_n) G(a, \theta)^2 d\theta$ . Next, for all  $a \in \mathcal{A}$  compute the optimal expected predicted loss

$$\begin{aligned} V^{**} &:= \min_{a \in \mathcal{A}} \mathbb{E}_{q_a^{**}} [G(a, \theta)] \\ &= \min_{a \in \mathcal{A}} \frac{P^2(a, \tilde{X}_n)}{P(a, \tilde{X}_n)}. \end{aligned}$$

Recognize that the expression on the right hand side above (for each  $a$ ) is the size-biased expected risk with respect to the true posterior distribution. The size-biased expectation can also be viewed as a “mean-standard deviation” type risk measure Furman and Zitikis (2008); note that

$$\frac{P^2(a, \tilde{X}_n)}{P(a, \tilde{X}_n)} = P(a, \tilde{X}_n) + \frac{\text{Var}(G(a, \theta))}{P(a, \tilde{X}_n)}.$$

Now, if the family  $\mathcal{Q}$  is restrictive,  $q_a^{**}$  only represents an approximation to the true posterior, and the corresponding expectation  $\mathbb{E}_{q_a^{**}} [G(a, \theta)]$  is only an approximation to the risk-measure above. Observe, further, that the lower bound calculation in (LCVB) is also upper bounded by  $\mathbb{E}_{q_a^{**}} [G(a, \theta)]$  but, in general, it is unclear when the latter equals the true posterior expected loss.

## 4 Finite Sample Bounds on the Optimality Gap

In this section we establish finite sample bounds on the Bayes optimality gap. Our results in here identify the regularity conditions on the data generating model  $\mathcal{P} := \{P_\theta, \theta \in \Theta\}$ , the prior

distribution  $\pi(\theta)$ , the variational family  $\mathcal{Q}$ , the risk function  $R(a, \theta)$  and the model risk function  $G(a, \theta)$  to obtain finite sample bounds on the (Bayes) optimality gap. Thus far, we have not placed any restrictions on the joint distribution of  $\tilde{X}_n$ . For the remainder of the paper, we operate under the following condition.

**Assumption 4.1.** The covariates  $\tilde{X}_n = (X_1, \dots, X_n)$  are independent and identically distributed (i.i.d).

Our finite sample bounds, in essence, depend on the ‘size’ of the model (sub-)space measured using covering numbers. First, recall the definition of covering numbers:

**Definition 4.1** (Covering numbers). Let  $\mathcal{P} := \{P_\theta, \theta \in \Theta\}$  be a parametric family of distributions and  $d : \mathcal{P} \times \mathcal{P} \mapsto [0, \infty)$  be a metric. An  $\epsilon$ -cover of a subset  $\mathcal{P}_K := \{P_\theta : \theta \in K \subset \Theta\}$  of the parametric family of distributions is a set  $K' \subset K$  such that, for each  $\theta \in K$  there exists a  $\theta' \in K'$  that satisfies  $d(P_\theta, P_{\theta'}) \leq \epsilon$ . The  $\epsilon$ -covering number of  $\mathcal{P}_K$  is

$$N(\epsilon, \mathcal{P}_K, d) = \min\{\text{card}(K') : K' \text{ is an } \epsilon\text{-cover of } K\},$$

where  $\text{card}(\cdot)$  represents the cardinality of the set.

Next, recall the definition of test function (Schwartz, 1965):

**Definition 4.2** (Test function). Any  $\mathcal{S}^n$ -measurable sequence of functions  $\{\phi_n\}$ ,  $\phi_n : \tilde{X}_n \mapsto [0, 1] \forall n \in \mathbb{N}$ , is a *test of a hypothesis* that a probability measure on  $\mathcal{S}$  belongs to a given set against the hypothesis that it belongs to an alternative set. The test  $\phi_n$  is *consistent* for hypothesis  $P_0$  against the alternative  $P \in \{P_\theta : \theta \in \Theta \setminus \{\theta_0\}\}$  if  $\mathbb{E}_{P_\theta}[\phi_n] \rightarrow \mathbb{1}_{\{\Theta \setminus \{\theta_0\}\}}(\theta)$ ,  $\forall \theta \in \Theta$  as  $n \rightarrow \infty$ .

A classic example could be a test function  $\phi_n^{\text{KS}} = \mathbb{1}_{\{\text{KS}_n > K_\nu\}}(\theta)$  that is constructed using the Kolmogorov-Smirnov statistic  $\text{KS}_n := \sup_t |\mathbb{F}_n(t) - \mathbb{F}_\theta(t)|$ , where  $\mathbb{F}_n(t)$  and  $\mathbb{F}_\theta(t)$  are the empirical and true distribution respectively, and  $K_\nu$  is the confidence level. If the null hypothesis is true, the Glivenko-Cantelli theorem shows that the KS statistic converges to zero as the number of samples increases to infinity.

Next, define a function  $L_n : \Theta \times \Theta \mapsto \mathbb{R}$  which measures the distance between pairs of distributions with parameters  $(\theta_1, \theta_2)$ . At the outset, we assume that  $L_n(\theta_1, \theta_2)$  is always positive, and  $C, C_0, C_1, C_2$ , and  $C_3$  are given positive constants. We use the following ‘control sequence’ to establish our probabilistic bounds.

**Definition 4.3** (Control Sequence).  $\{\epsilon_n\}$  is a sequence such that  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  and  $n\epsilon_n^2 \geq 1$ .

In order to bound the optimality gap, we require some control over how quickly the posterior distribution concentrates at the true parameter  $\theta_0$ . Our next assumption in terms of a verifiable test condition on the model (sub-)space is one of the conditions required to quantify this rate.

**Assumption 4.2.** Fix  $n \geq 1$ . Then, for any  $\epsilon > \epsilon_n$  in Definition 4.3,  $\exists$  a test function  $\phi_n : \tilde{X}_n \mapsto [0, 1]$  and set  $\Theta_n(\epsilon) \subseteq \Theta$  such that

$$(i) \quad \mathbb{E}_{P_0^n}[\phi_n] \leq C_0 \exp(-Cn\epsilon^2), \text{ and}$$

$$(ii) \quad \sup_{\{\theta \in \Theta_n(\epsilon) : L_n(\theta, \theta_0) \geq C_1 n \epsilon^2\}} \mathbb{E}_{P_\theta^n}[1 - \phi_n] \leq \exp(-Cn\epsilon^2).$$

Observe that Assumption 4.2(i) quantifies the rate at which a type 1 error diminishes with the sample size, while the condition in Assumption 4.2(ii) quantifies that of a type 2 error. Notice that both of these are stated through test functions; indeed, what is required are consistent test functions. Opportunely, (Ghosal et al., 2000, Theorem 7.1) (stated below in Lemma 4.1 for completeness) roughly implies that a bounded model subspace  $\mathcal{P}$  (the size of which is measured using covering numbers) guarantees the existence of *consistent* test functions, to test the null hypothesis that the true parameter is  $\theta_0$  against an alternate hypothesis – the alternate being defined using the ‘distance function’  $L_n(\theta_1, \theta_2)$ . Subsequently, we will use a specific distance function to obtain finite sample bounds for the optimal decisions and values. In some problem instances, it is also possible to construct consistent test functions directly without recourse to Lemma 4.1. We demonstrate this in Section 5 below.

Next, we assume a condition on the prior distribution that ensures that it provides sufficient mass to the set  $\Theta_n(\epsilon) \subseteq \Theta$ , as defined above in Assumption 4.2.

**Assumption 4.3.** Fix  $n \geq 1$ . Then, for any  $\epsilon > \epsilon_n$  in Definition 4.3 the prior distribution satisfies

$$\mathbb{E}_\Pi[\mathbb{1}_{\{\Theta_n(\epsilon)\}}] \leq \exp(-Cn\epsilon^2).$$

Notice that Assumption 4.3 is trivially satisfied if  $\Theta_n(\epsilon) = \Theta$ . The next assumption ensures that the prior distribution places sufficient mass around a neighborhood – defined using Rényi divergence – of the true parameter  $\theta_0$ .

**Assumption 4.4.** Fix  $n \geq 1$  and a constant  $\lambda > 0$ . Let  $A_n := \{\theta \in \Theta : D_{1+\lambda}(P_0^n \| P_\theta^n) \leq C_3 n \epsilon_n^2\}$ , where  $D_{1+\lambda}(P_0^n \| P_\theta^n) := \frac{1}{\lambda} \log \int \left( \frac{dP_0^n}{dP_\theta^n} \right)^\lambda dP_0^n$  is the Rényi Divergence between  $P_0^n$  and  $P_\theta^n$ , assuming  $P_0^n$  is absolutely continuous with respect to  $P_\theta^n$ . The prior distribution satisfies

$$\Pi\{A_n\} \geq \exp(-nC_2\epsilon_n^2).$$

Notice that the set  $A_n$  defines a neighborhood of the distribution corresponding to  $\theta_0$  in  $\mathcal{P}$ . The assumption guarantees that the prior distribution covers this neighborhood with positive mass. This is a standard assumption and if it is violated then the posterior too will place no mass in this neighborhood ensuring asymptotic inconsistency.

It is apparent by the first term in (RSVB) that in addition to Assumption 4.2, 4.3, and 4.4, we also require regularity conditions on the risk function  $R(a, \cdot)$ , that is on  $(F \circ G)(a, \cdot)$  for a given  $G(a, \cdot)$ . Our next assumption restricts the prior distribution with respect to  $R(a, \theta)$ .

**Assumption 4.5.** Fix  $n \geq 1$ . For any  $\epsilon > \epsilon_n$ ,  $a \in \mathcal{A}$ , and  $\bar{\gamma}' > 0$ ,

$$\int_{\{\bar{\gamma}' R(a, \theta) > C_4 n \epsilon_n^2\}} e^{\bar{\gamma}' R(a, \theta)} \pi(\theta) d\theta \leq \exp(-C_5 n \epsilon_n^2),$$

where  $C_4$  and  $C_5$  are positive constants.

Note that the set  $\{\bar{\gamma}' R(a, \theta) > C_4 n \epsilon_n^2\}$  represents the subset of the model space where the risk  $R(a, \theta)$  (for a fixed decision  $a$ ) is large, and the prior is assumed to place small mass over such sets. Finally, we also require the following condition lower bounding the risk function  $R$ .

**Assumption 4.6.** For any  $\bar{\gamma}' > 0$ ,  $R(a, \theta)$  is assumed to satisfy

$$W := \inf_{\theta \in \Theta} \inf_{a \in \mathcal{A}} e^{\bar{\gamma}' R(a, \theta)} > 0.$$

Note that any risk function which is bounded from below satisfies this condition.

We can now state our first result, establishing an upper bound on the expected deviation from the true model  $P_0$ , measured using distance function  $L_n(\cdot, \theta_0)$ , under the RSVB approximate posterior. We also note that the following result generalizes Theorem 2.1 of Zhang and Gao (2017), which is exclusively for their naive VB posterior.

**Theorem 4.1.** Fix  $a' \in \mathcal{A}$  and  $\bar{\gamma}' > 0$ . For any  $L_n(\theta, \theta_0) \geq 0$ , under Assumptions 4.2, 4.3, 4.4, 4.5, and 4.6, and for  $\min(C, C_4 + C_5) > C_2 + C_3 + C_4 + 2$  and

$$\eta_n^R(\bar{\gamma}') := \frac{1}{n} \inf_{q \in \mathcal{Q}} \mathbb{E}_{P_0^n}, \left[ \int_{\Theta} q(\theta) \log \frac{q(\theta)}{\pi(\theta|\tilde{X}_n)} d\theta - \bar{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta \right],$$

the RSVB approximator of the true posterior  $q_{a', \bar{\gamma}'}^*(\theta|\tilde{X}_n)$  satisfies,

$$\mathbb{E}_{P_0^n} \left[ \int_{\Theta} L_n(\theta, \theta_0) q_{a', \bar{\gamma}'}^*(\theta|\tilde{X}_n) d\theta \right] \leq M n (\epsilon_n^2 + \eta_n^R(\bar{\gamma}')), \quad (3)$$

for some constant  $M$  that depends on the  $C, C_0, C_1, C_4, W$ , and  $\lambda$ , and  $\epsilon_n^2 + \eta_n^R(\bar{\gamma}') \geq 0 \ \forall \bar{\gamma}' > 0$  and for all  $n \geq 1$ .

Since the result in Theorem 4.1 holds for any positive distance function, we now assume that

$$L_n(\theta, \theta_0) \equiv L_n^f(\theta, \theta_0) = n \left( \sup_{a \in \mathcal{A}} |f(G(a, \theta)) - f(G(a, \theta_0))| \right)^2, \quad (4)$$

where  $f(\cdot)$  defines  $\mathbf{a}_{\text{RS}}^*$  in Section 3.1. Notice that for a given  $\theta$ ,  $n^{-1/2} \sqrt{L_n^f(\theta, \theta_0)}$  is the uniform distance between the  $f(G(a, \theta))$  and  $f(G(a, \theta_0))$ . Intuitively, Theorem 4.1 implies that the expected uniform difference  $\frac{1}{n} L_n^f(\theta, \theta_0)$  with respect to the RSVB approximate posterior is  $O(\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))$ , and if  $\epsilon_n^2 + \eta_n^R(\bar{\gamma}') \rightarrow 0$  as  $n \rightarrow \infty$  then it converges to zero at that rate. The additional term  $\eta_n^R(\bar{\gamma}')$  emerges from the posterior approximation and depends on the choice of the variational family  $\mathcal{Q}$ , risk function  $R(\cdot, \cdot)$ , and the parameter  $\bar{\gamma}'$ . Later in this section, we specify the conditions on the family of distributions  $\mathcal{P}$ , the prior and the variational family  $\mathcal{Q}$  to ensure that  $\eta_n^R(\bar{\gamma}') \rightarrow 0$  as  $n \rightarrow \infty$ .

In order to use (4) we must demonstrate that it satisfies Assumption 4.2. This can be achieved by constructing bespoke test functions for a given  $f$  and  $G(a, \theta)$ . We demonstrate this approach by an example in Section 5. Below, we provide sufficient conditions for the existence of the test functions. These conditions are typically easy to verify when the loss functions  $\ell(\cdot, \cdot)$  are bounded, for instance.

First, recall the definition of the Hellinger distance

**Definition 4.4** (Hellinger distance). The Hellinger distance  $h(\theta_1, \theta_2)$  between the two probability distributions  $P_{\theta_1}$  and  $P_{\theta_2}$  is defined as

$$h(\theta_1, \theta_2)^2 = \int \left( \sqrt{dP_{\theta_1}} - \sqrt{dP_{\theta_2}} \right)^2.$$

To show the existence of test functions, as required in Assumption 4.2, we will use the following result from (Ghosal et al., 2000, Theorem 7.1), that is applicable only to distance measures that are bounded above by the Hellinger distance.

**Lemma 4.1** (Theorem 7.1 of (Ghosal et al., 2000)). Suppose that for some non-increasing function  $D(\epsilon)$ , some  $\epsilon_n > 0$  and for every  $\epsilon > \epsilon_n$ ,

$$N\left(\frac{\epsilon}{2}, \{P_\theta : \epsilon \leq m(\theta, \theta_0) \leq 2\epsilon\}, m\right) \leq D(\epsilon),$$

where  $m(\cdot, \cdot)$  is any distance measure bounded above by Hellinger distance. Then for every  $\epsilon > \epsilon_n$ , there exists a test  $\phi_n$  (depending on  $\epsilon > 0$ ) such that, for every  $j \geq 1$ ,

$$\begin{aligned} \mathbb{E}_{P_0^n}[\phi_n] &\leq D(\epsilon) \exp\left(-\frac{1}{2}n\epsilon^2\right) \frac{1}{1 - \exp\left(-\frac{1}{2}n\epsilon^2\right)}, \text{ and} \\ \sup_{\{\theta \in \Theta_n(\epsilon) : m(\theta, \theta_0) > j\epsilon\}} \mathbb{E}_{P_\theta^n}[1 - \phi_n] &\leq \exp\left(-\frac{1}{2}n\epsilon^2 j\right). \end{aligned}$$

In the subsequent paragraph, we state further assumptions on the model risk to show  $L_n^f(\cdot, \cdot)$  as defined in (4) satisfies Assumption 4.2. For brevity we denote  $n^{-1/2}\sqrt{L_n^f(\theta, \theta_0)}$  by  $d_f(\cdot, \cdot)$ , that is

$$d_f(\theta_1, \theta_2) := \sup_{a \in \mathcal{A}} |f(G(a, \theta_1)) - f(G(a, \theta_2))|, \quad \forall \{\theta_1, \theta_2\} \in \Theta \quad (5)$$

and the covering number of the set  $T(\epsilon) := \{P_\theta : d_f(\theta, \theta_0) < \epsilon\}$  as  $N(\delta, T(\epsilon), d_f)$ , where  $\delta > 0$  is the radius of each ball in the cover. We assume that the model risk  $G(a, \cdot)$  satisfies the following bound.

**Assumption 4.7.** The model risk satisfies

$$|f(G(a, \theta)) - f(G(a, \theta_0))| \leq K_1^f(a) d_{TV}(\theta, \theta_0),$$

where  $d_{TV}(\theta, \theta_0) = \frac{1}{2} \int |dP_\theta(x) - dP_{\theta_0}(x)|$  is the total variation distance. We further assume that  $\sup_{a \in \mathcal{A}} K_1^f(a) < \infty$ .

Using the definition of model risk  $G(a, \theta_0)$ , observe that Assumption 4.7 is trivially satisfied if  $f(\cdot)$  is Lipschitz continuous and the loss function  $\ell(x, a)$  is bounded in  $x$  for a given  $a \in \mathcal{A}$ , where  $\mathcal{A}$  is compact. Since, the total variation distance is bounded above by the Hellinger distance (Gibbs and Su, 2002), it follows that we can apply Lemma 4.1 to the metric  $d_f(\cdot, \cdot)$  defined in (5). In addition, we also assume a further regularity condition on the model risk.

**Assumption 4.8.** The model risk is locally Lipschitz in  $\theta$ ; that is for every  $\theta \in \mathbb{R}^d$ ,  $\exists$  a neighborhood  $O$  of  $\theta$  such that  $G(a, \tilde{\theta})$  is Lipschitz continuous for  $\tilde{\theta} \in O$ .

Since Euclidean space is locally compact,  $G(a, \theta)$  is locally Lipschitz if and only if it is Lipschitz in every compact subset  $\mathcal{C}$  of  $\mathbb{R}^s$ ; that is, for any  $\theta_1$  and  $\theta_2$  in  $\mathcal{C}$ ,

$$|G(a, \theta_1) - G(a, \theta_2)| \leq K_{\mathcal{C}}(a) \|\theta_1 - \theta_2\|,$$

where  $K_{\mathcal{C}}(a)$  is Lipschitz constant  $\forall \theta \in \mathcal{C} \subset \mathbb{R}^s$  and we assume that  $\sup_{a \in \mathcal{A}} K_{\mathcal{C}}(a) < \infty$ . Next, we impose the following condition on the monotonic transform function  $f(\cdot)$ .

**Assumption 4.9.** The monotonic function  $f: \mathbb{R} \mapsto \mathbb{R}$  is locally Lipschitz.

For example, in LCVB  $f(\cdot)$  is the logarithm function which is monotone and locally Lipschitz. We can now show that the covering number of the set  $T(\epsilon)$  satisfies

**Lemma 4.2.** Given  $\epsilon > \delta > 0$ , and under Assumption 4.8,

$$N(\delta, T(\epsilon), d_f) < \left( \frac{2\epsilon}{\delta} + 2 \right)^s. \quad (6)$$

Observe that the RHS in (6) is a decreasing function of  $\delta$ , infact for  $\delta = \epsilon/2$ , it is a constant in  $\epsilon$ . Therefore, using Lemmas 4.1 and 4.2, we show in the following result that  $L_n^f(\theta, \theta_0)$  in (4) satisfies Assumption 4.2.

**Lemma 4.3.** Fix  $n \geq 1$ . For a given  $\epsilon_n > 0$  and every  $\epsilon > \epsilon_n$ , such that  $n\epsilon_n^2 \geq 1$ . Under Assumption 4.7, 4.8, and 4.9,  $L_n^f(\theta, \theta_0) = n(\sup_{a \in \mathcal{A}} |f(G(a, \theta)) - f(G(a, \theta_0))|)^2$  satisfies

$$\mathbb{E}_{P_0^n}[\phi_n] \leq C_0 \exp(-Cn\epsilon^2), \quad (7)$$

$$\sup_{\{\theta \in \Theta: L_n^f(\theta, \theta_0) \geq C_1 n \epsilon^2\}} \mathbb{E}_{P_\theta^n}[1 - \phi_n] \leq \exp(-Cn\epsilon^2), \quad (8)$$

where  $C_0 = 2 * 10^s$  and  $C = \frac{C_1}{2(\sup_{a \in \mathcal{A}} K_1^f(a))^2}$  for any constant  $C_1 > 1$ .

Since this  $L_n^f(\theta, \theta_0)$  satisfies Assumption 4.2, Theorem 4.1 implies the following finite sample bound.

**Corollary 4.1.** Fix  $a' \in \mathcal{A}$  and  $\bar{\gamma}' > 0$ . Let  $\epsilon_n$  be a sequence such that  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ ,  $n\epsilon_n^2 \geq 1$  and

$$L_n^f(\theta, \theta_0) = n \left( \sup_{a \in \mathcal{A}} |f(G(a, \theta)) - f(G(a, \theta_0))| \right)^2.$$

Then under the Assumptions of Theorem 4.1 and Lemma 4.3 ; for  $C = \frac{C_1}{2(\sup_{a \in \mathcal{A}} K_1^f(a))^2}$ ,  $C_0 = 2 * 10^s$ ,  $C_1 > 1$  such that  $\min(C, C_4 + C_5) > C_2 + C_3 + C_4 + 2$ , and for  $\eta_n^R(\bar{\gamma}')$  as defined in Theorem 4.1, the RSVB approximator of the true posterior  $q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n)$  satisfies,

$$\mathbb{E}_{P_0^n} \left[ \int_{\Theta} L_n^f(\theta, \theta_0) q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta \right] \leq Mn (\epsilon_n^2 + \eta_n^R(\bar{\gamma}')), \quad (9)$$

where constant M depends only on  $C, C_0, C_1, C_4, W$  and  $\lambda$  and  $\epsilon_n^2 + \eta_n^R(\bar{\gamma}') \geq 0$  for all  $n \geq 1$ .

We now impose further conditions on the variational family  $\mathcal{Q}$  and the likelihood family  $\mathcal{P}$  to ensure that  $\epsilon_n^2 + \eta_n^R(\bar{\gamma}') \rightarrow 0$ , as  $n \rightarrow \infty$ . By definition  $\epsilon_n^2 \rightarrow 0$  as  $n \rightarrow \infty$ , and it remains to understand the conditions under which  $\eta_n^R(\bar{\gamma}')$  converges to zero.

**Assumption 4.10.**

- (i) Local asymptotic normality(LAN): Fix  $\theta \in \Theta$ . The sequence of log-likelihood functions  $\{\log P_\theta^n(\tilde{X}_n)\}$  (where  $\log P_\theta^n(\tilde{X}_n) = \sum_{i=1}^n \log p(x_i|\theta)$ ) satisfies a *local asymptotic normality* (LAN) condition, if there exist matrices  $r_n$  and  $I(\theta)$ , and random vectors  $\{\Delta_{n,\theta}\}$  such that  $\Delta_{n,\theta} \Rightarrow \mathcal{N}(0, I(\theta)^{-1})$  as  $n \rightarrow \infty$ , and for every compact set  $K \subset \mathbb{R}^d$

$$\sup_{h \in K} \left| \log P^n(\theta + r_n^{-1}h) - \log P^n(\theta) - h^T I(\theta) \Delta_{n,\theta} + \frac{1}{2} h^T I(\theta) h \right| \xrightarrow{P_{\theta_0}^n} 0 \text{ as } n \rightarrow \infty ,$$

- (ii) there exists a  $q(\theta) \in \mathcal{Q}$  such that  $\text{KL}(q(\theta) \parallel \pi(\theta|\tilde{X}_n)) < \infty, \forall n \geq 1$ , and
- (iii) there exists a sequence of distributions  $\{q_n(\theta)\} \in \mathcal{Q}$  such that it converges weakly to  $\delta_{\theta_0}$  at the rate of  $r_n$ .

The LAN condition, typical in asymptotic analyses of Bayesian models (Van der Vaart, 2000, Chapter 10), holds for a wide variety of models and affords significant flexibility in the analysis by allowing the likelihood to be asymptotically approximated by a scaled Gaussian centered around  $\theta_0$ . All models,  $P_\theta$ , that are twice differentiable in the parameter  $\theta$  satisfy the LAN condition with  $r_n = \sqrt{n}I$ , where  $I$  is an identity matrix. The second assumption, that is on  $\mathcal{Q}$ , ensures that the VB optimization problem is well defined for all  $n \geq 1$ . Since the variational family  $\mathcal{Q}$  is typically assumed to contain all possible distributions of a particular type (for instance, all Gaussians or all exponential family distributions), it must therefore contain a distribution that satisfies the third assumption. The existence of such  $\{q(\theta)\} \in \mathcal{Q}$  can be easily verified for a given family  $\mathcal{Q}$ .

**Proposition 4.1.** Fix  $\bar{\gamma}' > 0$ . Under Assumption 4.10,

$$\lim_{n \rightarrow \infty} \eta_n^R(\bar{\gamma}') = \lim_{n \rightarrow \infty} \frac{1}{n} \inf_{q \in \mathcal{Q}} \mathbb{E}_{P_0^n} \left[ \int_{\Theta} q(\theta) \log \frac{q(\theta)}{\pi(\theta|\tilde{X}_n)} d\theta - \bar{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta \right] = 0.$$

Our next result shows that, under the RSVB approximate posterior distribution  $q_{a', \bar{\gamma}'}^*(\theta|\tilde{X}_n)$ ,  $L_n^f(\cdot, \cdot)$  as defined in (4) converges to zero at the rate  $(\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))$  in  $P_0^n$ -probability. Here,  $Q_{a', \bar{\gamma}'}^*(S|\tilde{X}_n) := \int_S q_{a', \bar{\gamma}'}^*(\theta|\tilde{X}_n) d\theta$ .

**Corollary 4.2.** For any  $a' \in \mathcal{A}$ ,  $\bar{\gamma}' > 0$ , and diverging sequence  $M_n$ ,

$$\lim_{n \rightarrow \infty} Q_{a', \bar{\gamma}'}^* \left[ \left\{ \theta \in \Theta : \left( \sup_{a \in \mathcal{A}} |f(G(a, \theta)) - f(G(a, \theta_0))| \right)^2 > M_n(\epsilon_n^2 + \eta_n^R(\bar{\gamma}')) \right\} \middle| \tilde{X}_n \right] = 0$$

in  $P_0^n$ -probability.

Observe that if  $\sum_{n \geq 1} \frac{1}{M_n} < \infty$ , then the first Borel-Contelli lemma implies that the sequence converges almost-surely. First, recall from Theorem 4.1 that  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  and  $n\epsilon_n^2 \geq 1$ . The diverging sequence  $M_n$  can be chosen in three possible ways. First,  $M_n = o\left(\frac{1}{(\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))^b}\right)$ , for some  $b < 1$ , which ensures that the radius of the ball in Corollary 4.2 decreases to 0 as  $n \rightarrow \infty$ . Second,  $M_n = \left(\frac{1}{\epsilon_n^2 + \eta_n^R(\bar{\gamma}')} \right)$ , in this case ball will be of constant radius 1. Also observe that in the last two cases  $\sum_{n \geq 1} \frac{1}{M_n} = \infty$ , since  $\epsilon_n^2$  is not summable, therefore we do not have almost-sure convergence in these cases. In the final case,  $M_n = o\left(\frac{1}{(\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))^b}\right)$ , for some  $b > 1$  and summable (since,  $n\eta_n^R(\bar{\gamma}') < \infty$

due to Assumption 4.6 and 4.10(ii)). Note that, in this case the radius of the ball will diverge and hence we obtain almost-sure convergence.

In order to further characterize  $\eta_n^R(\bar{\gamma}')$ , we specify conditions on variational family  $\mathcal{Q}$  such that it  $\eta_n^R(\bar{\gamma}') = O(\epsilon_n'^2)$ , for some  $\epsilon_n' \geq \frac{1}{\sqrt{n}}$ . We impose following condition on the variational family  $\mathcal{Q}$  that lets us obtain the bound on  $\eta_n^R(\bar{\gamma}')$ .

**Assumption 4.11.** There exists a sequence of distribution  $\{q_n(\cdot)\}$  in the variational family  $\mathcal{Q}$  such that for a positive constant  $C_9$ ,

$$\frac{1}{n} [\text{KL}(q_n(\theta) \parallel \pi(\theta)) + \mathbb{E}_{q_n(\theta)} [\text{KL}(dP_0^n \parallel p(\tilde{X}_n | \theta))]] \leq C_9 \epsilon_n'^2. \quad (10)$$

Due to Assumption 4.1, that is  $X_i, i \in \{1, 2 \dots n\}$  are i.i.d, observe that

$$\frac{1}{n} \mathbb{E}_{q_n(\theta)} [\text{KL}(dP_0^n \parallel p(\tilde{X}_n | \theta))] = \mathbb{E}_{q_n(\theta)} [\text{KL}(dP_0 \parallel p(\xi | \theta))].$$

Therefore, intuitively the above assumption implies that the variational family must contain sequence of distributions that converges weakly to a dirac delta distribution at the true parameter  $\theta_0$  otherwise the second term in the LHS of (10) will be non-zero. Also, note that the above assumption does not imply that the minimizing sequence  $q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n)$  (automatically) converges weakly to a dirac-delta distribution at the true parameter  $\theta_0$ . It is remarkable that unlike Theorem 2.3 of Zhang and Gao (2017), our condition on  $\mathcal{Q}$  in Assumption 4.11, to obtain a bound on  $\eta_n^R(\bar{\gamma}')$ , does not require the support of the distributions in  $\mathcal{Q}$  to shrink to the true parameter  $\theta_0$ , at some rate, as the numbers of samples increases. Next, we show that

**Proposition 4.2.** Under Assumption 4.11 and for some positive constant  $C_8$  and  $C_9$ ,

$$\eta_n^R(\bar{\gamma}') \leq (C_8 + C_9) \epsilon_n'^2.$$

In section 5, where the likelihood is exponentially distributed, the prior is inverse-gamma (non-conjugate), and the variational family is the family of gamma distributions, we construct a sequence of distributions in the variational family that satisfies Assumption 4.11.

Next, we bound the optimality gap between the approximate optimal decision rule  $\mathbf{a}_{\text{RS}}^*$  and the true optimal decision. The bound, in particular, depends on the curvature of  $f(G(a, \theta_0))$  around the true optimal decision. First, recall the one-sided Hausdorff distance between sets  $A$  and  $B$  in  $\mathbb{R}^s$ :

$$d_H(A \| B) = \sup_{x \in A} d_h(x, B), \text{ where } d_h(x, B) = \inf_{y \in B} \|x - y\| \text{ and } \|\cdot\| \text{ is the Euclidean norm.} \quad (11)$$

Following Pflug (2003) we define a *growth condition* on  $f(G(a, \theta_0))$ .

**Definition 4.5. Growth condition:** Let  $\Psi^f(d) : [0, \infty) \mapsto [0, \infty)$  be a *growth function* if it is strictly increasing as  $d \rightarrow \infty$  and  $\lim_{d \rightarrow 0} \Psi^f(d) = 0$ . Then  $f(G(a, \theta_0))$  satisfies a growth condition with respect to  $\Psi^f(\cdot)$ , if

$$f(G(a, \theta_0)) \geq \inf_{z \in \mathcal{A}} f(G(z, \theta_0)) + \Psi^f \left( d_h \left( a, \arg \min_{z \in \mathcal{A}} f(G(z, \theta_0)) \right) \right). \quad (12)$$



The growth condition above is a generalization of strong-convexity. Indeed, if the  $f(\cdot)$  functional transform of the true model risk is strongly convex, then this condition is automatically satisfied.

**Theorem 4.2.** 1) Fix  $\bar{\gamma}' > 0$ . Suppose that the set  $\mathcal{A}$  is compact and  $f(G(a, \theta_0))$  satisfies the growth condition, with  $\Psi^f(d)$  such that  $\Psi^f(d)/d^\delta = C^f$ ,  $\forall \delta > 0$ . Then, for any  $\tau > 0$ , the  $P_0^n$ -probability of the following event

$$\left\{ \tilde{X}_n : d_h \left( \mathbf{a}_{\text{RS}}^*(\tilde{X}_n), \arg \min_{z \in \mathcal{A}} G(z, \theta_0) \right) \leq \left[ \frac{2\tau [M(\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))]^{1/2}}{C^f} \right]^{\frac{1}{\delta}} \right\}$$

is at least  $1 - \tau^{-1}$ , where  $M$  is the positive constant as defined in Theorem 4.1.

2) Fix  $\bar{\gamma}' > 0$ . Suppose that, there exists an  $n_0$  such that for all  $n \geq n_0$ ,  $\frac{\Psi^f(d_h(\mathbf{a}_{\text{RS}}^*, a^*))}{d_h(\mathbf{a}_{\text{RS}}^*, a^*)^\delta} = C_{n_0}^f$ ,  $\forall \delta > 0$ , where  $a^* = \arg \min_{z \in \mathcal{A}} G(z, \theta_0)$ . Then, for any  $\tau > 0$ , the  $P_0^n$ -probability of the following event

$$\left\{ \tilde{X}_n : d_h \left( \mathbf{a}_{\text{RS}}^*(\tilde{X}_n), \arg \min_{z \in \mathcal{A}} G(z, \theta_0) \right) \leq \left[ \frac{2\tau [M(\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))]^{1/2}}{C_{n_0}^f} \right]^{\frac{1}{\delta}} \right\}$$

is at least  $1 - \tau^{-1}$  for all  $n \geq n_0$ , where  $M$  is the positive constant as defined in Theorem 4.1.

To fix the intuition, suppose  $\delta = 2$  and  $\Psi^f(d) = \frac{C^f}{2}d^2$ , then  $C^f$  represents the Hessian of the true model risk,  $G(a, \theta_0)$ , near its optimizer. It is easy to see from the above result that for larger values of  $C^f$ ,  $\mathbf{a}_{\text{RS}}^*$  converges at a much faster rate than for the smaller values of  $C^f$ . That is, higher the curvature near the optimizer, the faster  $\mathbf{a}_{\text{RS}}^*$  converges. Next, we demonstrate a similar result for the true posterior decision rule  $\mathbf{a}_{\text{OS}}^*$ . The bound, in particular, depends on the curvature of  $G(a, \theta_0)$  around the true optimal decision, since  $f(x) = x$ . The growth function is denoted as  $\Psi^1(\cdot)$ .

**Corollary 4.3.** Suppose that the set  $\mathcal{A}$  is compact and  $G(a, \theta_0)$  satisfies the growth condition, with  $\Psi^1(d)$  such that  $\Psi^1(d)/d^\delta = C^1$ ,  $\forall \delta > 0$ . Then, for any  $\tau > 0$ , the  $P_0^n$ -probability of the following event

$$\left\{ d_h(\mathbf{a}_{\text{OS}}^*, \arg \min_{z \in \mathcal{A}} G(z, \theta_0)) \leq \left[ \frac{2\tau [M(\epsilon_n^2)]^{1/2}}{C^1} \right]^{\frac{1}{\delta}} \right\}$$

is at least  $1 - \tau^{-1}$ , where  $M$  is a positive constant as defined in Theorem 4.1.

Finally, we observe the following probabilistic bound on the gap between  $\mathbf{a}_{\text{RS}}^*$  and  $\mathbf{a}_{\text{OS}}^*$ .

**Corollary 4.4.** Fix  $\bar{\gamma}' > 0$ . Suppose that the set  $\mathcal{A}$  is compact and  $f(G(a, \theta_0))$  satisfies the growth condition, with  $\Psi^f(d)$  such that  $\Psi^f(d)/d^\delta = C^f$ ,  $\forall \delta > 0$  and  $G(a, \theta_0)$  satisfies the growth condition, with  $\Psi^1(d)$  such that  $\Psi^1(d)/d^\delta = C^1$ ,  $\forall \delta > 0$ . Then, for any  $\tau > 0$ , the  $P_0^n$ -probability of the following event

$$\left\{ d_h(\mathbf{a}_{\text{OS}}^*, \mathbf{a}_{\text{RS}}^*) \leq 2 \left[ \frac{2\tau [M(\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))]^{1/2}}{C^f} \right]^{\frac{1}{\delta}} + 2 \left[ \frac{2\tau [M(\epsilon_n^2)]^{1/2}}{C^1} \right]^{\frac{1}{\delta}} \right\}$$

is at least  $1 - 2\tau^{-1}$ , where  $M$  is a positive constant as defined in Theorem 4.1.

## 4.1 Naive VB

The naive VB method completely isolates the statistical estimation problem from the decision-making problem. Recall that , for  $R(\cdot, \cdot) = 0$ ,  $q_{a', \tilde{\gamma}'}^*(\theta|\tilde{X}_n)$  is same as  $q^*(\theta|\tilde{X}_n)$ , that is

$$q^*(\theta|\tilde{X}_n) = \operatorname{argmin}_{\tilde{q} \in \mathcal{Q}} \operatorname{KL}(\tilde{q}(\theta) \parallel \pi(\theta|\tilde{X}_n)) \quad (13)$$

and for  $f(x) = x$ , the optimal decision  $\mathbf{a}_{\text{RS}}^*$  is same as  $\mathbf{a}_{\text{NV}}^* \in \operatorname{argmin}_{a \in \mathcal{A}} \mathbb{E}_{q^*}[G(a, \theta)]$ . Since  $R(\cdot, \cdot) = 0$ , we do not require Assumption 4.5 and 4.6 to obtain analogous result to Theorem 4.1 for Naive VB algorithm. Therefore, the condition on the constants in Theorem 4.1, that is  $\min(C, C_4 + C_5) > C_2 + C_3 + C_4 + 2$  can be reduced to  $C > C_2 + C_3 + 2$  by choosing  $C_4$  to be very small and  $C_5$  to be very large. Also, it is straightforward to observe that  $f(x) = x$  satisfies Assumption 4.9 and for  $f(x) = x$  in Assumption 4.7, we replace  $K_1^f(a)$  with  $K_1^1(a)$ .

**Theorem 4.3.** Let  $\epsilon_n$  be a sequence such that  $\epsilon_n \rightarrow 0$  and  $n\epsilon_n^2 \rightarrow \infty$  as  $n \rightarrow \infty$  and

$$L_n^1(\theta, \theta_0) = n \left( \sup_{a \in \mathcal{A}} |G(a, \theta) - G(a, \theta_0)| \right)^2.$$

Then under Assumptions 4.2, 4.3, 4.4, 4.7, 4.8 for  $C = \frac{C_1}{2(\sup_{a \in \mathcal{A}} K_1^1(a))^2}$ ,  $C_0 = 2 * 10^8$ ,  $C_1 > 1$ , and positive constants  $C_2$  and  $C_3$  such that  $C > C_2 + C_3 + 2$ , the Naive VB approximation of the true posterior satisfies,

$$\mathbb{E}_{P_0^n} \left[ \int_{\Theta} L_n^1(\theta, \theta_0) q^*(\theta|\tilde{X}_n) d\theta \right] \leq Mn(\epsilon_n^2 + \kappa_n^2), \quad (14)$$

where constant M depends only on  $C, C_0, C_1$ , and  $\lambda$ , and

$$\kappa_n^2 := \eta_n^0(\tilde{\gamma}') = \frac{1}{n} \inf_{q \in \mathcal{Q}} \mathbb{E}_{P_0^n} [\operatorname{KL}(q(\theta) \parallel \pi(\theta|\tilde{X}_n))].$$

The next result establishes a bound on the optimality gap of the naive VB estimated optimal value  $V_{q^*}$  from the true optimal value  $V_0$ .

**Theorem 4.4.** For constant M as defined in Theorem 4.3,

$$\mathbb{E}_{P_0^n} [|V_{q^*} - V_0|] \leq [M(\epsilon_n^2 + \kappa_n^2)]^{\frac{1}{2}}, \quad (15)$$

where  $V_{q^*} = \min_{a \in \mathcal{A}} \int G(a, \theta) q^*(\theta|\tilde{X}_n) d\theta$  and  $V_0 = \min_{a \in \mathcal{A}} G(a, \theta_0)$ .

Next, we bound the optimality gap between the approximate optimal decision rule  $\mathbf{a}_{\text{NV}}^*$  and the true optimal decision. The bound, in particular, depends on the curvature of  $G(a, \theta_0)$  around the true optimal decision, since  $f(x) = x$ . The growth function is denoted as  $\Psi^1(\cdot)$ . The following theorem is a special case of the general result for  $\mathbf{a}_{\text{RS}}^*$  in Theorem 4.3.

**Theorem 4.5.** 1) Suppose that the set  $\mathcal{A}$  is compact and  $G(a, \theta_0)$  satisfies the growth condition, with  $\Psi^1(d)$  such that  $\Psi^1(d)/d^\delta = C^1$ ,  $\forall \delta > 0$ . Then, for any  $\tau > 0$ , the  $P_0^n$ - probability of the following event

$$\left\{ \tilde{X}_n : d_h \left( \mathbf{a}_{\text{NV}}^*(\tilde{X}_n), \arg \min_{z \in \mathcal{A}} G(z, \theta_0) \right) \leq \left[ \frac{2\tau [M(\epsilon_n^2 + \kappa_n^2)]^{\frac{1}{2}}}{C^1} \right]^{\frac{1}{\delta}} \right\}$$

is at least  $1 - \tau^{-1}$ , where  $M$  is the positive constant as defined in Theorem 4.3.

2) Suppose that, there exists an  $n_0$  such that for all  $n \geq n_0$ ,  $\frac{\Psi^1(d_h(\mathbf{a}_{\text{NV}}^*, a^*))}{d_h(\mathbf{a}_{\text{NV}}^*, a^*)^\delta} = C_{n_0}^1$ ,  $\forall \delta > 0$ , where  $a^* = \arg \min_{z \in \mathcal{A}} G(z, \theta_0)$ . Then, for any  $\tau > 0$ , the  $P_0^n$ -probability of the following event

$$\left\{ \tilde{X}_n : d_h \left( \mathbf{a}_{\text{NV}}^*(\tilde{X}_n), \arg \min_{z \in \mathcal{A}} G(z, \theta_0) \right) \leq \left[ \frac{2\tau [M(\epsilon_n^2 + \kappa_n^2)]^{\frac{1}{2}}}{C_{n_0}^1} \right]^{\frac{1}{\delta}} \right\}$$

is at least  $1 - \tau^{-1}$  for all  $n \geq n_0$ , where  $M$  is the positive constant as defined in Theorem 4.3.

## 4.2 Loss Calibrated VB

The *Loss-calibrated VB* (LCVB) algorithm, searches over the set  $\mathcal{Q}$  for approximations with the final decision-making task in mind, in essence *calibrating* the VB method by the model risk  $G(\cdot, \cdot)$ . Algorithm 2 summarizes this method. We formulate a min-max program that lower bounds the posterior predictive stochastic program, and jointly optimizes the decisions and the VB approximation to the posterior predictive distribution.

Observe that this method combines the posterior approximation and decision-making problems into one minimax optimization problem. The objective here can be directly contrasted with that in Algorithm 1. Note that the inner maximization will result in an approximate (loss calibrated) posterior distribution at each decision point  $a \in \mathcal{A}$ . Again, this objective is, in effect, uncomputable; we presented a computable version in Section 3.

In this section, we obtain a finite sample bound and convergence rate on the loss-calibrated optimal decision and optimal value. The analogous result to Theorem 4.1 can be obtained by substituting  $R(a, \theta) = \log G(a, \theta)$ , where  $G(\cdot, \cdot) > 0$  and with  $\bar{\gamma} = 1$ . Since,  $f(x) = \log(x)$  is locally Lipschitz and thus satisfy Assumption 4.9 and for  $f(x) = \log(x)$  in Assumption 4.7, we replace  $K_1^f(a)$  with  $K_1^{\log}(a)$ .

**Theorem 4.6.** Fix  $a_0 \in \mathcal{A}$  and let  $\epsilon_n$  be a sequence such that  $\epsilon_n \rightarrow 0$  and  $n\epsilon_n^2 \rightarrow \infty$  as  $n \rightarrow \infty$  and

$$L_n^{\log}(\theta, \theta_0) = n \left( \sup_{a \in \mathcal{A}} |\log G(a, \theta) - \log G(a, \theta_0)| \right)^2.$$

Then under Assumptions 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, and 4.9; for  $C = \frac{C_1}{2(\sup_{a \in \mathcal{A}} K_1^{\log}(a))^2}$ ,  $C_0 = 2 * 10^8$ ,  $C_1 > 1$ , some positive constants  $C_2, C_3, C_4$ , and  $C_5$  such that  $\min(C, (C_4 + C_5)) > C_2 + C_3 + C_4 + 2$ , and for

$$\eta_n^{\log G}(1) := \frac{1}{n} \inf_{q \in \mathcal{Q}} \mathbb{E}_{P_0^n} \left[ \int_{\Theta} q(\theta) \log \frac{q(\theta)}{\pi(\theta|\tilde{X}_n)} d\theta - \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) \log G(a, \theta) d\theta \right],$$

the Loss calibrated VB approximation of the true posterior satisfies,

$$\mathbb{E}_{P_0^n} \left[ \int_{\Theta} L_n^{\log}(\theta, \theta_0) q_{a_0}^*(\theta|\tilde{X}_n) d\theta \right] \leq Mn(\epsilon_n^2 + \eta_n^{\log G}(1)), \quad (16)$$

where constant  $M$  depends only on  $C, C_0, C_1, C_4, W$  and  $\lambda$  and  $\epsilon_n^2 + \eta_n^{\log G} \geq 0$  for all  $n \geq 1$ .

Now, observe that if  $G(a, \theta) = 1 \ \forall \{a, \theta\} \in \mathcal{A} \times \Theta$ , then  $\eta_n^{\log G}(1) = \kappa_n^2$ , and  $q_{a_0}^*(\theta|\tilde{X}_n) = q^*(\theta|\tilde{X}_n)$  and thus we recover the Naive VB result as stated in Theorem 4.3. Note that, the second term (inside the expectation) in the definition of  $\eta_n^{\log G}(1)$  could result in either  $\kappa_n^2 > \eta_n^{\log G}(1)$  or vice versa and therefore could play an important role in comparing the LCVB and naive VB approximations to the true optimal decision.

Next, we bound the optimality gap between the approximate LC optimal decision rule  $\mathbf{a}_{\text{LC}}^*$  and the true optimal decision. In contrast to the NV decision rule, the bound in LC decision-setting depends on the curvature of the logarithm of  $G(a, \theta_0)$  around the ‘true’ optimal decision. Note that the above growth condition is on the logarithm of the true model risk, therefore the curvature near the true optimizer will vary depending on its value near the true optimizer. We should expect a drastic change in the curvature near the true optimizer if the value near it is less than one. We denote the growth function for LCVB algorithm as  $\Psi^{\log}(\cdot)$ .

**Theorem 4.7.** 1) Suppose that the set  $\mathcal{A}$  is compact and  $\log G(a, \theta)$  has a growth function  $\Psi^{\log}(d)$  such that  $\Psi^{\log}(d)/d^\delta = C^{\log}$ . Then, for any  $\tau > 0$ , the  $P_0^n$ - probability of the following event

$$\left\{ d_h(\mathbf{a}_{\text{LC}}^*, \arg \min_{z \in \mathcal{A}} G(z, \theta_0)) \leq \left[ \frac{2\tau [M(\epsilon_n^2 + \eta_n^{\log}(1))]^{\frac{1}{2}}}{C^{\log}} \right]^{\frac{1}{\delta}} \right\}$$

is at least  $1 - \tau^{-1}$ , where  $M$  is a positive constant as defined in Theorem 4.6.

2) Suppose that, there exists an  $n_0$  such that for all  $n \geq n_0$ ,  $\frac{\Psi^{\log}(d_h(\mathbf{a}_{\text{LC}}^*, a^*))}{d_h(\mathbf{a}_{\text{LC}}^*, a^*)^\delta} = C_{n_0}^{\log}$ ,  $\forall \delta > 0$ , where  $a^* = \arg \min_{z \in \mathcal{A}} \log G(z, \theta_0)$ . Then, for any  $\tau > 0$ ,  $P_0^n$ - probability of the following event

$$\left\{ d_h(\mathbf{a}_{\text{LC}}^*, \arg \min_{z \in \mathcal{A}} G(z, \theta_0)) \leq \left[ \frac{2\tau [M(\epsilon_n^2 + \eta_n^{\log}(1))]^{\frac{1}{2}}}{C_{n_0}^{\log}} \right]^{\frac{1}{\delta}} \right\}$$

is at least  $1 - \tau^{-1}$  for all  $n \geq n_0$ , where  $M$  is a positive constant as defined in Theorem 4.6..

The next result combines Corollary 4.3 and Theorem 4.7 to obtain the probabilistic bound on the gap between  $\mathbf{a}_{\text{LC}}^*$  and  $\mathbf{a}_{\text{OS}}^*$ .

**Corollary 4.5.** Suppose that the set  $\mathcal{A}$  is compact and  $\log G(a, \theta)$  has a growth function  $\Psi^{\log}(d)$  such that  $\Psi^{\log}(d)/d^\delta = C^{\log}$  and  $G(a, \theta_0)$  satisfies the growth condition, with  $\Psi^1(d)$  such that  $\Psi^1(d)/d^\delta = C^1$ ,  $\forall \delta > 0$ . Then, for any  $\tau > 0$ , the  $P_0^n$ - probability of the following event

$$\left\{ d_h(\mathbf{a}_{\text{LC}}^*, \mathbf{a}_{\text{OS}}^*) \leq 2 \left[ \frac{2\tau [M(\epsilon_n^2)]^{\frac{1}{2}}}{C^1} \right]^{\frac{1}{\delta}} + 2 \left[ \frac{2\tau [M(\epsilon_n^2 + \eta_n^{\log}(1))]^{\frac{1}{2}}}{C^{\log}} \right]^{\frac{1}{\delta}} \right\}$$

is at least  $1 - 2\tau^{-1}$ , where  $M$  is a positive constant as defined in Theorem 4.6.

Symbol	Newsvendor	RVB
$\xi$	Demand	Forecast
$a$	Inventory level	Decision variable
$P_\theta$	Demand distribution	Likelihood
$G(a, \theta)$	Expected loss	model risk

Table 1: A dictionary for newsvendor terminology.

## 5 Application: Newsvendor Model

In this section, we study a canonical data-driven decision-making problem with a ‘well-behaved’  $G(a, \theta)$  viz., the data-driven newsvendor model. This problem has received extensive study in the literature, and remains a cornerstone of inventory management Scarf (1960b); Bertsimas and Thiele (2005); Levi et al. (2015) and is an ideal means to illustrate our theoretical insights. Recall that the newsvendor loss function is defined as

$$\ell(a, \xi) := h(a - \xi)^+ + b(\xi - a)^+$$

where  $h$  and  $b$  are given positive constants,  $\xi \in [0, \infty)$  the random demand, and  $a$  the inventory or decision variable, typically assumed to take values in a compact decision space  $\mathcal{A}$  with  $\underline{a} := \min\{a : a \in \mathcal{A}\}$  and  $\bar{a} := \max\{a : a \in \mathcal{A}\}$ , and  $\underline{a} > 0$ . The distribution over the random demand,  $P_\theta$  is assumed to be exponential with unknown rate parameter  $\theta \in (0, \infty)$ . Notice that this is precisely the model likelihood. The model risk can easily be derived as

$$G(a, \theta) = \mathbb{E}_{P_\theta}[\ell(a, \xi)] = ha - \frac{h}{\theta} + (b + h)\frac{e^{-a\theta}}{\theta}, \quad (17)$$

which is convex in  $a$ . We assume that  $\tilde{X}_n := \{\xi_1, \xi_2 \dots \xi_n\}$  be  $n$  observations of the random demand, assumed to be i.i.d random samples drawn from  $P_{\theta_0}$ . Table 1 lists a dictionary for translating standard terminology in the newsvendor literature to our current setting.

We now assume a non-conjugate inverse-gamma prior distribution over the parameter with shape and rate parameter  $\alpha$  and  $\beta$  respectively. We now verify that Assumptions 4.3, 4.2, 4.4, 4.6 and 4.5 (in that order) are satisfied in this newsvendor setting. The proofs of the lemma’s are delayed to the electronic companion for readability.

Since, for any  $a \in \mathcal{A}$ ,  $G(a, \theta)$  in (17) tends to infinity as  $\theta \rightarrow 0$ , we define the set  $\Theta_n(\epsilon) := \{\theta \in \Theta : \theta > e^{-n\epsilon^2}\} \forall \epsilon > \epsilon_n$  and verify that the inverse-gamma prior places ‘small’ mass on the complement of this set and therefore satisfies Assumption 4.3.

**Lemma 5.1.** Fix  $n \geq 1$ . For any  $\epsilon > \epsilon_n := \frac{1}{\sqrt{n}}$ , the inverse-gamma prior with shape parameter  $\alpha = 1$  and rate parameter  $\beta$ , satisfies

$$\Pi[\mathbb{1}_{\Theta_n^c}(\theta)] \leq \exp(-Cn\epsilon^2),$$

for a positive constant  $C = \beta$ .

Next, under the condition that the true demand distribution is exponential with parameter  $\theta_0$  (and  $P_0 \equiv P_{\theta_0}$ ), we demonstrate the existence of test functions satisfying Assumption 4.2. Recall the distance function  $L_n^f(\theta, \theta_0) = n \sup_{a \in \mathcal{A}} |f(G(a, \theta)) - f(G(a, \theta_0))|$ , and note that  $G(a, \theta)$  is the newsvendor model risk in (17).

**Lemma 5.2.** Fix  $n \geq 5$ . Then, for any  $\epsilon > \epsilon_n := \frac{1}{\sqrt{n}}$  with  $\epsilon_n \rightarrow 0$ , and  $n\epsilon_n^2 \geq 1$ , there exists a test function  $\phi_n : \tilde{X}_n \rightarrow [0, 1]$ , set  $\Theta_n(\epsilon) := \left\{ \theta \in \Theta : \theta > e^{-n\epsilon^2} \right\}$  and  $C > 2$  such that

$$(i) \quad \mathbb{E}_{P_0^n}[\phi_n] \leq \exp(-Cn\epsilon^2), \text{ and}$$

$$(ii) \quad \sup_{\{\theta \in \Theta_n(\epsilon) : L_n^f(\theta, \theta_0) \geq C_1 n\epsilon^2\}} \mathbb{E}_{P_\theta^n}[1 - \phi_n] \leq \exp(-Cn\epsilon^2).$$

Next, we show that there exist appropriate constants such that the inverse-gamma prior satisfies Assumption 4.4 when the demand distribution is exponential.

**Lemma 5.3.** Fix  $n_2 \geq 1$  and  $\lambda = 1$ . Let  $A_n := \left\{ \theta \in \Theta : D_{1+\lambda}(P_0^n \| P_\theta^n) \leq C_3 n\epsilon_n^2 \right\}$ , where  $D_{1+\lambda}(P_0^n \| P_\theta^n) := \frac{1}{\lambda} \log \int \left( \frac{dP_0^n}{dP_\theta^n} \right)^\lambda dP_0^n$  is the Rényi Divergence between  $P_0^n$  and  $P_\theta^n$  assuming  $P_0^n$  is absolutely continuous with respect to  $P_\theta^n$ . Then for any  $C_3 > 0$  and  $C_2 = \alpha C_3$ , the inverse-gamma prior distribution satisfies

$$\Pi\{A_n\} \geq \exp(-nC_2\epsilon_n^2), \forall n \geq n_2.$$

Note that, in the naive VB case  $R(\cdot, \cdot) = 0$ , therefore Assumption 4.5 and 4.6 are trivially satisfied. Whereas, in the LCVB setting  $R(a, \theta) = \log(G(a, \theta))$  and we show that the newsvendor model risk satisfies Assumption 4.5 and 4.6.

Next, it is straightforward to see that the newsvendor model risk  $G(a, \theta)$  is bounded below for a given  $a \in \mathcal{A}$ .

**Lemma 5.4.** For any  $a \in \mathcal{A}$  and positive constants  $h$  and  $b$ , the model risk

$$G(a, \theta) = ha - \frac{h}{\theta} + (b + h) \frac{e^{-a\theta}}{\theta} \geq \frac{h\underline{a}^2\theta^*}{(1 + a\theta^*)},$$

where  $\underline{a} := \min\{a \in \mathcal{A}\}$  and  $\underline{a} > 0$ .

Since,  $G(a, \theta)$  is bounded from below, any monotonic transform  $F(\cdot)$  of  $G(a, \theta)$  is also bounded from below. This implies that, in the LCVB case  $R(a, \theta) (= \log(G(a, \theta)))$  satisfies Assumption 4.6. Finally, we also show that the newsvendor model risk satisfies Assumption 4.5, when  $R(a, \theta) = \log(G(a, \theta))$ .

**Lemma 5.5.** Fix  $n \geq 1$  and  $\epsilon_n > 0$  such that  $n\epsilon_n^2 \geq 1$ . For any  $a \in \mathcal{A}$ ,  $\bar{\gamma}' = 1$  and,  $R(a, \theta) = \log G(a, \theta)$

$$\int_{\{G(a, \theta) > e^{C_4 n\epsilon_n^2}\}} G(a, \theta) \pi(\theta) d\theta \leq \exp(-C_5 n\epsilon_n^2),$$

with  $C_4 = C_5 > 0$ .

Note that Lemma 5.1 and 5.2 together implies that  $C = \beta > 2$  and  $\alpha = 1$ , where  $\alpha$  and  $\beta$  are the shape and rate parameters (resp.) of the inverse-gamma prior. Lemma 5.4 implies that  $C_5 (= C_4)$  is positive and can be chosen such that  $C > C_4 + C_5$ . Therefore, the condition on constants in Theorem 4.1 reduces to  $C_5 > 2 + C_2 + C_3$ . Next, Lemma 5.3 and 5.1 implies that  $C_2 = C_3$  and can be chosen to satisfy the simplified constraint with an inverse-gamma prior, that is  $C_5 > 2 + C_2 + C_3$ .

These lemma's show that when the demand distribution is exponential and with a non-conjugate inverse-gamma prior, a modeler can use our results in Theorem 4.5 and 4.7 for naive VB and LCVB algorithm(resp.) to bound the optimality gap in decisions. Note that Theorem 4.5 and 4.7 are special cases of more general Theorem 4.2. Recall that the bound obtained in Theorem 4.2 depends on  $\epsilon_n^2$  and  $\eta_n^R(\bar{\gamma}')$ , where

$$\eta_n^R(\bar{\gamma}') = \frac{1}{n} \inf_{q \in \mathcal{Q}} \mathbb{E}_{P_0^n} \left[ \int_{\Theta} q(\theta) \log \frac{q(\theta)}{\pi(\theta|\tilde{X}_n)} d\theta - \bar{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta \right].$$

Lemma 5.1 and 5.2 imply that  $\epsilon_n^2 = \frac{1}{n}$ , but in order to get complete bound, we further need to characterize  $\eta_n^R(\bar{\gamma}')$ . Also recall that, as a consequence of Assumption 4.11 in Proposition 4.2, we obtained  $\eta_n^R(\bar{\gamma}') \leq (C_8 + C_9)\epsilon_n'^2$ .

Therefore, in our next result, we show that in the newsvendor setting, we can construct a sequence  $\{q_n(\theta)\}$  in the variational family  $\mathcal{Q}$ , that is the family of gamma distributions such that the sequence  $\{q_n(\theta)\}$  satisfies Assumption 4.11 and thus identify  $\epsilon_n'$  and constant  $C_9$ .

**Lemma 5.6.** Let  $\{q_n(\theta)\}$  be a sequence of gamma distribution with shape parameter  $a = n$  and rate parameter  $b = \frac{n}{\theta_0}$ , then for inverse gamma prior  $\pi(\theta)$  and exponentially distributed likelihood model

$$\frac{1}{n} [\text{KL}(q_n(\theta) \parallel \pi(\theta)) + \mathbb{E}_{q_n(\theta)} [\text{KL}(dP_0^n \parallel p(\tilde{X}_n|\theta))]] \leq C_9 \epsilon_n'^2,$$

where  $\epsilon_n'^2 = \frac{\log n}{n}$  and  $C_9 = \frac{1}{2} + \max\left(0, 2 + \frac{2\beta}{\theta_0} - \log \sqrt{2\pi} - \log\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right) + \alpha \log \theta_0\right)$  and prior parameters are chosen such that  $C_9 > 0$ .

Now consider the naive VB case; since  $R(\cdot, \cdot) = 0$ , the term  $\kappa_n^2$  in Theorem 4.5 is bounded above by  $C_9 \epsilon_n'^2$ , where  $C_9$  and  $\epsilon_n'^2$  are derived in the result above. For the LCVB case, where  $R(\cdot, \cdot) = \log(G(\cdot, \cdot))$ , note that Lemma 5.4 implies that  $R(\cdot, \cdot)$  is bounded below, therefore  $C_8' = \log\left(\frac{h\underline{a}^2\theta^*}{(1+\bar{a}\theta^*)}\right)$ , where  $h, \underline{a}, \bar{a}$ , and  $\theta^*$  are given to the modeler or it is easily computable. Now if  $C_8' \geq 0$ , then it is straight forward to observe that  $\eta_n^{\log}(1)$  term in Theorem 4.7 is bounded above by  $C_9 \epsilon_n'^2$ . Otherwise, since  $\frac{\log n}{n} \geq \frac{1}{n}$ , it is bounded by  $(C_8 + C_9)\epsilon_n'^2$ , where  $C_8 = -C_8'$ , and  $C_9$  and  $\epsilon_n'^2$  are as derived in Lemma 5.6.

Finally, we conduct a simulation experiment using the newsvendor model described above. We fix  $\theta_0 = 0.68$ ,  $\underline{a} = 0.001$ ,  $\bar{a} = 75$ ,  $b = 0.1$ ,  $\alpha = 1$ , and  $\beta = 4.1$ . Next, we run naive VB and LCVB algorithms to obtain the respective optimal decision  $\mathbf{a}_{\text{NV}}^*$  and  $\mathbf{a}_{\text{LC}}^*$  for 9 different values of  $h \in \{0.001, 0.002, \dots, 0.009\}$  and repeat the experiment over 1000 sample paths. In Figure 1, we plot the 90<sup>th</sup> quantile of the  $|\mathbf{a}^* - \mathbf{a}_0^*|$ , where  $\mathbf{a}^* \in \{\mathbf{a}_{\text{NV}}^*, \mathbf{a}_{\text{LC}}^*\}$  and their respective upperbounds obtained using the results in the Theorem 4.5 and 4.7 for the above newsvendor model.

## 6 Conclusion

Data-driven decision-making has received significant research interest in the recent literature, in particular since the nature of the interplay between data and optimal decision-making can be quite different from the standard machine learning setting. While much of the literature focuses on empirical methods, Bayesian methods afford advantages particularly when making decisions

(a) Naive VB

(b) LCVB

Figure 1: Optimality gap in decisions (the 90<sup>th</sup> quantile over 1000 sample paths) against the number of samples ( $n$ ) for  $\mathbf{a}_{\text{NV}}^*$  and  $\mathbf{a}_{\text{LC}}^*$ . The solid lines present simulation results, while the dotted and dashed lines are theoretical bounds.

in context of stochastic models. However, Bayesian methods are also hampered by integration requirements that can be hard to satisfy in practice.

In this paper we presented the risk-sensitive variational Bayesian computational framework for Bayes-predictive data-driven decision-making, and analyzed the statistical performance of any computational algorithm derived from this framework by providing non-asymptotic bounds on the optimality gap. We also analyzed two specific algorithms, and for both the naive VB (NVB) and loss-calibrated VB (LCVB) algorithms we provide statistical analyses of the ‘goodness’ of the optimal decisions in terms of the true data generating model. We also compared the methods against the Bayes optimal solution on a newsvendor problem.

Our current methodology essentially relies on optimizing lower bounds to the ‘true’ problem at hand. One of our future objectives is to obtain sharp upper bounds on the true objective that can then provide a means of ‘squeezing’ the true optimal solution between these bounds. A second objective is to fully understand the interplay between robustness and our variational approximations. In some sense, robust methods aim to find the ‘worst’ distribution out of a set of distributions centered (in an appropriate sense) around a nominal distribution. On the other hand, VB methods find the closest distribution from a family that does not include the nominal distribution (if it did, then we could compute the posterior). There is almost a sense of duality between these perspectives that is worthy of further investigation. Third, from a methodological viewpoint, we are investigating the role of variational autoencoders (Doersch (2016)) in the context of data-driven decision-making. Currently, our decision-making model requires us to fully specify the likelihood and prior models, while in practice it would be beneficial to make this fully data-driven – precisely where autoencoder technology would be useful. To the best of our knowledge very little is known about the statistical properties of these models, or their role in decision-making contexts.



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## 7 Appendix

### 7.1 Alternative derivation of LCVB

We present the alternative derivation of LCVB. Consider the logarithm of the Bayes posterior risk,

$$\begin{aligned}\log \mathbb{E}_{\pi(\theta|\tilde{X}_n)}[G(a, \theta)] &= \log \int_{\Theta} G(a, \theta) \pi(\theta|\tilde{X}_n) d\theta \\ &= \log \int_{\Theta} \frac{q(\theta)}{q(\theta)} G(a, \theta) p(\theta|\tilde{X}_n) d\theta \\ &\geq - \int_{\Theta} q(\theta) \log \frac{q(\theta)}{G(a, \theta) p(\theta|\tilde{X}_n)} d\theta =: \mathcal{F}(a; q(\cdot), \tilde{X}_n)\end{aligned}\quad (18)$$

where the inequality follows from an application of Jensen's inequality (since, without loss of generality,  $G(a, \theta) > 0$  for all  $a \in \mathcal{A}$  and  $\theta \in \Theta$ ), and  $q \in \mathcal{Q}$ . Then, it follows that

$$\begin{aligned}\min_{a \in \mathcal{A}} \log \mathbb{E}_{\pi(\theta|\tilde{X}_n)}[G(a, \theta)] &\geq \min_{a \in \mathcal{A}} \max_{q \in \mathcal{Q}} \mathcal{F}(a; q(\theta), \tilde{X}_n) \\ &= \min_{a \in \mathcal{A}} \max_{q \in \mathcal{Q}} -\text{KL}(q(\theta) \parallel \pi(\theta|\tilde{X}_n)) + \int_{\Theta} \log G(a, \theta) q(\theta) d\theta.\end{aligned}\quad (19)$$

### 7.2 Proof of Theorem 4.1:

We prove our main result after series of important lemmas.

**Lemma 7.1.** For any  $a' \in \mathcal{A}$ ,  $\bar{\gamma}' > 0$ , and  $\beta > 0$ ,

$$\begin{aligned}&\mathbb{E}_{P_0^n} \left[ \beta \int_{\Theta} L_n^f(\theta, \theta_0) q_{a', \bar{\gamma}'}^*(\theta|\tilde{X}_n) d\theta \right] \\ &\leq \log \mathbb{E}_{P_0^n} \left[ \int_{\Theta} e^{\beta L_n^f(\theta, \theta_0)} \frac{e^{\bar{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta)}{\int_{\Theta} e^{\bar{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} d\theta \right] + \inf_{q \in \mathcal{Q}} \mathbb{E}_{P_0^n} \left[ \int_{\Theta} q(\theta) \log \frac{q(\theta)}{\pi(\theta|\tilde{X}_n)} d\theta \right. \\ &\quad \left. - \bar{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta \right] + \log \mathbb{E}_{P_0^n} \left[ \int_{\Theta} e^{\bar{\gamma}' R(a', \theta)} \frac{\prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta)}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} d\theta \right].\end{aligned}\quad (20)$$

*Proof.* Proof: For a fixed  $a' \in \mathcal{A}$ ,  $\bar{\gamma}' > 0$ , and  $\beta > 0$ , and using the fact that KL is non-negative, observe that the integral in the LHS of equation (20) satisfies,

$$\begin{aligned}
\beta \int_{\Theta} L_n^f(\theta, \theta_0) q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta &\leq \int_{\Theta} \log e^{\beta L_n^f(\theta, \theta_0)} q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta \\
&\quad + \text{KL} \left( q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) \left\| \frac{e^{\beta L_n^f(\theta, \theta_0)} e^{\bar{\gamma}' R(a', \theta)} \pi(\theta | \tilde{X}_n)}{\int_{\Theta} e^{\beta L_n^f(\theta, \theta_0)} e^{\bar{\gamma}' R(a', \theta)} \pi(\theta | \tilde{X}_n) d\theta} \right\| \right) \\
&= \int_{\Theta} \log e^{\beta L_n^f(\theta, \theta_0)} q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta + \log \int_{\Theta} e^{\beta L_n^f(\theta, \theta_0)} e^{\bar{\gamma}' R(a', \theta)} \pi(\theta | \tilde{X}_n) d\theta \\
&\quad + \int_{\Theta} q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) \log \frac{q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n)}{e^{\beta L_n^f(\theta, \theta_0)} e^{\bar{\gamma}' R(a', \theta)} \pi(\theta | \tilde{X}_n)} d\theta \\
&= \log \int_{\Theta} e^{\beta L_n^f(\theta, \theta_0)} e^{\bar{\gamma}' R(a', \theta)} \pi(\theta | \tilde{X}_n) d\theta + \int_{\Theta} q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) \log \frac{q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n)}{e^{\bar{\gamma}' R(a', \theta)} \pi(\theta | \tilde{X}_n)} d\theta.
\end{aligned}$$

Next, using the definition of  $q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n)$  in the second term of last equality, for any other  $q(\cdot) \in \mathcal{Q}$

$$\beta \int_{\Theta} L_n^f(\theta, \theta_0) q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta \leq \log \int_{\Theta} e^{\beta L_n^f(\theta, \theta_0)} e^{\bar{\gamma}' R(a', \theta)} \pi(\theta | \tilde{X}_n) d\theta + \int_{\Theta} q(\theta) \log \frac{q(\theta)}{e^{\bar{\gamma}' R(a', \theta)} \pi(\theta | \tilde{X}_n)} d\theta.$$

Finally, it follows from the definition of the posterior distribution

$$\begin{aligned}
&\beta \int_{\Theta} L_n^f(\theta, \theta_0) q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta \\
&\leq \log \int_{\Theta} e^{\beta L_n^f(\theta, \theta_0)} e^{\bar{\gamma}' R(a', \theta)} \frac{\prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta)}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta) d\theta} d\theta + \int_{\Theta} q(\theta) \log \frac{q(\theta)}{e^{\bar{\gamma}' R(a', \theta)} \pi(\theta | \tilde{X}_n)} d\theta, \\
&= \log \int_{\Theta} e^{\beta L_n^f(\theta, \theta_0)} \frac{e^{\bar{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta)}{\int_{\Theta} e^{\bar{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta) d\theta} d\theta + \int_{\Theta} q(\theta) \log \frac{q(\theta)}{e^{\bar{\gamma}' R(a', \theta)} \pi(\theta | \tilde{X}_n)} d\theta \\
&\quad + \log \int_{\Theta} e^{\bar{\gamma}' R(a', \theta)} \frac{\prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta)}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta) d\theta} d\theta, \tag{21}
\end{aligned}$$

where the last equality follows from adding and subtracting  $\log \int_{\Theta} e^{\bar{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta) d\theta$ . Now, taking expectation on either side of equation (21) and using Jensen's inequality on the first and the last term on the RHS,

$$\begin{aligned}
&\mathbb{E}_{P_0^n} \left[ \beta \int_{\Theta} L_n^f(\theta, \theta_0) q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta \right] \\
&\leq \log \mathbb{E}_{P_0^n} \left[ \int_{\Theta} e^{\beta L_n^f(\theta, \theta_0)} \frac{e^{\bar{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta)}{\int_{\Theta} e^{\bar{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta) d\theta} d\theta \right] + \inf_{q \in \mathcal{Q}} \mathbb{E}_{P_0^n} \left[ \int_{\Theta} q(\theta) \log \frac{q(\theta)}{\pi(\theta | \tilde{X}_n)} d\theta \right. \\
&\quad \left. - \bar{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta \right] + \log \mathbb{E}_{P_0^n} \left[ \int_{\Theta} e^{\bar{\gamma}' R(a', \theta)} \frac{\prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta)}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta) d\theta} d\theta \right] \tag{22}
\end{aligned}$$

□

First, we state a technical result that is important in proving our next lemma.

**Lemma 7.2** (Lemma 6.4 of Zhang and Gao (2017)). Suppose random variable  $X$  satisfies

$$\mathbb{P}(X \geq t) \leq c_1 \exp(-c_2 t),$$

for all  $t \geq t_0 > 0$ . Then for any  $0 < \beta \leq c_2/2$ ,

$$\mathbb{E}[\exp(\beta X)] \leq \exp(\beta t_0) + c_1.$$

*Proof.* Proof: Refer Lemma 6.4 of Zhang and Gao (2017). □

In the following result, we bound the first term on the RHS of equation (20).

**Lemma 7.3.** Under Assumptions 4.2, 4.3, 4.4, 4.5, and 4.6 and for  $\min(C, C_4 + C_5) > C_2 + C_3 + C_4 + 2$ ,

$$\mathbb{E}_{P_0^n} \left[ \int_{\Theta} e^{\beta L_n^f(\theta, \theta_0)} \frac{e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta)}{\int_{\Theta} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} d\theta \right] \leq e^{\beta C_1 n \epsilon_n^2} + (1 + C_0 + 3W^{-1}), \quad (23)$$

for  $0 < \beta \leq C_{10}/2$ , where  $C_{10} = \min\{\lambda, C, 1\}/C_1$  for any  $\lambda > 0$ .

*Proof.* Proof: First define the set

$$B_n := \left\{ \tilde{X}_n : \int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta \geq e^{-(1+C_3)n\epsilon_n^2} \Pi(A_n) \right\}, \quad (24)$$

where set  $A_n$  is defined in Assumption 4.4. We demonstrate that, under Assumption 4.4,  $P_0^n(B_n^c)$  is bounded above by an exponentially decreasing(in  $n$ ) term. First, note that, for  $A_n$  as defined in Assumption 4.4:

$$\begin{aligned} \mathbb{P}_0^n \left( \frac{1}{\Pi(A_n)} \int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta \leq e^{-(1+C_3)n\epsilon_n^2} \right) \\ \leq \mathbb{P}_0^n \left( \frac{1}{\Pi(A_n)} \int_{\Theta \cap A_n} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta \leq e^{-(1+C_3)n\epsilon_n^2} \right). \end{aligned} \quad (25)$$

Let  $\tilde{\pi}(\theta) d\theta := \frac{\mathbb{I}_{\{\Theta \cap A_n\}}(\theta)}{\Pi(A_n)} \pi(\theta) d\theta$ , and use this in the equation (25) for any  $\lambda > 0$  to obtain,

$$\begin{aligned} \mathbb{P}_0^n \left( \frac{1}{\Pi(A_n)} \int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta \leq e^{-(1+C_3)n\epsilon_n^2} \right) &\leq \mathbb{P}_0^n \left( \int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \tilde{\pi}(\theta) d\theta \leq e^{-(1+C_3)n\epsilon_n^2} \right) \\ &= \mathbb{P}_0^n \left( \left[ \int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \tilde{\pi}(\theta) d\theta \right]^{-\lambda} \geq e^{(1+C_3)\lambda n\epsilon_n^2} \right). \end{aligned}$$

Then, using the Chernoff's inequality in the last equality above, we have

$$\begin{aligned}
\mathbb{P}_0^n \left( \frac{1}{\Pi(A_n)} \int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta \leq e^{-(1+C_3)n\epsilon_n^2} \right) &\leq e^{-(1+C_3)\lambda n\epsilon_n^2} \mathbb{E}_{P_0^n} \left( \left[ \int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \tilde{\pi}(\theta) d\theta \right]^{-\lambda} \right) \\
&\leq e^{-(1+C_3)\lambda n\epsilon_n^2} \left[ \int_{\Theta} \mathbb{E}_{P_0^n} \left( \left[ \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \right]^{-\lambda} \right) \tilde{\pi}(\theta) d\theta \right] \\
&= e^{-(1+C_3)\lambda n\epsilon_n^2} \left[ \int_{\Theta} \exp(\lambda D_{\lambda+1}(P_0^n \| P_{\theta}^n)) \tilde{\pi}(\theta) d\theta \right] \\
&\leq e^{-\lambda n\epsilon_n^2}, \tag{26}
\end{aligned}$$

where the second inequality follows from first applying Jensen's inequality and then using Fubini's theorem, and the last inequality follows from Assumption 4.4.

Next, define the set  $K_n := \{\theta \in \Theta : L_n^f(\theta, \theta_0) > C_1 n\epsilon_n^2\}$ . Notice that set  $K_n$  is the set of alternate hypothesis as defined in Assumption 4.2. We bound the calibrated posterior probability of this set  $K_n$  to get a bound on the first term in the RHS of equation (20). Recall the sequence of test function  $\{\phi_n\}$  from Assumption 4.2. Observe that

$$\begin{aligned}
&\mathbb{E}_{P_0^n} \left[ \frac{\int_{K_n} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] \\
&= \mathbb{E}_{P_0^n} \left[ (\phi_n) \frac{\int_{K_n} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] + \mathbb{E}_{P_0^n} \left[ (1 - \phi_n) \frac{\int_{K_n} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] \\
&\leq \mathbb{E}_{P_0^n} \phi_n + \mathbb{E}_{P_0^n} [(1 - \phi_n) \mathbb{I}_{B_n^C}] + \mathbb{E}_{P_0^n} \left[ (1 - \phi_n) \mathbb{I}_{B_n} \frac{\int_{K_n} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] \\
&\leq \mathbb{E}_{P_0^n} \phi_n + \mathbb{E}_{P_0^n} [\mathbb{I}_{B_n^C}] + \mathbb{E}_{P_0^n} \left[ (1 - \phi_n) \mathbb{I}_{B_n} \frac{\int_{K_n} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right], \tag{27}
\end{aligned}$$

where, in the second inequality, we first divide the second term over set  $B_n$  and its complement, and then use the fact that  $\frac{\int_{K_n} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \leq 1$ . The third inequality is due the fact that  $\phi_n \in [0, 1]$ . Next, using Assumption 4.4 and 4.6 observe that on set  $B_n$

$$\begin{aligned}
\int_{\Theta} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta &\geq W \int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta \\
&\geq W e^{-(1+C_2+C_3)n\epsilon_n^2}.
\end{aligned}$$

Substituting the equation above in the third term of equation (27), we obtain

$$\begin{aligned}
&\mathbb{E}_{P_0^n} \left[ (1 - \phi_n) \mathbb{I}_{B_n} \frac{\int_{K_n} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] \\
&\leq W^{-1} e^{(1+C_2+C_3)n\epsilon_n^2} \mathbb{E}_{P_0^n} \left[ (1 - \phi_n) \mathbb{I}_{B_n} \int_{K_n} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta \right] \\
&\leq W^{-1} e^{(1+C_2+C_3)n\epsilon_n^2} \mathbb{E}_{P_0^n} \left[ (1 - \phi_n) \int_{K_n} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta \right]. \tag{*}
\end{aligned}$$

Now, using Fubini's theorem observe that,

$$\begin{aligned}
(\star) &= W^{-1} e^{(1+C_2+C_3)n\epsilon_n^2} \int_{K_n} e^{\tilde{\gamma}' R(a', \theta)} \mathbb{E}_{P_\theta^n} [(1 - \phi_n)] \pi(\theta) d\theta \\
&\leq W^{-1} e^{(1+C_2+C_3+C_4)n\epsilon_n^2} \left[ \int_{K_n \cap \{e^{\tilde{\gamma}' R(a', \theta)} \leq e^{C_4 n \epsilon_n^2}\}} \mathbb{E}_{P_\theta^n} [(1 - \phi_n)] \pi(\theta) d\theta \right. \\
&\quad \left. + e^{-C_4 n \epsilon_n^2} \int_{K_n \cap \{e^{\tilde{\gamma}' R(a', \theta)} > e^{C_4 n \epsilon_n^2}\}} e^{\tilde{\gamma}' R(a', \theta)} \pi(\theta) d\theta \right],
\end{aligned}$$

where, in the last inequality we first divide the integral over set  $\{\theta \in \Theta : e^{\tilde{\gamma}' R(a', \theta)} \leq e^{C_4 n \epsilon_n^2}\}$  and its complement and then use the upper bound on  $e^{\tilde{\gamma}' R(a', \theta)}$  in the first integral. Now, it follows that

$$\begin{aligned}
(\star) &\leq W^{-1} e^{(1+C_2+C_3+C_4)n\epsilon_n^2} \left[ \int_{K_n} \mathbb{E}_{P_\theta^n} [(1 - \phi_n)] \pi(\theta) d\theta + e^{-C_4 n \epsilon_n^2} \int_{\{e^{\tilde{\gamma}' R(a', \theta)} > e^{C_4 n \epsilon_n^2}\}} e^{\tilde{\gamma}' R(a', \theta)} \pi(\theta) d\theta \right] \\
&= W^{-1} e^{(1+C_2+C_3+C_4)n\epsilon_n^2} \left[ \int_{K_n \cap \Theta_n(\epsilon)} \mathbb{E}_{P_\theta^n} [(1 - \phi_n)] \pi(\theta) d\theta + \int_{K_n \cap \Theta_n(\epsilon)^c} \mathbb{E}_{P_\theta^n} [(1 - \phi_n)] \pi(\theta) d\theta \right. \\
&\quad \left. + e^{-C_4 n \epsilon_n^2} \int_{\{e^{\tilde{\gamma}' R(a', \theta)} > e^{C_4 n \epsilon_n^2}\}} e^{\tilde{\gamma}' R(a', \theta)} \pi(\theta) d\theta \right] \\
&\leq W^{-1} e^{(1+C_2+C_3+C_4)n\epsilon_n^2} \left[ \int_{K_n \cap \Theta_n(\epsilon)} \mathbb{E}_{P_\theta^n} [(1 - \phi_n)] \pi(\theta) d\theta + \Pi(\Theta_n(\epsilon)^c) \right. \\
&\quad \left. + e^{-C_4 n \epsilon_n^2} \int_{\{e^{\tilde{\gamma}' R(a', \theta)} > e^{C_4 n \epsilon_n^2}\}} e^{\tilde{\gamma}' R(a', \theta)} \pi(\theta) d\theta \right],
\end{aligned}$$

where the second inequality is obtained by dividing the first integral on set  $\Theta_n(\epsilon)$  and its complement, and the third inequality is due the fact that  $\phi_n \in [0, 1]$ . Now, using the equation above and Assumption 4.2, 4.3, and 4.5 observe that

$$\mathbb{E}_{P_0^n} \left[ (1 - \phi_n) \mathbb{I}_{B_n} \frac{\int_{K_n} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] \leq W^{-1} e^{(1+C_2+C_3+C_4)n\epsilon_n^2} \left[ 2e^{-Cn\epsilon_n^2} + e^{-(C_5+C_4)n\epsilon_n^2} \right].$$

Hence, choosing  $C, C_2, C_3, C_4$  and  $C_5$  such that  $-1 > 1 + C_2 + C_3 + C_4 - \min(C, (C_4 + C_5))$  implies

$$\mathbb{E}_{P_0^n} \left[ (1 - \phi_n) \mathbb{I}_{B_n} \frac{\int_{K_n} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] \leq 3W^{-1} e^{-n\epsilon_n^2}. \quad (28)$$

By Assumption 4.2, we have

$$\mathbb{E}_{P_0^n} \phi_n \leq C_0 e^{-Cn\epsilon_n^2}. \quad (29)$$

Therefore, substituting equation (26) from Lemma 7.3, equation (28), and (29) into (27), we obtain

$$\mathbb{E}_{P_0^n} \left[ \frac{\int_{K_n} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] \leq (1 + C_0 + 3W^{-1}) e^{-C_{10} C_1 n \epsilon_n^2}, \quad (30)$$

where  $C_{10} = \min\{\lambda, C, 1\}/C_1$ . Using Fubini's theorem, observe that the LHS in the equation (30) can be expressed as  $\mu(K_n)$ , where

$$d\mu(\theta) = \mathbb{E}_{P_0^n} \left[ \frac{\prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)}}{\int_{\Theta} e^{\bar{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] \pi(\theta) e^{\bar{\gamma}' R(a', \theta)} d\theta.$$

Next, recall that the set  $K_n = \{\theta \in \Theta : L_n^f(\theta, \theta_0) > C_1 n \epsilon_n^2\}$ . Applying Lemma 7.2 above with  $c_1 = (1 + C_0 + 3W^{-1})$ ,  $c_2 = C_{10}$ ,  $t_0 = C_1 n \epsilon_n^2$ , and for  $0 < \beta \leq C_{10}/2$ , we obtain

$$\mathbb{E}_{P_0^n} \left[ \int_{\Theta} e^{\beta L_n^f(\theta, \theta_0)} \frac{e^{\bar{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta)}{\int_{\Theta} e^{\bar{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} d\theta \right] \leq e^{\beta C_1 n \epsilon_n^2} + (1 + C_0 + 3W^{-1}). \quad (31)$$

□

Further, we have another technical lemma, that will be crucial in proving the subsequent lemma that upper bounds the last term in the equation (20).

**Lemma 7.4.** Suppose a positive random variable  $X$  satisfies

$$\mathbb{P}(X \geq t) \leq c_1 \exp(-c_2 t),$$

for all  $t \geq t_0 > 0$ ,  $c_1 > 0$ , and  $c_2 > 0$ . Then,

$$\mathbb{E}[X] \leq \exp(t_0) + \frac{c_1}{c_2}.$$

*Proof.* Proof: For any  $Z_0 > 0$ ,

$$\mathbb{E}[X] \leq Z_0 + \int_{Z_0}^{\infty} \mathbb{P}(X > x) dx \leq Z_0 + c_1 \int_{Z_0}^{\infty} \exp(-c_2 x) dx.$$

Therefore, choosing  $Z_0 = \exp(t_0)$ ,

$$\mathbb{E}[X] \leq \exp(t_0) + \frac{c_1}{c_2} \exp(-c_2 \exp(t_0)) \leq \exp(t_0) + \frac{c_1}{c_2}.$$

□

Next, we establish the following bound on the last term in the equation (20).

**Lemma 7.5.** Under Assumptions 4.2, 4.3, 4.4, 4.5, 4.6, and for  $C_4 + C_5 > C_2 + C_3 + 2$ ,

$$\mathbb{E}_{P_0^n} \left[ \int_{\Theta} \frac{e^{\bar{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta)}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} d\theta \right] \leq e^{C_4 n \epsilon_n^2} + \frac{2}{C_{11}}, \quad (32)$$

where  $C_{11} = \min\{\lambda, 1\}/C_4$  for any  $\lambda > 0$ .

*Proof.* Proof: Define the set

$$M_n := \{\theta \in \Theta : e^{\tilde{\gamma}' R(a', \theta)} > e^{C_4 n \epsilon_n^2}\}. \quad (33)$$

Using the set  $B_n$  in equation (24), observe that the measure of the set  $M_n$ , under the posterior distribution satisfies,

$$\mathbb{E}_{P_0^n} \left[ \frac{\int_{M_n} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] \leq \mathbb{E}_{P_0^n} [\mathbb{1}_{B_n^c}] + \mathbb{E}_{P_0^n} \left[ \mathbb{1}_{B_n} \frac{\int_{M_n} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right]. \quad (34)$$

Now, the second term of equation (34) can be bounded as follows: recall Assumption 4.5 and the definition of set  $B_n$ , both together imply that,

$$\begin{aligned} \mathbb{E}_{P_0^n} \left[ \mathbb{1}_{B_n} \frac{\int_{M_n} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] &\leq e^{(1+C_2+C_3)n\epsilon_n^2} \mathbb{E}_{P_0^n} \left[ \mathbb{1}_{B_n} \int_{M_n} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta \right] \\ &\leq e^{(1+C_2+C_3)n\epsilon_n^2} \mathbb{E}_{P_0^n} \left[ \int_{M_n} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta \right]. \quad (**) \end{aligned}$$

Then, using Fubini's Theorem  $(**) = e^{(1+C_2+C_3)n\epsilon_n^2} \Pi(M_n)$ . Next, using the definition of set  $M_n$  and then Assumption 4.5, we obtain

$$\begin{aligned} \mathbb{E}_{P_0^n} \left[ \mathbb{1}_{B_n} \frac{\int_{M_n} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] &\leq e^{(1+C_2+C_3)n\epsilon_n^2} e^{-C_4 n \epsilon_n^2} \int_{M_n} e^{\tilde{\gamma}' R(a', \theta)} \pi(\theta) d\theta \\ &\leq e^{(1+C_2+C_3)n\epsilon_n^2} e^{-C_4 n \epsilon_n^2} e^{-C_5 n \epsilon_n^2}. \end{aligned}$$

Hence, choosing the constants  $C_2, C_3, C_4$  and  $C_5$  such that  $-1 > 1 + C_2 + C_3 - C_4 - C_5$  implies

$$\mathbb{E}_{P_0^n} \left[ \mathbb{1}_{B_n} \frac{\int_{M_n} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] \leq e^{-n\epsilon_n^2} \quad (35)$$

Therefore, substituting (26) and (35) into (34)

$$\mathbb{E}_{P_0^n} \left[ \frac{\int_{M_n} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] \leq 2e^{-C_4 C_{11} n \epsilon_n^2}, \quad (36)$$

where  $C_{11} = \min\{\lambda, 1\}/C_4$ . Using Fubini's theorem, observe that the RHS in (36) can be expressed as  $\nu(M_n)$ , where the measure

$$d\nu(\theta) = \mathbb{E}_{P_0^n} \left[ \frac{\prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)}}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} \right] \pi(\theta) d\theta.$$

Applying Lemma 7.4 for  $c_1 = 2$ ,  $c_2 = C_{11}$ ,  $t_0 = C_4 n \epsilon_n^2$ , we obtain

$$\mathbb{E}_{P_0^n} \left[ \int_{\Theta} \frac{e^{\tilde{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta)}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta} d\theta \right] \leq e^{C_4 n \epsilon_n^2} + \frac{2}{C_{11}}. \quad (37)$$

□

*Proof.* Proof of Theorem 4.1: Finally, recall (20),

$$\begin{aligned}
& \beta \mathbb{E}_{P_0^n} \left[ \int_{\Theta} L_n^f(\theta, \theta_0) q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta \right] \\
& \leq \log \mathbb{E}_{P_0^n} \left[ \int_{\Theta} e^{\beta L_n^f(\theta, \theta_0)} \frac{e^{\bar{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta)}{\int_{\Theta} e^{\bar{\gamma}' R(a', \theta)} \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta) d\theta} d\theta \right] + \inf_{q \in \mathcal{Q}} \mathbb{E}_{P_0^n} \left[ \int_{\Theta} q(\theta) \log \frac{q(\theta)}{\pi(\theta | \tilde{X}_n)} d\theta \right. \\
& \quad \left. - \bar{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta \right] + \log \mathbb{E}_{P_0^n} \left[ \int_{\Theta} e^{\bar{\gamma}' R(a', \theta)} \frac{\prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta)}{\int_{\Theta} \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \pi(\theta) d\theta} d\theta \right]. \tag{38}
\end{aligned}$$

Substituting (32) and (23) into the above equation and then using the definition of  $\eta_n^R(\bar{\gamma}')$ , we get

$$\begin{aligned}
& \mathbb{E}_{P_0^n} \left[ \int_{\Theta} L_n^f(\theta, \theta_0) q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta \right] \\
& \leq \frac{1}{\beta} \left\{ \log(e^{\beta C_1 n \epsilon_n^2} + (1 + C_0 + 3W^{-1})) + \log \left( e^{C_4 n \epsilon_n^2} + \frac{2}{C_{11}} \right) + n \eta_n^R(\bar{\gamma}') \right\} \\
& \leq \left( C_1 + \frac{1}{\beta} C_4 \right) n \epsilon_n^2 + \frac{1}{\beta} n \eta_n^R(\bar{\gamma}') + \frac{(1 + C_0 + 3W^{-1}) e^{(-\beta C_1 n \epsilon_n^2)}}{\beta} + \frac{2 e^{-C_4 n \epsilon_n^2}}{C_{11} \beta},
\end{aligned}$$

where the last inequality uses the fact that  $\log x \leq x - 1$ . Choosing  $\beta = C_{10}/2 = \frac{\min(C, \lambda, 1)}{2C_1}$ ,

$$\begin{aligned}
& \mathbb{E}_{P_0^n} \left[ \int_{\Theta} L_n^f(\theta, \theta_0) q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta \right] \\
& \leq M' n (\epsilon_n^2 + \eta_n^R(\bar{\gamma}')) + \frac{2(1 + C_0 + 3W^{-1}) e^{(-\frac{C_{10}}{2} n \epsilon_n^2)}}{C_{10}} + \frac{4 e^{-C_4 n \epsilon_n^2}}{C_{11} C_{10}} \tag{39}
\end{aligned}$$

where  $M'$  depends on  $C, C_1, C_4, W$  and  $\lambda$ . Since the last two terms in (39) decrease and the first term increases as  $n$  increases, we can choose an  $M'$  large enough, such that for all  $n \geq 1$

$$M' n (\epsilon_n^2 + \eta_n^R(\bar{\gamma}')) > \frac{2(1 + C_0 + 3W^{-1})}{C_{10}} + \frac{4}{C_{11} C_{10}},$$

and therefore for  $M = 2M'$ ,

$$\mathbb{E}_{P_0^n} \left[ \int_{\Theta} L_n^f(\theta, \theta_0) q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta \right] \leq M n (\epsilon_n^2 + \eta_n^R(\bar{\gamma}')). \tag{40}$$

Also, observe that the LHS in the above equation is always positive, therefore  $(\epsilon_n^2 + \eta_n^R(\bar{\gamma}')) \geq 0 \forall n \geq 1$ . □

*Proof.* Proof of Lemma 4.1: Refer Theorem 7.1 of Ghosal et al. (2000). □

*Proof.* Proof of Lemma 4.2: For any positive  $k$  and  $\epsilon$ , let  $\theta \in [\theta_0 - k\epsilon, \theta_0 + k\epsilon]^s \subset \mathbb{R}^s$ . Now consider a set  $H_i = \{\theta_i^0, \theta_i^1, \dots, \theta_i^M, \theta_i^{M+1}\}$  and  $H = \otimes_s H_i$  with  $M = \lfloor \frac{2k\epsilon}{\delta'} \rfloor$ , where  $\theta_i^j = \theta_0 - k\epsilon + i\delta'$  for  $j = \{0, 1, \dots, M\}$  and  $\theta_i^{M+1} = \theta_0 + k\epsilon$ . Observe that for any  $\theta \in [\theta_0 - k\epsilon, \theta_0 + k\epsilon]^s$ , there exists a  $\theta^j \in H$  such that  $\|\theta - \theta^j\| < \delta'$ . Hence union of the  $\delta'$ -balls for each element in set  $H$  covers  $[\theta_0 - k\epsilon, \theta_0 + k\epsilon]^s$ , therefore  $N(\delta', [\theta_0 - k\epsilon, \theta_0 + k\epsilon]^s, \|\cdot\|) = (M + 2)^s$ .



Since,  $f(\cdot)$  is locally Lipschitz by Assumption 4.9 and  $G(a, \theta)$  is locally Lipschitz in  $\theta$  due to Assumption 4.8 for a given  $a \in \mathcal{A}$ , therefore for any  $\theta \in [\theta_0 - k\epsilon, \theta_0 + k\epsilon]^s$

$$\sup_{a \in \mathcal{A}} |f(G(a, \theta)) - f(G(a, \theta^j))| \leq C \|\theta - \theta^j\| \leq C\delta',$$

where  $C = \sup_{a \in \mathcal{A}} K(a)$  and is finite due to Assumption 4.8. Hence  $\delta'$ -cover of set  $[\theta_0 - k\epsilon, \theta_0 + k\epsilon]^s$  is  $C\delta'$  cover of set  $T(\epsilon)$  with  $k = 1/C$ .

Finally,

$$N(C\delta', T(\epsilon), d_f) \leq (M+2)^s \leq \left(\frac{2k\epsilon}{\delta'} + 2\right)^s = \left(\frac{2\epsilon}{C\delta'} + 2\right)^s$$

which implies for  $\delta = C\delta'$ ,

$$N(\delta, T(\epsilon), d_f) \leq \left(\frac{2\epsilon}{\delta} + 2\right)^s.$$

□

*Proof.* Proof of Lemma 4.3: Recall  $d_f(\theta, \theta_0) = \sup_{a \in \mathcal{A}} |f(G(a, \theta)) - f(G(a, \theta_0))|$  and  $T(\epsilon) = \{P_\theta : d_f(\theta, \theta_0) < \epsilon\}$ . Using Lemma 4.2, observe that for every  $\epsilon > \epsilon_n > 0$ ,

$$N\left(\frac{\epsilon}{2}, \{\theta : \epsilon \leq d_f(\theta, \theta_0) \leq 2\epsilon\}, d_f\right) \leq N\left(\frac{\epsilon}{2}, \{\theta : d_f(\theta, \theta_0) \leq 2\epsilon\}, d_f\right) < 10^s.$$

Using Assumption 4.7 and the fact that total variation distance is bounded above by Hellinger distance, we have

$$d_f(\theta, \theta_0) \leq \sup_{a \in \mathcal{A}} K_1^f(a) d_{TV}(\theta, \theta_0) \leq \sup_{a \in \mathcal{A}} K_1^f(a) h(\theta, \theta_0).$$

It follows from the above two observations and Lemma 2 that, for every  $\epsilon > \epsilon_n > 0$ , there exist tests  $\{\phi_n\}$  such that

$$\mathbb{E}_{P_0^n}[\phi_n] \leq 10^s \frac{\exp(-C'n\epsilon^2)}{1 - \exp(-C'n\epsilon^2)}, \quad (41)$$

$$\sup_{\{\theta \in \Theta : d_f(\theta, \theta_0) \geq \epsilon\}} \mathbb{E}_{P_\theta^n}[1 - \phi_n] \leq \exp(-C'n\epsilon^2), \quad (42)$$

where  $C' = \frac{1}{2(\sup_{a \in \mathcal{A}} K_1^f(a))^2}$ . Since these two conditions hold for every  $\epsilon > \epsilon_n$ , we can choose a constant  $K > 1$  such that for  $\epsilon = K\epsilon_n$

$$\mathbb{E}_{P_0^n}[\phi_n] \leq 10^s \frac{\exp(-C'K^2n\epsilon_n^2)}{1 - \exp(-C'K^2n\epsilon_n^2)} \leq 2 * 10^s \exp(-C'K^2n\epsilon_n^2), \quad (43)$$

$$\sup_{\{\theta \in \Theta : L_n^f(\theta, \theta_0) \geq K^2n\epsilon_n^2\}} \mathbb{E}_{P_\theta^n}[1 - \phi_n] = \sup_{\{\theta \in \Theta : d_f(\theta, \theta_0) \geq K\epsilon_n\}} \mathbb{E}_{P_\theta^n}[1 - \phi_n] \leq \exp(-C'K^2n\epsilon_n^2), \quad (44)$$

where the second inequality in (43) holds  $\forall n \geq n_0$ , where  $n_0 := \min\{n \geq 1 : C'K^2n\epsilon_n^2 \geq \log(2)\}$ . Hence the result follows for  $C_1 = K^2$  and  $C = C'K^2$ . □

*Proof.* Proof of Corollary 4.1: Using Lemma 4.3 observe that for any  $\Theta_n(\epsilon) \subseteq \Theta$ ,  $L_n^f(\theta, \theta_0)$  satisfies Assumption 4.2 with  $C_0 = 2 * 10^s$ ,  $C = \frac{C_1}{2(\sup_{a \in \mathcal{A}} K_1(a))^2}$  and for any  $C_1 > 1$ , since

$$\sup_{\{\theta \in \Theta_n(\epsilon) : L_n^f(\theta, \theta_0) \geq C_1n\epsilon_n^2\}} \mathbb{E}_{P_\theta^n}[1 - \phi_n] \leq \sup_{\{\theta \in \Theta : L_n^f(\theta, \theta_0) \geq C_1n\epsilon_n^2\}} \mathbb{E}_{P_\theta^n}[1 - \phi_n] \leq \exp(-Cn\epsilon_n^2).$$

Hence, applying Theorem 4.1 the proof follows. □

### 7.3 Proof of Proposition 4.1

*Proof.* Proof: Fix  $\bar{\gamma}' > 0$ . Now, recall,  $\eta_n^R(\bar{\gamma}') = \frac{1}{n} \inf_{q \in \mathcal{Q}} \mathbb{E}_{P_0^n} \left[ \int_{\Theta} q(\theta) \log \frac{q(\theta)}{\pi(\theta|\tilde{X}_n)} d\theta - \bar{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta \right]$ .

For brevity let us denote  $\text{KL}(q(\theta) \|\pi(\theta|\tilde{X}_n))$  as  $\text{KL}$ . First, observe that

$$\begin{aligned} \mathbb{E}_{P_0^n}[\text{KL}] &= \int_{\Theta} q(\theta) \log(q(\theta)) d\theta - \int_{\Theta} q(\theta) \log \pi(\theta) d\theta - \mathbb{E}_{P_0^n} \left[ \int_{\Theta} q(\theta) \log \left( \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \right) d\theta \right] \\ &\quad + \mathbb{E}_{P_0^n} \left[ \log \left( \int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta \right) \right]. \end{aligned} \quad (45)$$

Now, using Jensen's inequality and Fubini's theorem in the last term above,

$$\begin{aligned} \mathbb{E}_{P_0^n} \left[ \log \left( \int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta \right) \right] &\leq \log \left( \mathbb{E}_{P_0^n} \left[ \int_{\Theta} \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \pi(\theta) d\theta \right] \right) \\ &= \log \left( \int_{\Theta} \mathbb{E}_{P_0^n} \left[ \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \right] \pi(\theta) d\theta \right) = 0. \end{aligned}$$

Now, substituting the above result into (45) and dividing the third term in (45) over compact set  $K$ , containing the true parameter  $\theta_0$  and its complement in  $\Theta$ , it follows that

$$\begin{aligned} \mathbb{E}_{P_0^n}[\text{KL}] &\leq \int_{\Theta} q(\theta) \log(q(\theta)) d\theta - \int_{\Theta} q(\theta) \log \pi(\theta) d\theta - \mathbb{E}_{P_0^n} \left[ \int_K q(\theta) \log \left( \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \right) d\theta \right] \\ &\quad - \mathbb{E}_{P_0^n} \left[ \int_{\Theta \setminus K} q(\theta) \log \left( \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \right) d\theta \right] \end{aligned} \quad (46)$$

Next, we approximate the third term in the previous display using the LAN condition in Assumption 4.10. Let  $\Delta_{n,\theta_0} := r_n(\hat{\theta} - \theta_0)$ , with  $r_n = \sqrt{n}I$ . Re-parameterizing the expression with  $\theta = \theta_0 + r_n^{-1}h$  we have  $\int_K q(\theta) \log \left( \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \right) d\theta$

$$\begin{aligned} &= \det(r_n)^{-1} \int q(\theta_0 + r_n^{-1}h) \log \left( \prod_{i=1}^n \frac{p(X_i|(\theta_0 + r_n^{-1}h))}{p(X_i|\theta_0)} \right) dh \\ &= \det(r_n)^{-1} \int q(\theta_0 + r_n^{-1}h) \left( h^T I(\theta_0) \Delta_{n,\theta_0} - \frac{1}{2} h^T I(\theta_0) h + o_{P_{\theta_0}^n}(1) \right) dh \\ &= \int_K q(\theta) \left( [r_n(\theta - \theta_0)]^T I(\theta_0) [r_n(\hat{\theta} - \theta_0)] - \frac{1}{2} [r_n(\theta - \theta_0)]^T I(\theta_0) [r_n(\theta - \theta_0)] \right) d\theta + o_{P_{\theta_0}^n}(1) \\ &= -\frac{1}{2} \int_K q(\theta) [r_n(\theta - \hat{\theta})]^T I(\theta_0) [r_n(\theta - \hat{\theta})] d\theta + \frac{1}{2} [r_n(\hat{\theta} - \theta_0)]^T I(\theta_0) [r_n(\hat{\theta} - \theta_0)] \int_K q(\theta) d\theta + o_{P_{\theta_0}^n}(1) \\ &\geq -\frac{1}{2} \int_K q(\theta) [r_n(\theta - \hat{\theta})]^T I(\theta_0) [r_n(\theta - \hat{\theta})] d\theta + o_{P_{\theta_0}^n}(1), \end{aligned} \quad (47)$$

where the penultimate equality follows by adding and subtracting  $\frac{1}{2} [r_n(\hat{\theta} - \theta_0)]^T I(\theta_0) [r_n(\hat{\theta} - \theta_0)]$  and the last inequality is due to fact that the second term in the penultimate equality is non-negative.

Now, by substituting equation (47) into (46) we obtain,

$$\begin{aligned}
\mathbb{E}_{P_0^n}[\text{KL}] &\leq \int_{\Theta} q(\theta) \log q(\theta) d\theta - \int_{\Theta} q(\theta) \log \pi(\theta) d\theta + \frac{1}{2} \mathbb{E}_{P_0^n} \left[ \int_K q(\theta) [r_n(\theta - \hat{\theta})]^T I(\theta_0) [r_n(\theta - \hat{\theta})] d\theta \right] \\
&\quad - \mathbb{E}_{P_0^n} \left[ \int_{\Theta \setminus K} q(\theta) \log \left( \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \right) d\theta \right] + \mathbb{E}_{P_0^n} [o_{P_{\theta_0}^n}(1)] \\
&= \text{KL}(q(\theta) \|\pi(\theta)) + \frac{1}{2} \int_K \mathbb{E}_{P_0^n} [[r_n(\theta - \hat{\theta})]^T I(\theta_0) [r_n(\theta - \hat{\theta})]] q(\theta) d\theta + \mathbb{E}_{P_0^n} [o_{P_{\theta_0}^n}(1)] \\
&\quad - \int_{\Theta \setminus K} q(\theta) \mathbb{E}_{P_0^n} \left[ \log \left( \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \right) \right] d\theta, \tag{48}
\end{aligned}$$

where the second and fourth terms are due to Fubini's Theorem. Now, using the fact that  $\mathbb{E}_{P_0^n}[(\theta_0 - \hat{\theta})] = 0$ , we have

$$\begin{aligned}
&\mathbb{E}_{P_0^n} [[r_n(\theta - \hat{\theta})]^T I(\theta_0) [r_n(\theta - \hat{\theta})]] \\
&= \mathbb{E}_{P_0^n} [[r_n(\theta - \theta_0)]^T I(\theta_0) [r_n(\theta - \theta_0)]] + \mathbb{E}_{P_0^n} [[r_n(\theta_0 - \hat{\theta})]^T I(\theta_0) [r_n(\theta_0 - \hat{\theta})]] \\
&\quad + 2[r_n(\theta - \theta_0)]^T I(\theta_0) [r_n \mathbb{E}_{P_0^n}[(\theta_0 - \hat{\theta})]] \\
&= [[r_n(\theta - \theta_0)]^T I(\theta_0) [r_n(\theta - \theta_0)]] + \mathbb{E}_{P_0^n} [[r_n(\theta_0 - \hat{\theta})]^T I(\theta_0) [r_n(\theta_0 - \hat{\theta})]].
\end{aligned}$$

Substituting the equation above into (48),

$$\begin{aligned}
\mathbb{E}_{P_0^n}[\text{KL}] &\leq \frac{1}{2} \int_K [[r_n(\theta - \theta_0)]^T I(\theta_0) [r_n(\theta - \theta_0)]] q(\theta) d\theta - \int_{\Theta \setminus K} q(\theta) \mathbb{E}_{P_0^n} \left[ \log \left( \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \right) \right] d\theta \\
&\quad \text{KL}(q(\theta) \|\pi(\theta)) + \frac{1}{2} \mathbb{E}_{P_0^n} [[r_n(\theta_0 - \hat{\theta})]^T I(\theta_0) [r_n(\theta_0 - \hat{\theta})]] \int_K q(\theta) d\theta + \mathbb{E}_{P_0^n} [o_{P_{\theta_0}^n}(1)] \\
&\leq \frac{1}{2} \int_K [[r_n(\theta - \theta_0)]^T I(\theta_0) [r_n(\theta - \theta_0)]] q(\theta) d\theta - \int_{\Theta \setminus K} q(\theta) \mathbb{E}_{P_0^n} \left[ \log \left( \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \right) \right] d\theta \\
&\quad \text{KL}(q(\theta) \|\pi(\theta)) + \frac{1}{2} \mathbb{E}_{P_0^n} [[r_n(\theta_0 - \hat{\theta})]^T I(\theta_0) [r_n(\theta_0 - \hat{\theta})]] + \mathbb{E}_{P_0^n} [o_{P_{\theta_0}^n}(1)]. \tag{49}
\end{aligned}$$

Using (49), it follows that

$$\begin{aligned}
&\mathbb{E}_{P_0^n} \left[ \int_{\Theta} q(\theta) \log \frac{q(\theta)}{\pi(\theta|\tilde{X}_n)} d\theta - \bar{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta \right] \\
&\leq \text{KL}(q(\theta) \|\pi(\theta)) - \bar{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta + \frac{1}{2} \int_K [[r_n(\theta - \theta_0)]^T I(\theta_0) [r_n(\theta - \theta_0)]] q(\theta) d\theta \\
&\quad - \int_{\Theta \setminus K} q(\theta) \mathbb{E}_{P_0^n} \left[ \log \left( \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \right) \right] d\theta + \frac{1}{2} \mathbb{E}_{P_0^n} [[r_n(\theta_0 - \hat{\theta})]^T I(\theta_0) [r_n(\theta_0 - \hat{\theta})]] + \mathbb{E}_{P_0^n} [o_{P_{\theta_0}^n}(1)], \tag{50}
\end{aligned}$$

where matrices  $I(\theta_0)$  and  $r_n$  are as defined in Assumption 4.10. Now, it follows from substituting  $r_n = \sqrt{n}I$  and the definition of  $\eta_n^R(\bar{\gamma}')$  that,

$$\begin{aligned}
\eta_n^R(\bar{\gamma}') &\leq \inf_{q \in \mathcal{Q}} \left[ \frac{1}{n} \text{KL}(q(\theta) \|\pi(\theta)) - \frac{\bar{\gamma}'}{n} \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta + \frac{1}{2} \int_K [[(\theta - \theta_0)]^T I(\theta_0) [(\theta - \theta_0)]] q(\theta) d\theta \right. \\
&\quad \left. - \frac{1}{n} \int_{\Theta \setminus K} q(\theta) \mathbb{E}_{P_0^n} \left[ \log \left( \prod_{i=1}^n \frac{p(X_i|\theta)}{p(X_i|\theta_0)} \right) \right] d\theta \right] + \frac{1}{2} \mathbb{E}_{P_0^n} [[(\theta_0 - \hat{\theta})]^T I(\theta_0) [(\theta_0 - \hat{\theta})]] + \frac{1}{n} \mathbb{E}_{P_0^n} [o_{P_{\theta_0}^n}(1)]. \tag{51}
\end{aligned}$$

Using Cramer-Rao lower bound for  $\hat{\theta}$ , we know that  $\mathbb{E}_{P_0^n} [[(\theta_0 - \hat{\theta})]^T I(\theta_0) [(\theta_0 - \hat{\theta})]] \leq \frac{1}{n} \mathbb{E}_{P_0^n} [[(\theta_0 - \hat{\theta})]^T \text{Var}(\hat{\theta})^{-1} [(\theta_0 - \hat{\theta})]]$  for all  $n \geq 1$ . Notice that  $[(\theta_0 - \hat{\theta})]^T \text{Var}(\hat{\theta})^{-1} [(\theta_0 - \hat{\theta})]$  is Wald's test statistic and converges in distribution to  $\chi^2$  distribution with  $d$ -degrees of freedom, implying that

$$\limsup_{n \rightarrow \infty} \mathbb{E}_{P_0^n} [[(\theta_0 - \hat{\theta})]^T I(\theta_0) [(\theta_0 - \hat{\theta})]] \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_{P_0^n} [[(\theta_0 - \hat{\theta})]^T \text{Var}(\hat{\theta})^{-1} [(\theta_0 - \hat{\theta})]] = 0. \quad (52)$$

By definition,  $\frac{1}{n} \mathbb{E}_{P_0^n} [o_{P_{\theta_0}^n}(1)] \rightarrow 0$  as  $n \rightarrow \infty$ .

Next, observe that for any  $\hat{q}_n(\theta)$ , that degenerates to  $\delta_{\theta_0}$  at the rate of  $\sqrt{n}I$ , the first term in (51)

$$\begin{aligned} \inf_{q \in \mathcal{Q}} & \left[ \frac{1}{n} \text{KL}(q(\theta) \| \pi(\theta)) - \frac{\bar{\gamma}'}{n} \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta + \frac{1}{2} \int_K [[(\theta - \theta_0)]^T I(\theta_0) [(\theta - \theta_0)]] q(\theta) d\theta \right. \\ & \left. - \frac{1}{n} \int_{\Theta \setminus K} q(\theta) \mathbb{E}_{P_0^n} \left[ \log \left( \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \right) \right] d\theta \right] \\ & \leq \frac{1}{n} \text{KL}(\hat{q}_n(\theta) \| \pi(\theta)) - \frac{\bar{\gamma}'}{n} \inf_{a \in \mathcal{A}} \int_{\Theta} \hat{q}_n(\theta) R(a, \theta) d\theta + \frac{1}{2} \int_K [[(\theta - \theta_0)]^T I(\theta_0) [(\theta - \theta_0)]] \hat{q}_n(\theta) d\theta \\ & \quad - \frac{1}{n} \int_{\Theta \setminus K} \hat{q}_n(\theta) \mathbb{E}_{P_0^n} \left[ \log \left( \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \right) \right] d\theta \end{aligned} \quad (53)$$

$$\begin{aligned} & = \frac{1}{n} \int_{\Theta} \hat{q}_n(\theta) \log \hat{q}_n(\theta) d\theta - \frac{1}{n} \int_{\Theta} \hat{q}_n(\theta) \log \pi(\theta) d\theta + \frac{1}{2} \int_K [[(\theta - \theta_0)]^T I(\theta_0) [(\theta - \theta_0)]] \hat{q}_n(\theta) d\theta \\ & \quad - \frac{1}{n} \int_{\Theta \setminus K} \hat{q}_n(\theta) \mathbb{E}_{P_0^n} \left[ \log \left( \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \right) \right] d\theta - \frac{\bar{\gamma}'}{n} \inf_{a \in \mathcal{A}} \int_{\Theta} \hat{q}_n(\theta) R(a, \theta) d\theta \end{aligned} \quad (54)$$

Note that, by assumption,  $\hat{q}_n(\theta)$  converges weakly to  $\delta_{\theta_0}$ . Since,  $\log \pi(\theta_0) < \infty$ , the second term converges to zero as  $n \rightarrow \infty$ . It is straightforward to observe that the third term also converges to zero as  $n \rightarrow \infty$  because the compact set  $K$  contains the true parameter  $\theta_0$  and the last term also converges to zero, since  $\inf_{a \in \mathcal{A}} R(a, \theta_0) < \infty$ . Now consider the penultimate term, using Jensen's inequality,  $\mathbb{E}_{P_0^n} \left[ \log \left( \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \right) \right] \leq 0$ . Combined with the fact that set  $\Theta \setminus K$  does not contain true parameter  $\theta_0$ ,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \int_{\Theta \setminus K} \hat{q}_n(\theta) \mathbb{E}_{P_0^n} \left[ \log \left( \prod_{i=1}^n \frac{p(X_i | \theta)}{p(X_i | \theta_0)} \right) \right] d\theta = 0.$$

Re-parameterizing the first term in (54) by  $\mu = r_n(\theta - \theta_0)$  for  $r_n = \sqrt{n}I$  and using the definition of the rescaled density  $\check{q}_n(\mu) = \frac{1}{\det(r_n)} \hat{q}_n(r_n^{-1}\mu + \theta_0)$ ,

$$\begin{aligned} \frac{1}{n} \int_{\Theta} \hat{q}_n(\theta) \log \hat{q}_n(\theta) d\theta & = \frac{\log \det(\sqrt{n}I)}{n} + \frac{1}{n} \int \check{q}_n(\mu) \log \check{q}_n(\mu) d\mu \\ & = \frac{d \log n}{2} + \frac{1}{n} \int \check{q}_n(\mu) \log \check{q}_n(\mu) d\mu \end{aligned} \quad (55)$$

Observe that as  $n \rightarrow \infty$ ,  $\frac{\log n}{n} \rightarrow 0$  and the last term also converges to zero, since  $\int \check{q}_n(\mu) \log \check{q}_n(\mu) d\mu < \infty$ . Hence, the above observations combined together imply that

$$\limsup_{n \rightarrow \infty} \eta_n^R(\bar{\gamma}') \leq 0.$$

Since, Theorem 4.1 implies that  $\epsilon_n^2 + \eta_n^R(\bar{\gamma}') \geq 0$  for all  $n \geq 1$  and  $\epsilon_n^2 \rightarrow 0$  as  $n \rightarrow \infty$ , it follows that  $\lim_{n \rightarrow \infty} \eta_n^R(\bar{\gamma}') = 0$ .

□

*Proof.* Proof of Corollary 4.2: For any  $\delta > 0$ , using Markov inequality

$$\begin{aligned} P_0^n \left( Q_{a', \tilde{\gamma}'}^* \left[ \left\{ \frac{1}{n} L_n^f(\theta, \theta_0) > M_n(\epsilon_n^2 + \eta_n^R(\tilde{\gamma}')) \right\} \middle| \tilde{X}_n \right] > \delta \right) &\leq \frac{1}{\delta} \mathbb{E}_{P_0^n} Q_{a', \tilde{\gamma}'}^* \left[ \left\{ \frac{1}{n} L_n^f(\theta, \theta_0) > M_n(\epsilon_n^2 + \eta_n^R(\tilde{\gamma}')) \right\} \middle| \tilde{X}_n \right] \\ &\leq \frac{1}{n\delta M_n(\epsilon_n^2 + \eta_n^R(\tilde{\gamma}'))} \mathbb{E}_{P_0^n} \left[ \int_{\Theta} L_n^f(\theta, \theta_0) q_{a', \tilde{\gamma}'}^*(\theta | \tilde{X}_n) d\theta \right] \\ &\leq \frac{nM(\epsilon_n^2 + \eta_n^R(\tilde{\gamma}'))}{n\delta M_n(\epsilon_n^2 + \eta_n^R(\tilde{\gamma}'))} = \frac{M}{\delta M_n}, \end{aligned}$$

where the last inequality follows from Theorem 4.1. Since  $M_n$  is a diverging sequence, convergence in  $P_0^n$ -probability follows.  $\square$

## 7.4 Proof of Proposition 4.2

*Proof.* Proof:

Using the definition of  $\eta_n^R(\tilde{\gamma}')$ , following Zhang and Gao (2017), and the posterior distribution  $\pi(\theta | \tilde{X}_n)$ , observe that

$$\begin{aligned} n\eta_n^R(\tilde{\gamma}') &= \inf_{q \in \mathcal{Q}} \mathbb{E}_{P_0^n} \left[ \int_{\Theta} q(\theta) \log \frac{q(\theta)}{\pi(\theta | \tilde{X}_n)} d\theta - \tilde{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta \right] \\ &= \inf_{q \in \mathcal{Q}} \mathbb{E}_{P_0^n} \left[ \int_{\Theta} q(\theta) \log \frac{q(\theta)}{\pi(\theta)} d\theta + \int_{\Theta} q(\theta) \log \left( \frac{\int \pi(\theta) p(\tilde{X}_n | \theta) d\theta}{p(\tilde{X}_n | \theta)} \right) d\theta - \tilde{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta \right] \\ &= \inf_{q \in \mathcal{Q}} \left[ \text{KL}(q(\theta) \| \pi(\theta)) - \tilde{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta \right] + \mathbb{E}_{P_0^n} \left[ \int_{\Theta} q(\theta) \log \left( \frac{\int \pi(\theta) p(\tilde{X}_n | \theta) d\theta}{p(\tilde{X}_n | \theta)} \right) d\theta \right]. \end{aligned}$$

Now, using Fubini's in the last term of the equation above, we obtain

$$\begin{aligned} n\eta_n^R(\tilde{\gamma}') &= \inf_{q \in \mathcal{Q}} \left[ \text{KL}(q(\theta) \| \pi(\theta)) - \tilde{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta \right] \\ &\quad + \mathbb{E}_{q(\theta)} \left[ \text{KL}(dP_0^n \| p(\tilde{X}_n | \theta)) - \text{KL} \left( dP_0^n \middle| \middle| \int \pi(\theta) p(\tilde{X}_n | \theta) d\theta \right) \right] \end{aligned} \quad (56)$$

Observe that,  $\int_{\mathcal{X}^n} \int \pi(\theta) p(\tilde{X}_n | \theta) d\theta d\tilde{X}_n = 1$ . Since, KL is always non-negative, it follows from the equation above that

$$\begin{aligned} \eta_n^R(\tilde{\gamma}') &\leq \frac{1}{n} \inf_{q \in \mathcal{Q}} \left[ \text{KL}(q(\theta) \| \pi(\theta)) - \tilde{\gamma}' \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta + \mathbb{E}_{q(\theta)} [\text{KL}(dP_0^n \| p(\tilde{X}_n | \theta))] \right] \\ &\leq \frac{1}{n} \inf_{q \in \mathcal{Q}} [\text{KL}(q(\theta) \| \pi(\theta)) + \mathbb{E}_{q(\theta)} [\text{KL}(dP_0^n \| p(\tilde{X}_n | \theta))]] - \frac{1}{n} \tilde{\gamma}' \inf_{q \in \mathcal{Q}} \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta, \end{aligned} \quad (57)$$

where the last inequality follows from the following fact, for any functions  $f(\cdot)$  and  $g(\cdot)$ ,

$$\inf(f - g) \leq \inf f - \inf g.$$

Recall  $\epsilon'_n \geq \frac{1}{\sqrt{n}}$ . First consider the last term in (57). Notice that the coefficient of  $\frac{1}{n}$  is independent of  $n$  and is bounded from below. Therefore, there exist a positive constant  $C_8$ , such that

$$-\frac{1}{n}\bar{\gamma}' \inf_{q \in \mathcal{Q}} \inf_{a \in \mathcal{A}} \int_{\Theta} q(\theta) R(a, \theta) d\theta \leq C_8 \epsilon_n'^2. \quad (58)$$

Now, using Assumption 4.11, it is straightforward to observe that the first term in (57),

$$\frac{1}{n} \inf_{q \in \mathcal{Q}} [\text{KL}(q(\theta) \parallel \pi(\theta)) + \mathbb{E}_{q(\theta)} [\text{KL}(dP_0^n \parallel p(\tilde{X}_n | \theta))]] \leq C_9 \epsilon_n'^2. \quad (59)$$

Therefore, equation (58) and (59) together implies that  $\eta_n^R(\bar{\gamma}') \leq (C_8 + C_9) \epsilon_n'^2$  and the result follows.  $\square$

## 7.5 Proof of Theorem 4.2

**Lemma 7.6.** Given  $a' \in \mathcal{A}$  and for a constant  $M$ , as defined in Theorem 4.1

$$\mathbb{E}_{P_0^n} \left[ \sup_{a \in \mathcal{A}} \left| \int f(G(a, \theta)) q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta - f(G(a, \theta_0)) \right| \right] \leq [M(\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))]^{\frac{1}{2}}. \quad (60)$$

*Proof.* Proof: First, observe that

$$\begin{aligned} \left( \sup_{a \in \mathcal{A}} \left| \int f(G(a, \theta)) q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta - f(G(a, \theta_0)) \right| \right)^2 &\leq \left( \int \sup_{a \in \mathcal{A}} |f(G(a, \theta)) - f(G(a, \theta_0))| q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta \right)^2 \\ &\leq \int \left( \sup_{a \in \mathcal{A}} |f(G(a, \theta)) - f(G(a, \theta_0))| \right)^2 q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta, \end{aligned}$$

where the last inequality follows from Jensen's inequality. Now, using the Jensen's inequality again

$$\begin{aligned} &\left( \mathbb{E}_{P_0^n} \left[ \sup_{a \in \mathcal{A}} \left| \int f(G(a, \theta)) q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta - f(G(a, \theta_0)) \right| \right] \right)^2 \\ &\leq \mathbb{E}_{P_0^n} \left[ \left( \sup_{a \in \mathcal{A}} \left| \int f(G(a, \theta)) q_{a', \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta - f(G(a, \theta_0)) \right| \right)^2 \right]. \end{aligned}$$

Now, using Corollary 4.1 the result follows immediately.  $\square$

*Proof.* Proof of Theorem 4.2(1): Since, the above result holds for any  $a' \in \mathcal{A}$ , fix  $a' = \mathbf{a}_{\text{RS}}^*$  and observe that for any  $\bar{\gamma}' > 0$  and  $\tau > 0$ , the result in Lemma 7.6 implies that  $P_0^n$ - probability of

$$\left\{ [M(\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))]^{-\frac{1}{2}} \sup_{a \in \mathcal{A}} \left| \int f(G(a, \theta)) q_{\mathbf{a}_{\text{RS}}^*, \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta - f(G(a, \theta_0)) \right| > \tau \right\} \quad (61)$$

is at most  $\tau^{-1}$ . For  $\mathbf{a}_{\text{RS}}^*$ , it follows from the definition of  $\Psi^f(\cdot)$  that

$$\begin{aligned}
& \Psi \left( d_h(\mathbf{a}_{\text{RS}}^*, \arg \min_{a \in \mathcal{A}} f(G(a, \theta_0))) \right) \\
& \leq f(G(\mathbf{a}_{\text{RS}}^*, \theta_0)) - \inf_{z \in \mathcal{A}} f(G(z, \theta_0)) \\
& = f(G(\mathbf{a}_{\text{RS}}^*, \theta_0)) - \int f(G(\mathbf{a}_{\text{RS}}^*, \theta)) q_{\mathbf{a}_{\text{RS}}^*, \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta + \int f(G(\mathbf{a}_{\text{RS}}^*, \theta)) q_{\mathbf{a}_{\text{RS}}^*, \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta - \inf_{z \in \mathcal{A}} f(G(z, \theta_0)) \\
& \leq \left| f(G(\mathbf{a}_{\text{RS}}^*, \theta_0)) - \int f(G(\mathbf{a}_{\text{RS}}^*, \theta)) q_{\mathbf{a}_{\text{RS}}^*, \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta \right| + \left| \int f(G(\mathbf{a}_{\text{RS}}^*, \theta)) q_{\mathbf{a}_{\text{RS}}^*, \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta - \inf_{a \in \mathcal{A}} f(G(a, \theta_0)) \right| \\
& \leq 2 \sup_{a \in \mathcal{A}} \left| \int f(G(a, \theta)) q_{\mathbf{a}_{\text{RS}}^*, \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta - f(G(a, \theta_0)) \right|. \tag{62}
\end{aligned}$$

It follows from the above inequality that

$$\begin{aligned}
& \left\{ \left[ M(\epsilon_n^2 + \eta_n^R(\bar{\gamma}')) \right]^{-\frac{1}{2}} \Psi^{\log} \left( d_h(\mathbf{a}_{\text{RS}}^*, \arg \min_{z \in \mathcal{A}} f(G(z, \theta_0))) \right) > 2\tau \right\} \\
& \subseteq \left\{ \left[ M(\epsilon_n^2 + \eta_n^R(\bar{\gamma}')) \right]^{-\frac{1}{2}} \sup_{a \in \mathcal{A}} \left| \int f(G(a, \theta)) q_{\mathbf{a}_{\text{RS}}^*, \bar{\gamma}'}^*(\theta | \tilde{X}_n) d\theta - \log G(a, \theta_0) \right| > \tau \right\}. \tag{63}
\end{aligned}$$

Therefore, using the condition on the growth function in the statement of the theorem that,

$$\frac{\Psi^f \left( d_h(\mathbf{a}_{\text{RS}}^*, \arg \min_{a \in \mathcal{A}} f(G(a, \theta_0))) \right)}{d_h(\mathbf{a}_{\text{RS}}^*, \arg \min_{a \in \mathcal{A}} f(G(a, \theta_0)))^\delta} = C^f, \text{ the } P_0^n\text{-probability of the following event is at least } 1 - \tau^{-1}:$$

$$\left\{ d_h(\mathbf{a}_{\text{RS}}^*, \arg \min_{a \in \mathcal{A}} f(G(a, \theta_0))) \leq \tau^{\frac{1}{\delta}} \left[ \frac{2 \left[ M(\epsilon_n^2 + \eta_n^R(\bar{\gamma}')) \right]^{\frac{1}{2}}}{C^f} \right]^{\frac{1}{\delta}} \right\}. \tag{64}$$

Using the monotonicity of the function  $f(\cdot)$ , it follows that  $\arg \min_{a \in \mathcal{A}} f(G(a, \theta_0)) = \arg \min_{a \in \mathcal{A}} G(a, \theta_0)$  and hence the result follows.  $\square$

*Proof.* Proof of Theorem 4.2(2): The proof follows similar steps till equation (63) in the proof above and then uses the condition on the growth function  $\Psi^f(\cdot)$  given in the statement of the theorem for all  $n \geq n_0$ .  $\square$

*Proof.* Proof of Corollary 4.3:

Recall if  $\mathcal{Q} = \mathcal{M}$ ,  $R(\cdot, \cdot) = 0$ , and  $f(x) = x$ , then it is easy to observe that  $\mathbf{a}_{\text{RS}}^*$  is the Bayes posterior risk minimizer, where  $\mathcal{M}$  is large enough to include the true posterior. Therefore,  $\mathbf{a}_{\text{RS}}^* = \mathbf{a}_{\text{OS}}^*$ , since  $q_{\mathbf{a}_{\text{RS}}^*, \bar{\gamma}'}^*(\theta | \tilde{X}_n)$  coincides with the true posterior. Now, observe that, under the above conditions  $\eta_n^R(\bar{\gamma}') = \frac{1}{n} \inf_{q \in \mathcal{Q}} \mathbb{E}_{P_0^n} \left[ \int_{\Theta} q(\theta) \log \frac{q(\theta)}{\pi(\theta | \tilde{X}_n)} d\theta \right]$ . Since the variational family  $\mathcal{M}$ , is large enough to

include the true posterior for all  $n \geq 1$ , there exists a sequence of distributions  $\{q'_n(\theta)\} \subset \mathcal{Q}$ , such that  $\text{KL}(q'_n(\theta) \parallel \pi(\theta | \tilde{X}_n)) = 0 \ \forall n \geq 1$ . Now, it follows that

$$\eta_n^R(\bar{\gamma}') = \frac{1}{n} \inf_{q \in \mathcal{Q}} \mathbb{E}_{P_0^n} [\text{KL}(q(\theta) \parallel \pi(\theta | \tilde{X}_n))] \leq \frac{1}{n} \mathbb{E}_{P_0^n} [\text{KL}(q'_n(\theta) \parallel \pi(\theta | \tilde{X}_n))] = 0 \ \forall n \geq 1.$$

Since, the KL-divergence is always non-negative, the result follows immediately from the above inequality.  $\square$

*Proof.* Proof of Corollary 4.4: Using triangular inequality it is straightforward to see that

$$\begin{aligned} & P_0^n \left( d_h(\mathbf{a}_{\text{OS}}^*, \mathbf{a}_{\text{RS}}^*) > 2 \left[ \frac{2\tau [M_n \epsilon_n^2]^{\frac{1}{2}}}{C^1} \right]^{\frac{1}{\delta}} + 2 \left[ \frac{2\tau [M_n (\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))]^{\frac{1}{2}}}{C^f} \right]^{\frac{1}{\delta}} \right) \\ & \leq P_0^n \left( d_h(\mathbf{a}_{\text{OS}}^*, \arg \min_{z \in \mathcal{A}} G(z, \theta_0)) + d_h(\mathbf{a}_{\text{RS}}^*, \arg \min_{z \in \mathcal{A}} G(z, \theta_0)) > 2 \left[ \frac{2\tau [M_n \epsilon_n^2]^{\frac{1}{2}}}{C^1} \right]^{\frac{1}{\delta}} + 2 \left[ \frac{2\tau [M_n (\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))]^{\frac{1}{2}}}{C^f} \right]^{\frac{1}{\delta}} \right) \\ & \leq P_0^n \left( d_h(\mathbf{a}_{\text{OS}}^*, \arg \min_{z \in \mathcal{A}} G(z, \theta_0)) > \left[ \frac{2\tau [M_n \epsilon_n^2]^{\frac{1}{2}}}{C^1} \right]^{\frac{1}{\delta}} + \left[ \frac{2\tau [M_n (\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))]^{\frac{1}{2}}}{C^f} \right]^{\frac{1}{\delta}} \right) \\ & \quad + P_0^n \left( d_h(\mathbf{a}_{\text{RS}}^*, \arg \min_{z \in \mathcal{A}} G(z, \theta_0)) > \left[ \frac{2\tau [M_n \epsilon_n^2]^{\frac{1}{2}}}{C^1} \right]^{\frac{1}{\delta}} + \left[ \frac{2\tau [M_n (\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))]^{\frac{1}{2}}}{C^f} \right]^{\frac{1}{\delta}} \right) \\ & \leq P_0^n \left( d_h(\mathbf{a}_{\text{OS}}^*, \arg \min_{z \in \mathcal{A}} G(z, \theta_0)) > \left[ \frac{2\tau [M_n \epsilon_n^2]^{\frac{1}{2}}}{C^1} \right]^{\frac{1}{\delta}} \right) + P_0^n \left( d_h(\mathbf{a}_{\text{RS}}^*, \arg \min_{z \in \mathcal{A}} G(z, \theta_0)) > \left[ \frac{2\tau [M_n (\epsilon_n^2 + \eta_n^R(\bar{\gamma}'))]^{\frac{1}{2}}}{C^f} \right]^{\frac{1}{\delta}} \right) \end{aligned}$$

The proof follows immediately using Theorem 4.2 and Corollary 4.3.  $\square$

## 7.6 Proof of Theorem 4.3, 4.4, and 4.5

*Proof.* Proof of Theorem 4.3:

The proof follows immediately from Theorem 4.1 by substituting  $R(a, \theta) = 0$  and  $f(x) = x$ .  $\square$

Next, we obtain a finite sample bound on the regret, defined as the uniform difference between the Naive VB approximate posterior risk and the expected loss under the true data generating measure  $P_0$ .

**Lemma 7.7.** For a constant  $M$  as defined in Theorem 4.3

$$\mathbb{E}_{P_0^n} \left[ \sup_{a \in \mathcal{A}} \left| \int G(a, \theta) q^*(\theta | \tilde{X}_n) d\theta - G(a, \theta_0) \right| \right] \leq [M(\epsilon_n^2 + \kappa_n^2)]^{\frac{1}{2}}, \quad (65)$$



*Proof.* Proof: The result follows immediately from the following inequalities

$$\begin{aligned} \left( \sup_{a \in \mathcal{A}} \left| \int G(a, \theta) q^*(\theta | \tilde{X}_n) d\theta - G(a, \theta_0) \right| \right)^2 &\leq \left( \int \sup_{a \in \mathcal{A}} |G(a, \theta) - G(a, \theta_0)| q^*(\theta | \tilde{X}_n) d\theta \right)^2 \\ &\leq \int \left( \sup_{a \in \mathcal{A}} |G(a, \theta) - G(a, \theta_0)| \right)^2 q^*(\theta | \tilde{X}_n) d\theta, \end{aligned}$$

where the last inequality is a consequence of Jensens' inequality. Now, using Jensen's inequality again

$$\begin{aligned} &\left( \mathbb{E}_{P_0^n} \left[ \sup_{a \in \mathcal{A}} \left| \int G(a, \theta) q^*(\theta | \tilde{X}_n) d\theta - G(a, \theta_0) \right| \right] \right)^2 \\ &\leq \mathbb{E}_{P_0^n} \left[ \left( \sup_{a \in \mathcal{A}} \left| \int G(a, \theta) q^*(\theta | \tilde{X}_n) d\theta - G(a, \theta_0) \right| \right)^2 \right]. \end{aligned}$$

□

*Proof.* Proof of Theorem 4.4 : The result follows immediately from Lemma 7.7 and the following inequality

$$|V_{q^*} - V_0| \leq \sup_{a \in \mathcal{A}} \left| \int G(a, \theta) q^*(\theta | \tilde{X}_n) d\theta - G(a, \theta_0) \right|.$$

□

*Proof.* Proof of Theorem 4.5 (1) and (2):

The proof uses Lemma 7.7 and then follows similar steps as used in the proof of Theorem 4.2.

□

## 7.7 Proof of Theorem 4.6 and 4.7

*Proof.* Proof of Theorem 4.6:

The proof follows immediately from Theorem 4.6 by substituting  $\bar{\gamma}' = 1$ ,  $R(a, \theta) = \log G(a, \theta)$ , and  $f(x) = \log(x)$ .

□

Now, recall,  $\mathbf{a}_{\text{LC}}^* =: \operatorname{argmin}_{a \in \mathcal{A}} \max_{q \in \mathcal{Q}} \{ \mathbb{E}_q[\log G(a, \theta)] - \text{KL}(q(\theta) \| \pi(\theta | \tilde{X}_n)) \}$ . To obtain  $\mathbf{a}_{\text{LC}}^*$  iteratively, we first fix an  $a_0 \in \mathcal{A}$  and compute  $q_{a_0}^*(\theta | \tilde{X}_n)$  and then solve the outer optimization problem to compute  $a_1$ . Similarly, at each iteration  $k$ , we sequentially update  $q_{a_k}^*(\theta | \tilde{X}_n)$  and  $a_k$ , until convergence.

Next, we obtain a finite sample bound on the regret as defined in the lemma below:

**Lemma 7.8.** Given  $\mathbf{a}_{\text{LC}}^* \in \mathcal{A}$  and for a constant  $M$ , as defined in Theorem 4.6

$$\mathbb{E}_{P_0^n} \left[ \sup_{a \in \mathcal{A}} \left| \int \log G(a, \theta) q_{\mathbf{a}_{\text{LC}}^*}^*(\theta | \tilde{X}_n) d\theta - \log G(a, \theta_0) \right| \right] \leq [M(\epsilon_n^2 + \eta_n^{\log}(1))]^{\frac{1}{2}}. \quad (66)$$

*Proof.* Proof of Lemma 7.8: First, observe that

$$\begin{aligned} \left( \sup_{a \in \mathcal{A}} \left| \int \log G(a, \theta) q_{\mathbf{a}_{\text{LC}}^*}^*(\theta | \tilde{X}_n) d\theta - \log G(a, \theta_0) \right| \right)^2 &\leq \left( \int \sup_{a \in \mathcal{A}} |\log G(a, \theta) - \log G(a, \theta_0)| q_{\mathbf{a}_{\text{LC}}^*}^*(\theta | \tilde{X}_n) d\theta \right)^2 \\ &\leq \int \left( \sup_{a \in \mathcal{A}} |\log G(a, \theta) - \log G(a, \theta_0)| \right)^2 q_{\mathbf{a}_{\text{LC}}^*}^*(\theta | \tilde{X}_n) d\theta, \end{aligned}$$

where the last inequality follows from Jensen's inequality. Now, using the Jensen's inequality again

$$\begin{aligned} &\left( \mathbb{E}_{P_0^n} \left[ \sup_{a \in \mathcal{A}} \left| \int \log G(a, \theta) q_{\mathbf{a}_{\text{LC}}^*}^*(\theta | \tilde{X}_n) d\theta - \log G(a, \theta_0) \right| \right] \right)^2 \\ &\leq \mathbb{E}_{P_0^n} \left[ \left( \sup_{a \in \mathcal{A}} \left| \int \log G(a, \theta) q_{\mathbf{a}_{\text{LC}}^*}^*(\theta | \tilde{X}_n) d\theta - \log G(a, \theta_0) \right| \right)^2 \right]. \end{aligned}$$

Now, using Theorem 4.6 for  $a_0 = \mathbf{a}_{\text{LC}}^*$ , the result follows immediately.  $\square$

*Proof.* Proof of Theorem 4.7(1):

For any  $\tau > 0$ , observe that the result in Lemma 7.8 implies that  $P_0^n$ -probability of

$$\left\{ \left[ M(\epsilon_n^2 + \eta_n^{\log}(1)) \right]^{-\frac{1}{2}} \sup_{a \in \mathcal{A}} \left| \int \log G(a, \theta) q_{\mathbf{a}_{\text{LC}}^*}^*(\theta | \tilde{X}_n) d\theta - \log G(a, \theta_0) \right| > \tau \right\} \quad (67)$$

is at most  $\tau^{-1}$ . For  $\mathbf{a}_{\text{LC}}^*$ , it follows from the definition of  $\Psi^{\log}(\cdot)$  that

$$\begin{aligned} &\Psi^{\log} \left( d_h(\mathbf{a}_{\text{LC}}^*, \arg \min_{z \in \mathcal{A}} \log G(z, \theta_0)) \right) \\ &\leq \log G(\mathbf{a}_{\text{LC}}^*, \theta_0) - \inf_{z \in \mathcal{A}} \log G(z, \theta_0) \\ &= \log G(\mathbf{a}_{\text{LC}}^*, \theta_0) - \int \log G(\mathbf{a}_{\text{LC}}^*, \theta) q_{\mathbf{a}_{\text{LC}}^*}^*(\theta | \tilde{X}_n) d\theta + \int \log G(\mathbf{a}_{\text{LC}}^*, \theta) q_{\mathbf{a}_{\text{LC}}^*}^*(\theta | \tilde{X}_n) d\theta - \inf_{z \in \mathcal{A}} \log G(z, \theta_0) \\ &\leq \left| \log G(\mathbf{a}_{\text{LC}}^*, \theta_0) - \int \log G(\mathbf{a}_{\text{LC}}^*, \theta) q_{\mathbf{a}_{\text{LC}}^*}^*(\theta | \tilde{X}_n) d\theta \right| + \left| \int \log G(\mathbf{a}_{\text{LC}}^*, \theta) q_{\mathbf{a}_{\text{LC}}^*}^*(\theta | \tilde{X}_n) d\theta - \inf_{z \in \mathcal{A}} \log G(z, \theta_0) \right| \\ &\leq 2 \sup_{a \in \mathcal{A}} \left| \int \log G(a, \theta) q_{\mathbf{a}_{\text{LC}}^*}^*(\theta | \tilde{X}_n) d\theta - \log G(a, \theta_0) \right|. \end{aligned} \quad (68)$$

It follows from the above inequality that

$$\begin{aligned} &\left\{ \left[ M(\epsilon_n^2 + \eta_n^{\log}(1)) \right]^{-\frac{1}{2}} \Psi^{\log} \left( d_h(\mathbf{a}_{\text{LC}}^*, \arg \min_{z \in \mathcal{A}} \log G(z, \theta_0)) \right) > 2\tau \right\} \\ &\subseteq \left\{ \left[ M(\epsilon_n^2 + \eta_n^{\log}(1)) \right]^{-\frac{1}{2}} \sup_{a \in \mathcal{A}} \left| \int \log G(a, \theta) q_{\mathbf{a}_{\text{LC}}^*}^*(\theta | \tilde{X}_n) d\theta - \log G(a, \theta_0) \right| > \tau \right\}. \end{aligned} \quad (69)$$

Therefore, using the condition on the growth function in the statement of the theorem that,

$\frac{\Psi^{\log}\left(d_h\left(\mathbf{a}_{\text{LC}}^*, \arg \min_{z \in \mathcal{A}} \log G(z, \theta_0)\right)\right)}{d_h\left(\mathbf{a}_{\text{LC}}^*, \arg \min_{z \in \mathcal{A}} \log G(z, \theta_0)\right)^\delta} = C^{\log}$ , the  $P_0^n$ -probability of the following event is at least  $1 - \tau^{-1}$ :

$$\left\{d_h\left(\mathbf{a}_{\text{LC}}^*, \arg \min_{z \in \mathcal{A}} \log G(z, \theta_0)\right) \leq \tau^{\frac{1}{\delta}} \left[ \frac{2 \left[ M(\epsilon_n^2 + \eta_n^{\log}(1)) \right]^{\frac{1}{2}}}{C^{\log}} \right]^{\frac{1}{\delta}} \right\}. \quad (70)$$

Using the monotonicity of logarithm function, it follows that  $\arg \min_{z \in \mathcal{A}} \log G(z, \theta_0) = \arg \min_{z \in \mathcal{A}} G(z, \theta_0)$  and hence the result follows.  $\square$

*Proof.* Proof of Theorem 4.7(2): The proof follows similar steps till equation (69) in the proof above and then uses the condition on the growth function  $\Psi^{\log}(\cdot)$  given in the statement of the theorem for all  $n \geq n_0$ .  $\square$

*Proof.* Proof of Corollary 4.5: The proof follows immediately using triangular inequality, Corollary 4.5, Theorem 4.7, and using the monotonicity of logarithmic function.  $\square$

## 7.8 Newsvendor Problem

*Proof.* Proof of Lemma 5.1: Using the definition of inverse-gamma prior, observe that

$$\Pi\left[\theta \in \Theta : \theta \leq e^{-n\epsilon^2}\right] = \frac{\Gamma(\alpha, \beta e^{Cn\epsilon^2})}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)} \int_{\beta e^{Cn\epsilon^2}}^{\infty} e^{-t} t^{\alpha-1} dt,$$

where  $\Gamma(\cdot, \cdot)$  is the incomplete upper Gamma function and  $\Gamma(\cdot)$  is the Gamma function. Substituting  $\alpha = 1$  and using the fact that  $e^x > x$ , we have

$$\Pi\left[\theta \in \Theta : \theta \leq e^{-n\epsilon^2}\right] = e^{-\beta e^{Cn\epsilon^2}} < e^{-\beta n\epsilon^2}.$$

$\square$

Recall  $L_n^f(\theta, \theta_0) = n \sup_{a \in \mathcal{A}} |f(G(a, \theta)) - f(G(a, \theta_0))|$ . Next, we show that the exponentially distributed model  $P_\theta$  satisfies Assumption 4.2, for distance function  $L_n^f(\theta, \theta_0)$ , where  $G(a, \theta)$  is the newsvendor model risk.

*Proof.* Proof of Lemma 5.2:

First consider the following test function, constructed using  $\tilde{X}_n = \{\xi_1, \xi_2, \dots, \xi_n\}$ .

$$\phi_n := \mathbb{1} \left\{ \tilde{X}_n : \left| \frac{n}{\sum_{i=1}^n \xi_i} - \theta_0 \right| > \theta_0 \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2} \right\}.$$

We first verify that this test function satisfies condition (i) of the Lemma. Using Chebyshev's inequality

$$\begin{aligned}\mathbb{E}_{P_0^n}[\phi_n] &= P_0^n \left( \left\{ \tilde{X}_n : \left| \frac{n}{\sum_{i=1}^n \xi_i} - \theta_0 \right| > \theta_0 \sqrt{\frac{n+2}{(n-2)^2}} e^{Cn\epsilon^2} \right\} \right) \leq \frac{(n-2)^2}{\theta_0^2(n+2)} e^{-2Cn\epsilon^2} \mathbb{E}_{P_0^n} \left[ \left| \frac{n}{\sum_{i=1}^n \xi_i} - \theta_0 \right|^2 \right] \\ &= \frac{(n-2)^2}{\theta_0^2(n+2)} e^{-2Cn\epsilon^2} \mathbb{E}_{P_0^n} \left[ \left( \frac{n}{\sum_{i=1}^n \xi_i} \right)^2 + \theta_0^2 - \left( \frac{2n\theta_0}{\sum_{i=1}^n \xi_i} \right) \right]\end{aligned}$$

Now using the fact that the sum of  $n$  i.i.d exponential random variable with rate parameter  $\theta_0$  is Gamma distributed with rate and shape parameter  $\theta_0$  and  $n$  (respectively), we obtain

$$\begin{aligned}\mathbb{E}_{P_0^n}[\phi_n] &\leq \frac{(n-2)^2}{\theta_0^2(n+2)} e^{-2Cn\epsilon^2} \theta_0^2 \left[ \frac{n^2}{(n-1)(n-2)} + 1 - \frac{2n}{n-2} \right] \\ &= \frac{(n-2)^2}{n+2} e^{-2Cn\epsilon^2} \left[ \frac{n+2}{(n-1)(n-2)} \right] \\ &\leq e^{-2Cn\epsilon^2} \leq e^{-(C-1)n\epsilon^2},\end{aligned}\tag{71}$$

Thus, condition(i) is satisfied. Now to verify condition (ii) of the Lemma, observe that

$$\begin{aligned}\mathbb{E}_{P_\theta^n}[1 - \phi_n] &= P_\theta^n \left\{ \tilde{X}_n : \left| \frac{n}{\sum_{i=1}^n \xi_i} - \theta_0 \right| \leq \theta_0 \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2} \right\} \\ &= P_\theta^n \left\{ \tilde{X}_n : 1 - \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2} \leq \frac{n}{\sum_{i=1}^n \xi_i} \leq 1 + \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2} \right\} \\ &= P_\theta^n \left\{ \tilde{X}_n : \frac{n}{\theta_0} \left( 1 + \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2} \right)^{-1} \leq \sum_{i=1}^n \xi_i \leq \frac{n}{\theta_0} \left( 1 - \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2} \right)^{-1} \right\}\end{aligned}\tag{72}$$

Note that  $\left( 1 - \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2} \right)$  is always positive for  $n \geq 5$ . Since, the sum of  $n$  i.i.d exponential random variables with rate  $\theta$  is gamma distributed with rate  $\theta$  and shape parameter  $n$ , it follows that

$$\mathbb{E}_{P_\theta^n}[1 - \phi_n] = \int_{\frac{n}{\theta_0} \left( 1 + \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2} \right)^{-1}}^{\frac{n}{\theta_0} \left( 1 - \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2} \right)^{-1}} \frac{\theta^n}{\Gamma(n)} y^{n-1} e^{-\theta y} dy.\tag{73}$$

Observe that,  $\frac{\theta^n}{n\Gamma(n)} y^{n-1} e^{-\theta y} \leq \frac{e^{-n} n^n}{\Gamma(n)} y^{-1}$  for all  $y > 0$ . Next, using the inequality  $\sqrt{2\pi n} n^n e^{-n} \leq n\Gamma(n)$ ,  $\forall n \geq 1$ , we have  $\frac{\theta^n}{n\Gamma(n)} y^{n-1} e^{-\theta y} \leq \sqrt{\frac{n}{2\pi}} y^{-1}$  for all  $y > 0$ . Using this observation, we have

$$\begin{aligned}\mathbb{E}_{P_\theta^n}[1 - \phi_n] &= \int_{\frac{n}{\theta_0} \left( 1 + \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2} \right)^{-1}}^{\frac{n}{\theta_0} \left( 1 - \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2} \right)^{-1}} \frac{\theta^n}{\Gamma(n)} y^{n-1} e^{-\theta y} dy \\ &\leq \sqrt{\frac{n}{2\pi}} \log \left( \frac{1 + \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2}}{1 - \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2}} \right) \\ &\leq \sqrt{\frac{2}{\pi}} \sqrt{\frac{n(n+2)}{(n-2)^2}} \left( \frac{e^{-Cn\epsilon^2}}{1 - \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2}} \right),\end{aligned}\tag{74}$$

where the last inequality follows from the fact that  $\log(x) \leq x - 1$ . Since,  $n \geq 5$ , observe that

$$\begin{aligned}\mathbb{E}_{P_\theta^n}[1 - \phi_n] &\leq \sqrt{\frac{2}{\pi}} \sqrt{\frac{(1+2/n)}{(1-2/n)^2}} \left( \frac{e^{-Cn\epsilon^2}}{1 - \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2}} \right) \\ &\leq \sqrt{\frac{2}{\pi}} \sqrt{\frac{(1+2/5)}{(1-2/5)^2}} \left( \frac{e^{-Cn\epsilon^2}}{1 - \sqrt{\frac{n+2}{(n-2)^2}} e^{-Cn\epsilon^2}} \right) \\ &\leq \sqrt{\frac{2}{\pi}} \sqrt{\frac{(1+2/5)}{(1-2/5)^2}} \left( \frac{e^{-Cn\epsilon^2}}{1 - e^{-Cn\epsilon^2}} \right),\end{aligned}\tag{75}$$

where the last inequality follows from the fact that  $\sqrt{\frac{n+2}{(n-2)^2}} \leq 1$  for  $n \geq 5$ . Since,  $n\epsilon^2 > n\epsilon_n^2 \geq 1$ , we obtain

$$\mathbb{E}_{P_\theta^n}[1 - \phi_n] \leq \left( \sqrt{\frac{70}{9\pi}} \frac{1}{1 - e^{-C}} \right) e^{-1} e^{-(C-1)n\epsilon^2}.\tag{76}$$

Choosing  $C$  large enough  $\left( \sqrt{\frac{70}{9\pi}} \frac{1}{1 - e^{-C}} \right) e^{-1} < 1$ . Therefore,

$$\mathbb{E}_{P_\theta^n}[1 - \phi_n] \leq e^{-(C-1)n\epsilon^2}.\tag{77}$$

Next recall  $L_n^f(\theta, \theta_0) = n(\sup_{a \in \mathcal{A}} |f(G(a, \theta)) - f(G(a, \theta_0))|)^2$  and consider the case when  $f(x) = x$ . Observe that for the newsvendor model risk, for any  $a \in \mathcal{A}$  and on the set  $\Theta_n(\epsilon)$ , we have

$$\frac{\partial G(a, \theta)}{\partial \theta} = \frac{h}{\theta^2} - a(b+h) \frac{e^{-a\theta}}{\theta} - (b+h) \frac{e^{-a\theta}}{\theta^2} = \frac{1}{\theta^2} (h - (b+h)e^{-a\theta}(1+a\theta)) \leq h e^{n\epsilon^2}.$$

Therefore, on the set  $\Theta_n(\epsilon)$ ,  $G(a, \theta)$  is Lipschitz continuous for any  $a \in \mathcal{A}$ , that is

$$|G(a, \theta) - G(a, \theta_0)| \leq h e^{n\epsilon^2} |\theta - \theta_0|.$$

It implies that

$$\left\{ \theta \in \Theta_n(\epsilon) : |G(a, \theta) - G(a, \theta_0)| \geq C_1 \epsilon^2 \right\} \subseteq \left\{ \theta \in \Theta_n(\epsilon) : |\theta - \theta_0| \geq \frac{1}{h} C_1 \epsilon^2 e^{-n\epsilon^2} \right\}.$$

Hence choosing  $C_1 = h$ , we have

$$\sup_{\{\theta \in \Theta_n(\epsilon) : L_n^f(\theta, \theta_0) \geq C_1 n\epsilon^2\}} \mathbb{E}_{P_\theta^n}[1 - \phi_n] \leq \sup_{\{\theta \in \Theta_n(\epsilon) : |\theta - \theta_0| \geq \epsilon^2 e^{-n\epsilon^2}\}} \mathbb{E}_{P_\theta^n}[1 - \phi_n].\tag{78}$$

Since, the result in (77) holds for any  $\theta$ , it must hold for supremum over the set  $\left\{ \theta \in \Theta_n(\epsilon) : |\theta - \theta_0| \geq \frac{1}{h} C_1 \epsilon^2 e^{-n\epsilon^2} \right\}$  and therefore the result follows. □

*Proof.* Proof of Lemma 5.3:

First, we write the Rényi divergence between  $P_0^n$  and  $P_\theta^n$ ,

$$D_{1+\lambda}(P_0^n \| P_\theta^n) = \frac{1}{\lambda} \log \int \left( \frac{dP_0^n}{dP_\theta^n} \right)^\lambda dP_0^n = n \frac{1}{\lambda} \log \int \left( \frac{dP_0}{dP_\theta} \right)^\lambda dP_0 = n \left( \log \frac{\theta_0}{\theta} + \frac{1}{\lambda} \log \frac{\theta_0}{(\lambda+1)\theta_0 - \lambda\theta} \right),$$

when  $((\lambda + 1)\theta_0 - \lambda\theta) > 0$  and  $D_{1+\lambda}(P_0^n \| P_\theta^n) = \infty$  otherwise. Next observe that,

$$\Pi(A_n) = \Pi\left(0 \leq \left(\frac{\theta_0^{\lambda+1}}{\theta^\lambda((\lambda+1)\theta_0 - \lambda\theta)}\right) \leq e^{C_3\lambda\epsilon_n^2}\right), \quad (79)$$

and  $\inf_{\{\theta: \theta < \frac{\lambda+1}{\lambda}\theta_0\}} \left(\frac{\theta_0^{\lambda+1}}{\theta^\lambda((\lambda+1)\theta_0 - \lambda\theta)}\right) = 1$ , when  $((\lambda+1)\theta_0 - \lambda\theta) > 0$  and is attained at  $\theta = \theta_0$ . Therefore,

$$\begin{aligned} \Pi(A_n) &= \Pi\left(0 \leq \left(\frac{\theta_0^{\lambda+1}}{\theta^\lambda((\lambda+1)\theta_0 - \lambda\theta)}\right) \leq e^{C_3\lambda\epsilon_n^2}\right) \\ &= \Pi\left(1 \leq \left(\frac{\theta_0^{\lambda+1}}{\theta^\lambda((\lambda+1)\theta_0 - \lambda\theta)}\right) \leq e^{C_3\lambda\epsilon_n^2}\right). \end{aligned} \quad (80)$$

Now fixing  $\lambda = 1$ , we obtain

$$\begin{aligned} \Pi(A_n) &= \Pi\left(1 \leq \left(\frac{\theta_0^2}{\theta(2\theta_0 - \theta)}\right) \leq e^{C_3\epsilon_n^2}\right) \\ &= \Pi\left(\theta_0 - \theta_0\sqrt{1 - e^{-C_3\epsilon_n^2}} \leq \theta \leq \theta_0 + \theta_0\sqrt{1 - e^{-C_3\epsilon_n^2}}\right) \\ &= \Pi\left(|\theta - \theta_0| \leq \theta_0\sqrt{1 - e^{-C_3\epsilon_n^2}}\right). \end{aligned} \quad (81)$$

Observe that, for any prior distribution  $\pi(\theta)$ , as  $n \rightarrow \infty$ ,  $\Pi(A_n) \rightarrow 0$ . Now, consider an inverse-gamma prior, that is with cumulative distribution function  $\Pi(\{\theta \in \Theta : \theta < t\}) := \frac{\Gamma(\alpha, \frac{\beta}{t})}{\Gamma(\alpha)}$ , where  $\alpha(> 0)$  is the shape parameter,  $\beta(> 0)$  is the scale parameter,  $\Gamma(\cdot)$  is the Gamma function, and  $\Gamma(\cdot, \cdot)$  is the incomplete Gamma function. Using inverse-gamma prior in the last equation we have

$$\begin{aligned} \Pi(A_n) &= \frac{\Gamma\left(\alpha, \frac{\beta}{\theta_0(1 + \sqrt{1 - e^{-C_3\epsilon_n^2}})}\right) - \Gamma\left(\alpha, \frac{\beta}{\theta_0(1 - \sqrt{1 - e^{-C_3\epsilon_n^2}})}\right)}{\Gamma(\alpha)} \\ &\geq \frac{\exp\left(-\frac{\beta}{\theta_0}\left(1 - \sqrt{1 - e^{-C_3\epsilon_n^2}}\right)^{-1}\right)}{\alpha\Gamma(\alpha)} \left(\frac{\beta}{\theta_0}\right)^\alpha \left[\left(\frac{1}{1 - \sqrt{1 - e^{-C_3\epsilon_n^2}}}\right)^\alpha - \left(\frac{1}{1 + \sqrt{1 - e^{-C_3\epsilon_n^2}}}\right)^\alpha\right] \\ &= \frac{\exp\left(-\frac{\beta}{\theta_0}\left(1 - \sqrt{1 - e^{-C_3\epsilon_n^2}}\right)^{-1}\right)}{\alpha\Gamma(\alpha)} \left(\frac{\beta}{\theta_0}\right)^\alpha \left(\frac{1}{1 - \sqrt{1 - e^{-C_3\epsilon_n^2}}}\right)^\alpha \left[1 - \left(\frac{1 - \sqrt{1 - e^{-C_3\epsilon_n^2}}}{1 + \sqrt{1 - e^{-C_3\epsilon_n^2}}}\right)^\alpha\right] \end{aligned} \quad (82)$$

Now, consider the last term in the LHS of the equation above, multiplying and dividing by denominator, we obtain

$$\left[1 - \left(\frac{e^{-C_3\epsilon_n^2}}{(1 + \sqrt{1 - e^{-C_3\epsilon_n^2}})^2}\right)^\alpha\right] = \left[1 - \left(\frac{e^{-C_3\epsilon_n^2}}{(1 + \sqrt{1 - e^{-C_3\epsilon_n^2}})^2}\right)^\alpha\right] \geq [1 - e^{-\alpha C_3\epsilon_n^2}].$$

Since,  $\left(\frac{1}{1-\sqrt{1-e^{-C_3\epsilon_n^2}}}\right)^\alpha \geq 1$ , substituting the above equation into (82), we have

$$\begin{aligned}\Pi(A_n) &\geq \frac{\exp\left(-\frac{\beta}{\theta_0}\left(1-\sqrt{1-e^{-C_3\epsilon_n^2}}\right)^{-1}\right)}{\alpha\Gamma(\alpha)}\left(\frac{\beta}{\theta_0}\right)^\alpha\left[1-e^{-\alpha C_3\epsilon_n^2}\right] \\ &\geq \frac{\exp\left(-\frac{\beta}{\theta_0}\left(1-\sqrt{1-e^{-C_3\epsilon_n^2}}\right)^{-1}\right)}{\alpha\Gamma(\alpha)}\left(\frac{\beta}{\theta_0}\right)^\alpha e^{-\alpha C_3 n \epsilon_n^2},\end{aligned}\tag{83}$$

where the last inequality follows from the fact that,  $1-e^{-\alpha C_3\epsilon_n^2} \geq e^{-\alpha C_3 \frac{\epsilon_n^{-4}}{n} n \epsilon_n^2} \geq e^{-\alpha C_3 n \epsilon_n^2}$ , for any  $n \geq n_0$  such that  $n_0 := \min\left\{n \geq 1 : \frac{\epsilon_n^{-4}}{n} < 1\right\}$ . Since  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , observe that for any  $C_3, \beta, \theta_0$ , and  $\alpha$  there exists an  $n_1 \geq 1$ , such that for all  $n \geq n_1$ ,  $\frac{1}{\alpha\Gamma(\alpha)}\exp\left(-\frac{\beta}{\theta_0}\left(1-\sqrt{1-e^{-C_3\epsilon_n^2}}\right)^{-1}\right)\left(\frac{\beta}{\theta_0}\right)^\alpha \geq 1$ . Therefore, for inverse-Gamma prior  $C_2 = \alpha C_3$  and  $C_3$  is chosen such that and the result follows for all  $n \geq \max(n_0, n_1)$ .  $\square$

*Proof.* Proof of Lemma 5.4:

First, observe that for any  $a \in \mathcal{A}$ ,

$$\frac{\partial G(a, \theta)}{\partial \theta} = \frac{h}{\theta^2} - a(b+h)\frac{e^{-a\theta}}{\theta} - (b+h)\frac{e^{-a\theta}}{\theta^2} = \frac{1}{\theta^2}\left(h - (b+h)e^{-a\theta}(1+a\theta)\right).\tag{84}$$

Using the above equation the (finite) critical point  $\theta^*$  must satisfy,  $h - (b+h)e^{-a\theta^*}(1+a\theta^*) = 0$ . Therefore,

$$G(a, \theta) \geq G(a, \theta^*) = h\left(a - \frac{1}{\theta^*} + \frac{1}{\theta^*(1+a\theta^*)}\right) = \frac{ha^2\theta^*}{(1+a\theta^*)}.$$

Since  $h, b > 0$  and  $a\theta^* > 0$ , hence

$$G(a, \theta) \geq \frac{h\underline{a}^2\theta^*}{(1+a\theta^*)},$$

where  $\underline{a} := \min\{a \in \mathcal{A}\}$  and  $\underline{a} > 0$ .  $\square$

*Proof.* Proof of Lemma 5.5:

First, observe that  $G(a, \theta)$  is bounded above in  $\theta$  for a given  $a \in \mathcal{A}$

$$\begin{aligned}G(a, \theta) &= ha - \frac{h}{\theta} + (b+h)\frac{e^{-a\theta}}{\theta} \\ &\leq ha + \frac{b}{\theta}.\end{aligned}$$

Using the above fact and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\int_{\{G(a,\theta) > e^{C_4 n \epsilon_n^2}\}} G(a,\theta) \pi(\theta) d\theta &\leq \left( \int G(a,\theta)^2 \pi(\theta) d\theta \right)^{1/2} \left( \int \mathbb{I}_{\{G(a,\theta) > e^{C_4 n \epsilon_n^2}\}} \pi(\theta) d\theta \right)^{1/2} \\
&\leq \left( \int \left( ha + \frac{b}{\theta} \right)^2 \pi(\theta) d\theta \right)^{1/2} \left( \int \mathbb{I}_{\{ha + \frac{b}{\theta} > e^{C_4 n \epsilon_n^2}\}} \pi(\theta) d\theta \right)^{1/2} \\
&\leq e^{-C_4 n \epsilon_n^2} \left( \int \left( ha + \frac{b}{\theta} \right)^2 \pi(\theta) d\theta \right), \tag{85}
\end{aligned}$$

where the last inequality follows from using the Chebyshev's inequality. For the inverse-gamma prior

$$\int \left( ha + \frac{b}{\theta} \right)^2 \pi(\theta) d\theta = (ha)^2 + b^2 \left( \frac{\alpha^2}{\beta^2} + \frac{\alpha}{\beta^2} \right) + 2bha \frac{\alpha}{\beta}.$$

Choosing the model parameters  $h, b, \bar{a}, \alpha$ , and  $\beta$  appropriately the above expression can be made less than 1 and hence the lemma follows.  $\square$

*Proof.* Proof of Lemma 5.6: Since family  $\mathcal{Q}$  contains all gamma distributions, observe that  $\{q_n(\cdot) \in \mathcal{Q}\} \forall n \geq 1$ . By definition,  $q_n(\theta) = \frac{n^n}{\theta_0^n \Gamma(n)} \theta^{n-1} e^{-n \frac{\theta}{\theta_0}}$ . Now consider the first term; using the definition of the KL divergence it follows that

$$\text{KL}(q_n(\theta) \parallel \pi(\theta)) = \int q_n(\theta) \log(q_n(\theta)) d\theta - \int q_n(\theta) \log(\pi(\theta)) d\theta. \tag{86}$$

Substituting  $q_n(\theta)$  in the first term of the equation above and expanding the logarithm term, we obtain

$$\begin{aligned}
\int q_n(\theta) \log(q_n(\theta)) d\theta &= (n-1) \int \log \theta \frac{n^n}{\theta_0^n \Gamma(n)} \theta^{n-1} e^{-n \frac{\theta}{\theta_0}} d\theta - n + \log \left( \frac{n^n}{\theta_0^n \Gamma(n)} \right) \\
&= -\log \theta_0 + (n-1) \int \log \frac{\theta}{\theta_0} \frac{n^n}{\theta_0^n \Gamma(n)} \theta^{n-1} e^{-n \frac{\theta}{\theta_0}} d\theta - n + \log \left( \frac{n^n}{\Gamma(n)} \right) \tag{87}
\end{aligned}$$

Now consider the second term in the equation above. Substitute  $\theta = \frac{t\theta_0}{n}$  into the integral, we have

$$\begin{aligned}
\int \log \frac{\theta}{\theta_0} \frac{n^n}{\theta_0^n \Gamma(n)} \theta^{n-1} e^{-n \frac{\theta}{\theta_0}} d\theta &= \int \log \frac{t}{n} \frac{1}{\Gamma(n)} t^{n-1} e^{-t} dt \\
&\leq \int \left( \frac{t}{n} - 1 \right) \frac{1}{\Gamma(n)} t^{n-1} e^{-t} dt = 0. \tag{88}
\end{aligned}$$

Substituting the above result into (87), we get

$$\begin{aligned}
\int q_n(\theta) \log(q_n(\theta)) d\theta &\leq -\log \theta_0 - n + \log \left( \frac{n^n}{\Gamma(n)} \right) \\
&\leq -\log \theta_0 - n + \log \left( \frac{n^n}{\sqrt{2\pi n} n^{n-1} e^{-n}} \right) \\
&\leq -\log \sqrt{2\pi} \theta_0 + \frac{1}{2} \log n, \tag{89}
\end{aligned}$$



where the second inequality uses the fact that  $\sqrt{2\pi n}e^{-n} \leq n\Gamma(n)$ . Recall  $\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-\alpha-1}e^{-\frac{\beta}{\theta}}$ . Now consider the second term in (86). Using the definition of inverse-gamma prior and expanding the logarithm function, we have

$$\begin{aligned} - \int q_n(\theta) \log(\pi(\theta)) d\theta &= -\log\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right) + (\alpha+1) \int \log \theta \frac{n^n}{\theta_0^n \Gamma(n)} \theta^{n-1} e^{-n\frac{\theta}{\theta_0}} d\theta + \beta \frac{n}{(n-1)\theta_0} \\ &= -\log\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right) + (\alpha+1) \int \log \frac{\theta}{\theta_0} \frac{n^n}{\theta_0^n \Gamma(n)} \theta^{n-1} e^{-n\frac{\theta}{\theta_0}} d\theta \\ &\quad + \beta \frac{n}{(n-1)\theta_0} + (\alpha+1) \log \theta_0 \\ &\leq -\log\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right) + \beta \frac{n}{(n-1)\theta_0} + (\alpha+1) \log \theta_0, \end{aligned} \quad (90)$$

where the last inequality follows from the observation in (88). Substituting (90) and (89) into (86) and dividing either sides by  $n$ , we obtain

$$\begin{aligned} \frac{1}{n} \text{KL}(q_n(\theta) \parallel \pi(\theta)) &\leq \frac{1}{n} \left( -\log \sqrt{2\pi} \theta_0 + \frac{1}{2} \log n - \log\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right) + \beta \frac{n}{(n-1)\theta_0} + (\alpha+1) \log \theta_0 \right) \\ &= \frac{1}{2} \frac{\log n}{n} + \beta \frac{1}{(n-1)\theta_0} + \frac{1}{n} \left( -\log \sqrt{2\pi} - \log\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right) + (\alpha) \log \theta_0 \right). \end{aligned} \quad (91)$$

Now, consider the second term in the assertion of the lemma. Due to Assumption 4.1 that is  $\xi_i, i \in \{1, 2 \dots n\}$  are independent and identically distributed, we obtain

$$\frac{1}{n} \mathbb{E}_{q(\theta)} [\text{KL}(dP_0^n \parallel p(\tilde{X}_n | \theta))] = \mathbb{E}_{q_n(\theta)} [\text{KL}(dP_0 \parallel p(\xi | \theta))]$$

Now using the expression for KLdivergence between the two exponential distributions, we have

$$\frac{1}{n} \mathbb{E}_{q(\theta)} [\text{KL}(dP_0^n \parallel p(\tilde{X}_n | \theta))] = \int \left( \log \frac{\theta_0}{\theta} + \frac{\theta}{\theta_0} - 1 \right) \frac{n^n}{\theta_0^n \Gamma(n)} \theta^{n-1} e^{-n\frac{\theta}{\theta_0}} d\theta \leq \frac{n}{n-1} + 1 - 2 = \frac{1}{n-1}, \quad (92)$$

where second inequality uses the fact that  $\log x \leq x - 1$ . Combined together (92) and (91) for  $n \geq 2$  implies that

$$\begin{aligned} &\frac{1}{n} [\text{KL}(q(\theta) \parallel \pi(\theta)) + \mathbb{E}_{q(\theta)} [\text{KL}(dP_0^n \parallel p(\tilde{X}_n | \theta))]] \\ &\leq \frac{1}{2} \frac{\log n}{n} + \frac{1}{n} \left( 2 + \frac{2\beta}{\theta_0} - \log \sqrt{2\pi} - \log\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right) + \alpha \log \theta_0 \right) \leq C_9 \frac{\log n}{n}. \end{aligned} \quad (93)$$

where  $C_9 := \frac{1}{2} + \max\left(0, 2 + \frac{2\beta}{\theta_0} - \log \sqrt{2\pi} - \log\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right) + \alpha \log \theta_0\right)$  and the result follows.  $\square$

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