Asynchronous Stochastic Subgradient Methods for General Nonsmooth Nonconvex Optimization

Vyacheslav Kungurtsev [*], Malcolm Egan [†],
Bapi Chatterjee [‡], Dan Alistarh [§]

May 28, 2019

Abstract

Asynchronous distributed methods are a popular way to reduce the communication and synchronization costs of large-scale optimization. Yet, for all their success, little is known about their convergence guarantees in the challenging case of general non-smooth, non-convex objectives, beyond cases where closed-form proximal operator solutions are available. This is all the more surprising since these objectives are the ones appearing in the training of deep neural networks.

In this paper, we introduce the first convergence analysis covering asynchronous methods in the case of general non-smooth, non-convex objectives. Our analysis applies to stochastic sub-gradient descent methods both with and without block variable partitioning, and both with and without momentum. It is phrased in the context of a general probabilistic model of asynchronous scheduling accurately adapted to modern hardware properties. We validate our analysis experimentally in the context of training deep neural network architectures. We show their overall successful asymptotic convergence as well as exploring how momentum, synchronization, and partitioning all affect performance.

1 Introduction

Training parameters arising in Deep Neural Net architectures is a difficult problem in several ways [10]. First with multiple layers and standard activation functions such as sigmoid and softmax functions, the ultimate optimization problem is nonconvex. Second, with ReLU activation functions and max-pooling

[*]Department of Computer Science, Faculty of Electrical Engineering Czech Technical University in Prague (vyacheslav.kungurtsev@fel.cvut.cz). Support for this author was provided by the OP VVV project CZ.02.1.01/0.0/0.0/16_019/0000765 "Research Center for Informatics"

[†]Institut National des Sciences Appliquees de Lyon malcom.egan@insa-lyon.fr

[‡]IST Austria bapi.chatterjee@ist.ac.at

[§]IST Austria dan.alistarh@ist.ac.at

in convolutional structures, the problem is nonsmooth, i.e., it is not differentiable everywhere, although typically the set of non-differentiable points is a set of measure zero in the space of the parameters. Finally, in many applications it is unreasonable to load the whole sample size in memory to evaluate the objective function or (sub)gradient, thus samples must be taken, necessitating analysis in a probabilistic/statistical framework. In order to ensure reasonable performance in practice in solving this class of problems, it is desireable to take advantage of high performance parallel computing.

The analysis of parallel optimization algorithms using shared memory architecturs, in the contemporary era motivated by applications in machine learning, was ushered in with the seminar work [19] (although precursors exist, see the references therein). Other papers refined this analysis [15] and expanded it to nonconvex problems [14]. However, in all of these results, a very simplistic model of asynchronous computation is presented to analyze the problem. In particular, it is assumed that every block, among the set of blocks of iterates being optimized, has a set, finite, and equal probability of being chosen at every iteration, with a certain vector of delays that determine how old each block is that is stored in the cache relative to the shared memory. As one can surmise, this implies complete symmetry with regards to cores reading and computing the different blocks, which cannot correspond to asynchronous computation in practice. In particular, practical experience has shown that it can be effective for each core to control a set of blocks, thus the choice of blocks will depend on previous iterates, which core was last to update, and furthermore this creates probabilistic dependence between the delay vector and the choice of block. This is formalized in the work [5], which introduced a new probabilistic model of asynchronous parallel optimization and presented a coordinate-wise updating successive convex approximation algorithm.

In this paper we are interested in studying parallel asynchronous stochastic subgradient descent for general nonconvex nonsmooth objectives, as arising in the training of deep neural network architectures. Currently, there is no work in the literature specifically addressing this problem. The closest that appears is found in [23, 13], which considers asynchronous proximal gradient methods for solving problems of the form $f(x) + g(x)$ where f is smooth and nonconvex and $g(x)$ is nonsmooth with an easily computable closed form prox expression. This applies to the case of training a neural network which has no ReLUs or max pooling in the architecture itself, i.e., every activation is a smooth function, and there is an additional regularization term, such as an $l1$. In these papers they derive rates of convergence in expectation. In the general case where the activations themselves are nonsmooth (e.g., with the presence of ReLUs, etc.) there is no such additive structure, and no proximal operator exists to handle away the nonsmoothness and remove the necessity of computing and using subgradients explicitly in the optimization procedure.

This general nonsmooth nonconvex optimization problem is a difficult problem (see, e.g., [1]). The introduction of stochastically uncertain iterate updates creates an additional challenge. Classically, the framework of stochastic approximation, with stochastic estimates of the subgradient approximating elements in

a differential inclusion that defines a flow towards minimization of the objective function, is a standard, successful approach to analyzing algorithms for this class of problems. Some texts on the framework include [12], which we shall reference extensively in the paper, and [4]. See also [8] and [21] for some classical results in convergence of stochastic algorithms for nonconvex nonsmooth optimization. Interest in this class of problems has resurfaced recently with the advent and popularity of Deep Neural Network architectures, for instance see the analysis of nonconvex nonsmooth stochastic optimization with an eye towards such models in [6] and [16].

In this paper, using the state of the art model of parallel computation introduced in [5], we analyze nonsmooth nonconvex stochastic subgradient methods in a parallel asynchronous setting, from the stochastic approximation framework. Fitting the model into a framework and analysis developed in [12, Chapter 12], we show that the generic asynchronous stochastic subgradient method is convergent, with probability 1, for nonconvex nonsmooth functions. This is the first result for this class of algorithms. In short, this paper combines the state of the art in these two branches of work, on asynchronous updates and general stochastic nonsmooth nonconvex optimization, in an appropriate fashion while extending the scope of the results therein.

In addition we present numerical results on a few variations of asychronous subgradient descent, demonstrating their overall effectiveness as well as nuances in the practical convergence properties for different approaches. In the process of experimental verification, we noticed that strategies for momentum tuning, as in [22] were effective in improving the performance of the method. We show this in our numerical experiments, and also ensure the theoretical consistency of momentum with the stochastic approximation framework.

2 Problem Formulation

Consider the minimization problem

$$\min_x f(x), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is continuous (but could be nonconvex and nonsmooth) and furthermore, it is computationally infeasible to evaluate $f(x)$ or an element of the Clarke subdifferential $\partial f(x)$.

The problem (1) has many applications in machine learning, including the training of parameters in deep neural networks. In this setting, $f(x)$ is loss function evaluated on some model with x as its parameters, and is dependant on input data $A \in \mathbb{R}^{n\times m}$ and target values $y \in \mathbb{R}^m$ of high dimension, i.e., $f(x) = f(x; (A, y))$, with x a parameter to optimize with respect to the loss function. In cases of practical interest, f is decomposable in finite-sum form,

$$f(x) = \frac{1}{M}\sum_{i=1}^{M} l(m(x; A_i); y_i)$$

where $l : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ represents the training loss and $\{(A_i, y_i)\}$ is a partition of (A, y).

We are concerned with algorithms that solve (1) in a distributed fashion, i.e., using multiple processing cores. In particular, we are analyzing the following *inconsistent read* scenario: before computation begins, each core c is allocated a block of variables I^c, for which it is responsible to update. At each iteration the core modifies a block of variables i^k, chosen randomly among I^c. Immediately afer core c completes its k-th iteration, it updates the shared memory. A lock is only placed on the shared memory when a core writes to it, thus the process of reading may result in computations of the function evaluated variable values that never existed in memory, e.g., block 1 is read by core 1, then core 3 updates block 2, then core 1 reads block 2, etc. We shall index iterations by when a core writes a new set of values for the variable into memory. Thus at iteration k_c, the core is updating the variable block using a subgradient estimate that is evaluated at a point whose components are delayed relative to the current index.

We let $\mathbf{d}^k = \{d_1^{k_c}, ..., d_n^{k_c}\}$ be the vector of delays, thus the j-th component of x that is used in the computation of the update at k is actually not $x_j^{k_c}$ but $x_j^{d_j^{k_c}}$.

In this paper, we are interested in applying stochastic approximation methods, of which the classic stochastic gradient descent forms a special case. Since f in (1) is in general nonsmooth, we will exploit subgradient methods. Denote by ξ^k the set of mini-batches used to compute an element of the subgradient $g((x_1^{d_1^{k_c}}, ..., x_n^{d_n^{k_c}}); \xi^{k_c})$. The set of minibatches ξ^{k_c} is chosen uniformly at random from (A, y), independently at each iteration. By the central limit theorem, the error is asymptotically Gaussian as the total size of the data as well as the size of the mini-batches increases.

2.1 Algorithm Description

The main contribution of this paper is to establish convergence of the stochastic subgradient algorithm with multiple processing cores under asynchronous updates. To this end, we now recall the stochastic subgradient algorithm in Algorithm 1, from the perspective of the individual cores. The update of the iterate performed by

$$\begin{aligned} u_{i^{k_c}} &= m u_{i^{k_c}} + g_{i^{k_c}}^{k_c} \\ x_{i^{k_c}}^{k_c+1} &= x_{i^{k_c}}^{k_c} - (1-m)\gamma^{k_c} u_{i^{k_c}} \end{aligned}$$

where m is the momentum constant, required to satisfy $0 < m < 1$

Algorithm 1: Asynchronous Stochastic Subgradient Method for an Individual Core

Input: x_0, core c.

while Not converged **do**
  Sample i from the variables I_c corresponding to c.
  Sample ξ.
  Read x^{k_c} and from the shared memory
  Compute a subgradient estimate g^{k_c}
  Write, with a lock, $u_{i^{k_c}} = m u_{i^{k_c}} + g^{k_c}_{i^{k_c}}$
  Update, with a lock, $x_i = x_i - (1-m)\gamma^{k_c} u_{i^{k_c}}$
  $k_c = k_c + 1$
end while

3 Analysis

We first summarize the essential parts of the analysis,

1. **Probabilistic Model of Asynchronous Parallel Computation**: Each core only has access to a delayed version of the iterate, with uncertainty in the delays for each core. In order to be robust with respect to varying parallel architectures, we must use a general probabilistic model to study the convergence of the stochastic subgradient algorithm. Thus we present a model inspired by [5] that describes the process of multiple cores running Algorithm 1 in parallel that is general enough to encompass most architectures, while also establishing some minimal assumptions necessary to establish convergence.

2. **Introducing a Stochastic Process with Real-time Delay** In order to perform analysis from the stochastic approximation perspective, it is necessary to present the stochastic algorithm as a process that approximates a continuous time dynamic system which stabilizes at a solution of the original problem, and then prove that it asymptotically converges to such a process. Thus we expand the model of discrete iterations by introducing delays in real-time and introduce continuous-time interpolations of the sequence of iterates. We define a differential inclusion whose dynamics define a path which converges to stationary point of (1). Finally we introduce a set of assumptions that are either satisfied due to the presentation in the discrete time model, or are new ones that are reasonable in the context.

3. **Convergence Proof** Finally, we introduce the appropriate notions of convergence in this setting, and prove that the algorithm asymptotically converges in the appropriate probabilistic sense.

3.1 Probabilistic Model of Asynchronous Parallel Computation

For the discrete time probabilistic model of computation, as introduced in [5], we must present the basic requirements that must hold across cores. In particular, it is reasonable to expect that if some core is entirely faulty, or exponentially deccelerates in its computation, convergence should not be expected to be attained. Thus in this section, we present k as a *global* counter, indicating sequential updates of any block among the variables.

In iteration k, the updated iterate $x_{i^k}^{k+1}$ depends on a random vector $\zeta^k \triangleq (i^k, d^k, \xi^k)$. The distribution of ζ^k depends on the underlying scheduling or message passing protocol. We use the following formulation, which applies to a variety of architectures.

Let $\zeta^{0:t} \triangleq (\zeta^0, \zeta^1, ..., \zeta^t)$ be the stochastic process representing the evolution of the blocks and minibatches used, as well as the iterate delays. The σ-algebra $\mathcal{F}$ is obtained as follows. Let the cylinder $C^k(\zeta^{0:t}) \triangleq \{\omega \in \Omega : \omega^{0:k} = \zeta^{0:t}\}$ and define $\mathcal{F}^k \triangleq \sigma(C^k)$ and $\mathcal{F} \triangleq \sigma(\cup_{t=0}^{\infty} C^t)$ the cylinder σ-algebra on Ω.

Consider the conditional distribution of ζ^{k+1} given $\zeta^{0:k}$,

$$\mathbb{P}(\zeta^{k+1}|\zeta^{0:k}) = \frac{\mathbb{P}(C^{k+1}(\zeta^{0:k+1})}{\mathbb{P}(C^k(\zeta^{0:k}))},$$

we have the following assumptions, first on the probabilities and delays,

Assumption 3.1. *The random variables ζ^k satisfy,*

1. *There exists a δ such that $d_j^k \leq \delta$ for all j and k. Thus each $d_j^k \in \mathcal{D} \triangleq \{0, ..., \delta\}^n$.*
2. *For all i and $\zeta^{0:k-1}$ such that $p_{\zeta^{0:k-1}}(\zeta^{0:k-1}) > 0$, it holds that,*

$$\sum_{d \in \mathcal{D}} \mathbb{P}((i, d, \xi)|\zeta^{0:k-1}) \geq p_{min}$$

   for some $p_{min} > 0$.

3. *It holds that,*

$$\mathbb{P}\left(\left\{\zeta \in \Omega : \liminf_{k\to\infty} \mathbb{P}(\zeta|\zeta^{0:k-1}) > 0\right\}\right) = 1$$

The first condition indicates that there is some maximum possible delay in the vectors, that each element of x used in the computation of $x_{i^k}^{k+1}$ is not too old. The second is an irreducibility condition that there is a positive probability for any block or minibatch to be chosen, given any state of previous realizations of $\{\zeta^k\}$. The last assumption indicates that the set of events in Ω that asymptotically go to zero in conditional probability are of measure zero.

We have the standard assumption about the stochastic sub-gradient estimates. These assumptions hold under the standard stochastic gradient approach wherein

one samples some subset $\xi \subseteq \{1, ..., M\}$ of mini-batches uniformly from the set of size $|\xi|$ subsets of $\{1, ..., M\}$, done independently at each iteration. This results in independent noise at each iteration being applied to the stochastic subgradient term. From these mini-batches ξ, a subgradient is taken for each $j \in \xi$ and averaged, i.e.,

$$g(x, \xi) = \frac{1}{|\xi|} \sum_{j \in \xi} g_j(x) \tag{2}$$

where $g_j(x) \in \partial f_j(x)$.

Assumption 3.2. *The stochastic subgradient estimates $g(x, \xi)$ satisfy,*

1. $\mathbb{E}_\xi [g(x; \xi)] \in \partial f(x)$
2. $\mathbb{E}_\xi \left[dist(g(x; \xi), \partial f(x))^2 \right] \leq \sigma^2$
3. $\|g(x; \xi)\| \leq B_g$

In order to enforce global convergence, we wish to use a diminishing step-size. However, at the same time, as synchronization is to be avoided, there must not be a global counter indicating the rate of decrease of the step-size. In particular, each core will have its own local step size $\gamma^{\nu(c^k,k)}$ where c^k is the core, and, defining the random variable Z^k as the component of $\{1, ..., \bar{c}\}$ that is active at iteration k, the random variable denoting the number of updates performed by core c^k, denoted by $\nu(k)$ is given by $\nu(k) \triangleq \sum_{j=0}^{k} I(Z^j = c^k)$.

In addition, noting that it has been observed that in practice, partitioning variable blocks across cores is more efficient than allowing every processor to have the ability to choose across every variable block [15]. Thus we partition the blocks of variables across cores. We can thus denote c^k as being defined uniquely by i^k, the block variable index updated at iteration k.

Note that $\liminf_{k \to \infty} \frac{\gamma^{\nu(c^k,k)}}{k} = 0$ in probability is implied by

$$\sum_{i \in c^k, d \in \mathcal{D}, \xi \subseteq \{1,...,M\}} Pr((i, d, \xi) | \zeta^{0:k-1})) \to 0$$

for some subsequence, which is antithetical to Assumption 3.1, Part 2. Thus, note that the stepsizes $\gamma^{\nu(c^k,k)}$ satisfy, where the limit of the sequence is taken in probability,

$$\liminf_{k \to \infty} \frac{\gamma^{\nu(c^k,k)}}{k} > 0, \tag{3}$$

which is an assumption for the analysis of asynchronous parallel algorithms in [4].

We are now ready to present Algorithm 2. This is presented from the "global" iteration counter perspective.

Algorithm 2: Asynchronous Stochastic Subgradient Method

Input: x_0.

1: **while** Not converged and $k < k_{\max}$ **do**
2: Having realized $\zeta^{0:k-1}$, sample $\{\zeta^k = (i^k, d^k, \xi^k)\}|\zeta^{0:k-1}\}$.
3: Update $u_{i^k} = mu_{i^k} + g((x_1^{d_1^k}, x_2^{d_2^k}, ..., x_n^{d_n^k}), \xi^k)$
4: Update $x_{i^k}^{k+1} = x_{i^k}^k - (1-m)\gamma^{\nu(k)} u_{i^k}$
5: Set $k = k + 1$
6: **end while**

3.2 Continuous Time Model and Stochastic Process

In this section, we shall redefine the algorithm and its associated model presented in the previous section in a framework appropriate for analysis from the stochastic approximation perspective.

Consider the Algorithm described as such, for data block i with respective iteration k,

$$x_{k+1,i} = x_{k,i} + (1-m)\gamma^{k,i} \sum_{j=1}^{k} m^{k-j} Y_{j,i} \tag{4}$$

where $Y_{j,i}$ is the estimate of the partial subgradient with respect to block variables indexed by i at local iteration j.

In the context of Algorithm 1, the step size is defined to be the subsequence $\{\gamma^{k,i}\} = \{\gamma^{\nu(c(i),l)} : i = i^l\}$ where l is the iteration index for the core corresponding to block i. Thus it takes the subsequence of γ^k for which $i^k = i$ is the block of variables being modified.

The step $Y_{k,i}$ satisfies,

$$Y_{k,i} = g_i((x_{k-[d_i^k]_1,1}, ..., x_{k-[d_i^k]_j,j}, ..., x_{k-[d_i^k]_n,n})) + \delta M_{k,i}.$$

We denote $g_i(x)$ to denote a selection of some element of the subgradient, with respect to block i, of $f(x)$. The quantity $\delta M_{k,i}$ represents a Martingale difference, satisfying $\delta M_{k,i} = M_{k+1,i} - M_{k,i}$ for some Martingale M_k, a sequence of random variables which satisfies $\mathbb{E}[M_{k,i}] < \infty$ and $\mathbb{E}[M_{k+1,i}|M_{j,i}, j \leq k] = M_{k,i}$ with probability 1 for all k. It holds that $\mathbb{E}[|M_{k,i}|^2] < \infty$ and $\mathbb{E}[M_{k+1,i} - M_{k,i}][M_{j+1,i} - M_{j,i}]' = 0$. Finally, it holds that $\mathbb{E}_{k,i}[\delta M_{k,i}] = 0$. These are standard conditions implied by the sampling procedure in stochastic gradient methods, introduced by the original Robbins-Monro method [20].

In Stochastic Approximation, the standard approach is to formulate a dynamic system or differential inclusion that the sequence of iterates approaches asymptotically. For this reason, we introduce real time into the model of asynchronous computation, looking at the actual time elapsed between iterations for each block i.

Define $\delta\tau_{k,i}$ to be the real elapsed time between the k-th and $k+1$-st iteration for block i. We let $T_{k,i} = \sum_{j=0}^{k-1} \delta\tau_{j,i}$ and define for $\sigma \geq 0$, $p_l(\sigma) = \min\{j : T_{j,i} \geq \sigma\}$ the first iteration at or after σ.

We assume now that the step-size sequence comes from an underlying real function, i.e.,

$$\gamma^{k,i} = \frac{1}{\delta\tau_{k,i}} \int_{T_{k,i}}^{T_{k,i}+\delta\tau_{k,i}} \gamma(s)ds$$

satisfying

$$\begin{array}{l} \int_0^\infty \gamma(s)ds = \infty, \text{ where } 0 < \gamma(s) \to 0 \text{ as } s \to \infty, \\ \text{There are } T(s) \to \infty \text{ as } s \to \infty \text{ such that} \\ \quad \lim_{s\to\infty} \sup_{0\le t\le T(s)} \left| \frac{\gamma(s)}{\gamma(s+t)} - 1 \right| = 0 \end{array} \tag{5}$$

We now define new σ-algebras $\mathcal{F}_{k,i}$ and $\mathcal{F}^+_{k,i}$ defined to measure the random variables

$$\begin{array}{l} \{\{x_0\}, \{Y_{j-1,i} : j, i \text{ with } T_{j,i} < T_{k+1,i}\}, \\ \quad \{T_{j,i} : j, i \text{ with } T_{k,i} \le T_{k+1,i}\}\} \end{array}$$

and

$$\begin{array}{l} \{\{x_0\}, \{Y_{j-1,i} : j, i \text{ with } T_{j,i} \le T_{k+1,i}\}, \\ \quad \{T_{j,i} : j, i \text{ with } T_{k,i} \le T_{k+1,i}\}\}, \end{array}$$

indicating the set of events up to, and up to and including the computed noisy update at k, respectively.

Note that each of these constructions is still consistent with a core updating different blocks at random, with $\delta\tau_{k,i}$ arising from an underlying distribution for $\delta\tau_{k,c(i)}$.

Let us relate these σ-algebras to those in the previous section. Note that this takes subsets of random variables (i^k, d^k, ξ^k) for which k is such that i^k is i (in the original notation of k). The form of $Y_{k,i}$ defined above incorporates the random variable d^k and i^k, as in which components are updated and the age of the information used by where the subgradient is evaluated, as well as ξ^k by the presence of the Martingale difference noise.

For any sequence $Z_{k,i}$ we write $Z^\sigma_{k,i} = Z_{p_i(\sigma)+k,i}$. Thus, let $\delta\tau^\sigma_{k,i}$ denote the inter-update times for block i starting at the first update at or after σ, and $\gamma^\sigma_{k,i}$ the associated step sizes.

Now let $x^\sigma_{0,i} = x_{p_i(\sigma),\sigma}$ and for $k \ge 0$, $x^\sigma_{k+1,i} = x^\sigma_{k,i} + (1-m)\gamma^\sigma_{k,i} \sum_{j=1}^k m^{k-j} Y^\sigma_{j,i}$.

We consider $t^\sigma_{k,i} = \sum_{j=0}^{k-1} \gamma^\sigma_{j,i}$ and $\tau^\sigma_{k,i} = \sum_{j=0}^{k-1} \gamma^\sigma_{j,i} \delta\tau^\sigma_{j,l}$.

We introduce piecewise constant interpolations of the vectors in real-time given by,

$$\begin{array}{ll} x^\sigma_i(t) = x^\sigma_{k,i}, & t \in [t^\sigma_{k,i}, t^\sigma_{k+1,i}], \\ \hat{x}^\sigma_i(t) = x^\sigma_{k,i}, & t \in [\tau^\sigma_{k,i}, \tau^\sigma_{k+1,i}], \\ N^\sigma_i(t) = t^\sigma_{k,i}, & t \in [\tau^\sigma_{k,i}, \tau^\sigma_{k+1,i}], \\ \tau^\sigma_i(t) = \tau^\sigma_{k,i}, & t \in [t^\sigma_{k,i}, t^\sigma_{k+1,i}] \end{array}$$

The step $Y_{k,i}$ satisfies,

$$Y_{k,i} = g_i((x_{k-[d^k_i]_1,1}, ..., x_{k-[d^k_i]_j,j}, ..., x_{k-[d^k_i]_n,n})) + \delta M_{k,i},$$

with associated process $Y^\sigma_{k,l}$. We denote $g_i(x)$ to denote a selection of some element of the subgradient, with respect to block i, of $f(x)$. The quantity $\delta M_{k,i}$

represents a Martingale difference, satisfying $\delta M_{k,i} = M_{k+1,i} - M_{k,i}$ for some Martingale M_k, a sequence of random variables which satisfies $\mathbb{E}[M_{k,i}] < \infty$ and $\mathbb{E}[M_{k+1,i}|M_{j,i}, j \leq k] = M_{k,i}$ with probability 1 for all k. It holds that $\mathbb{E}[|M_{k,i}|^2] < \infty$ and $\mathbb{E}[M_{k+1,i} - M_{k,i}][M_{j+1,i} - M_{j,i}]' = 0$. Finally, it holds that $\mathbb{E}_{k,i}[\delta M_{k,i}] = 0$. These are standard conditions implied by the sampling procedure in stochastic gradient methods, introduced by the original Robbins-Monro method [20].

We also have,

$$N_i^\sigma(\tau_i^\sigma(t)) = t_{k,i}^\sigma,\ t \in [t_{k,i}^\sigma, t_{k+1,i}^\sigma],$$
$$x_i^\sigma(t) = \hat{x}_i^\sigma(\tau_l^\sigma(t)),\ \hat{x}_i^\sigma(t) = x_l^\sigma(N_i^\sigma(t))$$

We make the following assumptions on the real delay times. These ensure that the real-time delays do not grow without bound, either on average, or on relevantly substantial probability mass. Intuitively, this means that it is highly unlikely that any core deccelerates exponentially in its computation speed.

Assumption 3.3. *It holds that $\{\delta\tau_{k,i}^\sigma; k, i\}$ is uniformly integrable.*

Assumption 3.4. *There exists a function $u_{k+1,i}^\sigma$ and random variables $\Delta_{k+1,i}^{\sigma,+}$ and a random sequence $\{\psi_{k+1,i}^\sigma\}$ such that*

$$\mathbb{E}_{k,i}^+[\delta\tau_{k+1,i}^\sigma] = u_{k+1,i}^\sigma(\hat{x}_i^\sigma(\tau_{k+1,i}^\sigma - \Delta_{k+1,i}^{\sigma,+}), \psi_{k+1,i}^\sigma)$$

and there is a $\bar{u}$ such that for any compact set A,

$$\lim_{m,k,\sigma} \frac{1}{m} \sum_{j=k}^{k+m-1} E_{k,i}^\sigma[u_{j,i}^\sigma(x, \psi_{k+1,i}^\sigma) - \bar{u}_i(x)] I_{\{\psi_{k+1,i}^\sigma \in A\}} = 0$$

Lemma 3.1. *It holds that $\{Y_{k,i}, Y_{k,i}^\sigma; k, i\}$ is uniformly integrable. Thus, so is $\left\{\sum_{j=1}^k m^{k-j} Y_{j,i}, \sum_{j=1}^k m^{k-j} Y_{j,i}^\sigma; k, i\right\}$*

Proof. Uniform integrability of $\{Y_{k,i}, Y_{k,i}^\sigma; k, i\}$ follows from Assumption 3.2, part 3. The uniform integrability of $\left\{\sum_{j=1}^k m^{k-j} Y_{j,i}, \sum_{j=1}^k m^{k-j} Y_{j,i}^\sigma; k, i\right\}$ follows from $0 < m < 1$ and the fact that a geometric sum of a uniformly integrable sequence is uniformly integrable. □

Lemma 3.2. *It holds that, for any $K > 0$, and all l,*

$$\sup_{k<K} \sum_{j=k-[d_i^k]_l}^{k} \gamma_{j,i}^\sigma \to 0$$

in probability as $\sigma \to \infty$.

Proof. As $\sigma \to \infty$, by the definition of $\gamma_{k,i}^\sigma$, $\gamma_{k,i}^\sigma \to 0$ and since by Assumption 3.1 $\max d_i^k \leq \delta$, for all $k < K$, $\sum_{j=k-[d_i^k]_l}^k \gamma_{j,i}^\sigma \leq \delta\gamma_{k-\delta,i}^\sigma \to 0$. □

3.3 Convergence

As mentioned earlier, the primary goal of the previous section is to define a stochastic process that approximates some real-time process asymptotically, with this real-time process defined by dynamics for which at the limit the path converges to a stationary point. In particular, we shall see that the process defined approximates the path of a differential inclusion,

$$\dot{x}_i(t) \in \partial_i f(x(t))$$

and we shall see that this path defines stationary points of $f(\cdot)$.

3.3.1 Weak Convergence

In this section we prove weak convergence of the iterates, to be defined shortly. Much of the proof of the Theorem can be taken from the analagous result in Chapter 12 of [12], which considers a particular model of asynchronous stochastic approximation. As we introduced a slightly different model from the literature, some of the details of the procedure are now different, and thus in this section we indicate how to treat the distinctions in the proof and show that the result still holds.

Weak convergence is defined in terms of the Skorohod topology, a technical topology weaker than the topology of uniform convergence on bounded intervals, defined in [2]. Convergence of a function $f_n(\cdot)$ to $f(\cdot)$ in the Skorohod topology is equivalent to uniform convergence on each bounded time interval. We denote by $D^j[0,\infty)$ the j-fold product space of real-valued functions on the interval $[0,\infty)$ that are right continuous with left-hand limits, with the Skorohod topology. It is a complete and separable metric space.

Now we define some terminology arising in the theory of weak convergence. We present a result indicating sufficient conditions for a property called tightness.

Theorem 3.1. *[12, Theorem 7.3.3] Consider a sequence of processes $\{A_k(\cdot)\}$ with paths in $D(-\infty,\infty)$ such that for all $\delta > 0$ and each t in a dense set of $(-\infty,\infty)$ there is a compact set $K_{\delta,t}$ such that,*

$$\inf_n \mathbb{P}\left[A_n(t)| \in K_{\delta,t}\right] \geq 1-\delta,$$

and for any $T > 0$,

$$\lim_{\delta\to 0} \limsup_n \sup_{|\tau|\leq T} \sup_{s\leq\delta} \mathbb{E}\left[\min\left[|A_n(\tau+s) - A_n(\tau)|, 1\right]\right] = 0$$

then $\{A_n(\cdot)\}$ is tight in $D(-\infty,\infty)$.

If a sequence is tight then every weak sense limit process is also a continuous time process. We say that $A_k(t)$ *converges weakly* to A if,

$$\mathbb{E}\left[F(A_k(t))\right] \to \mathbb{E}\left[F(A(t))\right]$$

for any bounded and continuous real-valued function $F(\cdot)$ on $\mathbb{R}^n$.

Finally we must define the notion of an invariant set for a differential inclusion (DI).

Definition 3.1. *A set $\Lambda \subset \mathbb{R}^n$ is an invariant set for a DI $\dot{x} \in g(x)$ if for all $x_0 \in \Lambda$, there is a solution $x(t)$, $-\infty < t < \infty$ that lies entirely in Λ and satisfies $x(0) = x_0$.*

Now we present our main Theorem of the paper.

Theorem 3.2. *Let all Assumptions 3.1, 3.2, 3.3, and 3.4.*
Then, the following system of differential inclusions,

$$\tau_i(t) = \int_0^t \bar{u}_i(\hat{x}(\tau_i(s)))ds, \tag{6}$$

$$\dot{x}_i(t) \in \partial_i f(\hat{x}(\tau_i(t))), \tag{7}$$

$$\dot{\hat{x}}_i \bar{u}_i(\hat{x}) \in \partial_i f(\hat{x}) \tag{8}$$

holds for any $\overline{u}$ satisfying 3.4. On large intervals $[0,T]$, $\hat{x}^\sigma(\cdot)$ spends nearly all of its time, with the fraction going to one as $T \to \infty$ and $\sigma \to \infty$ in a small neighborhood of a bounded invariant set of (8).

Proof. By Theorem 8.6, Chapter 3 in [9] a sufficient condition for tightness of a sequence $\{A_n(\cdot)\}$ is that for each $\delta > 0$ and each t in a dense set in $(-\infty, \infty)$, there is a compact set $K_{\delta,t}$ such that $\inf_n \mathbb{P}[A_n(t) \in K_{\delta,t}] \geq 1 - \delta$ and for each positive T, $\lim_{\delta \to 0} \limsup_n \sup_{|\tau| \leq T,\, s \leq \delta} \mathbb{E}\left[|A_n(\tau + s) - A_n(\tau)|\right] = 0$.

Now since $Y_{k,i}$ is uniformly bounded, and $Y^\sigma_{k,i}(\cdot)$ is its interpolation with jumps only at t being equal to some $T_{k,i}$, it holds that for all i,

$$\lim_{\delta \to 0} \limsup_\sigma \mathbb{P}\left[\sup_{t \leq T,\, s \leq \delta} |Y^\sigma_{k,i}(t+s) - Y^\sigma_{k,i}(t)| \geq \eta\right] = 0$$

and so by the definition of the algorithm,

$$\lim_{\delta \to 0} \limsup_\sigma \mathbb{P}\left[\sup_{t \leq T,\, s \leq \delta} |x^\sigma_{k,i}(t+s) - x^\sigma_{k,i}(t)| \geq \eta\right] = 0$$

which implies,

$$\lim_{\delta \to 0} \limsup_\sigma \mathbb{E}\left[\sup_{t \leq T,\, s \leq \delta} |x^\sigma_{k,i}(t+s) - x^\sigma_{k,i}(t)|\right] = 0$$

and the same argument implies tightness for $\{\tau^\sigma_i(\cdot), N^\sigma_i(\cdot)\}$ by the uniform boundedness of $\{\delta\tau^\sigma_{i,k}\}$ and bounded, decreasing $\gamma^\sigma_{k,i}$ and positive $u^\sigma_{k,i}(x, \psi^\sigma_{k+1,i})$, along with Assumption 3.4. Lipschitz continuity follows from the properties of the interpolation functions. Specifically, the Lipschitz constant of $x^\sigma_i(\cdot)$ is B_g.

All of these together imply tightness of $\hat{x}^\sigma_i(\cdot)$ as well. Thus,

$$\{x^\sigma_i(\cdot), \tau^\sigma_i(\cdot), \hat{x}^\sigma_i(\cdot), N^\sigma_i(\cdot); \sigma\}$$

is tight in $D^{4n}[0,\infty)$. This implies the Lipschitz continuity of the subsequence limits with probability one, which exist in the weak sense by Prohorov's Theorem, Theorems 6.1 and 6.2 [3].

As $\sigma \to \infty$ we denote the weakly convergent subsequence's weak sense limits by,

$$(x_i(\cdot), \tau_i(\cdot), \hat{x}_i(\cdot), N_i(\cdot))$$

Note that,

$$\begin{aligned} x_i(t) &= \hat{x}_i(\tau_i(t)), \\ \hat{x}_i(t) &= x_i(N_i(t)), \\ N_i(\tau_i(t)) &= t. \end{aligned}$$

For more details, see the proof of Theorem 8.2.1 [12].

Let,

$$\begin{aligned} M_i^\sigma(t) &= \textstyle\sum_{k=0}^{k=p(\sigma)} (1-m)\delta\tau_{k,i} \left(\sum_{j=0}^{k} m^j \delta M_{k-j,i}^\sigma \right) \\ \tilde{G}_i^\sigma(t) &= \textstyle\sum_{k=0}^{k=p(\sigma)} \delta\tau_{k,i} \Big[(1-m) \sum_{j=0}^{k} m^j g_i((x^\sigma_{k-j-[d_i^{k-j}]_1,1}(t), ..., \\ &\qquad x^\sigma_{k-j-[d_i^{k-j}]_j,j}(t), ..., x_{k-j-[d_i^{k-j}]_N,N}))(t) - g_i(\hat{x}_i^\sigma(t)) \Big] \\ \bar{G}_i^\sigma(t) &= \textstyle\sum_{k=0}^{k=p(\sigma)} \delta\tau_{k,i} g_i(\hat{x}^\sigma(t)) \\ W_i^\sigma(t) &= \hat{x}_i^\sigma(\tau_i^\sigma(t)) - x_{i,0}^\sigma - \bar{G}_i^\sigma(t) = \tilde{G}_i^\sigma(t) + M_i^\sigma(t) \end{aligned}$$

Now for any bounded continuous and real-alued function $h(\cdot)$, an arbitrary integer p, and t and τ, and $s_j \geq t$ real, we have

$$\begin{aligned} &\mathbb{E}\left[h(\tau_i^\sigma(s_j), \hat{x}^\sigma(\tau_i^\sigma(s_j)) \left(W_i^\sigma(t+\tau) - W_i^\sigma(t) \right) \right] \\ &\quad -\mathbb{E}\left[h(\tau_i^\sigma(s_j), \hat{x}^\sigma(\tau_i^\sigma(s_j)) \left(\tilde{G}_i^\sigma(t+\tau) - \tilde{G}_i^\sigma(t) \right) \right] \\ &\quad -\mathbb{E}\left[h(\tau_i^\sigma(s_j), \hat{x}^\sigma(\tau_i^\sigma(s_j)) \left(M_i^\sigma(t+\tau) - M_i^\sigma(t) \right) \right] = 0, \end{aligned}$$

Now the term involving M^σ equals zero from the Martingale property.

We now claim that the term involving $\tilde{G}_i^\sigma$ goes to zero as well. Since $x_{k,i}^\sigma \to x_i^\sigma$ it holds that, by Lemma 3.2, $(x^\sigma_{k-[d_i^k]_1,1}(t), ..., x^\sigma_{k-[d_i^k]_j,j}(t), ..., x^\sigma_{k-[d_i^k]_N,N}) \to \hat{x}^\sigma(t)$ as well. By the upper semicontinuity of the subgradient, it holds that there exists a $g_i(\hat{x}_i^\sigma(t)) \in \partial_i f(\hat{x}_i^\sigma(t))$ such that

$$\begin{aligned} &g_i((x^\sigma_{k-[d_i^k]_1,1}(t), ..., x^\sigma_{k-[d_i^k]_j,j}(t), ..., x_{k-[d_i^k]_N,N}))(t) \\ &\qquad \to g_i(\hat{x}_k^\sigma(t)) \end{aligned}$$

as $\sigma \to \infty$. Thus each term in the sum converges to $g_i(\hat{x}_{k-j}^\sigma(t))$. Now, given j, as $k \to \infty$, the boundedness assumptions and stepsize rules imply that $g_i(\hat{x}_{k-j}^\sigma(t)) \to g_i(\hat{x}_k^\sigma(t))$. On the other hand as $k \to \infty$ and $j \to \infty$, $m^j g_i(\hat{x}_{k-j}^\sigma(t)) \to 0$. Thus $\sum_{j=0}^{k} m^j g_i(\hat{x}_{k-j}^\sigma(t)) \to \frac{1-m^k}{1-m} g_i(\hat{x}_k^\sigma(t)) \to \frac{1}{1-m} g_i(\hat{x}_k^\sigma(t))$, and the claim has been shown.

Thus the weak sense limit of $\lim_{\sigma\to\infty} W_i^\sigma(\cdot) = W_i(\cdot)$ satisfies

$$\mathbb{E}\left[h(\tau_i(s_j), \hat{x}(\tau_i(s_j)) \left(W_i(t+\tau) - W_i(t) \right) \right]$$

and thus by Theorem 7.4.1 [12] is a martingale and is furthermore a constant with probability one by the Lipschitz continuity of x by Theorem 4.1.1 in [12]. Thus,

$$W(t) = \hat{x}(t) - \hat{x}(0) - \int_0^t g(\hat{x}(s))ds = 0,$$

where $g(\hat{x}(s)) \in \partial f(\hat{x}(s))$, and (7) holds. $\square$

3.3.2 Convergence With Probability One

The previous Theorem showed that under the conditions described for the algorithm, there is a weakly convergent subsequence to an invariant set. We can now use the results in [7] to infer from weak convergence, probability one convergence of the sequence of iterates.

For this, we shall use the machinery developed in [7], which establishes conditions for which a weakly convergent stochastic approximation algorithm approximating a continuous ODE converges with probability one, under certain conditions. One can study the proof structure to quickly reveal that with minor modifications the results carry through. In particular, when $\dot{b}$ appears in the proof, one can replace it with an element of the differential inclusion, and the limit point is replaced by the invariant set. Assumption 2.1 in [7] is now associated with a set-valued map $S(x, T, \phi)$, and by the noise structure of the assumptions, it can easily be seen that $\bar{L}$ exists for all possible values of x, T and ϕ in the notation of the paper. One can see that the uniqueness appears once in the beginning of the proof of Theorem 3.1 with the existence of this T_1 such that the trajectory lies in a specific ball around the limit point for $t \geq T_1$. This can be replaced by the trajectory lying in this ball around the invariant set, for T_1 defined as the supremum of such $\hat{T}_1$ associated with every possible subgradient, i.e., element of the DI. Since the subgradient is a compact set and is upper semicontinuous, this supremum exists. Finally, note that Assumption 3.2 is as Assumption 4.1 in [7] and thus similarly implies Theorem 4.1 and Theorem 5.3. This proves that as $\sigma \to \infty$, w.p.1 $x^{\sigma}(\cdot)$ converges to an invariant set of (1).

3.3.3 Properties of the Limit Point

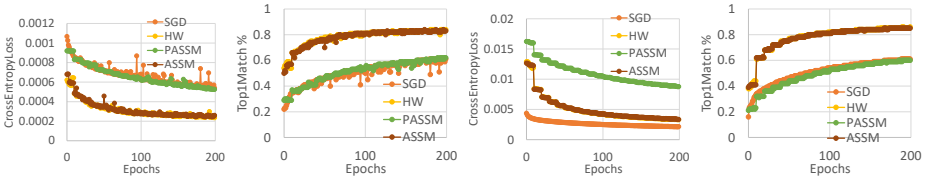
Finally, we wish to characterize the properties of this invariant set. From Corollary 5.11 [6], we can conclude that problems arising in training of deep neural network architectures, wherein $f(x) = l(y_j, a_L)$ with $l(\cdot)$ one of several standard loss functions, including logistic or Hinge loss, and $a_i = \rho_i(V_i(x)a_{i-1})$ or $i = 1, ..., L$ layers, are activation functions, which are piece-wise defined to be $\log x$, e^x, $\max(0, x)$ or $\log(1 + e^x)$, are such that their set of invariants $\{x^*\}$ for its associated differential inclusion satisfies $0 \in \partial f(x^*)$, and furthermore the values $f(x^k)$ for any iterative algorithm generating $\{x^k\}$ such that $x^k \to x^*$, an invariant of $f(x)$, converge.

Note that the differential inclusions defined above ensure asymptotic convergence to block-wise stationarity, i.e.,

$$0 \in \partial_i f(x) \text{ for all } i$$

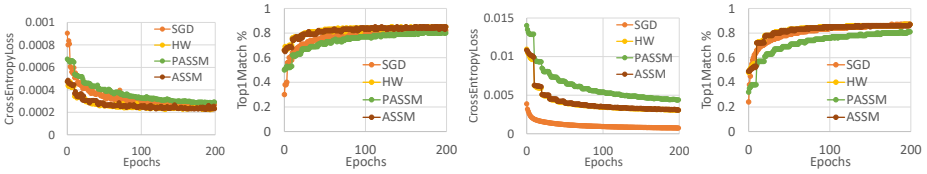
It is clear, however, that every stationary point is also block-wise stationary, i.e., that $0 \in \partial f(x)$ implies $0 \in \partial_i f(x)$ for all i.

One can alternatively consider a *consistent read* variant of the algorithm, wherein every core updates the entire vector (thus there is no block partitioning)



(a) Test Loss. (b) Test Accuracy. (c) Train Loss. (d) Train Accuracy.

Figure 1: We plotted the loss and accuracy trajectory for test and train experiments of the methods. **SGD** runs a single process, whereas the asynchronous methods run 10 concurrent processes. In this set of experiments we have no momentum correction. The **HW** and **ASSM** demonstrate better convergence per epoch compared to **PASSM**.



(a) Test Loss. (b) Test Accuracy. (c) Train Loss. (d) Train Accuracy.

Figure 2: Same setting as in the Fig 1. We used a momentum = 0.9. It can be observed that with momentum correction the convergence of **PASSM** improves significantly. Mitliagkas et al. [17] experimentally showed that the degree of asynchrony directly relates to momentum; our experiments show that the relative gain in terms of convergence per epoch by momentum correction is better for **PASSM** that exhibits more asynchrony compared to **ASSM**, which uses locks for write consistency.

but locks the shared memory whenever it either reads of writes from it, presents, primarily, a simplification of the method analyzed. Thus, the same general approach applies to such a procedure. In particular, this amounts to $i^k = \{1, ..., n\}$ for all k, for which it is clear that the above analysis is a simplification for (in addition, every delay vector has the same quantity in each component). Thus this implies that every limit of $x^\sigma(t)$ as either $\sigma \to \infty$ or $t \to \infty$ is a critical point of $f(x)$ and, with probability one, asymptotically the algorithm converges to a critical point of $f(x)$.

In practice, the set of block-wise stationary points which are not stationary is not large, and we shall see in the numerical results section that partitioning in the manner described is practically efficient, thus we chose to perform the analysis on the more general case.

4 Numerical Results

In this section, we describe an experimental evaluation comparing the following algorithms:

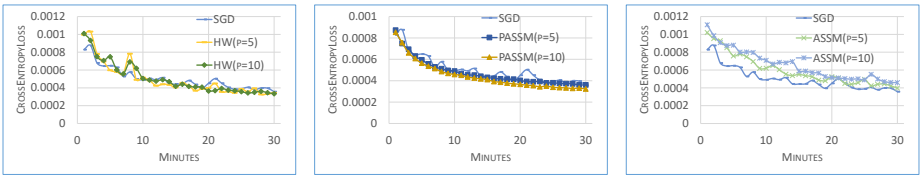
1. **SGD:** Sequential **S**tochastic **G**radient **D**escent method.
2. **HW:** Hogwild [19] with lock-free read and updates of $x_{k,i}$. **HW** has no provable convergence guarantee for nonsmooth nonconvex models.
3. **ASSM:** **A**synchronous **S**tochastic **S**ubgradient **M**ethod. **ASSM** differs from **HW** in its use of locks to update $x_{k,i}$ to make consistent writes.
4. **PASSM:** The presented **P**artitioned **A**synchronous **S**tochastic **S**ubgradient **M**ethod. We read as well as update $x_{k,i}$ lock-free asynchronously.

The implementation is based on the open-source Pytorch library [18] and the multi-processing framework of Python. We used the standard Resnet18 [11] neural network as a well-known example of a nonsmooth nonconvex model. We trained the network over the CIFAR-10[1] dataset consisting of 50000 labeled images for training and 10000 labeled images for testing. For each of the methods, we adopt a decreasing step size strategy $\gamma^{k,i} = (\alpha^j \times \gamma)/\sqrt{k}$, where $\alpha^j > 0$ is a constant for the j^{th} processing core. γ is fixed initially. This satisfies $\lim_{k\to\infty} \gamma^{k,i} = 0$ and $\sum_{k=1}^{\infty} \gamma^{k,i} = \infty$. In each of the methods we use an $L2$ penalty in form of the weight-decay of 0.0005. Additionally, we introduced an $L1$ penalty of 0.0001 that simply gets added to the gradients after it has been put through the $L2$ penalty.

We benchmarked the implementations on a NUMA workstation – 2 sockets, 10 cores apiece, running at 2.4GHz (Intel(R) Xeon(R) E5- 2640), HT enabled 40 logical cores, Linux 4.18.0-0.bpo.1-amd64 (Debian 9) – containing 4 Nvidia GeForce GTX 1080 GPUs. The processes running an asynchronous method use the available GPU resources – memory and threads – concurrently. The Pytorch implementation by default uses all the available cores for computations. To evaluate the scalability with cores, we bind the processes restricting their computations to individual CPU cores. The experimental results are presented and discussed in Figures 1 to 3.

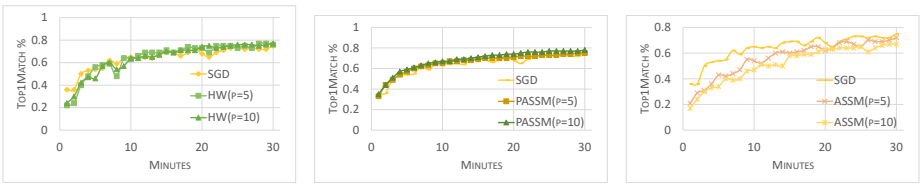
Experimental Observation Summary. The block partitioning design of **PASSM** is helpful in reducing potential write conflicts in an asynchronous shared-memory setting and thereby results in better convergence per unit time compared to a well-known asynchronous variant of SGD: **HW**. In addition, the momentum correction comparatively better stimulates the convergence per epoch of block partitioning approach which offers improved asynchronous model update.

[1] `https://www.cs.toronto.edu/~kriz/cifar.html`



(a) SGD vs. HW. (b) SGD vs. PASSM. (c) SGD vs. ASSM.

Figure 3: This set of figures presents the test-loss trajectory against time (in minutes) while comparing asynchronous methods – running 5 and 10 concurrent processes – with sequential SGD. We used momentum = 0.9 in each of them. A separate concurrent process keeps on saving a snapshot of the shared model on an interval of 1 minute, simultaneously with the training processes. Firstly, it can be observed that the convergence of **PASSM** is faster compared to the other two asynchronous methods for identical number of processes. This can be understood in terms of block partitioning the model across processes: it helps reducing the synchronization cost and thereby potentially speeds up the data processing per unit time. Furthermore, we clearly gain in terms of convergence per unit time when we increase the number of processes in **PASSM**. In contrast, we note that the use of locks by **ASSM** actually slows it down when we increase the number of processes. This set of experiments demonstrate that **PASSM** has better convergence with respect to wall-clock time in addition to the scalability with parallel resources.



(a) SGD vs. HW. (b) SGD vs. PASSM. (c) SGD vs. ASSM.

Figure 4: Same setting as in the Fig 3, momentum = 0.9. We plotted test-accuracy in terms of Top1 correct match % vs time (in minutes). In can be observed that **PASSM** offers faster convergence per unit time in accuracy as well compared to the other two asynchronous methods.

5 Discussion and Conclusion

In this paper we analyzed the convergence theory of asynchronous stochastic subgradient descent. We found that 1) the state of the art probabilistic model on asynchronous parallel architecture applied to the stochastic subgradient method is largely consistent with the existing theory and assumptions with regards to stochastic approximation, and 2) the consistency carried over to being sufficient to maintain the canonical weak, and subsequently with probability one convergence results. Thus, the existing developments of stochastic approximation as applied to stochastic subgradient methods, namely [6], were able to be extended in a straightforward manner to the setting of being performed asynchronously, as modeled faithfully in the contemporary literature.

Finally, we presented numerical results that indicate some possible performance variabilities in three types of asynchrony: block partitioning inconsistent read (for which the above convergence theory applies), full-variable-update consistent read (for which the above convergence theory also applies), and full-variable-update inconsistent read (for which no convergence theory exists, at the moment).

References

[1] Adil Bagirov, Napsu Karmitsa, and Marko M Mäkelä. *Introduction to Nonsmooth Optimization: theory, practice and software*. Springer, 2014.

[2] P. Billingsley. *Convergence of probability measures*. Wiley, 1968.

[3] Patrick Billingsley. *Convergence of probability measures*. John Wiley & Sons, 2013.

[4] Vivek S Borkar. *Stochastic approximation: a dynamical systems viewpoint*. Baptism's 91 Witnesses, 2008.

[5] Loris Cannelli, Francisco Facchinei, Vyacheslav Kungurtsev, and Gesualdo Scutari. Asynchronous parallel algorithms for nonconvex big-data optimization. part i: Model and convergence. *arXiv preprint arXiv:1607.04818*, 2016.

[6] Damek Davis, Dmitriy Drusvyatskiy, Sham Kakade, and Jason D Lee. Stochastic subgradient method converges on tame functions. *arXiv preprint arXiv:1804.07795*, 2018.

[7] Paul Dupuis and Harold J Kushner. Stochastic approximation and large deviations: Upper bounds and wp 1 convergence. *SIAM Journal on Control and Optimization*, 27(5):1108–1135, 1989.

[8] Yu M Ermol'ev and VI Norkin. Stochastic generalized gradient method for nonconvex nonsmooth stochastic optimization. *Cybernetics and Systems Analysis*, 34(2):196–215, 1998.

[9] Stewart N Ethier and Thomas G Kurtz. *Markov processes: characterization and convergence*, volume 282. John Wiley & Sons, 2009.

[10] Ian Goodfellow, Yoshua Bengio, Aaron Courville, and Yoshua Bengio. *Deep learning*, volume 1. MIT press Cambridge, 2016.

[11] Kaiming He, Xiangyu Zhang, Shaoqing Ren, and Jian Sun. Deep residual learning for image recognition. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pages 770–778, 2016.

[12] Harold Kushner and G George Yin. *Stochastic approximation and recursive algorithms and applications*, volume 35. Springer Science & Business Media, 2003.

[13] Zhize Li and Jian Li. A simple proximal stochastic gradient method for nonsmooth nonconvex optimization. In *Advances in Neural Information Processing Systems*, pages 5564–5574, 2018.

[14] Xiangru Lian, Yijun Huang, Yuncheng Li, and Ji Liu. Asynchronous parallel stochastic gradient for nonconvex optimization. In *Advances in Neural Information Processing Systems*, pages 2737–2745, 2015.

[15] Ji Liu and Stephen J Wright. Asynchronous stochastic coordinate descent: Parallelism and convergence properties. *SIAM Journal on Optimization*, 25(1):351–376, 2015.

[16] Szymon Majewski, Błażej Miasojedow, and Eric Moulines. Analysis of nonsmooth stochastic approximation: the differential inclusion approach. *arXiv preprint arXiv:1805.01916*, 2018.

[17] Ioannis Mitliagkas, Ce Zhang, Stefan Hadjis, and Christopher Ré. Asynchrony begets momentum, with an application to deep learning. In *2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 997–1004, 2016.

[18] Adam Paszke, Sam Gross, Soumith Chintala, Gregory Chanan, Edward Yang, Zachary DeVito, Zeming Lin, Alban Desmaison, Luca Antiga, and Adam Lerer. Automatic differentiation in pytorch. 2017.

[19] Benjamin Recht, Christopher Re, Stephen Wright, and Feng Niu. Hogwild: A lock-free approach to parallelizing stochastic gradient descent. In *Advances in neural information processing systems*, pages 693–701, 2011.

[20] Herbert Robbins and Sutton Monro. A stochastic approximation method. In *Herbert Robbins Selected Papers*, pages 102–109. Springer, 1985.

[21] Andrzej Ruszczyński. A linearization method for nonsmooth stochastic programming problems. *Mathematics of Operations Research*, 12(1):32–49, 1987.

[22] Jian Zhang and Ioannis Mitliagkas. Yellowfin and the art of momentum tuning. *arXiv preprint arXiv:1706.03471*, 2017.

[23] Rui Zhu, Di Niu, and Zongpeng Li. Asynchronous stochastic proximal methods for nonconvex nonsmooth optimization. *arXiv preprint arXiv:1802.08880*, 2018.