Decorous Combinatorial Lower Bounds for Row Layout Problems

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In this paper we consider the Double-Row Facility Layout Problem (DRFLP). Given a set of departments and pairwise transport weights between them the DRFLP asks for a non-overlapping arrangement of the departments along both sides of a common path such that the weighted sum of the center-to-center distances between the departments is minimized. Despite its broad applicability in factory planning, only small instances can be solved to optimality in reasonable time. Apart from this even deriving good lower bounds using existing integer programming formulations and branch-and-cut methods is a challenging problem. We focus here on deriving combinatorial lower bounds which can be computed very fast. These bounds generalize the star inequalities of the Minimum Linear Arrangement Problem. Furthermore we exploit a connection of the DRFLP to some parallel identical machine scheduling problem. Our lower bounds can be further improved by combining them with a new distance-based mixed-integer linear programming model, which is not a formulation for the DRFLP, but can be solved close to optimality quickly. We compare the new lower bounds to some heuristically determined upper bounds on medium-sized and large DRFLP instances. Special consideration is given to the case when all departments have the same length. Furthermore we show that the lower bounds that we derive using adapted variants of our approaches for the Parallel Row Ordering Problem, a DRFLP variant where the row assignment of the departments is given in advance and spaces between neighboring departments are not allowed, are even better with respect to the gaps.

Keywords. Facilities planning and design; Integer programming; Row layout problem; Lower bounds

1 Introduction

In this paper we consider special facility layout problems which have several applications, in particular in factory planning. For recent surveys on facility layout problems in general we refer, e. g., to [1, 14, 16, 26, 37, 47]. An instance of the *Multi-Row Facility Layout Problem* (`MRFLP`)

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consists of n one-dimensional departments $[n] := \{1, \dots, n\}$ with given positive lengths ℓ_i, $i \in [n]$, pairwise non-negative weights $w_{ij} = w_{ji} \in \mathbb{R}_+$, $i, j \in [n]$, $i < j$, which usually correspond to the amount of transport between the departments, and a set $\mathcal{R} := [m]$, $m \in \mathbb{N}$, of rows. The objective is to find an assignment $r\colon [n] \to \mathcal{R}$ of departments to rows and horizontal positions for the centers of the departments such that departments in the same row do not overlap and such that the total weighted sum of the center-to-center distances, measured in horizontal direction, between all pairs of departments is minimized. So we look for a vector $q \in \mathbb{R}^n$ of positions and a vector $r \in \mathcal{R}^n$ of the assignment of the departments to the m rows solving the following optimization problem

$$\min_{r \in \mathcal{R}^n, q \in \mathbb{R}^n} \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} |q_i - q_j|$$

$$\text{s. t. } |q_i - q_j| \geq \frac{\ell_i + \ell_j}{2}, \qquad i, j \in [n], i < j, \text{ if } r_i = r_j.$$

The special case of the MRFLP with $m = 2$ is called *Double-Row Facility Layout Problem* (DRFLP), see, e. g., [11, 29]. The DRFLP is in particular relevant for real-world applications because material handling and thus real factory layouts most often reduce to double-row layouts and variants thereof. Indeed, it was noted by several authors that in factory planning the costs of the production are highly influenced by the layout of the departments, see, e. g., [18, 36, 65]. Besides its applications in factory planning, the DRFLP can be used to find an arrangement of rooms in hospitals [19, 28, 32], office centers or schools [7]. Further applications include setting books on a shelf [8], balancing hydraulic turbine runners and optimal data memory layout generation for digital signal processors [16]. We refer to [42] for further applications.

In the following, we denote the center-to-center distance between two departments $i, j \in [n], i < j$, by $d_{ij} = d_{ji}$. To illustrate the structure of double-row layouts and the corresponding distance calculation we give an example. Note that in an optimal double-row layout there might be free space between two neighboring departments in the same row.

Example 1 We consider four departments with lengths $\ell_i = i$, $i = 1, 2, 3$, $\ell_4 = 1$ and pairwise non-zero weights $w_{13} = w_{23} = 1, w_{24} = 2$. Figure 1 illustrates an optimal double-row layout with solution value $1 \cdot 2 = 2$.

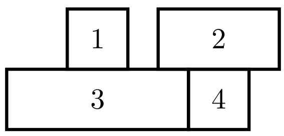


Figure 1: An optimal double-row layout for an instance with $\ell_i = i$, $i = 1, 2, 3$, $\ell_4 = 1$, and non-zero weights $w_{13} = w_{23} = 1$, $w_{24} = 2$. Note that there is some free space between the neighboring departments 1 and 2 in row 1.

1.1 Literature Review

A well-studied special case of the MRFLP is the *Single-Row Facility Layout Problem* (SRFLP) with $m = 1$, i. e., all departments are assigned to the same row. Exact optimization approaches for the SRFLP are based on relaxations of integer linear programming (ILP) and semidefinite programming (SDP) formulations, see, e. g., [3, 4, 5, 10] as well as [13, 15, 17, 43, 44]. The strongest ILP approach is an linear programming based cutting plane algorithm using betweenness variables that can solve instances with up to 35 departments within a few hours [5]. The strongest SDP approach to date, using products of ordering variables, is even stronger and allows to solve instances with up to 42 departments within one hour and to obtain small gaps for instances with up to 81 departments within 51 hours [43, 44]. Additionally, several heuristic algorithms have been suggested that are able to obtain good layouts [22, 25, 49, 50, 58, 60]. One of the leading heuristics was presented in [60], where a multi-start simulated annealing heuristic obtains the best known solutions or small gaps for instances from the literature with 60 to 80 departments. Furthermore this heuristic is tested on instances with up to 1000 departments. A recent survey on the SRFLP is given in [47].

In contrast to the SRFLP, the DRFLP has received much less attention in the literature. From a practical point of view the DRFLP seems much harder than the SRFLP. The ILP-based approach in [21] (see also the corresponding corrections by [68]) can handle instances with up to 10 departments whereas the exact ILP approach of [8] can solve instances with up to 12 departments to optimality. The latter model was improved in [63] such that one is able to solve a DRFLP instance with 15 departments in at most 11 hours. Recently, [29] presented an algorithm which can solve DRFLP instances with up to 16 departments in less than 12 hours.

To the best of our knowledge there has not been research on computing non-trivial lower bounds for the DRFLP. The enumeration scheme of [29] cannot be used to obtain lower bounds for larger instances because one would have to calculate a lower bound for each of the exponentially many row assignments, which is out of scope for n large. The mixed-integer programming models, see, e. g., [8, 63], are based on big-M-type constraints to couple continuous position variables with binary ordering variables. Thus, their linear relaxations are rather weak. So using them in a branch-and-cut approach leads to weak lower bounds and so to large gaps for medium-sized and large DRFLP instances, even after a longer time limit because the root node gaps are hardly improved. For detailed computational results we refer to Section 4.

For the DRFLP only few problem-specific heuristics were presented in the literature [21, 31, 56, 69], partially handling some extended versions that include, e. g., clearance conditions between departments in the same row. But without the knowledge of good lower bounds it is hard to evaluate the quality of these heuristics.

Because the MRFLP and the DRFLP are very challenging problems in practice, several special cases have been studied in the literature. There are two main classes of simplifications. First one reduces the freedom in the arrangement of the departments. In the *Space-Free MRFLP* and *DRFLP* (SF-MRFLP and SF-DRFLP) one restricts to a common left border of the rows and spaces between neighboring departments in the same row are not allowed. For the SF-DRFLP, which is also known as *Corridor Allocation Problem*, heuristics and exact approaches were presented in [2, 48] and [7, 29, 41]. Similar to the general DRFLP the enumeration approach of [29] can solve space-free double-row instances with up to 16 departments in less than 12 hours. If one additionally fixes the row assignment of each of the departments we derive the *k-Parallel Row Ordering Problem* (kPROP) (in our notation k equals the number of rows m) and the *Parallel Row Ordering Problem* (PROP) for $m = 2$ [9, 38, 55, 66]. The best approach for these problems in [29] is the basis for the enumerative approach for the DRFLP. Instances with up to 25 departments are solved to optimality. For larger n one can derive lower bounds via the SDP approach in [38].

The second area of simplifications for the DRFLP considers the departments and not their arrangement. The *Multi-Row Equidistant Facility Layout Problem* (MREFLP) is a special case of the MRFLP with all departments equal in shape [6] and the DRFLP with departments of equal length is called (DREFLP). Recently, in [12] it is shown that in the MREFLP the departments can be arranged on an integer grid and an ILP and an SDP model are presented. As a result, equidistant double-row and equidistant multi-row instances with up to 25 departments were solved to optimality for $2 \leq m \leq 5$ and gaps with less than 4 % were obtained for instances with up to 50 departments and $2 \leq m \leq 5$. Due to the grid structure of optimal solutions the MREFLP can be seen as a special case of the *Quadratic Assignment Problem* (QAP), see, e. g., [54]. In [39] it is shown that the best method for the SRFLP is better than methods especially tailored to the equidistant SRFLP, see, e. g., [57, 59].

If we restrict the SRFLP with departments of equal length to binary weights $w_{ij} \in \{0, 1\}, i, j \in$

$[n], i < j$, we obtain the well-studied *Minimum Linear Arrangement Problem* (LA), see, e. g., [34, 35]. Given a graph $G = (V, E)$ with, w. l. o. g., $V = \{1, \dots, n\}$, the LA looks for a bijection $q\colon V \to V$ such that

$$\sum_{ij \in E} |q_i - q_j|$$

is minimized. The LA is already an $\mathcal{NP}$-hard problem [30] and hence all other row layout problems mentioned above are also $\mathcal{NP}$-hard. For the LA combinatorial lower bounds were presented in [20]. These results were the starting point for our investigations together with a research question in [40]. In [40] the so called *Checkpoint Ordering Problem* (CPOP) was introduced. Given a set of n departments with lengths ℓ_i and weights $w_i, i \in [n]$, the CPOP asks for a space-free non-overlapping arrangement of the departments in one row such that the sum of the weighted distances of the centers of the departments to a checkpoint whose position is given in advance is minimized. The CPOP is closely related to the SRFLP and it was asked in [40] whether some partial relation of the SRFLP and certain scheduling problems can be exploited further in the row layout setting.

1.2 Our Contribution

The main contributions of this paper are the following:

- We indicate a relation between some special DRFLP, where we only explicitly measure the (weighted) distance of some specific department to the others, to the parallel identical machine scheduling problem with minimum (weighted) completion time.
- We develop the first non-trivial combinatorial lower bounds for the DRFLP, the DREFLP and the PROP. These bounds can also be extended to the multi-row case, i. e., to the MRFLP, the MREFLP and the kPROP.
- We show how to combine these lower bounds with a new mixed-integer linear programming model to compute even stronger lower bounds for the DRFLP, the DREFLP and the PROP via some branch-and-cut algorithm within a given time limit of a few minutes.
- We present a corrected and short proof of a result of Samarghandi and Eshghi [61] which states that the SRFLP with weights $w_{ij} = 1, i, j \in [n], i < j$, can be solved to optimality in polynomial time by using some specific order of the departments. We use this result to further strengthen our lower bounding model for the PROP.
- In a computational study we compare our lower bounds for DRFLP instances from the literature as well as for medium-sized and large randomly generated instances with lower bounds received via some branch-and-cut algorithm within a time limit of one hour for a well-known DRFLP formulation [8]. Furthermore we compare them to some heuristically determined upper bounds. Apart from this we investigate the strength of our DREFLP and PROP lower bounds.

This paper is structured as follows. In Section 2 we present combinatorial lower bounds for the DRFLP and prove their correctness. In Section 3 we introduce a distance-based ILP model to further improve these bounds. Furthermore we shortly explain in both sections which adaptations are needed for deriving lower bounds for the PROP (and partially the kPROP). In Section 4 we computationally investigate the strength of our newly derived lower bounds for medium-sized and large DRFLP, DREFLP as well as PROP instances by comparing them to some bounds from the literature and heuristically determined upper bounds. We conclude this paper in Section 5 and present directions for future work.

2 Combinatorial Lower Bounds

To the best of our knowledge combinatorial lower bounds specialized to the DRFLP are not known in the literature and lower bounds received via some branch-and-cut algorithm within a given time limit of one hour for some DRFLP formulation from the literature [8] are rather weak as we will see in Section 4. In this section we present three possibilities to compute combinatorial lower bounds for the DRFLP. To simplify the presentation we concentrate on lower bounds for the DRFLP and show at the end of this section how to extend these lower bounds to the MRFLP and to the kPROP and the PROP as well. Apart from this we will have a closer look at the equidistant case of the DRFLP.

In the following we generalize the so called *star inequalities* of the LA, see, e. g., [20], and we indicate a connection of a special DRFLP to the parallel identical machine scheduling problem with minimum weighted completion time with four machines (an exact definition is given below). With these results we partially answer a research question in [40] whether one can use ideas from the scheduling literature for row layout problems.

2.1 Weighted Star Lower Bound

We start with a description of the star inequalities, which are used for determining lower bounds for the optimal solution value of the LA in [20] given some graph $G = (V, E)$. Let q be a solution of the LA. Then the star inequalities for a fixed node $i \in V$ and a set $S \subseteq V \setminus \{i\}$ read as follows

$$\sum_{j \in S} |q_i - q_j| \geq \left\lfloor \tfrac{(|S|+1)^2}{4} \right\rfloor, \tag{1}$$

see, e. g., [20]. One can derive this formula by arranging all nodes in S as close as possible to node i. With $S_i = \{j \in V : ij \in E\}$ a lower bound for the optimal solution value of the LA is given by

$$\tfrac{1}{2} \sum_{i \in V} \left\lfloor \tfrac{(|S_i|+1)^2}{4} \right\rfloor,$$

because we count the minimal contribution of each node (each pairwise absolute difference is counted twice and so we have to divide the sum by two).

In the following we present three different ways to measure the contribution of each department to the sum of the weighted distances in the DRFLP. These three approaches are related to the *Parallel Identical Machine Scheduling Problem with minimum weighted completion time*, see, e. g., [33, 46, 53, 64], often called $P||\sum w_k C_k$ where C_k denotes the completion time of some job k.

Definition 2 *Given a set of jobs J with processing times $p_k \in \mathbb{R}_+$ and weights $w_k \in \mathbb{R}_+, k \in J$, one looks for an assignment of start times $t_k \in \mathbb{R}_+$ of the jobs J to $u \in \mathbb{N}$ parallel identical machines such that no two jobs overlap on one machine and such that the sum of the weighted completion times $\sum_{k \in J} w_k C_k$ with $C_k = t_k + p_k$ is minimized. For constant u we denote this problem by $P_u||\sum w_k C_k$ and for u part of the input by $P||\sum w_k C_k$.*

The scheduling problem $P_u||\sum w_k C_k$ is weakly $\mathcal{NP}$-hard, see, e. g., [52], and $P||\sum w_k C_k$ is $\mathcal{NP}$-hard in the strong sense [52]. For $u = 1$, this problem is a single machine scheduling problem and can be solved in polynomial time by the so called *Smith rule* [64]. The Smith rule states that in an optimal solution the jobs are ordered non-increasingly by their relative weights $\frac{w_k}{p_k}$ for $k \in J$. In the literature the Smith rule has also been extended to the parallel machine case, i. e., the jobs are ordered non-increasingly by their relative weights $\frac{w_k}{p_k}$ for $k \in J$ and we assign each of the jobs using this order to the next machine that gets idle. As we will see below, in general optimality might be lost for a schedule determined like this. Further, it is well known that the unweighted case, i. e., $P||\sum C_k$ with $w_k = 1$ for $k \in J$, can be solved to optimality in polynomial time by the *Shortest Processing Time rule* (SPT), where one processes the jobs in increasing order of their processing time. We will show next how to use these rules for deriving combinatorial lower bounds for the optimal value of some DRFLP instance. For this we will frequently use the following notation.

Definition 3 *Let (n, w, ℓ) be a DRFLP instance. We denote by*

$$\mathcal{Q}(n, w, \ell) = \{(r, q) \colon r, q \text{ is a feasible solution for the DRFLP instance } (n, w, \ell)\}$$

the set of feasible solutions. For a fixed department $i \in [n]$, a set $S \subseteq [n] \setminus \{i\}$ and some $(r, q) \in \mathcal{Q}(n, w, \ell)$ we denote the sum of the weighted distances from all departments in S to i by

$$W_i(q, S) := \sum_{j \in S} w_{ij} \left| q_i - q_j \right| .$$

The best possible value of $W_i(\cdot, S)$ for some fixed set S over all feasible solutions is denoted by

$$\widehat{W}_i(S) := \min_{(r,q) \in \mathcal{Q}(n,w,\ell)} W_i(q, S),$$

and the optimal value of the DRFLP is then

$$\widehat{W} := \tfrac{1}{2} \min_{(r,q) \in \mathcal{Q}(n,w,\ell)} \sum_{i \in [n]} W_i(q, [n] \setminus \{i\}).$$

Note that in the calculation of $\widehat{W}$ we have to divide the sum of the $W_i(.,.)$ by two because each pairwise distance is counted twice.

The common idea for our combinatorial bounding procedure is to find lower bounds for $\widehat{W}_i([n] \setminus \{i\})$, which will give rise to lower bounds for $\widehat{W}$: for each feasible solution $(r, q) \in \mathcal{Q}(n, w, \ell)$ we have

$$\tfrac{1}{2} \sum_{i \in [n]} \widehat{W}_i([n] \setminus \{i\}) = \tfrac{1}{2} \sum_{i \in [n]} \min_{(r,q) \in \mathcal{Q}(n,w,\ell)} W_i(q, [n] \setminus \{i\}) \leq \tfrac{1}{2} \min_{(r,q) \in \mathcal{Q}(n,w,\ell)} \sum_{i \in [n]} W_i(q, [n] \setminus \{i\}) = \widehat{W}.$$

The following proposition is essential for our considerations. It reduces the set of possibly optimal solutions.

Proposition 4 *Let (n, w, ℓ) be a DRFLP instance and let $i \in [n], S \subseteq [n] \setminus \{i\}$. Then there exists a solution $(r, q) \in \mathcal{Q}(n, w, \ell)$ for which $\widehat{W}_i(S)$ is attained such that there is some $j \in S$ with $q_i = q_j$, i. e., j lies directly opposite i.*

Proof. Let $i \in [n]$, $S \subseteq [n] \setminus \{i\}$ and let $(r, q) \in \mathcal{Q}(n, w, \ell)$ be a solution minimizing $W_i(\cdot, S)$. Assume, w. l. o. g., that $r_i = 1$ and that $\{j \in S \colon r_j = 2\} \neq \emptyset$, otherwise we can easily place one department $\hat{\jmath}$ opposite to i and reduce the distance of i and $\hat{\jmath}$. We get

$$W_i(q, S) = \sum_{\substack{j \in S \\ r_j = 1}} w_{ij} \left| q_i - q_j \right| + \sum_{\substack{j \in S \\ q_j < q_i \\ r_j = 2}} w_{ij} (q_i - q_j) + \sum_{\substack{j \in S \\ q_i < q_j \\ r_j = 2}} w_{ij} (q_j - q_i) + \underbrace{\sum_{\substack{j \in S \\ q_i = q_j}} w_{ij} (q_j - q_i)}_{=0} .$$

If there does not exist a $j \in S$ with $q_i = q_j$, then by the optimality of (r, q) shifting all departments $j \in S$ with $r_j = 2$ to the left or to the right by some small $\varepsilon > 0$ does not change the objective value. So we can shift all departments in row 2, w. l. o. g., to the left until one department lies opposite i. □

This result shows that in order to determine a lower bound for $\widehat{W}_i(S)$ for $i \in [n], S \subseteq [n] \setminus \{i\}$ it suffices to determine a lower bound for varying $j \in S$ opposite i. This motivates the following definition.

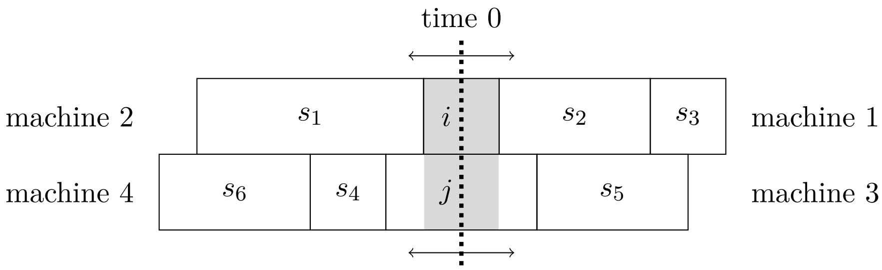


Figure 2: Visualization of the connection of the DRFLP and parallel machine scheduling on four machines. Here departments i and j lie opposite and we have to arrange departments $\{s_1, \dots, s_6\}$. In the lower bound calculations we will partially adjust the start of the jobs (departments) at a machine by half the length of i (see gray area) or half the length of j. Furthermore we have to keep in mind that in scheduling one considers the completion times of the jobs but in the DRFLP we measure the distances between the centers of the departments.

Definition 5 *Let (n, w, ℓ) be a DRFLP instance, and let $i \in [n]$, $S \subseteq [n] \setminus \{i\}$ and $j \in S$. Denote*

$$\widehat{W}_{(i,j)}(S) := \min \{W_i(q, S) \colon (r, q) \in \mathcal{Q}(n, w, \ell), q_i = q_j\}. \tag{2}$$

An immediate consequence of Proposition 4 is the following corollary.

Corollary 6 *Let (n, w, ℓ) be a DRFLP instance and let $i \in [n], S \subseteq [n] \setminus \{i\}$. Then $\widehat{W}_i(S) = \min_{j \in S} \widehat{W}_{(i,j)}(S)$.*

Thus, in order to compute lower bounds for $\widehat{W}$, it suffices to determine lower bounds for $\widehat{W}_{(i,j)}(S)$ for all valid choices of i, j and S. In the following we determine three different lower bounds for $\widehat{W}_{(i,j)}(S)$ given some DRFLP instance. In all three variants we interpret the optimization problem (2) for computing $\widehat{W}_{(i,j)}(S)$ as a scheduling problem $P_4 || \sum w_k C_k$ with weights $w_k = w_{ik}$. The departments correspond to the jobs in the $P_4 || \sum w_k C_k$ and the lengths of the departments to the processing times, i. e., $p_k = \ell_k, k \in S \setminus \{j\}$. Given a feasible solution of the optimization problem (2), then, as illustrated in Figure 2, machine 1 and machine 2 of the scheduling problem correspond to row 1 in this solution and machine 3 and machine 4 to row 2.

Thus we are able to use methods from the scheduling literature to compute lower bounds for the DRFLP. All lower bound calculations have in common that we sort the jobs in $S \setminus \{j\}$ by some given order. Respecting some machine-dependent non-availability times from zero to $a = (a_1, \dots, a_4) \in \mathbb{R}^4_+ \cup \{\infty\}$ (i. e., no job on machine k may start before a_k, $k = 1, \dots, 4$), the jobs are assigned in a greedy manner. Whenever a machine becomes idle and is available one assigns the next unscheduled job in the list non-preemptively. Our basic algorithm is summarized in Algorithm 1.

Definition 7 *Let $S = (s_1, \dots, s_{|S|})$ be an* ordered *sequence of jobs (departments) with processing times (lengths) $\ell^S \in \mathbb{R}^{|S|}_+$ and let $a \in \mathbb{R}^4_+$ denote four non-availability times. Then we denote by $C^{basic}(S, \ell^S, a)$ the greedy solution returned by Algorithm 1 when scheduling the jobs in this order.*

For the first lower bound we use the SPT rule, i. e., we order the jobs (departments) by increasing length. Furthermore, machine 1 and machine 2 are non-available from 0 to $\frac{\ell_i}{2}$ and machine 3 and machine 4 from 0 to $\frac{\ell_j}{2}$.

Definition 8 *Let (n, w, ℓ) be a DRFLP instance. Let $i \in [n]$, $S \subseteq [n] \setminus \{i\}$, $j \in S$ with $S^{spt}_j = (s_1, \dots, s_{|S|-1})$ a sequence of departments in $S \setminus \{j\}$ with length $\ell^{S^{spt}_j} = (\ell_{s_1}, \dots, \ell_{s_{|S|-1}})$ ordered by increasing lengths and let*

$$C^{spt,(i,j)}(S, \ell) := C^{basic}(S^{spt}_j, \ell^{S^{spt}_j}, (\tfrac{\ell_i}{2}, \tfrac{\ell_i}{2}, \tfrac{\ell_j}{2}, \tfrac{\ell_j}{2})).$$

Algorithm 1: Basic($S = (s_1, \dots, s_{|S|}), \ell^S, a$)

Input : parallel machine scheduling problem with ordered jobs $S = (s_1, \dots, s_{|S|})$, processing times $\ell^S \in \mathbb{R}_+^{|S|}$, non-availability times from zero to $a = (a_1, \dots, a_4)$ on the 4 machines

Output : completion times $C_{s_k}, s_k \in S$, as $C^{\text{basic}}(S, \ell^S, a)$.

Initialize $(\bar{\ell}_1, \bar{\ell}_2, \bar{\ell}_3, \bar{\ell}_4) \leftarrow (a_1, \dots, a_4)$.

for $k = 1, \dots, |S|$ **do**

  Choose $\bar{m} \in \arg\min\{\bar{\ell}_o \colon o \in \{1, 2, 3, 4\}\}$.

  $\bar{\ell}_{\bar{m}} \leftarrow \bar{\ell}_{\bar{m}} + \ell^S_{s_k}$.

  $C_{s_k} \leftarrow \bar{\ell}_{\bar{m}}$.

return C_{s_k}, $s_k \in S$.

Furthermore, let $w'_{i\bullet} = (w'_{i1}, \dots, w'_{i(|S|-1)})$ *be the weights* w_{ik} *of* $k \in S \setminus \{j\}$, *ordered* decreasingly. *Then the* SPT-lower-bound *is*

$$W^{spt}_{(i,j)}(S) := \sum_{k=1}^{|S|-1} w'_{ik} \left(C^{spt,(i,j)}_{s_k}(S, \ell) - \frac{\ell_{s_k}}{2} \right).$$

In the special case of all weights being equal to one the SPT-distance-bound *is*

$$W^{dst}_{(i,j)}(S) := \sum_{k=1}^{|S|-1} \left(C^{spt,(i,j)}_{s_k}(S, \ell) - \frac{\ell_{s_k}}{2} \right).$$

The SPT-distance-bound cannot be used to derive bounds for the optimal value of the DRFLP. However, it can be used to derive lower bounds for the (geometric) distances between the departments themselves without regarding the amount of transports. We will make use of them later in the lower bound ILP model presented in Section 3. In $W^{\text{spt}}_{(i,j)}(S)$ we assign the highest weights to the earliest jobs (the departments closest to department i) in order to get a lower bound for $\widehat{W}_{(i,j)}(S)$. For an illustration we refer to Figure 3.

Proposition 9 *Let* (n, w, ℓ) *be a DRFLP instance, and let* $i \in [n]$, $S \subseteq [n] \setminus \{i\}$, $j \in S$ *and* $W^{dst}_{(i,j)}(S)$ *and* $W^{spt}_{(i,j)}(S)$ *as defined above. Then*

$$W^{dst}_{(i,j)}(S) = \min \{ \sum_{k=1}^{|S|-1} |q_i - q_{s_k}| : (r, q) \in \Omega(n, w, \ell), q_i = q_j \}, \tag{3}$$

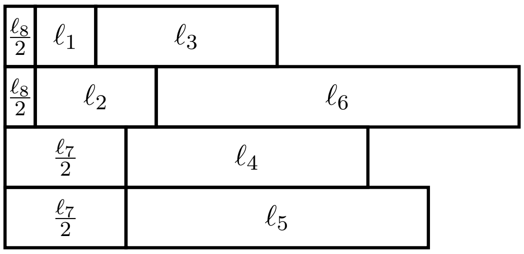
$$W^{spt}_{(i,j)}(S) \leq \widehat{W}_{(i,j)}(S). \tag{4}$$

Proof. Let $i \in [n], S \subseteq [n] \setminus \{i\}, j \in S$. The correctness of (3) follows directly by the correctness of the SPT rule for the problem $P_4 || \sum C_k$. Note that in comparison to the scheduling problem in the DRFLP the distances are measured between the centers of the departments, i. e., we obtain $d_{ik} = C_k - \frac{\ell_k}{2}$, $k \in S \setminus \{j\}$, and $d_{ij} = 0$.

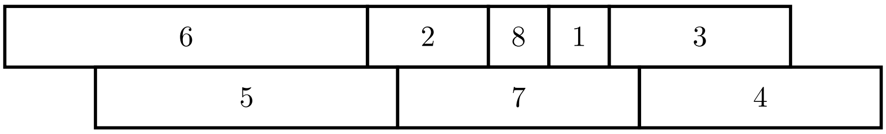
The only difference in (4) is that we additionally assign the highest weights to the departments closest to i which maintains correctness. □

Example 10 Consider a double-row instance with $n = 8$, $\ell_k = k$, $k \in [6]$, $\ell_7 = 4$, $\ell_8 = 1$, and non-zero weights $w_{18} = \frac{1}{2}$, $w_{28} = 1$, $w_{38} = 3$, $w_{48} = 3$, $w_{58} = 1$, $w_{68} = 7$, $w_{78} = 5$. Our aim is to compute $W^{\text{dst}}_{(8,7)}([7])$ and $W^{\text{spt}}_{(8,7)}([7])$. Therefore we consider the problem $P_4 || \sum_{k \in [6]} w_{k8} C_k$ where the non-availability times range from zero to $\ell_8/2 = 0.5$ on machines 1 and 2 and to $\ell_7/2 = 2$ on machines 3 and 4. We apply the SPT rule for the jobs (departments) [6] and obtain the schedule illustrated in Figure 3a. Then we arrange the departments on machine 1 and machine 2 space-free to row 1 in the double-row layout by respecting their order and the departments in machine 3 and machine 4 space-free to row 2 such that department 8 lies directly opposite 7 as illustrated in Figure 3b. So we get $W^{\text{dst}}_{(8,7)}([7]) = 1 + 1.5 + 3 + 4 + 4.5 + 5.5 = 19.5$. Next, we assign the highest weights to departments closest to department 8. For instance, w_{68} is assigned to department 1 and w_{38} to department 2. In total, we obtain

$$W^{\text{spt}}_{(8,7)}([7]) = 7 \cdot 1 + 3 \cdot 1.5 + 3 \cdot 3 + 1 \cdot 4 + 1 \cdot 4.5 + 0.5 \cdot 5.5 = 31.75.$$



(a) A schedule obtained by the SPT rule for set of jobs [6] where $\frac{\ell_8}{2}$ and $\frac{\ell_7}{2}$ are fixed.



(b) Double-row layout obtained by arranging the departments from Figure 3a to row 1 and row 2 such that department 8 lies directly opposite department 7.

Figure 3: Consider an instance with $n = 8$, $\ell_k = k$, $k \in [6]$, $\ell_7 = 4$, $\ell_8 = 1$, and non-zero weights $w_{18} = \frac{1}{2}$, $w_{28} = 1$, $w_{38} = 3$, $w_{48} = 3$, $w_{58} = 1$, $w_{68} = 7$, $w_{78} = 5$. Then we obtain $W^{\text{dst}}_{(8,7)}([7]) = 19.5$ and $W^{\text{spt}}_{(8,7)}([7]) = 31.75$.

Note that in general the value $\min_{j \in S} W^{\text{dst}}_{(i,j)}(S)$ (and thus the value $\min_{j \in S} W^{\text{spt}}_{(i,j)}(S)$) is not obtained by arranging $i \in [n]$ directly opposite a shortest department of $S \subseteq [n] \setminus \{i\}$.

Example 11 Consider a DRFLP instance with $\ell_1 = \ldots = \ell_4 = 1$, $\ell_5 = 5$ and non-zero weights $w_{ij} = 1$, $i, j \in [5], i < j$. Then $W^{\text{dst}}_{(1,j)}(S) = 5$ for $j = 2, 3, 4$ and $S = \{2, \ldots, 5\}$, but $W^{\text{dst}}_{(1,5)}(S) = W^{\text{spt}}_{(1,5)}(S) = 1 + 2 + 1 = 4$. So it is the best to assign the largest department directly opposite department i in this example. The corresponding layout is illustrated in Figure 4.

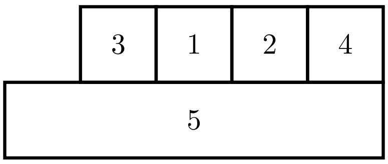


Figure 4: Consider a double-row instance with $\ell_1 = \ldots = \ell_4 = 1$, $\ell_5 = 5$ and weights $w_{ij} = 1$, $i, j \in [5], i < j$. The sum of the (weighted) distances of department 1 to the remaining departments is minimized by arranging it directly opposite department 5, which is the largest department in this instance.

2.2 Scheduling Lower Bound

In this section we suggest two further possibilities to bound $\widehat{W}_{(i,j)}(S)$ with $i \in [n], S \subseteq [n] \setminus \{i\}, j \in S$ from below. Our main tool is the approximation algorithm in [46] for the $P||\sum w_k C_k$. The associated algorithm determines a schedule by applying the Smith rule in the parallel machine case. The jobs are ordered non-increasingly by $\frac{w_k}{p_k}$ and the corresponding schedule is determined by Algorithm 1.

Theorem 12 *([46]) We consider the problem $P||\sum w_k C_k$ with jobs J. Then using the Smith rule for sorting the jobs leads to a $\frac{1+\sqrt{2}}{2}$-approximation algorithm for the $P||\sum w_k C_k$ with the running time $\mathcal{O}(|J| \cdot \log(|J|))$. Moreover, in the case with at least two machines, the bound $\frac{1+\sqrt{2}}{2}$ is tight.*

We will write $\alpha^{KK} := \frac{1+\sqrt{2}}{2} \approx 1.207$. In order to determine a lower bound for $\widehat{W}_{(i,j)}(S)$ for $i \in [n], S \subseteq [n] \setminus \{i\}, j \in S$, we again interpret the departments $S \setminus \{j\}$ as jobs of $P_4||\sum_{k \in S\setminus\{j\}} w_{ik} C_k$ and apply the approximation algorithm of Kawaguchi and Kyan [46]. However, in the lower bound calculation we have to take care of two facts. First, the distance calculations for the `DRFLP` are center-to-center whereas the algorithm by Kawaguchi and Kyan is based on completion times. Second, because the $P_4||\sum w_k C_k$ solution is not exact but only approximate, we must respect the approximation factor α^{KK}.

Definition 13 *Let (n, w, ℓ) be a `DRFLP` instance, and let $i \in [n]$, $S \subseteq [n] \setminus \{i\}$ and $j \in S$. We denote by $S_j^{sc} = (s_1, \ldots, s_{|S|-1})$ a sequence of departments $S \setminus \{j\}$ with length vector $\ell^{S_j^{sc}}$ ordered according to the Smith rule, i. e., non-increasingly by $\frac{w_{ik}}{\ell_k}$. Denote by*

$$C^{sc,(i,j)}(S, \ell) := C^{basic}(S_j^{sc}, \ell^{S_j^{sc}}, (0, 0, 0, 0))$$

the completion times returned by Algorithm 1 for this ordering. Then the `SCHED1`-lower-bound is

$$W^{sc}_{(i,j)}(S) := \frac{1}{\alpha^{KK}} \sum_{k=1}^{|S|-1} w_{is_k} \cdot C^{sc,(i,j)}_{s_k}(S, \ell) + \sum_{k=1}^{|S|-1} w_{is_k} \cdot (\tfrac{1}{2} \min\{\ell_i, \ell_j\} - \tfrac{\ell_{s_k}}{2}). \tag{5}$$

Proposition 14 *Let (n, w, ℓ) be a `DRFLP` instance and $i \in [n], j \in S \subseteq [n]\setminus\{i\}$. Then $W^{sc}_{(i,j)}(S) \leq \widehat{W}_{(i,j)}(S)$.*

Proof. Let $i \in [n], j \in S \subseteq [n] \setminus \{i\}$ be given. Here i lies opposite j and we want to bound the sum of the weighted distances of i to all other departments. We want to interpret this as a variant of $P_4||\sum_{k\in S\setminus\{j\}} w_{ik} C_k$. Let $(r, q) \in \mathcal{Q}(n, w, \ell)$ be an optimal solution of (2). A corresponding solution of $P_4||\sum_{k\in S\setminus\{j\}} w_{ik} C_k$ is then $C_k = |q_k - q_i| + \frac{\ell_k}{2} - \frac{\ell_{\hat{\imath}_k}}{2}$, $k \in S \setminus \{j\}$, where $\hat{\imath}_k = i$ if k is in the same row as i and $\hat{\imath}_k = j$ otherwise. The completion times of the scheduling problem are formed by taking into account that the `DRFLP` measures center-to-center distances. Let v^* denote the optimal value of the scheduling problem $P_4||\sum_{k\in S\setminus\{j\}} w_{ik} C_k$, then

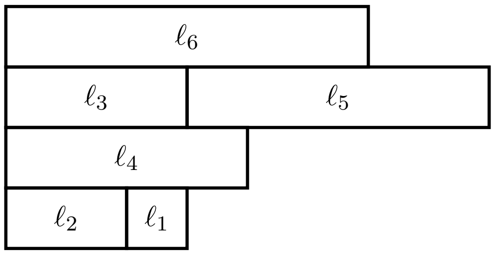
$$\begin{aligned}
\widehat{W}_{(i,j)}(S) &= \sum_{k\in S\setminus\{j\}} w_{ik}\,|q_k - q_i| \geq \sum_{k\in S\setminus\{j\}} w_{ik} C_k + \tfrac{1}{2} \sum_{k\in S\setminus\{j\}} w_{ik}(\min\{\ell_i, \ell_j\} - \ell_k) \\
&\geq v^* + \tfrac{1}{2} \sum_{k\in S\setminus\{j\}} w_{ik}(\min\{\ell_i, \ell_j\} - \ell_k) \\
&\geq \tfrac{1}{\alpha^{KK}} \sum_{k=1}^{|S|-1} w_{is_k} C^{\mathrm{sc},(i,j)}_{s_k}(S, \ell) + \tfrac{1}{2} \sum_{k=1}^{|S|-1} w_{is_k}(\min\{\ell_i, \ell_j\} - \ell_{s_k}) = W^{\mathrm{sc}}_{(i,j)}(S),
\end{aligned}$$

where the last inequality follows by Theorem 12. □

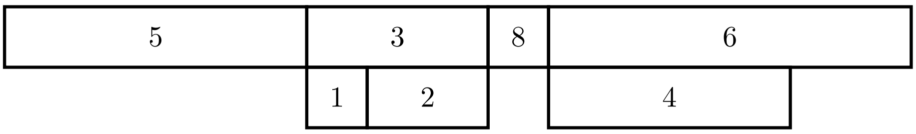
The following Example 15 illustrates the differences to the calculation of the star lower bounds.

Example 15 We consider again Example 10 and our aim is to compute $W^{\text{sc}}_{(8,7)}([7])$. We sort the departments by decreasing relative weights, i. e., $S^{\text{sc}}_j = (6, 3, 4, 2, 1, 5)$, and compute $C^{\text{sc},(i,j)}(S, \ell)$. The obtained schedule is illustrated in Figure 5a and the corresponding double-row layout is illustrated in Figure 5b, where we do not show department 7 because it overlaps with departments 2 and 4 while the distance calculation is done as illustrated here. Then we obtain

$$W^{\text{sc}}_{(8,7)}([7]) = \frac{1}{\alpha^{KK}} \left(w_{18}(\ell_1 + \ell_2) + w_{28}\ell_2 + w_{38}\ell_3 + w_{48}\ell_4 + w_{58}(\ell_5 + \ell_3) + w_{68}\ell_6\right) + \sum_{k\in[6]} \tfrac{w_{ik}}{2} \left(\min\{\ell_7, \ell_8\} - \ell_k\right) \approx 34.2 > 31.75 = W^{\text{spt}}_{(8,7)}([7]).$$



(a) Schedule on four machines obtained using the Smith rule.



(b) Double-row layout deduced from the schedule above where 8 lies directly opposite 7. Note that department 7 is not drawn because it is larger than department 8 and would overlap with departments 2 and 4.

Figure 5: Consider an instance with $n = 8$, $\ell_k = k$, $k \in [6]$, $\ell_7 = 4$, $\ell_8 = 1$, and non-zero weights $w_{18} = \frac{1}{2}$, $w_{28} = 1$, $w_{38} = 3$, $w_{48} = 3$, $w_{58} = 1$, $w_{68} = 7$, $w_{78} = 5$. We get $W^{\text{sc}}_{(8,7)}([7]) \approx 34.2$.

By Proposition 14 we obtain a lower bound for $\widehat{W}_{(i,j)}(S), i \in [n], S \subseteq [n] \setminus \{i\}, j \in S$. But, as shown in Example 15 and Figure 5b, we do not receive a valid double-row layout. The reason for this is that we only use $\frac{\min\{\ell_i,\ell_j\}}{2}$ and neglect that one of the two departments might be longer. For calculating the next bound we introduce two artificial jobs (departments) with length $\frac{1}{2} \cdot (\max\{\ell_i, \ell_j\} - \min\{\ell_i, \ell_j\}) = \frac{|\ell_i - \ell_j|}{2}$ and weights such that they are chosen first by the Smith rule. In order to get a best possible lower bound afterwards the weight is chosen minimal with respect to the desired property.

Definition 16 *Let (n, w, ℓ) be a* DRFLP *instance, and let $i \in [n]$, $S \subseteq [n] \setminus \{i\}$ and $j \in S$. Define $\beta := \max\{\frac{w_{ik}}{\ell_k} : k \in S \setminus \{j\}\}$. We add two dummy departments $n + 1$ and $n + 2$ with lengths $\ell_{n+1} = \ell_{n+2} = \frac{|\ell_i - \ell_j|}{2}$ and weights $w_{i(n+1)} = w_{i(n+2)} = \beta \frac{|\ell_i - \ell_j|}{2}$. Let $S^{sc2}_j = (s_1, \ldots, s_{|S|+1})$ be a sequence of departments $S \cup \{n + 1, n + 2\} \setminus \{j\}$ ordered according to the Smith rule with $s_1 = n + 1, s_2 = n + 2$ and with length vector $\ell^{S^{sc2}_j}$ and denote by*

$$C^{sc2,(i,j)}(S, \ell) := C^{basic}(S^{sc2}_j, \ell^{S^{sc2}_j}, (0, 0, 0, 0))$$

the completion times returned by Algorithm 1 for this ordering. Then the SCHED2*-lower-bound is*

$$W^{sc2}_{(i,j)}(S) := \tfrac{1}{\alpha^{KK}} \sum_{k=1}^{|S|+1} w_{is_k} \cdot C^{sc2,(i,j)}_{s_k}(S, \ell) + \sum_{k=3}^{|S|+1} w_{is_k} \cdot \left(\tfrac{1}{2} \min\{\ell_i, \ell_j\} - \tfrac{\ell_{s_k}}{2}\right) - \beta \tfrac{(\ell_i - \ell_j)^2}{2}. \quad (6)$$

Proposition 17 *Let (n, w, ℓ) be a* DRFLP *instance and $i \in [n], j \in S \subseteq [n] \backslash \{i\}$. Then $W^{sc2}_{(i,j)}(S) \leq \widehat{W}_{(i,j)}(S)$.*

Proof. Let $i \in [n], j \in S \subseteq [n] \setminus \{i\}$ be given. The proof of this result is similar to the proof of Proposition 14. In contrast to this proof we introduce two dummy departments $n+1, n+2$ here to level different lengths of i and j. In the associated scheduling problem we then also count the completion times of the dummy departments. So we have to subtract this value afterwards. Let all objects be as defined in Definition 16 and let $(r, q) \in \mathcal{Q}(n, w, \ell)$ be an optimal solution of (2). The corresponding scheduling solution is $C_{s_k} = |q_{s_k} - q_i| + \frac{1}{2}(\ell_{s_k} - \min\{\ell_i, \ell_j\})$ for $k = 3, \ldots, |S^{\text{sc2}}_j|$. Let v^* be the optimal solution value of $P_4|| \sum_{k \in S^{\text{sc2}}_j} w_{ik} C_k$, then

$$
\begin{aligned}
\widehat{W}_{(i,j)}(S) &= \sum_{k \in S \setminus \{j\}} w_{ik} \, |q_k - q_i| \\
&\geq \sum_{k=1}^{|S|+1} w_{is_k} C_{s_k} + \frac{1}{2} \sum_{k=3}^{|S|+1} w_{is_k} (\min\{\ell_i, \ell_j\} - \ell_{s_k}) - w_{i(n+1)} \ell_{n+1} - w_{i(n+2)} \ell_{(n+2)} \\
&\geq v^* + \frac{1}{2} \sum_{k=3}^{|S|+1} w_{is_k} (\min\{\ell_i, \ell_j\} - \ell_{s_k}) - \beta \frac{(\ell_i - \ell_j)^2}{2} \\
&\geq \frac{1}{\alpha^{KK}} \sum_{k=1}^{|S|+1} w_{is_k} C^{\text{sc2},(i,j)}_{s_k}(S, \ell) + \frac{1}{2} \sum_{k=3}^{|S|+1} w_{is_k} (\min\{\ell_i, \ell_j\} - \ell_{s_k}) - \beta \frac{(\ell_i - \ell_j)^2}{2} \\
&= W^{\text{sc2}}_{(i,j)}(S),
\end{aligned}
$$

where the last inequality follows by Theorem 12. Furthermore note that by the choice of $\beta, \ell_{n+1} = \ell_{n+2}, w_{i(n+1)} = w_{i(n+2)}$ using the Smith rule it is possible to set $s_1 = n+1, s_2 = n+2$. □

The combination of the previous results leads to one of our main results – a first combinatorial lower bound for the optimal value of the DRFLP.

Theorem 18 *Let (n, w, ℓ) be a* DRFLP *instance. Let $V_i := \{k \in [n] \setminus \{i\} \colon w_{ik} > 0\}$ for $i \in [n]$. Then*

$$
\frac{1}{2} \sum_{i \in [n]} \min_{j \in V_i} \max\{W^{spt}_{(i,j)}(V_i), W^{sc}_{(i,j)}(V_i), W^{sc2}_{(i,j)}(V_i)\} \tag{7}
$$

is a lower bound for the optimal value $\widehat{W}$ of the DRFLP*. The bound (7) can be computed in $\mathcal{O}\left(n^3 \cdot \log(n)\right)$.*

Proof. The correctness follows from propositions 9, 14 and 17 and the definition of $\widehat{W}$. The running time for fixed $i \in [n], j \in V_i$, is $\mathcal{O}\left(n \cdot \log(n)\right)$ because one has to sort the jobs (departments) in order to apply Algorithm 1. Since there are $\mathcal{O}\left(n^2\right)$ such summands the total running time is $\mathcal{O}\left(n^3 \cdot \log(n)\right)$. □

2.3 Extensions

In this section we discuss extensions of the combinatorial lower bounds to the general MRFLP, the DREFLP and the PROP.

2.3.1 MRFLP

In general, all the lower bounds presented above for the DRFLP can be extended to lower bounds for the MRFLP. Indeed, for the bounds in Proposition 9 we can use the same approach but we have to check all $\binom{|S|}{m-1}$ (with $S \subseteq [n] \setminus \{i\}$) choices for departments directly opposite the fixed department $i \in [n]$. The same is true for the scheduling bounds SCHED1 and SCHED2. For the MRFLP we have to slightly extend Algorithm 1 to handle scheduling problems on $2m$ parallel machines. However, the running time for the calculation of (7) is increased significantly in comparison to the double-row case, but remains polynomial if m is fixed.

2.3.2 DREFLP

In the calculation of the SPT-lower-bound we assign high weights to small departments. So the question arises if we can simplify the calculation of the combinatorial lower bounds in the equidistant case, because there we do not need Algorithm 1 to determine an optimal arrangement of the departments depending on their lengths. We start with the special case of all weights being equal to one. Note that we assume as done in the literature, see, e. g., [6, 12], that the department lengths are equal to one.

Proposition 19 *Let $(n, w, \mathbf{1})$ be a DREFLP instance. Let $i \in [n]$ and $S \subseteq [n] \setminus \{i\}$, then for all $j \in S \setminus \{i\}$ and all solutions $(r, q) \in \Omega(n, w, \mathbf{1})$ we get*

$$W^{dst}_{(i,j)}(S) = \left\lfloor \frac{\left(\left\lceil \frac{|S|-1}{2} \right\rceil + 1\right)^2}{4} \right\rfloor + \left\lfloor \frac{\left(\left\lfloor \frac{|S|-1}{2} \right\rfloor + 1\right)^2}{4} \right\rfloor \leq \sum_{k \in S} |q_i - q_k|.$$

Proof. For $i \in [n]$ we arrange one department of $S \subseteq [n] \setminus \{i\}$ directly opposite i and we assign the remaining $\left\lceil \frac{|S|-1}{2} \right\rceil$ departments to row 1 and $\left\lfloor \frac{|S|-1}{2} \right\rfloor$ departments to row 2. The result follows from the star inequalities (1) for the LA. □

Consequently, we can also simplify the calculation of the SPT-lower bound. For $i \in [n]$ we sort the departments in $S \subseteq [n] \setminus \{i\}$ by decreasing weights $w_{ik}, k \in S$, and assign the departments in that order as close as possible to i, i. e., a department with highest weight $w_{ik}, k \in S$, lies directly opposite i. We denote this lower bound by $W_i^{\text{sort}}(S)$.

If we know that two departments $i, j \in [n], i < j$, overlap and so lie exactly opposite due to the grid structure [12], we can determine a lower bound for the weighted distances of i and j to the departments $S \subseteq [n] \setminus \{i, j\}$. For this we order the departments in S by decreasing weight $w_{ik} + w_{jk}$, $k \in S$, and get a sequence $S_{i,j}^{\text{E-spt}} = (s_1, \ldots, s_{|S|})$. With

$$C^{\text{E-spt},(i,j)}(S) = C^{\text{basic}}(S_{i,j}^{\text{E-spt}}, (1, \ldots, 1), (0, 0, 0, 0))$$

we get

$$W_{(i,j)}^{\text{E-spt}}(S) := \sum_{k=1}^{|S|} (w_{is_k} + w_{js_k})(C_{s_k}^{\text{E-spt},(i,j)}(S)). \tag{8}$$

Proposition 20 *Let $(n, w, \mathbf{1})$ be a DREFLP instance. Let $i, j \in [n], i < j$, and $S \subseteq [n] \setminus \{i, j\}$, then for all DREFLP solutions $(r, q) \in \Omega(n, w, \mathbf{1})$ with $q_i = q_j$ we have*

$$W_{(i,j)}^{\text{E-spt}}(S) \leq \sum_{k \in S} (w_{ik} + w_{jk}) |q_i - q_k|.$$

Proof. The result follows directly by Proposition 9 and its proof. □

2.3.3 PROP

Finally we have a look at row layout problems where the assignment of the departments to the rows is already known like the SRFLP, the PROP and the kPROP. We concentrate on the PROP in the description, but the other cases follow analogously.

For the PROP the lower bound calculation of $W^{\text{dst}}_{(i,j)}$ and $W^{\text{spt}}_{(i,j)}$, $i, j \in [n], i \neq j$, can be adapted as follows. Because the row assignment is fixed, we can split the calculation of the distances in inner-row and inter-row distances. Let $i \in [n]$ and $S \subseteq [n] \setminus \{i\}$. We first order the departments $S_1 := \{j \in S : r_i = r_j\} \subseteq S$ which are in the same row as i increasingly by their lengths and get $S^{\text{inn}} = (s_1, \dots, s_{|S_1|})$ with length vector $\ell^{S^{\text{inn}}}$. Applying Algorithm 1 we get the completion times

$$C^{\text{inn},i}(S, \ell) = C^{\text{basic}}(S^{\text{inn}}, \ell^{S^{\text{inn}}}, (\tfrac{\ell_i}{2}, \tfrac{\ell_i}{2}, \infty, \infty))$$

and with $(w'_{i1}, \dots, w'_{i|S_1|})$ being a decreasingly sorted list of the weights $w_{ik}, k \in S_1$, the bounds

$$W^{\text{dst-inn}}_i(S) := \sum_{k=1}^{|S_1|} (C^{\text{inn},i}_{s_k}(S, \ell) - \tfrac{\ell_{s_k}}{2}), \qquad W^{\text{spt-inn}}_i(S) := \sum_{k=1}^{|S_1|} w'_{ik}(C^{\text{inn},i}_{s_k}(S, \ell) - \tfrac{\ell_{s_k}}{2}),$$

For the inter-row distances we have to consider all possible departments lying opposite i. So let $j \in S$ with $r_j \neq r_i$ be fixed. Now order the remaining departments in the other row, i. e. $S_{2,j} := \{k \in S : j \neq k, r_i \neq r_k\} \subseteq S$, by increasing length and get $S^{\text{int}}_j = (s'_1, \dots, s'_{|S_{2,j}|})$ with length vector $\ell^{S^{\text{int}}_j}$. Applying Algorithm 1 we get the completion times

$$C^{\text{int},(i,j)}(S, \ell) = C^{\text{basic}}(S^{\text{int}}_j, \ell^{S^{\text{int}}_j}, (\infty, \infty, \tfrac{\ell_j}{2}, \tfrac{\ell_j}{2})).$$

As before, with $(w''_{i1}, \dots, w''_{i|S_2|})$ being a decreasingly sorted list of the weights w_{ik}, $k \in S_{2,j}$, we get the bounds

$$W^{\text{dst-int}}_{(i,j)}(S) := \sum_{k=1}^{|S_{2,j}|} (C^{\text{int},(i,j)}_{s'_k}(S, \ell) - \frac{\ell_{s'_k}}{2}),$$
$$W^{\text{spt-int}}_{(i,j)}(S) := \sum_{k=1}^{|S_{2,j}|} w''_{ik}(C^{\text{int},(i,j)}_{s'_k}(S, \ell) - \frac{\ell_{s'_k}}{2}).$$

Combining the inner-row and inter-row bounds leads to $W^{\text{P-dst}}_{(i,j)}(S)$ and $W^{\text{P-spt}}_{(i,j)}(S)$. Similar adaptions are possible for improving $W^{\text{sc}}_{(i,j)}(S)$ ($W^{\text{sc2}}_{(i,j)}(S)$ for the PROP is then the same as $W^{\text{sc}}_{(i,j)}(S)$) in the case of fixed row assignments. We denote the improved PROP bounds by prepending “P-” to the name.

In order to compute a lower bound for the PROP we can sum up the lower bounds for inner-row distances of each of the departments to the others and divide this sum by two. To obtain a global lower bound for the inter-row distances in the PROP we sum up the weighted distances of each department in row 1 to row 2, i. e., for each $i \in [n]$ with $r_i = 1$ we compute $\min_{j \in [n], r_j = 2} W^{\text{spt-int}}_{(i,j)}(S)$ and vice versa and we take the maximum value of these. By this method we do not have to divide the obtained value by two.

3 A Distance-Based Lower Bounding ILP Model

For the SRFLP a distance-based model was introduced in [10] to compute a lower bound for the optimal solution value. The lower bound calculation was combined with some branch-and-cut algorithm. These results were based on investigations of the LA in [20] where the authors combined a distance model with combinatorial bounds. In Section 3.1 we introduce an ILP model consisting of distance variables and so called overlap variables, which is not a formulation for the DRFLP. The optimal solution value of that model is a lower bound for the optimal value of the DRFLP. Our model is based on our combinatorial lower bounds presented in the previous section. In the equidistant case of the DRFLP the model can be strengthened and we will also mention which adaptions are possible in the case of PROP or kPROP. We want to use the newly derived cutting planes in a branch-and-cut algorithm. So we describe in Section 3.2 appropriate separators.

3.1 The Lower Bounding Model

In the description of our ILP model we start with the variables. We use distance variables $d_{ij} = d_{ji} \geq 0, i, j \in [n], i < j$. In contrast to the literature, see, e. g., [29, 63], where left-right ordering variables were used, we use binary overlap variables $x_{ij} = x_{ji} \in \{0, 1\}, i, j \in [n], i < j$. Two departments i and j overlap if their positions satisfy $|q_i - q_j| < \frac{\ell_i + \ell_j}{2}$. The associated variables have the following interpretation

$$x_{ij} = \begin{cases} 1, & \text{departments } i \text{ and } j \text{ lie in different rows and overlap,} \\ 0, & \text{otherwise.} \end{cases}$$

We want to note that the model does not contain position variables for the departments.

It was proven in [29] that there always exists an optimal double-row layout where the distance from the left border of the leftmost department to the right border of the rightmost department is at most $M := \sum_{i=\lfloor \frac{n+1}{3} \rfloor + 1}^{n} \ell_i$ where the departments are sorted in ascending order according to their length. Apart from this we define a parameter $\iota \in \{0, 1\}$ which is one if and only if all department lengths are integral. This is the case in almost all test instances in the literature. Our lower bounding model for the DRFLP reads as follows.

$$\min \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} d_{ij}$$

$$\sum_{\substack{i,j \in S \\ i<j}} x_{ij} \leq |S| - 1, \qquad S \subseteq [n], |S| \geq 2, \tag{9}$$

$$\sum_{j \in S \cup T} x_{ij} \leq |S| + 1, \qquad i \in [n], S \subset [n] \setminus \{i\} \text{ with } \sum_{j \in S} \ell_j \geq \ell_i, \; T := \{j \in [n] \setminus (S \cup \{i\}) \colon \ell_j \geq \max_{k \in S} \ell_k\}, \tag{10}$$

$$d_{ij} + \left(\tfrac{\ell_i + \ell_j}{2}\right) x_{ij} \geq \tfrac{\ell_i + \ell_j}{2}, \qquad i, j \in [n], i < j, \tag{11}$$

$$d_{ij} + \left(M - \ell_i - \ell_j + \tfrac{1}{2}\iota\right) x_{ij} \leq M - \tfrac{\ell_i + \ell_j}{2}, \qquad i, j \in [n], i < j, \tag{12}$$

$$d_{ij} + d_{jk} - d_{ik} \geq 0, \qquad i, j, k \in [n], |\{i, j, k\}| = 3, i < k, \tag{13}$$

$$\sum_{j \in S} d_{ij} \geq \min_{j \in S} W^{\text{dst}}_{(i,j)}(S), \qquad i \in [n], S \subseteq [n] \setminus \{i\}, \tag{14}$$

$$\sum_{j \in S} w_{ij} d_{ij} \geq \min_{j \in S} \max \left\{ \begin{array}{c} W^{\text{spt}}_{(i,j)}(S), \\ W^{\text{sc}}_{(i,j)}(S), \\ W^{\text{sc2}}_{(i,j)}(S) \end{array} \right\}, \qquad i \in [n], S \subseteq V_i, \tag{15}$$

$$\sum_{\substack{i,j \in S \\ i<j}} d_{ij} + o \sum_{\substack{i,j \in S \\ i<j}} x_{ij} \geq o, \qquad S \subset [n], |S| = 3, o = \sum_{i \in S} \ell_i + \min_{i \in S} \ell_i, \tag{16}$$

$$\sum_{\substack{i,j \in S \\ i<j}} d_{ij} + o \sum_{\substack{i,j \in S \\ i<j}} x_{ij} \geq o, \qquad S \subset [n], |S| = 4, \; o = \tfrac{3}{2} \sum_{i \in S} \ell_i + 2 \cdot \min_{\substack{i_1, i_2 \in S \\ i_1 \neq i_2}} (\ell_{i_1} + \ell_{i_2}), \tag{17}$$

$$\sum_{\substack{i,j \in S \\ i<j}} \ell_i \ell_j d_{ij} + o \sum_{\substack{i,j \in S \\ i<j}} x_{ij} \geq o, \qquad S \subseteq [n], |S| \geq 3, \; o = \tfrac{1}{6} \left(\left(\sum_{i \in S} \ell_i \right)^3 - \sum_{i \in S} \ell_i^3 \right), \tag{18}$$

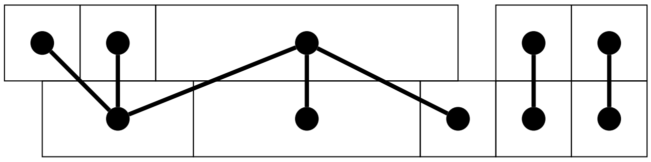


Figure 6: Visualization of the forest associated to the overlap variables of a double-row layout. Each node represents a department and there exists an edge between two different departments if and only if both overlap.

$$x_{ij} = x_{ji} \in \{0,1\}, \qquad i,j \in [n], i<j, \tag{19}$$
$$d_{ij} = d_{ji} \geq 0, \qquad i,j \in [n], i<j.$$

The overlap variables in a double-row layout have to build a forest if we interpret them as edges in a graph where each department represents a single node of the associated graph and two nodes (departments) are connected by an edge if both overlap in the layout, see Figure 6. We ensure this by the well-known subtour elimination constraints (9), see, e. g., [24, 27], for the complete description of the forest polytope. But the forest also has to satisfy further properties concerning the degree of certain nodes. Let $i \in [n]$ be fixed and let $S \subset [n] \setminus \{i\}$ with $\sum_{j \in S} \ell_j \geq \ell_i$, i. e., the departments in S are in total at least as long as department i, then at most $|S|+1$ departments of the set $S \cup \{k \in [n] \setminus (S \cup \{i\})\colon \ell_k \geq \max_{j \in S} \ell_j\}$ can overlap with i. This results in (10). So, for instance, a department i can overlap with at most two departments that are at least as long as i itself.

The distance and the overlap variables are coupled via (11) and (12). On the one hand, if two departments do not overlap, then the distance between the centers of both is at least the sum of half the lengths of the departments. On the other hand the distance of two departments that overlap cannot be larger than the sum of half the lengths of both departments. Assuming integral department length we can even enforce that this value is $\frac{1}{2}$ less because the overlap is then at least one half (the departments are arranged on the half grid according to [42]). As used in previous layout models, see, e. g., [10], the distance variables have to satisfy the triangle inequalities (13). Furthermore, we use our combinatorial bounds to bound the sum of the (weighted) distances between all departments of some set $S \subset [n]$, see (14)–(15).

If we know that certain departments do not overlap pairwise, then we can treat them as departments in an `SRFLP` instance and use constraints known to be valid for the `SRFLP`, see (16) and (17). For the validity of these inequalities and especially for the calculation of the right-hand side o one compares all different orderings of the associated departments and counts how often the length of each single department appears. Inequalities (18) are an adapted version of the so called *clique inequalities* presented in [10] for the `SRFLP`. Note, inequalities (16)–(18) are trivially satisfied if one of the associated x-variables equals one. Finally, we have the integrality of the overlap variables (19).

3.1.1 Adaptations of Our Lower Bounding Model for the `DREFLP`

For the `DREFLP` there always exists an optimal solution on the grid [12]. Therefore we can restrict to solutions where two departments overlap if and only if they lie directly opposite each other. So the interpretation of our overlap variables changes to

$$x^e_{ij} = x^e_{ji} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ lie directly opposite each other,} \\ 0, & \text{otherwise,} \end{cases}$$

$i,j \in [n], i<j$. Our model specialized to the `DREFLP` reads as follows.

$$\min \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} d_{ij}$$

(13),

$$\sum_{\substack{j\in[n]\\ j\neq i}} x^e_{ij} \leq 1, \qquad i \in [n], \tag{20}$$

$$\sum_{\substack{i,j\in[n]\\ i<j}} x^e_{ij} \leq \left\lfloor \frac{n}{2} \right\rfloor, \tag{21}$$

$$\sum_{\substack{i,j\in[n]\\ i<j}} x^e_{ij} \geq n - \left\lceil \frac{2n}{3} \right\rceil + 1, \qquad n \geq 9, \tag{22}$$

$$d_{ij} + x^e_{ij} \geq 1, \qquad i,j \in [n], i<j, \tag{23}$$

$$d_{ij} + Mx^e_{ij} \leq M, \qquad i,j \in [n], i<j, \tag{24}$$

$$\sum_{\substack{i,j\in[n]\\ i<j}} d_{ij} \geq \begin{cases} \frac{(n+1)n(n-1)}{12}, & n \text{ odd},\\ \frac{(n+2)n(n-2)}{12}, & n \text{ even}, \end{cases} \tag{25}$$

$$\sum_{j\in[n]\setminus\{i\}} d_{ij} \geq \left\lfloor \frac{\left(\left\lceil \frac{n-2}{2} \right\rceil + 1\right)^2}{4} \right\rfloor + \left\lfloor \frac{\left(\left\lfloor \frac{n-2}{2} \right\rfloor + 1\right)^2}{4} \right\rfloor, \qquad i \in [n], \tag{26}$$

$$\sum_{j\in S} w_{ij} d_{ij} \geq W_i^{\text{sort}}(S), \qquad i \in [n], S \subseteq V_i, \tag{27}$$

$$\sum_{k\in S} (w_{ik}d_{ik} + w_{jk}d_{jk}) - x^e_{ij}(W^{\text{E-spt}}_{(i,j)}(S)) \geq 0, \qquad i,j \in [n], i<j, S \subseteq [n] \setminus \{i,j\}, \tag{28}$$

$$\sum_{\substack{i,j\in S\\ i<j}} d_{ij} + o \sum_{\substack{i,j\in S\\ i<j}} x^e_{ij} \geq o, \qquad S \subseteq [n], |S| \geq 3, o = \frac{1}{6}\left(|S|^3 - |S|\right), \tag{29}$$

$$x^e_{ij} = x^e_{ji} \in \{0,1\}, \qquad i,j \in [n], i<j. \tag{30}$$

There always exists an optimal solution to the DREFLP on the grid [12]. So each department may overlap with at most one department, see (20). Additionally, we can bound the total sum of the overlap variables by $\lfloor \frac{n}{2} \rfloor$. We can also generalize (21) to $\sum_{i,j\in S, i<j} x^e_{ij} \leq \left\lfloor \frac{|S|}{2} \right\rfloor$ for sets $S \subseteq [n]$. Constraints (20) and (21) can be seen as a strengthened version of inequalities (9) and (10). For $n \geq 9$ there always exists an optimal equidistant double-row layout which uses at most $\left\lceil \frac{2n}{3} \right\rceil - 1$ columns of the grid [12]. It follows that at least $n - \left\lceil \frac{2n}{3} \right\rceil + 1$ columns contain two departments, see (22). If two departments overlap, then their distance is zero, see (24), and at least one otherwise, see (23). In the unweighted case of the DREFLP, i. e., if all weights are equal to one, an optimal solution can be determined directly [23] and this value is a lower bound for the sum of the distances in the DREFLP, see (25). Note that we are not aware of a similar result for the DRFLP. So we take advantage of the DREFLP structure here. Apart from this we can bound the sum of the weighted distances of some $i \in [n]$ to all departments $S \subseteq V_i$ from below using our combinatorial bounds, see (27). If two departments $i, j \in [n], i < j$, overlap, we can use $W^{\text{E-spt}}_{(i,j)}(S)$ defined in (8) as a lower bound for the weighted distances of i and j to the departments $S \subseteq [n] \setminus \{i, j\}$, see (28). If i and j do not overlap, inequality (28) is redundant. Inequalities (29) are the clique inequalities (18) used before with $\ell_i = 1, i \in [n]$.

3.1.2 Adaptations of Our Model for the PROP

For the PROP further improvements of our lower bounding model (9)–(18) are possible because the row assignment of the departments is given. So one can hope to achieve smaller gaps for the PROP in comparison to the DRFLP, especially if the rows are balanced, i. e., the sum of the lengths

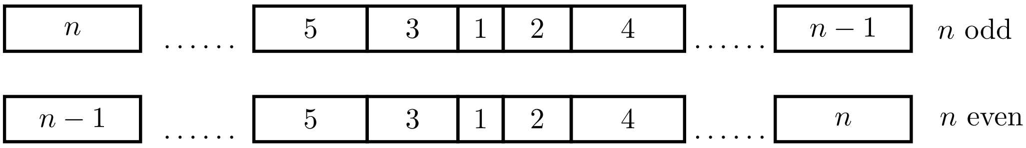


Figure 7: Visualization of optimal solutions (and permutations π^* of $[n]$) for SRFLP instances with weights $w_{ij} = 1, i, j \in [n], i < j$, and $\ell_i \leq \ell_{i+1}$ for $i \in [n-1]$ for the different parities of n.

of the departments in row 1 is close to the corresponding sum in row 2. Balanced layouts are of special interest in practice because in factory planning the size of the factory influences the production costs [51].

For the departments in some row $r \in \mathcal{R}$ we can use results from the SRFLP literature because the departments are arranged without spaces. For bounding the sum of the distances of the departments we use the fact that the SRFLP with $w_{ij} = 1, i, j \in [n], i < j$, can easily be solved in polynomial time.

Theorem 21 ([61]) *Let (n, w, ℓ) with $w_{ij} = 1$, $i, j \in [n], i < j$, be an SRFLP instance. Let a single-row layout with associated permutation π^* be derived by sorting the departments in an ascending order according to their lengths and placing the first department in the middle and the remaining ones right and left in an alternating manner to the ones already assigned, see Figure 7. Then this layout is optimal with objective value $\frac{n-1}{2}\sum_{i=1}^{n} \ell_i + \sum_{k=2}^{n-1}(k-1)(n-k)\ell_{\pi^*(k)}$.*

This result was stated in [61] but not proven correctly. So we present a new proof here.

Proof. Let an arbitrary permutation $\pi\colon [n] \to [n]$ of the departments be given where $\pi(k)$ denotes the kth department from the left border. The distance $d_{\pi(i)\pi(j)}$ of two departments $\pi(i), \pi(j), i, j \in [n], \pi(i) < \pi(j)$, equals $d_{\pi(i)\pi(j)} = \frac{\ell_{\pi(i)}+\ell_{\pi(j)}}{2} + \sum_{k=i+1}^{j-1} \ell_{\pi(k)}$. Then with $C := \sum_{i,j=1, i<j}^{n} \frac{\ell_i+\ell_j}{2} = \frac{n-1}{2}\sum_{i=1}^{n} \ell_i$ we get

$$\sum_{\substack{i,j=1\\i<j}}^{n} w_{ij} d_{ij} = \sum_{\substack{i,j=1\\i<j}}^{n} d_{ij} = \sum_{\substack{i,j=1\\i<j}}^{n} d_{\pi(i)\pi(j)} = C + \sum_{\substack{i,j=1\\i<j}}^{n} \sum_{k=i+1}^{j-1} \ell_{\pi(k)} = C + \sum_{k=2}^{n-1} (k-1)(n-k)\ell_{\pi(k)}. \quad (31)$$

For proving the optimality of π^* we consider the quadratic function $f\colon [2, n-1] \to \mathbb{R}$ with $f(k) = (k-1)(n-k)$. The function f is strongly concave with its unique maximum point at $k^* = \frac{n+1}{2} \in [2, n-1]$ and is symmetric to k^*. Then (31) is minimized by π^* and the result follows. □

Let $R_k, k \in \mathcal{R}$, denote the indices of the departments which are assigned to row k. Then we can use the following constraints for PROP.

$$\sum_{\substack{i,j \in R_k\\i<j}} \ell_i \ell_j d_{ij} = \frac{1}{6}\left((\sum_{i \in R_k} \ell_i)^3 - \sum_{i \in R_k} \ell_i^3 \right), \qquad k \in \mathcal{R}, \quad (32)$$

$$\sum_{\substack{i,j \in R_k\\i<j}} d_{ij} \geq \sum_{\substack{i,j \in R_k\\i<j}} \frac{\ell_i + \ell_j}{2} + \sum_{z=2}^{|R_k|-1} (z-1)(|R_k| - z)\ell_{\pi^{*,k}(z)}, \qquad k \in \mathcal{R}, \quad (33)$$

$$d_{ij} \geq \frac{\ell_i + \ell_j}{2}, \qquad k \in \mathcal{R}, i, j \in R_k, i < j, \quad (34)$$

$$d_{ij} \leq \left(\sum_{z \in R_k} \ell_z \right) - \frac{\ell_i + \ell_j}{2}, \qquad k \in \mathcal{R}, i, j \in R_k, i < j, \quad (35)$$

$$d_{ij} \leq \max\{\sum_{z\in R_1} \ell_z, \sum_{z\in R_2} \ell_z\} - \frac{\ell_i + \ell_j}{2}, \qquad i \in R_1, j \in R_2. \tag{36}$$

Treating the departments in the same row as a SRFLP we can use the clique equation (32) as shown to be valid for the SRFLP in [10] (we can still use (18)). As already mentioned the sum of the distances between departments in the same row can be bounded using Theorem 21 where in (33) an optimal layout of departments R_k of the unweighted SRFLP according to Theorem 21 is denoted by $\pi^{*,k}$.

Of course, two departments in the same row satisfy a minimal distance condition, see (34). So we do not need (11) for departments in the same row. Note that in the PROP one can bound distances between two departments in the same row (35) and also in different rows (36) because the departments are arranged without spaces and with a fixed left border. So we do not need (12) for departments lying in the same row. Apart from this we can use the improved combinatorial lower bounds in (14) and (15). For a summary of the complete PROP model we refer to the appendix.

3.2 Separation

In this section we describe a branch-and-cut algorithm which is based on the inequalities (9)–(18) and we explain for which subsets $S \subseteq [n]$ the constraints are indeed used in the calculation of the lower bounds. We include inequalities (11) from the beginning as well as inequalities (9) for $S = [n]$ and (15) for $S = V_i$. Inequalities (14) are included for $S = V_i$ and $S = [n]$. Furthermore, we add all triangle inequalities (13).

The remaining constraints are separated in the following way. We separate inequalities (12) by complete enumeration. It is well-known that the problem to decide whether the vector of the x-variables is contained in the forest polytope, i. e., the convex hull over all incidence vectors of forests of a complete undirected graph with n nodes, can be solved in polynomial time, see, e. g., [62] pp. 880–881. Indeed, assuming non-negativity of the x-variables, one can determine a maximally violated subtour elimination constraint (9) solving special minimum cut problems on an associated directed graph.

There are potentially exponentially many inequalities of type (10), so we use the following heuristic approach. First we sort the departments according to their lengths in ascending order, i. e., $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_i \leq \ldots \leq \ell_n$. For each $i \in [n]$ we determine the set $\bar{S}_i := \{j \in [n] \setminus \{i\} \colon \ell_j \geq \ell_i\}$ and separate constraints

$$\sum_{j\in\bar{S}_i} x_{ij} \leq 2$$

explicitly. These inequalities imply all inequalities of (10) with $|S| = 1$. For $2 \leq |S| \leq 3$ we add an inequality if S satisfies $\sum_{j\in S} \ell_k \geq \ell_i$ and $\sum_{j\in S\setminus\{k\}} \ell_j < \ell_i$ for all $k \in S$, because the inequality is redundant otherwise.

We separate inequalities (16) and (17) by brute-force enumeration. In [10] it is conjectured that the separation problem of the general clique inequalities of the SRFLP is $\mathcal{NP}$-hard. For this reason we restrict to sets S of size three and four in (18) and check all these inequalities by complete enumeration.

It remains the usage of our combinatorial lower bounds in (14) and (15). Given a relaxation $\bar{x}, \bar{d}$, we construct two sets for fixed $i \in [n]$. At first, we consider all departments which are according to the distance variables $\bar{d}$ close to i, i. e., $S_1^i = \{j \in [n] \setminus \{i\} \colon \bar{d}_{ij} \leq \frac{\ell_i+\ell_j}{2}\}$ and previous tests, which are not included in this paper, show that it is worth to check $S_2^i = \{j \in [n] \setminus \{i\} \colon \bar{d}_{ij} \leq \ell_i + \ell_j\}$ as well.

For the PROP similar separation strategies were used. Testing our DREFLP model with a branch-and-cut algorithm we include inequalities (13), (20)–(23), (25) and (26) from the beginning as well as inequalities (27) for $S = V_i$. We separate inequalities (24) and (28) by brute force enumeration and as done in our `DRFLP` lower bounding model we restrict inequalities (29) to sets $|S| = 3$ and $|S| = 4$ and check all these inequalities by complete enumeration.

4 Computational Results

In this section we present our computational experiments implemented in C++. We used Cplex 12.8 [45] as an `ILP` solver. All results were conducted on a 2.30GHz dual-core computer running on Debian GNU/Linux 8 in single processor mode.

We compare our combinatorial lower bounds (7) as well as our branch-and-cut algorithm, described in Section 3.2, interrupted after a time limit of a few minutes on instances from the literature as well as on randomly generated medium-sized and large double-row instances. All instances are available from the authors. Because the use of the enumerative approach of [29] for computing lower bounds is out of scope due to the exponential number of subproblems that have to be solved (approximately), we compare our lower bounds to the lower bounds that can be achieved via branch-and-cut on the model presented in [8] within a given time limit of one hour. We decided to use the weaker model presented in [8] and not to use the model in [63] because this contains a huge number of variables and constraints for larger n and so even the calculation of the root node value was rather time-consuming in our tests. Further note that we do not test instances from [21, 56] because there only double-row instances with clearance conditions were considered.

Since only smaller instances were solved to optimality in the literature (see Table 1), we generate random instances with $n \in \{20, 30, 40, 50\}$. To obtain a wide set of random instances we set the transport density to 10 %, 50 % and 100 % and we choose integer transport weights randomly between 1 and 10. The integral lengths of the departments are chosen randomly between 1 and 15 (see Table 2) as well as between 5 and 10 (see Table 3). For each type we created ten instances. We denote these instances by n_k where n is the number of departments and k is the transport density. The first column "Instance" of the tables displays the instance name. The lower bound value obtained by applying a branch-and-cut algorithm with a given time limit of one hour for the `DRFLP` formulation in [8] is given in column "Amaral". Our combinatorial bound (7) is presented as well as the lower bounds derived via branch-and-cut within a given time limit of three resp. ten minutes, see columns "ILP_{3min}" and "ILP_{10min}". In order to show that our combinatorial lower bounds significantly strengthen our ILP model we tested our ILP without using the lower bounds. The results can be found in column $ILP^{pure}_{3\text{min}}$. The best known upper bound is given in column "heuristic" (or "optimal" in Table 1) and the time spent for the heuristic in seconds is given in column "time heur.". For the convenience of the reader we also calculate the average gaps. The gaps are calculated via

$$Gap = \frac{\text{upper bound} - \text{lower bound}}{\text{upper bound}} \cdot 100,$$

and are given in percent. "$Gap_{Ama.}$" refers to the average gaps of the model in [8] after a time limit of one hour and "$Gap_{(7)}$", "$Gap_{ILP^{pure}}$" and "Gap_{ILP}" to the average gaps of our combinatorial lower bound and of our lower bounding model without and with the use of the combinatorial lower bounds, respectively.

In order to obtain upper bounds we use a heuristic approach similar to the one in [21]. Note that, given the row assignment and the order of the departments in each row, we only need to solve a small linear program to obtain the exact position of the departments and hence possible spaces between neighboring departments, see, e. g., [56]. In [21] five ways for determining a `DRFLP` start solution were presented. For each instance we test all five variants and afterwards apply a 1-OPT and a 2-OPT heuristic to a best start layout, combined with determining the best positions via the associated linear program.

Instance	Amaral	(7)	$ILP_{3\min}$	optimal	heuristic	$Gap_{\text{Ama.}}$	$Gap_{(7)}$	Gap_{ILP}
$HK15$	9059.0	7900.0	11777.0	16570.0	16740.0	45.33	52.32	28.93
$Am14a$	1757.0	1483.5	2082.0	2904.0	2920.0	39.50	48.92	28.31
$Am14b$	1451.5	1386.0	1893.5	2736.0	2736.0	46.95	49.34	30.79
$Am15_1$	1554.5	1543.0	2209.0	3195.0	3272.0	51.35	51.71	30.86
$P16_a$	2247.5	3676.5	4937.0	7365.5	7466.5	69.49	50.08	32.97
$P16_b$	1598.5	2968.0	4053.0	5870.5	6306.0	72.77	49.44	30.96

Table 1: Results for instances from the literature where the optimal solution values are known [29]. The gaps are given in percent.

Instance	Amaral	(7)	$ILP^{pure}_{3\min}$	$ILP_{3\min}$	$ILP_{10\min}$	heuristic	$Gap_{Ama.}$	$Gap_{(7)}$	$Gap_{ILP^{pure}}$	Gap_{ILP}	time heur.
20_{10}	338.79	289.86	468.14	500.60	505.45	617.35	38.13	51.29	19.11	14.21	3.15
20_{50}	1527.12	3202.19	3626.43	5469.71	5495.22	8506.05	81.89	62.45	57.23	35.39	5.21
20_{100}	3310.24	10431.77	8529.66	13986.62	14058.16	20943.10	83.95	50.29	59.15	32.84	7.09
30_{10}	331.84	802.17	1018.18	1514.81	1520.63	2544.90	86.45	68.37	58.93	39.48	15.30
30_{50}	2101.71	9839.95	8233.34	17484.33	17498.97	30996.05	93.22	68.33	73.41	43.56	30.81
30_{100}	4291.18	34059.95	18287.52	43916.72	43931.98	69736.45	93.85	51.20	73.75	37.03	58.69
40_{10}	432.03	1796.76	1904.74	3801.49	3973.69	8007.10	94.42	77.08	75.90	49.43	55.09
40_{50}	3043.79	22713.90	13841.45	42319.98	42332.02	76055.40	95.99	70.16	81.78	44.39	142.88
40_{100}	6090.44	80325.81	29964.70	102936.41	102974.79	167635.30	96.36	52.06	82.11	38.56	280.10
50_{10}	17.82	3184.06	3053.28	6667.47	8725.34	18006.80	99.89	82.23	82.89	51.39	159.35
50_{50}	52.36	42486.84	22313.42	79545.26	79550.13	149788.90	99.97	71.61	85.08	46.84	427.47
50_{100}	63.18	156449.80	48166.27	199949.20	199949.20	328566.70	99.98	52.37	85.32	39.13	1009.80

Table 2: Results for randomly generated double-row instances with integral department lengths between 1 and 15. We display the average values over ten instances each. The average gaps are given in percent. Note that for six instances with $n = 50$ and density 10 % we had to enlarge the time limit to five minutes for ILP^{pure}.

4.1 Results for the DRFLP

In Table 1 we show results for some DRFLP instances from the literature, see, e. g., [29], where optimal solutions are known. Since the instances are rather small, the heuristic needs less than 20 seconds for each instance. The gap of our combinatorial lower bounds (7) are close to 50 % and by our branch-and-cut algorithm we reduce the gap to 28 % to 33 %. The heuristically determined solutions are rather good, but even for these small instances the heuristic could only determine one optimal solution.

Tables 2 and 3 show that our combinatorial lower bounds, which were computed in less than one second, clearly outperform the lower bounds obtained via branch-and-cut algorithm within a time limit of one hour for the DRFLP formulation in [8] on the randomly generated instances. These lower bounds are rather weak and so the gaps are close to 100 % for large n. Using branch-and-cut to improve our bounds allows a significant strengthening to final gaps between 14 and 55 %. For the ILP variant that does not use the combinatorial bounds the gaps are much higher. For instances with at least 40 departments the average gaps are higher than 70 %. Regarding (7) and our ILP the gaps are smaller for dense instances. Enlarging the time limit for our ILP approach from 3 to 10 minutes usually has only a very small effect on the bound. So three minutes seem to be a good value (for the larger instances this is even faster than our heuristic). Comparing the gaps in Tables 2 and 3 the instance type does not seem to have a large impact on the quality of our lower bounding approach, especially for the non-sparse instances.

4.2 Results for the DREFLP

We consider the results of our lower bounding model (13) and (20)–(30) specialized to the DREFLP. In Table 4 we compare our lower bounding model with a time limit of three minutes with an ILP model for the DREFLP (denoted by "$Gap_{Anjos\ ILP\ 3h}$") and an SDP approach for the DREFLP [12] (denoted by "$Gap_{SDP\ 3h}$") with a time limit of three hours. The upper bounds ("best ub")

Instance	Amaral	(7)	$ILP^{pure}_{3\text{min}}$	$ILP_{3\text{min}}$	$ILP_{10\text{min}}$	heuristic	$Gap_{Ama.}$	$Gap_{(7)}$	$Gap_{ILP^{pure}}$	Gap_{ILP}	time heur.
20_{10}	251.69	334.18	402.79	502.13	502.13	674.85	54.01	47.03	33.92	21.12	2.80
20_{50}	1276.73	4015.68	3428.96	6047.42	6054.10	8832.65	85.52	54.61	61.15	31.52	5.08
20_{100}	2424.67	13225.15	8506.21	15833.20	15842.39	22573.75	89.24	41.45	62.31	29.85	7.20
30_{10}	310.58	950.50	890.62	1723.57	1729.66	2905.65	88.49	65.80	67.78	39.01	15.73
30_{50}	1838.96	12560.22	7744.96	21068.41	21070.50	32132.35	94.27	60.95	75.88	34.42	31.74
30_{100}	3663.20	43408.32	18239.38	53278.66	53278.66	78339.90	95.32	44.57	76.71	31.98	50.78
40_{10}	446.25	2072.82	1628.15	4171.48	4176.62	8168.50	94.32	74.07	79.66	50.86	54.25
40_{50}	3039.25	28568.01	13403.33	50194.18	50194.18	79917.90	96.20	64.26	83.22	37.15	135.46
40_{100}	5760.78	101189.12	29338.09	125597.20	125597.20	187431.80	96.93	46.00	84.35	32.98	242.43
50_{10}	11.56	3652.35	2802.76	7695.32	7943.25	17620.70	99.94	79.24	84.04	54.58	176.94
50_{50}	21.36	53907.31	21815.19	96871.04	96871.04	160570.10	99.99	66.42	86.41	39.64	383.10
50_{100}	51.45	194844.10	47001.25	243675.60	243675.60	368747.80	99.99	47.15	87.25	33.90	886.15

Table 3: Results for randomly generated double-row instances with integral department lengths between 5 and 10. We display the average values over ten instances each. The average gaps are given in percent.

Instances	(7) for DREFLP	ILP_{3min}	best ub	$Gap_{(7)}$	Gap_{ILP}	$Gap_{Anjos\ ILP\ 3h}$	$Gap_{SDP\ 3h}$
Y_{20}	4301	5821	6046	28.86	3.72	0.00	0.00
Y_{25}	7032	9887	10170	30.86	2.78	1.22	0.36
Y_{30}	9237	13315	13790	33.02	3.44	2.78	0.14
Y_{35}	12607	18595	19087	33.95	2.58	21.27	0.26
Y_{40}	15332	22809	23739	35.41	3.92	23.88	0.37
Y_{45}	19952	29639	31442	36.54	5.73	26.35	0.65
Y_{50}	25839	39450	41517	37.76	4.98	28.35	0.62

Table 4: Results for equidistant instances from the literature [12, 67]. The upper bounds "best ub" are taken from [12]. We compared our lower bounding model with the ILP and the SDP from [12] with a given time limit of three hours. The value of (7) and ILP_{3min} are rounded to integers.

in Table 4 are taken from [12]. For benchmark instances from the literature, see, e. g., [39, 67], with 20 to 50 departments the gaps of our combinatorial bounds are around 35 % and the gaps of our lower bounding model are between 2.58 % and 5.73 %. While our lower bounding model outperforms the ILP approach of [12] for $n \geq 35$, the SDP approach provides the best lower bounds, but with a higher running time.

Additionally we test the equidistant so called "sko" benchmark instances of [17] with up to 81 departments. As n is rather large we increase the time limit of our lower bounding model to 15 minutes and 60 minutes, respectively. In order to simplify a comparison with the results in Table 4 we used the program and the computer of [12] for the SDP and ILP lower bounds of [12]. The results are shown in Tables 5 and 6. One can see that even for such large instances all gaps of our lower bounding model are less than 13.14 % and usually smaller. Our combinatorial bounds (7), which lie between 53.21 % and 58.06 %, are significantly improved by our ILP. Similarly to the Y-instances the SDP bounds are better.

Instances	(7) for DREFLP	ILP_{15min}	best ub	$Gap_{(7)}$	Gap_{ILP}	$Gap_{SDP\ 3h}$
sko42-1	5957	11717	12731	53.21	7.96	0.72
sko49-1	9142	18736	20512	55.43	8.66	1.91
sko56-1	13942	29201	31988	56.41	8.71	1.95
sko64-1	20705	43408	48574	57.37	10.64	4.27

Table 5: Results for equidistant instances from the literature [17]. We compare to the best upper bounds, gaps and SDP lower bounds that are derived using the approach presented in [12] with a time limit of three hours.

Instances	(7) for DREFLP	ILP_{1h}	best ub	$Gap_{(7)}$	Gap_{ILP}	$Gap_{SDP\ 3h}$
sko72-1	29912	61905	69621	57.04	11.08	4.87
sko81-1	43114	89288	102793	58.06	13.14	8.78

Table 6: Results for equidistant instances from the literature [17]. We compare to the best upper bounds, gaps and SDP lower bounds that are derived using the approach presented in [12] with a time limit of three hours. Because of the large n we use a time limit of one hour for our ILP lower bounding approach.

Comparing the results of the standard DRFLP and the DREFLP, one can see that the gaps of our lower bounding model are much better in the equidistant case. One reason for this behavior is that in the equidistant case we have a lower bound for the sum of the distances (25).

4.3 Results for the PROP

For the PROP we test the same instances as in [29]. The optimal solution values of all the instances are given in [29]. Looking at the strength of our lower bounding approach adapted to the PROP one sees in Table 7 that the average gaps are even better than for the DRFLP. Most often they are less than 30 % and for the balanced instances with (approximately) half of the departments in each row the average gaps are at most 16 %. So the approach seems to work well for balanced instances. One reason for the worse performance on unbalanced instances is that the lower bound calculation cannot take into account that some of the departments in a longer row will not overlap with some of the departments in the other row at all. For this remember that the left border of the layout is fixed and free spaces are not allowed between departments in the same row.

5 Conclusion

The Double-Row Facility Layout Problem (DRFLP) is a very challenging problem with various application areas, including factory planning. Despite its broad applicability it can only be solved to optimality in reasonable time for rather small instances. Apart from this, using the integer-programming based solution approaches from the literature one derives even with high running times very large gaps for large instances. So heuristics are currently the only way to determine solutions for larger instances. In order to evaluate the quality of heuristically determined solutions, we developed in this paper combinatorial lower bounds for the optimal solution value of the DRFLP. Indeed, interpreting some subproblem of the DRFLP as a parallel identical machine scheduling problem we computed the first known non-trivial combinatorial lower bounds for the DRFLP. Furthermore we combined these bounds with a new mixed-integer linear programming model, which is indeed not a formulation for the DRFLP, to obtain even better lower bounds. Only few heuristics are present in the literature for the standard DRFLP. We compare our lower bounds to upper bounds derived by some construction heuristic presented in [21] which is combined with a 1-opt and a 2-opt improvement heuristic. Our computational results show that we were able to obtain non-trivial lower bounds for large double-row instances. We received average gaps of 32 % to 46 % for large dense instances and of about 50 to 55 % for large sparse instances using our lower bounding approach. Note that the pure combinatorial bounds, which can be determined very fast, could usually be strengthened significantly, but with average gaps from 40 % to 80 % they are still much better than the model [8] from the literature after a time limit of one hour for larger instances. Additionally, we showed how our bounds can be specialized to the equidistant DRFLP and can be extended to the (k-) parallel row ordering problem. In both cases the gaps are better than for the DRFLP.

As already mentioned, only few heuristic approaches are known for the DRFLP in the literature.

Instance	i	(7) for `PROP`	$ILP_{10\text{sec}}$	optimal	Gap_{ILP}
P16	2	3756.80	6554.99	6934.75	5.46
P16	3	4598.28	7298.76	9452.25	22.99
P16	4	5028.68	8103.06	10522.75	23.33
P16	5	5350.14	8797.24	11767.75	25.44
P20	2	6587.88	11322.10	12772.75	11.36
P20	3	8455.90	12799.60	17520.75	26.69
P20	4	9184.90	14052.45	20493.25	31.12
P20	5	9815.33	15375.70	22558.75	31.76
P21	2	5722.76	9870.59	11109.40	10.98
P21	3	6792.91	11066.58	13297.20	16.64
P21	4	7887.36	13195.82	17310.80	23.63
P21	5	8203.20	13665.52	19159.40	28.33
P22	2	7203.15	12479.41	14090.30	11.48
P22	3	8729.44	14212.50	18295.50	22.24
P22	4	9941.98	16823.44	22567.10	25.48
P22	5	10232.11	17523.40	24132.10	27.25
P23	2	7590.92	13483.86	15048.40	10.52
P23	3	9604.89	15550.08	20248.40	23.19
P23	4	10777.65	18001.96	25070.60	28.11
P23	5	11171.86	19012.92	27313.80	30.13
P24	2	8604.50	15281.90	17563.20	13.03
P24	3	9808.99	16746.94	20632.00	18.59
P24	4	11445.04	19266.70	25632.40	24.58
P24	5	12543.23	21904.52	30316.80	27.45
P25	2	9507.26	17015.60	19393.30	12.40
P25	3	11746.93	19965.92	25477.90	21.32
P25	4	13478.91	23248.80	31095.10	25.13
P25	5	14043.55	24771.06	33228.50	25.34
AV25	2	6227.73	11166.12	13394.00	15.44
AV25	3	7635.86	13241.86	17278.20	22.94
AV25	4	8223.55	14455.58	19577.80	25.60
AV25	5	8868.04	15681.86	21808.00	27.26

Table 7: Results for `PROP` instances from the literature [9, 29]: The instance name includes the number of departments n. The first $\lfloor \frac{n}{i} \rfloor$ departments of an instance are assigned to row 1. The table shows the average values of our lower bounds using the variant of (7) adapted for the `PROP` and using our lower bounding approach ($ILP_{10\text{sec}}$), average optimal solution values ("optimal") and average gaps ("Gap_{ILP}") over five instances each (and over two instances for $n = 16$ and $n = 20$). The gaps are given in percent.

Now, with this new possibility for evaluation, it remains for future work to construct new heuristic approaches. Another interesting research question is to exploit which is the best way to extend our lower bounding model to a DRFLP formulation. For good solution times the development of sophisticated branching strategies in a branch-and-cut algorithm might be important.

From a practical point of view, it is important to extend the Single-Row Facility Layout Problem and the DRFLP in order to handle more characteristics important in practice. For instance, the standard models do not allow for individual input and output positions of the departments and certain clearance conditions. Additionally, it remains for future work to investigate more complex path structures in the shape of a U, a T or an X. Here the lower-bounding approaches developed in the current paper might help in determining non-trivial lower bounds as well.

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Appendix

For the PROP with given row assignment r and sets $R_i = \{j \in [n] \colon r_j = i\}, i = 1, 2$, we use the following lower bounding model with $M = \max\{\sum_{z \in R_1} \ell_z, \sum_{z \in R_2} \ell_z\}$.

$$\min \sum_{\substack{i,j \in [n] \\ i<j}} w_{ij} d_{ij}$$

$$\sum_{\substack{i \in S \cap R_1 \\ j \in S \cap R_2}} x_{ij} \leq |S| - 1, \qquad S \subseteq [n], |S| \geq 2,$$

$$\sum_{j \in S \cup T} x_{ij} \leq |S| + 1, \qquad i \in [n], S \subset \{j \in [n] \colon r_j \neq r_i\} \text{ with } \sum_{j \in S} \ell_j \geq \ell_i,$$

$$T := \{j \in \{k \in [n] \colon r_k \neq r_i\} \colon \ell_j \geq \max_{k \in S} \ell_k\},$$

$$d_{ij} + \left(\tfrac{\ell_i + \ell_j}{2}\right) x_{ij} \geq \tfrac{\ell_i + \ell_j}{2}, \qquad i \in R_1, j \in R_2,$$

$$d_{ij} + \left(M - \ell_i - \ell_j + \tfrac{1}{2}\iota\right) x_{ij} \leq M - \tfrac{\ell_i + \ell_j}{2}, \qquad i \in R_1, j \in R_2,$$

$$d_{ij} + d_{jk} - d_{ik} \geq 0, \qquad i, j, k \in [n], |\{i, j, k\}| = 3, i < k,$$

$$\sum_{j \in S} d_{ij} \geq \min_{j \in \{k \in [n] \colon r_k \neq r_i\}} W^{\text{P-dst}}_{(i,j)}(S), \qquad i \in [n], S \subseteq [n] \setminus \{i\},$$

$$\sum_{j \in S} w_{ij} d_{ij} \geq \min_{j \in \{k \in [n] \colon r_k \neq r_i\}} \max \left\{ \begin{array}{c} W^{\text{P-spt}}_{(i,j)}(S), \\ W^{\text{P-sc}}_{(i,j)}(S) \end{array} \right\}, \qquad i \in [n], S \subseteq V_i,$$

$$\sum_{\substack{i,j \in S \\ i<j}} d_{ij} + o \sum_{\substack{i \in S \cap R_1 \\ j \in S \cap R_2}} x_{ij} \geq o, \qquad S \subset [n], |S| = 3, o = \sum_{i \in S} \ell_i + \min_{i \in S} \ell_i,$$

$$\sum_{\substack{i,j \in S \\ i<j}} d_{ij} + o \sum_{\substack{i \in S \cap R_1 \\ j \in S \cap R_2}} x_{ij} \geq o, \qquad S \subset [n], |S| = 4,$$

$$o = \tfrac{3}{2} \sum_{i \in S} \ell_i + 2 \cdot \min_{\substack{i_1, i_2 \in S \\ i_1 \neq i_2}} (\ell_{i_1} + \ell_{i_2}),$$

$$\sum_{\substack{i,j \in S \\ i<j}} \ell_i \ell_j d_{ij} + o \sum_{\substack{i \in S \cap R_1 \\ j \in S \cap R_2}} x_{ij} \geq o, \qquad S \subseteq [n], |S| \geq 3,$$

$$o = \tfrac{1}{6} \left(\left(\sum_{i \in S} \ell_i \right)^3 - \sum_{i \in S} \ell_i^3 \right),$$

$$\sum_{\substack{i,j \in R_k \\ i<j}} \ell_i \ell_j d_{ij} = \frac{1}{6} \left((\sum_{i \in R_k} \ell_i)^3 - \sum_{i \in R_k} \ell_i^3 \right), \qquad k \in \mathfrak{R},$$

$$\sum_{\substack{i,j \in R_k \\ i<j}} d_{ij} \geq \sum_{\substack{i,j \in R_k \\ i<j}} \frac{\ell_i + \ell_j}{2} + \sum_{z=2}^{|R_k|-1} (z-1)(|R_k| - z) \ell_{\pi^{*,k}(z)}, \qquad k \in \mathfrak{R},$$

$$d_{ij} \geq \frac{\ell_i + \ell_j}{2}, \qquad k \in \mathfrak{R}, i, j \in R_k, i < j,$$

$$d_{ij} \leq \left(\sum_{z \in R_k} \ell_z \right) - \frac{\ell_i + \ell_j}{2}, \qquad k \in \mathfrak{R}, i, j \in R_k, i < j,$$

$$d_{ij} \leq M - \frac{\ell_i + \ell_j}{2}, \qquad i \in R_1, j \in R_2,$$

$$x_{ij} = x_{ji} \in \{0, 1\}, \qquad i \in R_1, j \in R_2,$$

$$d_{ij} = d_{ji} \geq 0, \qquad i, j \in [n], i < j.$$