Globalized Robust Optimization with Gamma-Uncertainties*

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Abstract

Globalized robust optimization has been proposed as a generalization of the standard robust optimization framework in order to allow for a controlled decrease in protection depending on the distance of the realized scenario from the predefined uncertainty set.

In this work, we specialize the notion of globalized robustness to Γ-uncertainty in order to extend its usability for discrete optimization. We show that in this case, the generalized robust counterpart possesses algorithmically tractable reformulations for mixed-integer linear nominal problems. Under mild assumptions, we derive a reformulation which uses only slightly more variables and constraints than the standard robust counterpart under Γ-uncertainty. For combinatorial problems, our globalized robust counterpart remains fixed-parameter tractable, although with a runtime exponential in Γ. Furthermore, we show that globalized robust optimization under scaling of the Γ-uncertainty-sets is NP-hard already in simple cases. We support our theoretical findings by experimental results on the globalized robust versions of the minimum-spanning-tree, shortest-path and knapsack problems. It turns out that our algorithmically tractable reformulations are not more difficult to

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solve than the respective standard robust counterparts while globalized robustness is guaranteed. Moreover, they produce solutions which offer suitable protection against uncertainty while performing very well in the nominal objective function.

Keywords: Globalized Robust Optimization, Gamma-Uncertainties, Mixed-Integer Programming, Combinatorial Optimization

Mathematics Subject Classification: 90C31 - 90C57 - 90C11 - 90C27

1 Motivation

In standard robust optimization, optima are determined among solutions that are feasible for all data realizations within some predefined uncertainty set. For example, given a linear constraint $a^T x \leq b$ with an uncertain left-hand side $a \in \mathbb{R}^n$, strict robust optimization would demand that this constraint be fulfilled for all $a \in S$, where $S \subseteq \mathbb{R}^n$ is the corresponding uncertainty set. Whereas robust solutions are fully protected against data realizations within the predefined uncertainty set, no guarantee for feasibility or maximum degree of violation whatsoever can be given if the uncertain data realizes outside these sets. However, it is desirable to obtain robust solutions which exhibit a controlled decay in solution quality depending on the distance of the actual realization of the uncertain data from the uncertainty set. This is where the paradigm of globalized robust optimization comes into play.

Globalized robustness has first been proposed by Ben-Tal et al. (2006), then under the name of "comprehensive robustness", later renamed to *globalized robust optimization* in Ben-Tal et al. (2009), and has recently been taken up by Ben-Tal et al. (2017). The common idea was the introduction of a distance measure $\phi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ which is closed, jointly convex in both arguments and fulfils $\phi(a,a) = 0$ for all $a \in \mathbb{R}^n$. It is considered together with the "actual", "inner" or "core" uncertainty set S_1, i.e. a set of data realizations expected by the planner, and its superset $S_2 \supseteq S_1$, which is the "outer" uncertainty set, representing the set of all "physically" possible realizations of the data. Both inner and outer uncertainty set are assumed to be convex. The globalized robust version of the uncertain constraint $a^T x \leq b$ from above is

$$a^T x \leq b + \min_{a' \in S_1} \phi(a, a') \quad (\forall a \in S_2). \tag{1}$$

Clearly, the minimum on the right-hand size of Constraint (1) is 0 whenever $a \in S_1$ holds. For $a \in S_2 \setminus S_1$, the right-hand size increases gradually with a moving further away from S_1 in the distance measure given by ϕ – thus allowing a higher violation of the constraint the higher the distance between a and S_1 becomes. A typical choice for the distance measure is $\phi(a, a') = \alpha(\|a - a'\|)$ for a convex, non-negative function α with $\alpha(0) = 0$ and some norm $\|\cdot\|$ on $\mathbb{R}^n$.

The original globalized robust counterpart by Ben-Tal et al. (2006, 2009) was limited to outer uncertainty sets which could be represented as the inner uncertainty set plus a suitable cone, distance measures of the form $\phi(a, a') = \lambda \|a - a'\|$ for some $\lambda \in \mathbb{R}_+$ and linear uncertain constraints which depend linearly on the uncertain data. These restrictions could be lifted by the redefinition of the globalized robust counterpart by Ben-Tal et al. (2017), who also derive tractable reformulations of the resulting models for special cases. Altogether, this greatly widens the applicability of the globalized robust approach in the realm of continuous non-linear optimization.

For applications in the context of robust discrete and combinatorial optimization, an important questions is if the concept of globalized robustness can suitably be used in conjunction with Γ-uncertainty-sets, which are very popular in this field. These polyhedral uncertainty sets were introduced by Bertsimas and Sim (2004) to allow for tractable reformulations of linear and mixed-integer linear optimization problems. They characterize scenarios for realizations of uncertain data by the number of deviating coefficients, defining an uncertainty set by specifying a maximum number of deviations Γ. If the globalized robust variant of a linear objective or constraint is to remain linear for inner and outer Γ-uncertainty-sets S_1 and S_2, an obvious choice for the distance measure in the framework by Ben-Tal et al. (2017) is $\phi(a, a') = \lambda \|a - a'\|_p$, with $\lambda \in \mathbb{R}_+$, $p \in \{1, \infty\}$, which directly allows for linear reformulation using standard duality arguments. In the case $p = 1$, ϕ counts the number of deviating coefficients with respect to some nominal scenario $\underline{a}$, which is exactly what we do when using Γ-uncertainty. Now, it is desirable to generalize this distance measure to $\phi(a, a') = \alpha(\|a - a'\|_1)$ for some function α with the same properties as above while still maintaining the linearity of the resulting robust counterpart. The derivations in Ben-Tal et al. (2017), however, lead to non-linear robust counterparts. This is the motivation behind the present study of a dedicated framework for globalized robustness using Γ-uncertainty-sets.

For the ease of notation, we replace the distance measure ϕ by a so-called *uncertainty reserve* $f\colon S_2 \to \mathbb{R}_+$. The function f is shall be zero for all $a \in S_1$, such that strict robustness is ensured for all these scenarios. A globally robustified constraint then reads as follows

$$a^T x \leq b + f(a) \quad (\forall a \in S_2).$$

We interpret the value $f(a)$ as an additional reserve or funds for the case of a very adverse scenario realizing. It can also be seen as a punishment for the adversary determining the data realization for picking overly adverse scenarios. The framework by Ben-Tal et al. (2017) is now recovered by choosing $f(a) = \min_{a' \in S_1} \phi(a', a)$. In this paper, we mainly study uncertainty reserves of the form $f(a) = \alpha(\gamma(a))$, where $\gamma(a)$ denotes the number of deviating coefficients with respect to some given nominal scenario. For this case, we present efficient reformulations of the resulting globalized robust counterpart which will indeed retain the linearity of the nominal problem.

For a combinatorial nominal problem, we show that it is fixed-parameter tractable, although with an exponential runtime in Γ in general. In the case of a linear uncertainty reserve $f(a) = \lambda\gamma(a)$ for some $\lambda \in \mathbb{R}_+$, however, our framework reduces to an ordinary Γ-robust counterpart as studied by Bertsimas and Sim (2004), which allows for the use of the linear-time fixed-parameter approach studied in Bertsimas and Sim (2003). For an uncertainty reserve based on the factor by which the inner uncertainty set is scaled up to measure the distance to the realized scenario, we present complexity results. Indeed, the resulting globalized robust counterpart is NP-hard already for common polynomial-time solvable combinatorial problems under simplifying assumptions for the allowed deviations of constraint or cost coefficients.

This paper is rounded off by a broad computational study on several combinatorial optimization problems, examining the solution time performance of our reformulations as well as the solution quality. compared to the classical robust counterpart.

Our work is to be distinguished from that presented in Bienstock (2007) and Büsing and D'Andreagiovanni (2012) who give another generalization of Γ-uncertainty, the so-called *multi-band uncertainty*. The latter allows for the incorporation of an empirical distribution into the uncertainty set by prescribing a separate value Γ_i for each *band*, i.e. range, of parameter deviation, demanding that $\Gamma = \sum_i \Gamma_i$ hold.

The overall idea of specializing certain robust paradigms to the case of Γ-uncertainty in order to obtain stronger results with a higher applicability in discrete problem settings is also present, for example, in Buchheim and Kurtz (2017) who develop a discrete analogue of adaptive robustness. It allows to choose not only one, but multiple robust solutions, from which final solution can be selected after the uncertainty has realized. This idea has been further specialized by Chassein (2017) with yet stronger results when exactly two such solutions may be specified. Another example of this principle is the work of Gottschalk et al. (2018) who consider uncertain flows over time based on Γ-robustness. For a comprehensive summary of the current state-of-the-art and best practices in robust optimization, we refer to the surveys by Gorissen et al. (2015) and Sözüer and Thiele (2016).

The remainder of this paper is structured as follows. Section 2 introduces the concept of globalized Γ-robustness. In Section 3, we investigate uncertainty reserves measuring the number of deviating coefficients. We give efficient reformulations and establish a fixed-parameter tractability result. Section 4 is then devoted to a reformulation and NP-hardness results for uncertainty reserves measuring the metric distance of a given scenario to the inner Γ-uncertainty set. Section 5 presents a detailed computational study involving the reformulations from Section 3, comparing them in terms of computation time and protection against uncertainty and offsetting them against the ordinary Γ-robust counterpart. Finally, in Section 6 we give our conclusions on the practicability and protective power of the globalized Γ-robust counterpart.

2 The Concept of Globalized Γ-Robustness

Let us consider a mixed-integer linear optimization problem (MIP) in n variables and m constraints of the form

$$\begin{aligned} \min \quad & c^T x && \text{(2a)} \\ \text{s.t.} \quad & Ax \leq b && \text{(2b)} \\ & x \in \mathbb{Z}^{n-p} \times \mathbb{R}^p, && \text{(2c)} \end{aligned}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $0 \leq p \leq n$. Throughout this paper, we will assume that the cost vector c is affected by data uncertainty, while the constraint matrix A and the right-hand side b are not. Results obtained in this setting can easily be extended to uncertain constraints, see Remark 2.1. A realization of the uncertain cost vector c is called a *scenario*. Let the *scenario* or *uncertainty set* $\mathcal{S}$ denote the index set of all possible realizations of the uncertain cost.

In any given scenario $S \in \mathcal{S}$, the actual cost of a solution x is $c^S(x) := \sum_{i=1}^n c_i^S x_i$ with $c^S \in \mathbb{R}^n$. We call x a *robust optimal solution* if it solves the so-called *(strict) robust counterpart* that is given by the optimization problem

$$\begin{aligned} \min \quad & \max_{S \in \mathcal{S}} c^S(x) && \text{(3a)} \\ \text{s.t.} \quad & Ax \leq b && \text{(3b)} \\ & x \in \mathbb{Z}^{n-p} \times \mathbb{R}^p. && \text{(3c)} \end{aligned}$$

The Γ-model for data uncertainty by Bertsimas and Sim (2004) now posits that the cost coefficient c_i^S of any variable x_i is from the interval $[\underline{c}_i, \underline{c}_i + \hat{c}_i]$, where $\underline{c}_i$ is the expected cost coefficient and $\hat{c}_i$ is its maximal expected upwards deviation. They introduced a parameter $0 \leq \Gamma \leq n$ which can be chosen by the planner to decide against how many deviating objective coefficients a solution to Problem (3) should be protected. Let

$$\gamma(S) := |\{i \in \{1, \dots, n\} \mid c_i^S > \underline{c}_i\}|$$

denote the number of deviating coefficients in a scenario S. The set of considered scenarios is then given by

$$\mathcal{S}_\Gamma := \{S \in \mathcal{S} \mid c_i^S \in [\underline{c}_i, \underline{c}_i + \hat{c}_i], \gamma(S) \leq \Gamma\}.$$

We call a solution x that is optimal for the robust counterpart (3) with scenario set $\mathcal{S}^\Gamma$ *strictly* Γ*-robust optimal*. In Bertsimas and Sim (2004) and in somewhat more detail in Bertsimas and Sim (2003), the authors show that this choice of scenario set allows a compact reformulation of the robust counterpart as a mixed-integer linear program. Their reformulation of the strict robust counterpart (3) takes the following form:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} \underline{c}_i x_i + \Gamma u + \sum_{i=1}^{n} v_i && && \text{(4a)}\\
\text{s.t.} \quad u + v_i &\geq \hat{c}_i x_i && (\forall i = 1, \dots, n) && \text{(4b)}\\
u &\in \mathbb{R}_+ && && \text{(4c)}\\
v_i &\in \mathbb{R}_+ && (\forall i = 1, \dots, n) && \text{(4d)}\\
Ax &\leq b && && \text{(4e)}\\
x &\in \mathbb{Z}^{n-p} \times \mathbb{R}^p. && && \text{(4f)}
\end{aligned}$$

Furthermore, they show that combinatorial algorithms for a nominal combinatorial optimization problem can be used to solve the corresponding Γ-robust counterpart. Especially, combinatorial problems in the complexity class P remain in this class after Γ-robustification. These are major reasons why this approach has become widely popular. However, this approach as well as all strict robust counterparts, e.g. those based on ellipsoidal uncertainty sets due to Ben-Tal and Nemirovski (2000), possess a serious drawback when it comes to scenarios that lie outside the prescribed uncertainty set. In case a scenario realizes in which there are more deviating coefficients than assumed or higher deviations than assumed, the solution might be vastly suboptimal without any guarantee of quality whatsoever.

2.1 The New Globalized Robust Concept

In the following, we introduce a concept for globalized robustness with an underlying idea that is similar to those of Ben-Tal et al. (2006) and Ben-Tal et al. (2017). It allows to completely protect the computed solution against scenarios within a given core uncertainty set $\mathcal{S}' \subseteq \mathcal{S}$ while enforcing guarantees on the behaviour under scenarios from the larger uncertainty set $\mathcal{S}$. In Ben-Tal et al. (2006) and Ben-Tal et al. (2017), this is realized via the introduction of a function $\phi \colon \mathcal{S} \times \mathcal{S} \to \mathbb{R}_+$ which measures the distance between any two scenarios from $\mathcal{S}$. Here, we work with a function $f \colon \mathcal{S} \to \mathbb{R}_+$, the so-called *uncertainty reserve*, which models the acceptable additional costs for each scenario $S \in \mathcal{S}$ in the case of its realization. The approach of Ben-Tal et al. (2006) and Ben-Tal et al. (2017) is retrieved by setting $f(S) := \min_{S' \in \mathcal{S}'} \phi(S, S')$ for all $S \in \mathcal{S}$. Independent of the particular choice of f, the cost of a solution x which is to be minimized is then given by

$$\text{grob}(x) := \max_{S \in \mathcal{S}} \{c^S(x) - f(S)\}.$$

With this definition, the globalized robust counterpart of an uncertain optimization problem with feasible region $X \subseteq \mathbb{R}^n$ reads

$$\min \quad \text{grob}(x) = \max_{S \in \mathcal{S}} \{c^S(x) - f(S)\} \tag{5a}$$
$$\text{s.t.} \quad x \in X. \tag{5b}$$

An optimal solution to Problem (5) is called *globally robust optimal*. In this paper, we will concentrate on on scenario sets S based on Γ-uncertainty, in which case we call such a solution *globally Γ-robust optimal*.

The uncertainty reserve f allows to assign different weights to the scenarios included in the optimization. In the case of a likely scenario in which the costs hardly deviate from their expected realization, we might choose a smaller weight. In the more unlikely case that they deviate strongly, we expect a deterioriation of solution quality anyway. A suitable choice of f will help to keep these effects in control, as it enables us to tolerate more deterioriation the higher the deviations in the costs are.

The framework we propose can be seen as a compromise between the nominal optimization problem and the strict robust counterpart which protects against all scenarios in $\mathcal{S}$. In fact, it includes these two problems as special cases. If S_N denotes the nominal scenario, then choosing

$$f(S) := \begin{cases} 0, & \text{if } S = S_N \\ \infty, & \text{otherwise} \end{cases}$$

amounts to solving the nominal problem. On the other hand, if $\mathcal{S}' \subseteq \mathcal{S}$ is the subset against which we want to have full protection, this can be achieved by setting

$$f(S) := \begin{cases} 0, & \text{if } S \in \mathcal{S}' \\ \infty, & \text{otherwise.} \end{cases}$$

Scenarios from $\mathcal{S} \setminus \mathcal{S}'$ are then disregarded in the resulting solution. The compromise between these two extremes can be realized by choosing f equal to 0 within some core uncertainty set $\mathcal{S}'$ and having it increase in a monotonous and convex fashion outside of $\mathcal{S}'$ when passing to scenarios with more or higher deviations. We remark that our framework also includes the case of regret-robust optimization which strives to minimize the highest loss that can occur under any scenario by not choosing the solution that would have been optimal under this scenario (see Kouvelis and Yu (1997) for a comprehensive treatment as well as Aissi et al. (2009) for an extensive survey). This goal can be achieved by choosing $f(S) := \min_{x \in X} c^S(x)$ for each scenario S that is to be considered in the objective function. A globally robust optimal solution then coincides with a *regret-robust optimal* solution

$$\bar{x} = \underset{x \in X}{\text{argmin}} \left\{ \max_{S \in \mathcal{S}} \{c^S(x) - \min_{x' \in X} c^S(x')\} \right\}.$$

Remark 2.1. *We remark that the concept of globalized robustness as defined here for uncertain objective functions readily extends to uncertain constraints. Let $\mathcal{S}$ denote the index set of all possible realizations of the left-hand side of an uncertain constraint $a^T x \leq b$. We then introduce a corresponding uncertainty reserve $f\colon \mathcal{S} \to \mathbb{R}_+$ as before and define the new robustified version of the constraint by*

$$\operatorname{grob}(x) := \max_{S \in \mathcal{S}} \{a^S(x) - f(S)\} \leq b,$$

where $a^S(x) = \sum_{i=1}^n a_i^S x_i$. This extension of the concept to uncertain constraints can be used for tolerating a higher violation in the case of a more unlikely scenario, for example.

2.2 Controlling the Number and the Magnitude of the Deviations

In addition to the parameter Γ for the maximal number of deviating coefficients in any scenario, we will use a second parameter $\delta \leq 0$ to control the magnitude of these deviations against which a solution shall be protected. This means we consider scenario sets $\mathcal{S}_\Gamma(\delta)$ given by

$$\mathcal{S}_\Gamma(\delta) := \{S \mid c_i^S \in [\underline{c}_i, \underline{c}_i + \delta \hat{c}_i], \gamma(S) \leq \Gamma\},$$

which gives the flexibility to have the uncertainty reserve f depend on either Γ or δ. Observe that from the definition it follows that $\mathcal{S}_{\Gamma'}(\delta) \subseteq \mathcal{S}_\Gamma(\delta)$ for $\Gamma' \leq \Gamma$ and $\mathcal{S}_\Gamma(\delta') \subseteq \mathcal{S}_\Gamma(\delta)$ for $\delta' \leq \delta$.

In the next two sections, we will consider two different cases for the choice of f. In the first case, we will fix a value of δ, w.l.o.g. $\delta = 1$. Furthermore, f will only depend on the number of deviating coefficients in a given scenario, which means $f(S) = f(S')$ whenever $\gamma(S) = \gamma(S')$. We will then use the simplified notation $f(\Gamma) := f(S)$ if $\gamma(S) = \Gamma$ and write $S_\Gamma(1) := S_\Gamma$. In the second case, we will fix a value for Γ and consider the case of lower and upper bounds $\underline{\delta}, \bar{\delta} \in \mathbb{R}_+$ such that $\delta \in [\underline{\delta}, \bar{\delta}]$. Defining $S_\Gamma([\underline{\delta}, \bar{\delta}]) := \bigcup_{\delta \in [\underline{\delta}, \bar{\delta}]} S_\Gamma(\delta)$, we consider an uncertainty reserve $f\colon S_\Gamma([\underline{\delta}, \bar{\delta}]) \to \mathbb{R}$ which only depends on the magnitude δ.

For both cases, we will show how efficient reformulations can be achieved to equivalently rewrite the globalized Γ-robust counterpart as an MIP.

3 Uncertainty Reserves Depending only on Γ

In this section, we consider globalized robustness under fixed maximal deviations $\hat{c}_i$ for all $i = 1, \dots, n$ and an uncertainty reserve f that is a function of the number of deviating coefficients. More precisely, we will consider a subset $\mathit{\Gamma} \subseteq \{0, 1, \dots, n\}$ of possible values for the parameter $\Gamma \in \mathit{\Gamma}$ that limits the number of deviating coefficients a robust solution takes into account. Consequently, we will only consider scenarios S with $\gamma(S) \in \mathit{\Gamma}$.

Let $\mathit{\Gamma} := \{\Gamma_1, \dots, \Gamma_k\} \subseteq \{0, 1, \dots, n\}$ be a discrete set with $\Gamma_1 \leq \dots \leq \Gamma_k$ for some $k \geq 1$. Using the notation $\bar{\Gamma} := \Gamma_k$, we consider $\mathcal{S}_{\bar{\Gamma}}$ as the set of all scenarios that could possibly materialize. For each $\Gamma \in \mathit{\Gamma}$ and the corresponding subsets $\mathcal{S}_\Gamma$ of these scenarios, the inclusion $\mathcal{S}_{\Gamma_1} \subseteq \dots \subseteq \mathcal{S}_{\Gamma_k} = \mathcal{S}_{\bar{\Gamma}}$ holds. We are now interested in possible reformulations of the globalized Γ-robust counterpart (5), which in this special case takes the form

$$\min_{x\in X} \operatorname{grob}(x) = \min_{x\in X} \left\{ \max_{\substack{S\in\mathcal{S}_{\bar{\Gamma}}:\\ \gamma(S)\in\Gamma}} \{c^S(x) - f(S)\} \right\}. \tag{6}$$

Note that the condition $\gamma(S) \in \mathit{\Gamma}$ in Problem (6) for the choice of scenarios S is necessary for a correct modelling of the globalized robust counterpart. Omitting it would allow the choice of scenarios with a number of deviating coefficients outside of $\mathit{\Gamma}$, which can make a difference depending on the uncertainty reserve f. If we were to restrict ourselves to uncertainty reserves whose value $f(\Gamma)$ for a $\Gamma \notin \mathit{\Gamma}$ is always equal to $f(\Gamma')$ for the next-highest $\Gamma' \in \mathit{\Gamma}$ if there is one, and equal to ∞ otherwise, for example, this adition could be left out. In this case, scenarios outside of $\mathcal{S}_\Gamma$ are always dominated by scenarios inside $\mathcal{S}_\Gamma$ in the inner optimization. For our first reformulation, however, we will not impose this condition on f.

The following lemma is a step towards a compact reformulation of Problem (6) in the case that f only depends on the number of deviating coefficients.

Lemma 3.1. *Let f be chosen such that $f(S) = f(S')$ for all $S, S' \in \mathcal{S}_{\bar{\Gamma}}$ with $\gamma(S) = \gamma(S')$. Then the worst-case cost of any feasible solution $x \in X$ is given by*

$$\operatorname{grob}(x) = \max_{\substack{S\in\mathcal{S}_{\bar{\Gamma}}:\\ \gamma(S)\in\Gamma}} \{c^S(x) - f(S)\} = \max_{\Gamma\in\mathit{\Gamma}} \left\{ \max_{S\in\mathcal{S}_\Gamma} \{c^S(x) - f(\Gamma)\} \right\}.$$

Proof. With f chosen as assumed, we can sort the scenarios in $\mathcal{S}_{\bar{\Gamma}}$ by the number of deviating coefficients, which yields

$$\max_{\substack{S\in\mathcal{S}_{\bar{\Gamma}}:\\ \gamma(S)\in\Gamma}} \{c^S(x) - f(S)\} = \max \left\{ \max_{\substack{S\in\mathcal{S}_{\bar{\Gamma}}:\\ \gamma(S)=\Gamma_1}} \{c^S(x) - f(\Gamma_1)\}, \dots, \max_{\substack{S\in\mathcal{S}_{\bar{\Gamma}}:\\ \gamma(S)=\Gamma_k}} \{c^S(x) - f(\Gamma_k)\} \right\}.$$

As the scenarios $S \in \mathcal{S}_{\bar{\Gamma}}$ with $\gamma(S) = \Gamma_j$, $j = 1, \dots, k$, are exactly those from $\mathcal{S}_{\Gamma_j}$, this can be rewritten to

$$\max_{j=1,\dots,k} \left\{ \max_{\substack{S\in\mathcal{S}_{\Gamma_j}:\\ \gamma(S)=\Gamma_j}} \{c^S(x) - f(\Gamma_j)\} \right\}.$$

As a last step, we can now leave out the addition $\gamma(S) = \Gamma_j$, as the inner maximization and the constant uncertainty reserve for each j will ensure the choice of a scenario with the maximum possible number of deviations for the optimal j. Thus, we reformulate the above as

$$\max_{\Gamma\in\mathit{\Gamma}} \left\{ \max_{S\in\mathcal{S}_\Gamma} \{c^S(x) - f(\Gamma)\} \right\},$$

which proves the claim. □

This result will be the basis for the reformulations derived in the remainder of this section.

3.1 A First Algorithmically Tractable Reformulation

In general, the globalized Γ-robust counterpart is a nonlinear optimization problem even if the nominal problem is linear. However, considering only a finite number of possible Γ-values and having f only depend on Γ allows us to obtain an equivalent reformulation as a linear optimization problem.

Theorem 3.2. *Let f be chosen such that $f(S) = f(S')$ for all $S, S' \in \mathcal{S}_{\bar{\Gamma}}$ with $\gamma(S) = \gamma(S')$. Then the globalized Γ-robust counterpart*

$$\min_{x \in X} \operatorname{grob}(x) = \min_{x \in X} \max_{\Gamma \in \mathit{\Gamma}} \left\{ \max_{S \in \mathcal{S}_\Gamma} \{c^S(x) - f(\Gamma)\} \right\}$$

can be reformulated as

$$\begin{aligned}
\min \quad & \omega && && (7a)\\
\text{s.t.} \quad & \sum_{i=1}^n \underline{c}_i x_i + \Gamma u^\Gamma + \sum_{i=1}^n v_i^\Gamma - f(\Gamma) \le \omega && (\forall \Gamma \in \mathit{\Gamma}) && (7b)\\
& u^\Gamma + v_i^\Gamma \ge \hat{c}_i x_i && (\forall \Gamma \in \mathit{\Gamma})(\forall i = 1, \dots, n) && (7c)\\
& u^\Gamma \ge 0 && (\forall \Gamma \in \mathit{\Gamma}) && (7d)\\
& v_i^\Gamma \ge 0 && (\forall \Gamma \in \mathit{\Gamma})(\forall i = 1, \dots, n) && (7e)\\
& x \in X. && && (7f)
\end{aligned}$$

Proof. As a first step, we linearize the objective function via an auxiliary variable $\omega \in \mathbb{R}$, which yields

$$\begin{aligned}
\min \quad & \omega && (8a)\\
\text{s.t.} \quad & \max_{\Gamma \in \mathit{\Gamma}} \max_{S \in \mathcal{S}_\Gamma} \{c^S(x) - f(\Gamma)\} \le \omega && (8b)\\
& x \in X. && (8c)
\end{aligned}$$

Then we reformulate Constraint (8b) as a finite set of constraints

$$\max_{S \in \mathcal{S}_\Gamma} \{c^S(x) - f(\Gamma)\} \le \omega \quad (\forall \Gamma \in \mathit{\Gamma}),$$

using the finiteness of the set $\mathit{\Gamma}$. Furthermore, for every fixed $x \in X$ and $\Gamma \in \mathit{\Gamma}$, we have

$$\max_{S \in \mathcal{S}_\Gamma} \{c^S(x) - f(\Gamma)\} = \max_{S \in \mathcal{S}_\Gamma} \{c^S(x)\} - f(\Gamma). \tag{9}$$

The remaining maximization subproblem over the set $\mathcal{S}_\Gamma$ can then be rewritten as an MIP. To do this, we choose binary variables $y_i^\Gamma \in \{0,1\}$ for each $i = 1, \dots, n$, with $y_i^\Gamma = 1$ if and only if the worst-case scenario for up to Γ-many deviating coefficients has the i-th coefficient deviate from its nominal value. The corresponding optimization task is then equivalent to

$$\begin{aligned}
\max \quad & \sum_{i=1}^n \underline{c}_i x_i + \sum_{i=1}^n \hat{c}_i x_i \cdot y_i^\Gamma \\
\text{s.t.} \quad & \sum_{i=1}^n y_i^\Gamma \le \Gamma \\
& y_i^\Gamma \in \{0,1\} \quad (\forall i = 1, \dots, n).
\end{aligned}$$

We now use arguments similar to those used in standard Γ-robust optimization as follows. As the constraint system of this problem is totally unimodular, we can as well choose $y_i^\Gamma \in [0,1]$. The dual problem – leaving out the constant term $\sum_{i=1}^n \underline{c}_i x_i$ in the objective function – takes the form

$$\begin{aligned}
\min \quad & \Gamma u^\Gamma + \sum_{i=1}^n v_i^\Gamma \\
\text{s.t.} \quad & u^\Gamma + v_i^\Gamma \geq \hat{c}_i x_i \quad (\forall i = 1, \ldots, n) \\
& u^\Gamma \geq 0 \\
& v_i^\Gamma \geq 0 \quad (\forall i = 1, \ldots, n),
\end{aligned}$$

where u^Γ is the dual variable to the single constraint and v_i^Γ that to the upper bound of y_i^Γ. We replace the maximization subproblem in Equation (9) by its dual. Carrying out the indicated substitutions in Problem (8) and considering the direction of Constraint (8b), which allows us to leave out the minimization on the left-hand side, we obtain the desired reformulation. □

Theorem 3.2 gives us a formulation that has $|\mathit{\Gamma}| \cdot (n+1) + 1$ additional decision variables, but which is linear in return. Similar to the standard reformulation of the strict Γ-robust counterpart, we have achieved an algorithmically tractable reformulation in only a moderately larger number of variables and constraints when compared to the nominal problem that ignores uncertainties. We have shown hereby that analogous results hold true for the more general global Γ-robust counterpart. Note that this step required us to choose $\mathit{\Gamma}$ as a finite set.

3.2 A More Compact Linear Reformulation

Restricting the choice of the uncertainty sets slightly further, and considering nominal optimization problems with a feasible set $X \subseteq \mathbb{R}_+^n$, we will see in this section that the reformulation of Theorem 3.2 can be improved significantly. To this end, we assume in the following that $\mathit{\Gamma}$ takes the form of an integer interval $\mathit{\Gamma} = [\underline{\Gamma}, \bar{\Gamma}] := \{\underline{\Gamma}, \underline{\Gamma}+1, \ldots, \bar{\Gamma}\} \subseteq \{0, 1, \ldots, n\}$. As a further helpful notation, we introduce $\Delta_\Gamma f := f(\Gamma) - f(\Gamma - 1)$, $\underline{\Gamma} + 1 \leq \Gamma \leq \bar{\Gamma}$, for the increase in the uncertainty reserve when passing from a scenario with Γ deviating coefficients to one with $\Gamma + 1$.

We then obtain a more compact reformulation of the globalized Γ-robust counterpart via the following reasoning. For a given $x \in X$, we order the values $\hat{c}_i x_i$, $i = 1, \ldots, n$, such that $\hat{c}_{i_1} x_{i_1} \geq \ldots \geq \hat{c}_{i_n} x_{i_n}$ holds. The worst-case scenario under exactly Γ-many deviating coefficients has the coefficients $i_1, \ldots, i_\Gamma$ deviate, which means it is constructed by adding the deviating coefficients in the order established above. This property can be used to determine the optimum number of deviating coefficients from the point of view of the adversary in the following fashion: start with the nominal scenario and add one deviating coefficient in iteration γ according to the order above, as long as its additional cost outweighs the corresponding increase $\Delta_\gamma f$ in the uncertainty reserve, but choosing at least $\underline{\Gamma}$- and at most $\bar{\Gamma}$-many coefficients.

This idea can be modelled as an MIP by introducing a decision variable $y_{\gamma i} \in \{0,1\}$, $\gamma = 1, \ldots, \bar{\Gamma}$ and $i = 1, \ldots, n$, that takes a value of 1 if and only if in iteration γ we choose coefficient i as the additional deviating coefficient. The worst-case cost of a solution $x \in X$ can then be determined as

$$\text{grob}(x) = \sum_{i=1}^{n} \underline{c}_i x_i + \max \quad \sum_{\gamma=1}^{\bar{\Gamma}} \sum_{i=1}^{n} \hat{c}_i x_i \cdot y_{\gamma i} - \sum_{\gamma=1}^{\bar{\Gamma}} \Delta_\gamma f \cdot \left(\sum_{i=1}^{n} y_{\gamma i} \right) \tag{10a}$$

$$\text{s.t.} \quad \sum_{\gamma=1}^{\bar{\Gamma}} y_{\gamma i} \leq 1 \qquad (\forall i = 1, \dots, n) \tag{10b}$$

$$\sum_{i=1}^{n} y_{\gamma i} \leq 1 \qquad (\forall \gamma = 1, \dots, \bar{\Gamma}) \tag{10c}$$

$$\sum_{i=1}^{n} y_{\gamma i} = 1 \qquad (\forall \gamma = 1, \dots, \underline{\Gamma}) \tag{10d}$$

$$\sum_{i=1}^{n} y_{\gamma i} \leq \sum_{i=1}^{n} y_{\gamma-1,i} \qquad (\forall \gamma = \underline{\Gamma}+1, \dots, \bar{\Gamma}) \tag{10e}$$

$$y_{\gamma i} \in \{0,1\} \qquad \begin{array}{l} (\forall \gamma = 1, \dots, \bar{\Gamma}) \\ (\forall i = 1, \dots, n). \end{array} \tag{10f}$$

Problem (10) uses the *incremental* or *δ-method* (see e.g. Vielma (2015)) to model the cost of a solution x in Objective Function (10a) depending on the number of deviating coefficients. Constraint (10b) ensures that each coefficent is chosen at most once, while Constraint (10c) enforces that in each iteration at most one element is added to the scenario – via Constraint (10d) we *have* to choose one in each of the first $\underline{\Gamma}$ iterations. Constraints (10e) and (10f) are required for the correctness of the cost computation according to the δ-method.

In the following, we study the properties of Problem (10) in more detail. We start with a simple observation about the uncertainty reserve.

Lemma 3.3. *Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a convex function. Then we have $\Delta_i f \leq \Delta_{i+1} f$ for all $i \in \mathbb{R}_+$.*

Proof. Due to the convexity of f, we have $f(\lambda \cdot (i-1) + (1-\lambda) \cdot (i+1)) \leq \lambda \cdot f(i-1) + (1-\lambda) \cdot f(i+1)$ for all $i \in \mathbb{R}_+$ and $\lambda \in [0,1]$. For $\lambda = \frac{1}{2}$, and with slight a rearrangement, this yields $f(i) - f(i-1) \leq f(i+1) - f(i)$, which proves the claim. □

Using Lemma 3.3, we can remove Constraint (10e) as redundant whenever f is convex. This is because the maximization will then always choose all the coefficients $i_1, \dots, i_\Gamma$ for some $\Gamma \in [\underline{\Gamma}, \bar{\Gamma}]$, leaving out none in between, as the first $\Delta_\gamma f$-values are the smallest ones. This way, the filling condition of the δ-method is automatically satisfied.

Problem (10) without Constraints (10d) and (10e) can be interpreted as a maximum-weight matching problem on a complete bipartite graph $G = (U \cup W, E)$ with $|U| = \bar{\Gamma}$ and $|W| = n$ as well as $E = \{\{\gamma, i\} \mid \gamma \in U, i \in W\}$. The decision variables $y_{\gamma i}$ then model which edges are chosen for the matching. From this interpretation it immediately follows that the constraint matrix corresponding to Constraints (10b)– (10c) is totally unimodular, which still holds when adding Constraint (10d) and the bounds on y.

The above reasoning allows us to equivalently consider the linear programming

(LP) relaxation again, which can be stated as

$$
\begin{aligned}
\operatorname{grob}(x) = \sum_{i=1}^{n} \underline{c}_i x_i + \max \quad & \sum_{\gamma=1}^{\bar{\Gamma}} \sum_{i=1}^{n} \hat{c}_i x_i \cdot y_{\gamma i} - \sum_{\gamma=1}^{\bar{\Gamma}} \Delta_\gamma f \cdot \left(\sum_{i=1}^{n} y_{\gamma i} \right) && \text{(11a)}\\
\text{s.t.} \quad & \sum_{\gamma=1}^{\bar{\Gamma}} y_{\gamma i} \le 1 \quad (\forall i = 1, \dots, n) && \text{(11b)}\\
& \sum_{i=1}^{n} y_{\gamma i} \le 1 \quad (\forall \gamma = \underline{\Gamma} + 1, \dots, \bar{\Gamma}) && \text{(11c)}\\
& \sum_{i=1}^{n} y_{\gamma i} = 1 \quad (\forall \gamma = 1, \dots, \underline{\Gamma}) && \text{(11d)}\\
& y_{\gamma i} \ge 0 \quad (\forall \gamma = 1, \dots, \bar{\Gamma})(\forall i = 1, \dots, n). && \text{(11e)}
\end{aligned}
$$

Note that we used here that part of Constraint (10c) is already contained in Constraint (10d) and that the upper bound on y is redundant to Constraints (10b) and (10c). As before, we replace this linear program for the computation of the worst-case cost by its dual to obtain a compact reformulation of the globalized Γ-robust counterpart. This dual is given by

$$
\begin{aligned}
\min \quad & \sum_{\gamma=1}^{\bar{\Gamma}} u_\gamma + \sum_{i=1}^{n} v_i && \text{(12a)}\\
\text{s.t.} \quad & u_\gamma + v_i \ge \hat{c}_i x_i - \Delta_\gamma f \quad (\forall \gamma = 1, \dots, \bar{\Gamma})(\forall i = 1, \dots, n) && \text{(12b)}\\
& u_\gamma \ge 0 \quad (\forall \gamma = \underline{\Gamma} + 1, \dots, \bar{\Gamma}) && \text{(12c)}\\
& v_i \ge 0 \quad (\forall i = 1, \dots, n). && \text{(12d)}
\end{aligned}
$$

Note that part of the u-variables are unrestricted due to the equality in Constraints (11d). Replacing Subproblem (11) with Subproblem (12) in Problem (6) yields the following result.

Theorem 3.4. *For a feasible set $X \subseteq \mathbb{R}^n_+$, an integer interval $\Gamma = [\underline{\Gamma}, \bar{\Gamma}]$, $0 \le \underline{\Gamma} \le \bar{\Gamma}$, for the allowable* Γ*-values and a convex uncertainty reserve f, the globalized* Γ*-robust counterpart* (6) *can be stated as:*

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} \underline{c}_i x_i + \sum_{\gamma=1}^{\bar{\Gamma}} u_\gamma + \sum_{i=1}^{n} v_i && \text{(13a)}\\
\text{s.t.} \quad & u_\gamma + v_i \ge \hat{c}_i x_i - \Delta_\gamma f \quad (\forall \gamma = 1, \dots, \bar{\Gamma})(\forall i = 1, \dots, n) && \text{(13b)}\\
& u_\gamma \ge 0 \quad (\forall \gamma = \underline{\Gamma} + 1, \dots, \bar{\Gamma}) && \text{(13c)}\\
& v_i \ge 0 \quad (\forall i = 1, \dots, n) && \text{(13d)}\\
& x \in X. && \text{(13e)}
\end{aligned}
$$

With the above theorem, we have obtained an alternative formulation to that of Theorem 3.2 which can be used under mild additional assumptions. It has the advantage that much fewer additional variables and constraints are required. We will see in our computational study that the impact on the run-time is enormous.

3.3 A Parametrization for 0/1-Optimization Problems

In this section, we study optimization problems in which the variables are restricted to take binary values only. In this case, it is known that the determination of standard Γ-robust counterparts can elegantly be reduced to solving finitely-many auxiliary problems of the same type as the nominal optimization problem. Although the concept of globalized Γ-robustness is more general and powerful, we show that analogous results are true for solving globalized Γ-robust counterparts. In the following, we assume in addition to the assumptions of the previous section that $X \subseteq \{0,1\}^n$, that $\underline{\Gamma} = 0$ and that f is not only convex but also monotonically increasing. The latter implies $\Delta_\gamma f \geq 0$ for all $\gamma = 1, \ldots, \bar{\Gamma}$. For any fixed $x \in X$ and given values u_γ, $\gamma = 1, \ldots, \bar{\Gamma}$, the structure of Problem (13) then allows us to compute all variables v_i, $i = 1, \ldots, n$, as

$$v_i = \max_{1,\ldots,\bar{\Gamma}} \{\hat{c}_i x_i - \Delta_\gamma f - u_\gamma, 0\},$$

which follows from Constraints (13b) and (13d) as well as the minimization of the objective function. Using $x_i \in \{0,1\}$, this can also be written as $v_i = \max_{1,\ldots,\bar{\Gamma}} \{\hat{c}_i - \Delta_\gamma f - u_\gamma, 0\} \cdot x_i$. Substituting in Problem (13), and making use of $\underline{\Gamma} = 0$, this yields

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} \underline{c}_i x_i + \sum_{\gamma=1}^{\bar{\Gamma}} u_\gamma + \sum_{i=1}^{n} \max_{\gamma=1,\ldots,\bar{\Gamma}} \{\hat{c}_i - \Delta_\gamma f - u_\gamma, 0\} \cdot x_i && \text{(14a)} \\ \text{s.t.} \quad & u_\gamma \geq 0 \quad (\forall \gamma = 1, \ldots, \bar{\Gamma}) && \text{(14b)} \\ & x \in X. && \text{(14c)} \end{aligned}$$

We see that the u-variables – besides being non-negative – do not occur in any of the constraints of this reformulated problem, but instead only affect the objective function. Thus, in any optimal solution to Problem (14), the u-variables can be chosen from a finite set of possible values that can be computed beforehand. In other words, Problem (14) can be solved by solving finitely many version of this problem with fixed u-values. This amounts to solving a finite number of problems of the same structure as the original nominal problem with varying objective function coefficients.

To this end, consider again a fixed solution $x \in X$ and assume that the values $\hat{c}_i x_i$, $i = 1, \ldots, n$, are ordered such that $\hat{c}_{i_1} x_{i_1} \geq \ldots \geq \hat{c}_{i_n} x_{i_n}$ holds. Going back to Problem (13), its minimization subproblem can be stated as

$$\begin{aligned} \min \quad & \sum_{\gamma=1}^{\bar{\Gamma}} u_\gamma + \sum_{i=1}^{n} v_i && \text{(15a)} \\ \text{s.t.} \quad & u_\gamma + v_i \geq \hat{c}_i x_i - \Delta_\gamma f \quad (\forall \gamma = 1, \ldots, \bar{\Gamma})(\forall i = 1, \ldots, n) && \text{(15b)} \\ & u_\gamma \geq 0 \quad (\forall \gamma = 1, \ldots, \bar{\Gamma}) && \text{(15c)} \\ & v_i \geq 0 \quad (\forall i = 1, \ldots, n). && \text{(15d)} \end{aligned}$$

As both variables u and v are non-negative, Constraint (15b) can be replaced by

$$u_\gamma + v_i \geq \max\{\hat{c}_i x_i - \Delta_\gamma f, 0\}$$

for all $\gamma = 1, \ldots, \bar{\Gamma}$ and $i = 1, \ldots, n$ as this does not affect the set of feasible solutions. To simplify the exposition, we define a new parameter $d_{\gamma i} \in \mathbb{R}_+$ for all $\gamma = 1, \ldots, \bar{\Gamma}$ and $i = 1, \ldots, n$ as

$$d_{\gamma i} := \max\{\hat{c}_i x_i - \Delta_\gamma f, 0\}. \qquad (16)$$

Considering again the primal version of this problem, we come back to Problem (11), which can now be written as

$$\max \quad \sum_{\gamma=1}^{\bar{\Gamma}} \sum_{i=1}^{n} d_{\gamma i} y_{\gamma i} \tag{17a}$$

$$\text{s.t.} \quad \sum_{\gamma=1}^{\bar{\Gamma}} y_{\gamma i} \leq 1 \quad (\forall i = 1, \ldots, n) \tag{17b}$$

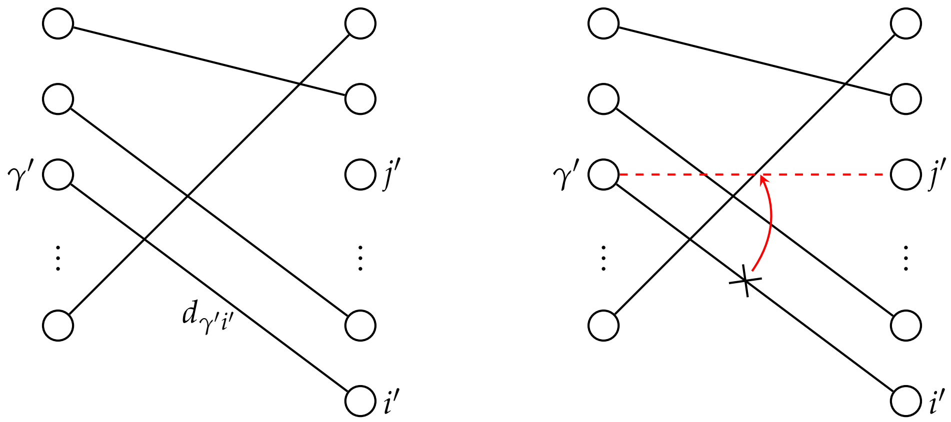
$$\sum_{i=1}^{n} y_{\gamma i} \leq 1 \quad (\forall \gamma = 1, \ldots, \bar{\Gamma}) \tag{17c}$$

$$y_{\gamma i} \geq 0 \quad (\forall \gamma = 1, \ldots, \bar{\Gamma})(\forall i = 1, \ldots, n). \tag{17d}$$

As we have already discussed, this problem can be interpreted as a maximum-weight matching problem on a complete bipartite graph $G = (U \cup W, E)$. The node subset U contains $\bar{\Gamma}$-many nodes, W contains n nodes, and the weights on the edges are exactly the values $d_{\gamma i}$. In an integer solution to this problem, each node will be matched to at most one other node. The structure of the weights $d_{\gamma i}$ allows us to deduce even more information:

Lemma 3.5. *Let the weights $d_{\gamma i}$ be defined as in Equation* (16). *Then there is an optimal solution $\bar{y} \in \{0,1\}^{\bar{\Gamma} \cdot n}$ to Problem* (17) *with $\bar{y}_{\gamma i} = 0$ for all $\gamma = 1, \ldots, \bar{\Gamma}$ and $i > \bar{\Gamma}$.*

Proof. Let $\bar{y} \in \{0,1\}^{\bar{\Gamma} \cdot n}$ be an optimal solution to Problem (17). Suppose there is a $\gamma' \in \{1, \ldots, \bar{\Gamma}\}$ and an i' such that $\bar{\Gamma} < i' \leq n$ with $\bar{y}_{\gamma' i'} = 1$. This solution $\bar{y}$ then describes a maximum-weight matching $\bar{M}$ in G as shown in Figure 1. As all weights $d_{\gamma i}$



(a) Matching $\bar{M}$ according to $\bar{y}$ (b) Matching M' according to y'

Figure 1: Illustration of the matching construction of Lemma 3.5

are non-negative, the optimal matching will contain excactly $|U| = \bar{\Gamma}$-many edges. This means that there is at least one node $j' \in U$ with $j' \leq \bar{\Gamma}$ which is unmatched. Now let an alternative solution $y' \in \{0,1\}^{\bar{\Gamma} \cdot n}$ corresponding to another matching M' be defined by

$$y'_{\gamma i} := \begin{cases} 1, & \text{if } \gamma = \gamma' \wedge i = j' \\ 0, & \text{if } \gamma = \gamma' \wedge i = i' \\ \bar{y}_{\gamma i}, & \text{otherwise.} \end{cases}$$

This new matching is feasible as it deviates from $\bar{M}$ only in the edge $\{\gamma', i'\}$ which has been replaced by the edge $\{\gamma', j'\}$. As we have $\hat{c}_{j'}^T x_{j'} \geq \hat{c}_{i'}^T x_{i'}$, we can conclude

$$d_{\gamma' j'} y'_{\gamma' j'} = \max\{\hat{c}_{j'}^T x_{j'} - \Delta_{\gamma'} f, 0\} \cdot y'_{\gamma' j'} \geq \max\{\hat{c}_{i'}^T x_{i'} - \Delta_{\gamma'} f, 0\} \cdot \bar{y}_{\gamma' i'} = d_{\gamma' i'} \bar{y}_{\gamma' i'}.$$

It follows

$$\sum_{\gamma=1}^{\bar{\Gamma}} \sum_{i=1}^{n} d_{\gamma i} \bar{y}_{\gamma i} \leq \sum_{\gamma=1}^{\bar{\Gamma}} \sum_{i=1}^{n} d_{\gamma i} y'_{\gamma i},$$

which means that the new matching M' is at least as good as $\bar{M}$. By iterating this construction, we come to an optimal matching in which the first $\bar{\Gamma}$ nodes in W are matched and the others not, which proves the claim. □

The application of the above lemma leads to a reduction in the number of decision variables in Problem (17) as we only have to consider $y_{\gamma i}$ for $i \in \{1, \dots, \bar{\Gamma}\}$ instead of $i \in \{1, \dots, n\}$. Accordingly, this also leads to a reduction in the number of constraints of the dual subproblem (15). Using strong duality and complementary slackness, we will now show that there always is an optimal dual solution in which each u_γ takes values from within a precomputable finite set. The following theorem uses the notation $[a]^+ := \max\{a, 0\}$ for the non-negative part of a number $a \in \mathbb{R}$.

Theorem 3.6. *For the dual subproblem* (15)*, there always exists an optimal solution* $(\bar{u}, \bar{v})$*, where*

$$\bar{u}_\gamma \in U_\gamma := \left\{ \bigcup_{k \in \{1,\dots,\bar{\Gamma}\}} \{[\hat{c}_k - \Delta_\gamma f]^+\}, \bigcup_{k \in \{1,\dots,\bar{\Gamma}\}} \{[\hat{c}_k - \Delta_\gamma f]^+ - [\hat{c}_k - \Delta_k f]^+\}, 0 \right\}$$

for all $\gamma \in \{1, \dots, \bar{\Gamma}\}$.

Proof. We prove the theorem by explicitly constructing a primal-dual pair of optimal solutions. For the primal problem, let

$$\bar{y}_{\gamma i} := \begin{cases} 1, & \text{if } i = \gamma \\ 0, & \text{otherwise} \end{cases}$$

for all $\gamma, i \in \{1, \dots, \bar{\Gamma}\}$, which corresponds to a matching that always matches opposite nodes. The edge weights $d_{\gamma i}$ are always non-negative. We define

$$k := \operatorname*{argmin}_{\gamma \in \{1,\dots,\bar{\Gamma}\}} \{d_{\gamma i} \mid d_{\gamma i} > 0 \wedge i = \gamma\},$$

which exactly yields the number of edges in the matching that have a positive weight as the sorting of the deviations and the convex uncertainty reserve imply

$$d_{\gamma i} = \max\{\hat{c}_i x_i - \Delta_\gamma f, 0\} \geq \max\{\hat{c}_{i+1} x_{i+1} - \Delta_{\gamma+1}\} = d_{\gamma' i'}$$

for all $\gamma, \gamma', i, i' \in \{1, \dots, \bar{\Gamma}\}$ with $\gamma' \geq \gamma$ and $i' \geq i$. For the dual problem, we define

$$\bar{u}_{\gamma i} := \begin{cases} d_{\gamma k} - d_{k+1,k}, & \text{if } \gamma \leq k \\ 0, & \text{otherwise} \end{cases}$$

for $\gamma \in \{1, \dots, \bar{\Gamma}\}$ and

$$\bar{v}_{\gamma i} := \begin{cases} \hat{c}_i x_i - \hat{c}_k x_k + d_{k+1,k}, & \text{if } i \leq k \\ 0, & \text{otherwise} \end{cases}$$

for $i \in \{1, \dots, \bar{\Gamma}\}$. If $k = \bar{\Gamma}$, we define $d_{k+1,k} := 0$. In the following, we will first show the dual feasibility of $(\bar{u}, \bar{v})$ and will then see that complementary slackness holds together with $\bar{y}$. By construction $\bar{u}$ and $\bar{v}$ are non-negative. Thus, the single dual constraint to check is $u_\gamma + v_i \geq \hat{c}_i x_i - \Delta_\gamma f$, which we will do via a case distinction:

Case 1: We first consider $\gamma \leq k$ and make another distinction for i.

Case 1.1: For $i \leq k$, we get

$$\begin{aligned} \bar{u}_\gamma + \bar{v}_i &= d_{\gamma k} - d_{k+1,k} + \hat{c}_i x_i - \hat{c}_k x_k + d_{k+1,k} \\ &\geq \hat{c}_k x_k - \Delta_\gamma f + \hat{c}_i x_i - \hat{c}_k x_k = \hat{c}_i x_i - \Delta_\gamma f. \end{aligned}$$

Case 1.2: For $i < k$ (which implies $k < \bar{\Gamma}$), we obtain

$$\bar{u}_\gamma + \bar{v}_i = d_{\gamma k} - d_{k+1,k} \geq \hat{c}_k x_k - \Delta_\gamma f - \max\{\hat{c}_k x_k - \Delta_{k+1} f, 0\}.$$

If $\hat{c}_k x_k - \Delta_{k+1} f \geq 0$, we can continue as follows:

$$\begin{aligned} \bar{u}_\gamma + \bar{v}_i &\geq \hat{c}_k x_k - \Delta_\gamma f - \hat{c}_k x_k + \Delta_{k+1} f \\ &= -\Delta_\gamma f + \Delta_{k+1} f \\ &\geq \hat{c}_{k+1} x_{k+1} - \Delta_\gamma f \\ &\geq \hat{c}_i x_i - \Delta_\gamma f. \end{aligned}$$

In the second step, $\hat{c}_{k+1} x_{k+1} \leq \Delta_{k+1} f$ holds by definition of k. If $\hat{c}_k x_k - \Delta_{k+1} f < 0$, we can immediately conclude $\bar{u}_\gamma + \bar{v}_i \geq \hat{c}_k x_k - \Delta_\gamma f \geq \hat{c}_i x_i - \Delta_\gamma f$.

Case 2: Then we consider $\gamma > k$ (which again implies $k < \bar{\Gamma}$) and distinguish two cases for i as well.

Case 2.1: For $i \leq k$, we see

$$\bar{u}_\gamma + \bar{v}_i = \hat{c}_i x_i - \hat{c}_k x_k + d_{k+1,k} \geq \hat{c}_i x_i - \hat{c}_k x_k + \hat{c}_k x_k - \Delta_{k+1} f \geq \hat{c}_i x_i - \Delta_\gamma f,$$

where $\Delta_{k+1} f \leq \Delta_\gamma f$ is due to the convexity of f.

Case 2.2: For $i > k$, we have

$$\bar{u}_\gamma + \bar{v}_i = 0 \geq \hat{c}_{k+1} x_{k+1} - \Delta_{k+1} f \geq \hat{c}_i x_i - \Delta_\gamma f.$$

This completes the proof of dual feasibility. With respect to optimality, we see that $\bar{y}$ fulfils all primal inequalities with equality. Thus, we only have to check the complementary slackness condition for the dual constraints, which reads

$$(u_\gamma + v_i - d_{\gamma i}) \cdot y_{\gamma i} = 0 \quad (\forall \gamma \in \{1, \dots, \bar{\Gamma}\})(\forall i \in \{1, \dots, \bar{\Gamma}\}).$$

As $y_{\gamma i} = 0$ whenever $\gamma \neq i$, it remains to show $u_\gamma + v_i - d_{\gamma i} = 0$ whenever $\gamma = i$. We do this via a further case distinction:

Case 1: For $\gamma = i \leq k$, we can use $d_{\gamma k} \geq d_{kk} > 0$ to obtain

$$\begin{aligned} u_\gamma + v_i - d_{\gamma i} &= d_{\gamma k} - d_{k+1,k} + \hat{c}_i x_i - \hat{c}_k x_k + d_{k+1,k} - d_{\gamma i} \\ &= \hat{c}_k x_k - \Delta_\gamma f + \hat{c}_i x_i - \hat{c}_k x_k - \hat{c}_k x_k - \hat{c}_i x_i + \Delta_\gamma f \\ &= 0. \end{aligned}$$

Case 2: For $\gamma = i > k$, complementary slackness is fulfilled trivially as $u_\gamma = v_i = 0$ and $d_{ll} = 0$ for all $k < l \leq \bar{\Gamma}$.

This proves the theorem. □

We have now shown that it is possible to choose the values of u_γ, $\gamma = 1, \ldots, \bar{\Gamma}$, from the finite set U_γ of size $2\bar{\Gamma} + 1$ to construct an optimal solution to Model (14). This leads to the following result:

Theorem 3.7. *For a feasible set $X \subseteq \{0,1\}^n$, an integer interval $\mathit{\Gamma} = [0, \bar{\Gamma}]$, $\bar{\Gamma} \geq 0$, and a convex uncertainty reserve f, the globalized* Γ*-robust counterpart* (6) *can be solved by solving $(2\bar{\Gamma} + 1)^{\bar{\Gamma}}$-many deterministic versions of the original uncertain optimization problem.*

In other words, under the conditions of Theorem 3.7 the globalized Γ-robust counterpart is fixed-parameter tractable in $\bar{\Gamma}$ for any polynomial-time solvable nominal problem.

3.4 Reformulation for a Linear Uncertainty Reserve

In this final section on uncertainty reserves depending only on the number of deviating coefficients, we will consider the case of a linear uncertainty reserve. For combinatorial optimization problems, globalized Γ-robustness reduces to strict Γ-robustness in this case.

Theorem 3.8. *Let $f\colon \mathit{\Gamma} \to \mathbb{R}_+, \Gamma \mapsto \lambda \cdot \Gamma$, for some $\lambda \in \mathbb{R}_+$ and $\mathit{\Gamma} = [\underline{\Gamma}, \bar{\Gamma}]$, $0 \leq \underline{\Gamma} \leq \bar{\Gamma}$ as well as $X \subseteq \{0,1\}^n$. Then an optimal solution to the globalized* Γ*-robust counterpart* (6) *is given by any optimal solution to the strict robust counterpart* (3) *when choosing*

$$\mathcal{S} := \{S \mid c_i^S \in [\underline{c}_i, \underline{c}_i + \max\{\hat{c}_i - \lambda, 0\}], \gamma(S) \leq \bar{\Gamma}\}.$$

as the scenario set.

Proof. If we introduce binary variables y for the choice of deviating objective coefficients as in the proof of Theorem 3.2, the Γ-robust counterpart can be written as

$$\begin{aligned} \min \quad & \omega && && \text{(18a)} \\ \text{s.t.} \quad & \sum_{i=1}^n \underline{c}_i x_i + \sum_{i=1}^n \hat{c}_i x_i \cdot y_i^\Gamma - \lambda\Gamma \leq \omega && (\forall \Gamma \in \mathit{\Gamma}) && \text{(18b)} \\ & \sum_{i=1}^n y_i^\Gamma \leq \Gamma && (\forall \Gamma \in \mathit{\Gamma}) && \text{(18c)} \\ & x \in X && && \text{(18d)} \\ & y^\Gamma \in \{0,1\}^n && (\forall \Gamma \in \mathit{\Gamma}). && \text{(18e)} \end{aligned}$$

As observed before, for any given solution $x \in X$ the globalized robust Γ-counterpart will choose a scenario where the deviations are added in decreasing order $\hat{c}_{i_1} x_{i_1} \geq \ldots \geq \hat{c}_{i_n} x_{i_n}$, stopping when reaching the first $i_j = \underline{\Gamma}, \ldots, \bar{\Gamma}$ with $\hat{c}_{i_j} x_{i_j} < \lambda \cdot i_j$. Thus, the above problem is equivalent to

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} \underline{c}_i x_i + \sum_{i=1}^{n} \hat{c}_i x_i \cdot y_i - \lambda \sum_{i=1}^{n} y_i && \text{(19a)}\\
& \sum_{i=1}^{n} y_i \leq \bar{\Gamma} && \text{(19b)}\\
& x \in X && \text{(19c)}\\
& y \in \{0,1\}^n. && \text{(19d)}
\end{aligned}
$$

The objective can now be rewritten as $\sum_{i=1}^{n} \underline{c}_i x_i + \sum_{i=1}^{n} (\hat{c}_i x_i - \lambda) \cdot y_i$. If we introduce the additional constraint $y_i = 0$ for all $i = 1, \ldots, n$ with $(\hat{c}_i x_i - \lambda) \leq 0$, it can further be rewritten as

$$
\sum_{i=1}^{n} \underline{c}_i x_i + \sum_{i=1}^{n} \max\{\hat{c}_i x_i - \lambda, 0\} \cdot y_i = \sum_{i=1}^{n} \underline{c}_i x_i + \sum_{i=1}^{n} \max\{\hat{c}_i - \lambda, 0\} \cdot x_i y_i,
$$

where we have used in the second step that $X \subseteq \{0,1\}^n$. Altogether, we have the equivalent formulation

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} \underline{c}_i x_i + \sum_{i=1}^{n} \max\{\hat{c}_i - \lambda, 0\} \cdot x_i y_i && \text{(20a)}\\
& \sum_{i \in I}^{n} y_i \leq \bar{\Gamma} && \text{(20b)}\\
& x \in X && \text{(20c)}\\
& y_i \in \{0,1\}^n \quad (\forall i \in I), && \text{(20d)}
\end{aligned}
$$

where $I := \{i = 1, \ldots, n \mid \hat{c}_i x_i - \lambda > 0\}$. This precisely represents the strict $\bar{\Gamma}$-robust solution with deviations $\hat{c}_i - \lambda$, $i \in I$. □

With this result, we can use the parametrization derived by Bertsimas and Sim (2003), which solves a sequence of $n+1$ deterministic versions of the uncertain combinatorial optimization problem to determine the strictly Γ-robust solution:

$$
\min_{1,\ldots,n+1} \left\{ \bar{\Gamma} \cdot d_l + \min_{x \in X} \left\{ \underline{c}(x) + \sum_{j=1}^{l} (d_j - d_l) \cdot x_i \right\} \right\},
$$

where we assume w.l.o.g. that the deviations $d_i := \hat{c}_i - \lambda$, $i = 1, \ldots, n$, are sorted in descending order and that they are all positive. An optimal solution to this problem will automatically be an optimal solution to the globalized Γ-robust counterpart.

4 Uncertainty Reserves Depending only on δ

Until now, we have considered the maximum number of deviating coefficients in a scenario as variable, while the magnitude of the deviations was fixed. Now, we will discuss the opposite case: a given maximum number $\Gamma \in \{0,1,\dots,n\}$ of deviating coefficients and a variable parameter $\delta \in \mathbb{R}_+$ that allows the adversary to control the magnitude of the deviations. As defined in Section 2.2, the scenario set $\mathcal{S}_\Gamma(\delta)$ contains all scenarios S for which the scenario costs fulfil $c_i^S \in [\underline{c}_i, \underline{c}_i + \delta\hat{c}_i]$ and $|\{i \in \{1,\dots,n\} \mid c_i^S > \underline{c}_i\}| \leq \Gamma$. This implies $\mathcal{S}_\Gamma(\delta) \subseteq \mathcal{S}_\Gamma(\delta')$ for all $\delta' \geq \delta$ – indeed, increasing δ increases $\mathcal{S}_\Gamma(\delta)$ proportionally. In the following, we consider the case of uncertainty reserves that only depend on this proportionality factor to derive compact formulations for the globalized Γ-robust counterpart. To be precise, let $\eth(S) := \min\{\delta \in \mathbb{R}_+ \mid (\forall i = 1,\dots,n)\, c_i^S \leq \underline{c}_i + \delta\hat{c}_i\}$ denote the minimum factor δ such that $S \in \mathcal{S}_\Gamma(\delta)$. For any two scenarios S, S' with $\eth(S) = \eth(S')$, we then have $f(S) = f(S')$. Similar as before, we define $f(\eth(S)) := f(S)$ to simplify the notation.

While a discrete set of possible δ-values can be handled in a similar fashion as in Lemma 3.1 and Theorem 3.2, we will see that a continuous set of δ-values entails a higher problem complexity. We show this upon deriving a tractable formulation for the case of interval scenarios together with a linear uncertainty reserve.

In the case of interval scenarios, all nominal objective function coefficients can increase by the specified maximum deviation at once, or in other words $\Gamma = n$. This means that the corresponding scenario set $S(\delta) := S_n(\delta)$ for some (not necessarily integer) value $\delta \in [\underline{\delta}, \bar{\delta}]$ with $0 \leq \underline{\delta} \leq \bar{\delta}$ is given by those scenarios whose cost for coefficient $i = 1,\dots,n$ is $c_i^S = \underline{c}_i + \delta\hat{c}_i$. In addition, we will assume that the uncertainty reserve is linear in δ, i.e. $f\colon [\underline{\delta}, \bar{\delta}] \to \mathbb{R}, \delta \mapsto \lambda\delta$, for some $\lambda \in \mathbb{R}_+$. In this setting, we can reduce the globalized Γ-robust counterpart to two separate optimization problems that depend on the lower and the upper bound of δ, respectively.

Theorem 4.1. *Let $X \subset \mathbb{R}^n$, $\delta \in [\underline{\delta}, \bar{\delta}]$ and $\Gamma = n$. Then the globalized Γ-robust counterpart under scenarios $\bigcup_{\delta \in [\underline{\delta}, \bar{\delta}]} S_n(\delta)$ and linear uncertainty reserve with cost factor λ can be stated as*

$$\min\left\{\min_{x \in X}\{\underline{c}(x) + \underline{\delta} \cdot \hat{c}(x) \mid \hat{c}(x) \leq \lambda\}, \min_{x \in X}\{\underline{c}(x) + \bar{\delta} \cdot \hat{c}(x) \mid \hat{c}(x) \geq \lambda\}\right\}.$$

Proof. With the assumption of interval scenarios, the globalized Γ-robust counterpart takes the form

$$\min_{x \in X} \max_{\delta \in [\underline{\delta}, \bar{\delta}]} \max_{S \in \mathcal{S}_n(\delta)} \left\{c^S(x) - \lambda \cdot \delta\right\} = \min_{x \in X} \max_{\delta \in [\underline{\delta}, \bar{\delta}]} \left\{\sum_{i=1}^n (\underline{c}_i + \delta\hat{c}_i) \cdot x_i - \lambda \cdot \delta\right\},$$

which can be rearranged to

$$\min_{x \in X} \max_{\delta \in [\underline{\delta}, \bar{\delta}]} \left\{\underline{c}(x) + \delta\left(\sum_{i=1}^n \hat{c}_i) \cdot x_i - \lambda\right)\right\}.$$

For any fixed $x \in X$, the inner maximization problem is linear in δ, such that optima are attained at the interval borders. Thus, the problem can be simplified to

$$\begin{aligned}
&\min_{x \in X} \max\{\underline{c}(x) + \underline{\delta} \cdot (\hat{c}(x) - \lambda), \underline{c}(x) + \bar{\delta} \cdot (\hat{c}(x) - \lambda)\} \\
=& \begin{cases} \min_{x \in X} \underline{c}(x) + \underline{\delta} \cdot (\hat{c}(x) - \lambda), & \text{if } \hat{c}(x) \leq \lambda \\ \min_{x \in X} \underline{c}(x) + \bar{\delta} \cdot (\hat{c}(x) - \lambda), & \text{if } \hat{c}(x) \geq \lambda. \end{cases}
\end{aligned}$$

Leaving away the constants $-\lambda\underline{\delta}$ and $-\lambda\bar{\delta}$, solving the globalized Γ-robust counterpart consequently is equivalent to solving the two optimization problems

$$\begin{array}{rrcl} \min & \underline{c}(x) + \underline{\delta} \cdot \hat{c}(x) & & \\ \text{s.t.} & \hat{c}(x) & \leq & \lambda \\ & x & \in & X \end{array}$$

and

$$\begin{array}{rrcl} \min & \underline{c}(x) + \bar{\delta} \cdot \hat{c}(x) & & \\ \text{s.t.} & \hat{c}(x) & \geq & \lambda \\ & x & \in & X \end{array}$$

respectively. □

This theorem shows us that we can solve the globalized Γ-robust counterpart under the stated conditions via two separate optimization problems with fixed values of δ. However, the additional constraints restricting the deviations makes solving these two problems NP-hard for many classical combinatorial optimization problems. We will show this in the following for shortest path problem.

Theorem 4.2. *The globalized* Γ*-robust shortest path problem under scenarios* $\bigcup_{\delta \in [\underline{\delta}, \bar{\delta}]} S_n(\delta)$ *and linear uncertainty reserve is weakly NP-hard.*

Proof. We prove the claim via a two-way reduction from and to the resource-constrained shortest-path problem (RCSP). Let I be an instance of RCSP consisting of a directed graph $G = (V, A)$, a source $s \in V$, a sink $t \in V$, arc costs $c_a \in \mathbb{R}_+$ and arc resource consumptions $r_a \in \mathbb{R}^+$ for each arc $a \in A$ as well as a resource bound $R \in \mathbb{R}_+$. The objective is to minimize the total arc cost $c(p) := \sum_{a \in p} c_a$ over all s-t-paths p whose resource consumption $r(p) := \sum_{a \in p} r_a$ fulfils the resource bound R We will transform I to an instance I' of the globalized Γ-robust shortest-path problem. Without loss of generality, we can assume that I has a resource-feasible path. For I' we use the same graph G, source s and sink t. The nominal arc costs are given by the arc costs c of I, while the deviations are given by its arc resource consumptions r. The value of λ is set to R. Let $\underline{c}^{\max} := \max_{a \in A}\{\underline{c}_a\}$ be the heighest arc cost. Then the lower and the upper bounds for δ shall be $\underline{\delta} := 0$ and $\bar{\delta} := \frac{1}{\lambda}\underline{c}^{\max}(n-1)$ respectively. For the feasible set, we define $X := \{\text{all feasible } s\text{-}t\text{-paths in } G\}$. Finally, let $C \in \mathbb{R}_+$. We will now show that I' possesses a feasible path $\bar{p} \in X$ with $\text{grob}(p) \leq C$ if I has a feasible path $\bar{q} \in X$ with $c(\bar{q}) \leq C$.

First, we consider the reduction from RCSP. Let $\bar{p}$ be an optimal path for the globalized Γ-robust shortest-path problem with $\text{grob}(\bar{p}) \leq C$. We show that this path satisfies the resource bound $r(\bar{p}) \leq R$ with $c(\bar{p}) = \text{grob}(\bar{p})$. By Theorem 4.1, $\bar{p}$ is an optimal solution to

$$\min_{x \in X}\{\underline{c}(x) \mid \hat{c}(x) \leq \lambda\} \tag{21}$$

or

$$\min_{x \in X}\left\{\underline{c}(x) + \left(\frac{1}{\lambda}\underline{c}^{\max}(n-1)\right) \cdot \hat{c}(x) \mid \hat{c}(x) \geq \lambda\right\}. \tag{22}$$

On the one hand, we have assumed that I posseses a feasible path, which in turn is also feasible for Problem (21). On the other hand, the optimal value of Problem (22) is bounded from below by $\min_{x \in X}\{\underline{c}(x) + \underline{c}^{\max}(n-1)\}$. As $\underline{c}(x) \geq 0$ for all $x \in X$, this expression takes a value of at least $\underline{c}^{\max}(n-1) > C \geq \operatorname{grob}(\bar{p})$. Altogether, this means that $\bar{p}$ is optimal for Problem (21), which means $\operatorname{grob}(\bar{p}) = \underline{c}(\bar{p})$ and $\hat{c}(\bar{p}) \leq \lambda$. Thus, $r(\bar{p}) \leq R$ and fulfils $c(\bar{p}) = \operatorname{grob}(\bar{p}) < C$.

For the reverse reduction, let $\bar{q}$ be an optimal solution to RCSP with $c(\bar{q})$. We show that this path is feasible for the globalized Γ-robust shortest path problem with $\operatorname{grob}(\bar{q})$. As we have already shown, there is no path in X for which Problem (22) has an objective value smaller than C. As $r(\bar{q}) \leq R$, $\bar{q}$ is feasible for Problem (21). For the global minimum we then have

$$\min_{x \in X} \operatorname{grob}(x) \leq \min_{x \in X}\{\underline{c}(x) \mid \hat{c}(x) \leq \lambda\} \leq \underline{c}(\bar{q}) = c(\bar{q}) \leq C,$$

which proves the claim. □

In extension of the above theorem, it is even possible to show that the problem stays NP-hard when fixing the upper bound $\bar{\delta} \in \mathbb{R}_+$ beforehand. To see this, we perform a scaling of the arc costs by setting

$$\underline{c}_a := \frac{\bar{\delta} \cdot \lambda}{c^{\max}(n-1)} \cdot c_a$$

with

$$c^{\max} := \max_{a \in A}\{c_a\}.$$

Under this scaling, the arc with the highest cost remains the most expensive one. The value $\underline{c}^{\max}$ defined in the above proof is then given by

$$\underline{c}^{\max} = \frac{\bar{\delta} \cdot \lambda}{c^{\max}(n-1)} \cdot c^{\max} = \frac{\bar{\delta} \cdot \lambda}{(n-1)}.$$

The proof now works as before, which shows that the problem is still NP-hard for a fixed upper bound $\bar{\delta}$.

An analoguous consideration shows that the globalized Γ-robust version of the minimum-spanning-tree problem is also weakly NP-hard, which can be shown by a reduction from the diameter-constrained minimum-spanning-tree problem. The same holds true for the globalized Γ-robust version of the minimum-perfect-matching problem.

5 Computational Experiments

In the following, we will conduct a broad computational study comparing our new concept of the globalized Γ-robust counterpart to the traditional strict Γ-robust counterpart. We will investigate both the computational efficiency of our model formulations for globalized Γ-robust counterparts as well as the benefit of using globalized Γ-robustness to protect optimization problems from data deviations in a more flexible fashion than it is possible via strict Γ-robustness. To this end, we test our concepts on several indicative combinatorial optimization problems, namely the minimum-spanning-tree problem, the shortest-path problem and the knapsack problem.

Our experiments have been run on a server comprising Intel Xeon E5-2690 3.00 GHz computers with 25 MB cache and 128 GB RAM, using 5 threads and a time limit of

1 hour per instance. For our implementation, we have used the Python-API of the MIP solver Gurobi 7.5.1 (see Gurobi Optimization, Inc. (2017)). When we compare the performances of the nominal problem, the globalized Γ-robust counterpart in the equivalent formulations of Models 7 and 13 as well the strict Γ-robust counterpart, we will abbreviate them to **Nom**, **GR 1**, **GR 2** and **SR** respectively. In the appendix, we give detailed information about the nominal instances we used for the experiments as well as the minimal and maximal computation times of the above models for each instance.

5.1 Minimum-Spanning-Trees

The first benchmark set consists of instances of the minimum-spanning-tree probles (MSTP). We obtained the data for our test instances from SNDlib (see Orlowski et al. (2007)), a library of 26 telecommunication networks to evaluate algorithms for the survivable network design problem. From each SNDlib network, we generated nominal MSTP instances by maintaining the original network topology, interpreting each network as an undirected graph $G = (V, E)$ and using the costs of the cheapest *additional module* as the edge costs (resorting to the cost of the *preinstalled module* if none was available). Out of these, we defined 1 instance with deterministic edge cost deviations and 5 more instances with random edge cost deviations. The former type has edge cost deviations $\hat{c}_{ij} := 0.5c_{ij}$ for each edge $\{i, j\}$ with nominal edge costs c_{ij}, while the latter type possesses cost deviations $\hat{c}_{ij}$ that have been drawn uniformly from the interval $[\min_{\{i',j'\}\in E} c_{i'j'}, 4c_{ij}]$. Each instance was considered together with four different uncertainty reserves which were chosen as

$$f_{\text{lin},\alpha}(\Gamma) := \frac{\alpha}{|E|} \sum_{\{i,j\}\in E} c_{ij} \cdot \Gamma, \quad \alpha \in \{0.2, 0.4\}$$

and

$$f_{\text{quad},\beta}(\Gamma) := \frac{\beta}{|E|} \sum_{\{i,j\}\in E} c_{ij} \cdot \Gamma^2, \quad \beta \in \{0.1, 0.2\}.$$

For the globalized Γ-robust problem, we chose $\mathit{\Gamma} = \{0, \ldots, 20\}$ while for the strict robust problem we considered the cases $\Gamma = 10$ and $\Gamma = 20$.

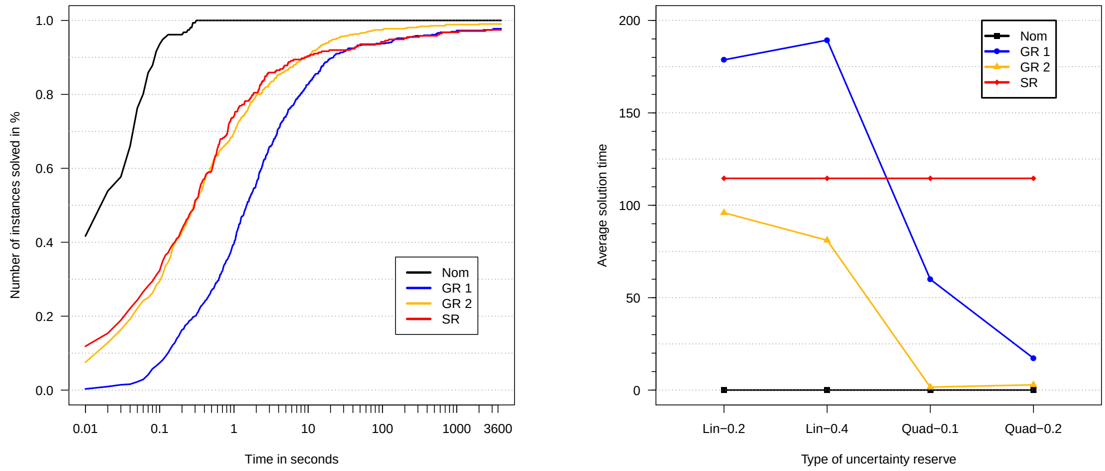
Table 1 in the appendix lists all the network topologies together with their respective size, ordered by the number of nodes, as well as minimal and maximal computations times for any of the instances created out of a given network. It shows that the nominal instances of the problem are all trivial to solve, but that the robustified versions are partly significantly harder. Among the latter, the more compact formulation GR 2 of the globalized robust counterpart performs much better then the equivalent formulation GR 1 while it is competitive to the strict robust counterpart SR.

We will examine these first findings in more detail in the following section. Varying between six choices for the edge cost deviations, we consider 156 uncertain instances in our computational experiments. To compare Nom, four types of uncertainty reserves for the globalized robust counterparts GR 1 and GR 2 as well as two different Γ-values for SR, the total number of problems to be solved is 1716.

5.2 Comparison of Computation Times

In the following, we compare the computation times of Nom, GR 1, GR 2 and SR on the instances we created. To solve them, we used an MIP formulation for MSTP where we separated the anti-cycle constraints during branch-and-bound. Note that for reasons of comparability this also applies to Nom, which naturally could have been solved using a combinatorial algorithm as well.

Figure 2 shows the computational performance for the MSTP instances up to the time limit of 3600 seconds. In Figure 2a, we plot the cumulative percentage of instances



(a) Performance profile

(b) Average solution times per uncertainty reserve

Figure 2: Computational perfomance of the different problems and formulations for MSTP

whose solution needed at most the computation time stated on the x-axis. As an assessment of the computational complexity of the nominal problem, we can see that all its instances together are solved within 4 seconds. With respect to the globalized Γ-robust counterpart, it is obvious that GR 2 outperforms the original formulation GR 1, as the former solves almost 40% more instances in up to 1 seconds and about 2% instances more overall – a strictly dominant performance curve in total. Moreover, we observe that GR 2 is about as computationally complex as SR, despite the linearly-many additional variables the former contains. Indeed, GR 2 seems to be slightly dominant for the more difficult instances. The most plausible explanation for this behaviour is that subtracting the increment of the uncertainty reserve in Constraint (13b) as compared to Constraint (4b) tends to make the problem easier to solve. The higher the uncertainty reserve is, the easier the problem becomes, as more and more constraints of type (13b) become redundant, with both GR 1 and GR 2 benefitting. This assessment is confirmed by Figure 2b, from which we recognize that computation times fall sharply when passing from a linear to a quadratic uncertainty reserve. In the same sense, a higher α or β lets them drop in many cases, too. As a reference, we show the average solution times for Nom and SR as constant lines in the graph. The figure underlines that the globalized robust counterpart performs an interpolation between the strict robust counterpart and the nominal problem – both in terms of computation time as degree of conservativeness; the latter aspect will be investigated in Section 5.2.1. We point out that for the choice $f(\gamma) = 0$ for all $\gamma = 1, \ldots, \bar{\Gamma}$, not only the two problems, globalized and strict robust counterpart, but also the respective model formulations themselves are essentially equivalent.

When examining the scalability of the formulations depending on instance size as given by the number of nodes, our findings were the following: while the nominal problem is largely unaffected by instance size (in the range of sizes we consider), there are marked differences for the robust counterparts. Here it is important to note that the SNDlib networks are not homogeneous in graph structure, which is presumably why the computation times fluctuate highly with instance size instead of growing more uniformly. With the exception of a single network (*brain* with 161 nodes), GR 2 is always superior to GR 1 over a given network topology. This is most likely explainable by the much lower number of variables in the second formulation which observably leads to better bounds and thus to less nodes in the branch-and-bound tree that need to be solved. Indeed, GR 2 is about as efficient to solve as SR, as we observed before.

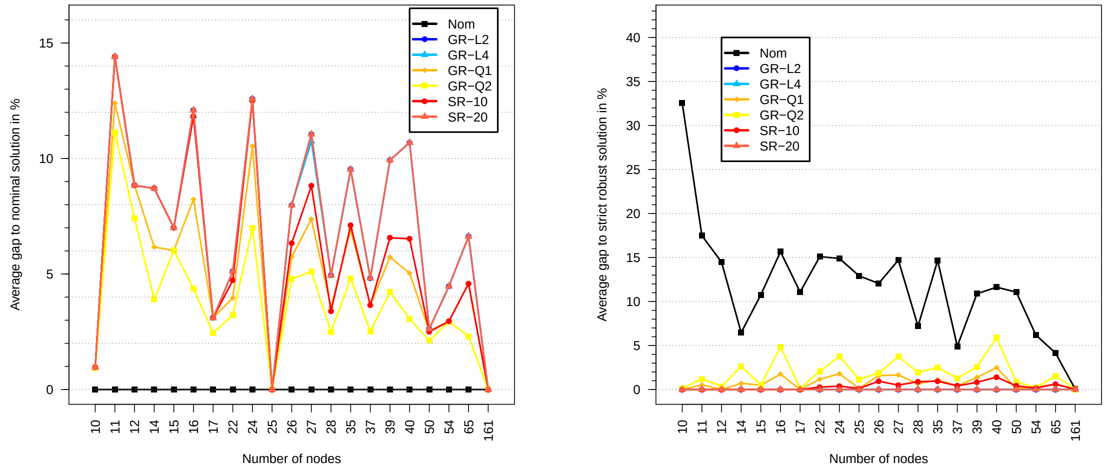
A further observation was that the instances with fixed deviations of the edge costs are significantly harder to solve than those with random deviations. The average computation times of the former are higher by more than a factor of two for GR 1 and SR, while they are about five times as high in the case of GR 2. Independent from the type of deviation, GR 2 is generally solved much faster than either GR 1 or SR. When distinguishing between the two types of deviations, fixed and random, we found that in both cases, there are actually only few network topologies for which the robust problem variants were significantly harder to solve than the corresponding nominal versions. Most notably, the instance with fixed deviations on network *dfn-bwin* with 10 nodes was solved by several orders of magnitude faster by GR 2 than by the other two robust models. For the random deviations, GR 2 performed much better in two cases.

Finally, we remark that both formulations of the globalized robust counterpart are solved much more easily for more progressive uncertainty reserves, which holds for both fixed and random deviations. Here we observed again that under a linear uncertainty reserve the average solution time for fixed deviations is roughly five times higher for random deviations.

5.2.1 Protection against Uncertainties

Now we compare the degrees of protection against uncertainties that the different models provide. A major motivation for devising our globalized robust counterpart was to alleviate the conservativeness entailed by the strict counterpart by punishing the adversary for choosing more improbable and antagonistic scenarios – this way incorporating our hedging against these cases using an uncertainty reserve. To this end, we evaluate the suitedness of our framework for this purpose by comparing the costs attained by the different robust models in the nominal objective function and the worst-case objective function respectively.

Figure 3 shows this comparison over all instances.



(a) Gaps to the nominal optima

(b) Gaps to the worst-case optima for $\Gamma = 20$

Figure 3: Comparison of all MSTP models evaluated in the nominal and worst-case objective function respectively

Figure 3a has a separate curve for each model which depicts the average optimality gap of its optimal solution in the nominal objective function over all instances. While **GR -L2** and **GR -L4** represent the globalized Γ-robust counterpart with uncertainty reserves $f_{\text{lin},0.2}$ and $f_{\text{lin},0.4}$ respectively, **GR-Q1** and **GR-Q2** are short for the globalized Γ-robust counterpart with uncertainty reserves $f_{\text{quad},0.1}$ and $f_{\text{quad},0.2}$ respectively. In a similar fashion, **SR-10** and **SR-20** stand for the strict Γ-robust counterpart with $\Gamma = 10$ and $\Gamma = 20$ respectively. Note that each instance size in the figure corresponds almost uniquely to one network topology listed in Table 1 in the appendix.

Obviously, and as expected, the most progressive uncertainty reserve in the comparison, $f_{\text{quad},0.2}$, is the least conservative choice. We observe that the nominal objective values produced by GR-Q2 are always the closest to those of the nominal problem itself, with GR-Q1 being the second best in all cases. While there are few instance sizes for which the nominal values of all model coincide, there also many which show marked differences. For the instance with 16 nodes (*newyork*), for example, the nominal value obtained from GR-Q2 is only about 4% of the value of the nominal problem, while the strict robust solution value produced by SR-20 is about 12% off, with SR-10 being not much better. For 60% of all instances, the resulting solutions are never worse than the nominal cost by more than 4% – as compared to a deterioriation of 9% for the strict robust counterpart. Overall, $f_{\text{quad},0.2}$ strictly dominates all other uncertainty reserves – and even dominates the strict robust counterpart with $\Gamma = 10$, too – with $f_{\text{quad},0.1}$ in second and practically no difference between the two linear uncertainty reserves and the strict robust counterpart with $\Gamma = 20$. The latter is most probably due to the relatively low uncertainty reserve values they entail compared to the relatively high deviations. It is an indication that the factor α needs to be chosen somewhat higher to make a difference.

We remark that for the fixed deviations all robust models found solutions whose nominal value is the same as that of the solution of nominal model. However, this is not a major distortion of the two figures as the instances with random deviations are overrepresented by 5 to 1 in our test set. In addition, excluding the instances with fixed deviations would only tend to make the described effects more pronounced. Finally, we see that the size of an instance has little to no effect on the nominal solution quality of either robust model, which means that our observations are largely indepent of this factor.

A second important measure of quality is the performance of the computed solutions in worst-case cost, i.e. their objective value in the case that all allowed deviations actually occur. The level of protection each of the different models offers in this case is demonstrated in Figure 3b in the same fashion as we did for the nominal objective function. It shows the average gaps of the solutions compared to the objective value obtain by the strict robust counterpart for $\Gamma = 20$. While we see that the nominal solutions can be far off the optimum and offer little to no protection, all robust models perform about equally well. Even the least conservative of the models, GR-Q2, always stays within 6% of the optimum.

Finally, we add a small observation we made when comparing the globalized robust solutions obtained by GR-Q2 and GR-L4. This point of view corresponds to the assumption that the planner actually intended to calculate with an uncertainty reserve – either in order to be less conservative than with the strict robust model or because the globalized robust objective is the actual objective function he wants to optimize anyway. We saw that the nominal solution is usually far from optimal with respect to the globalized robust objective – often by double-digit percentages – while the strict robust models deviate by up to 4%. This finding establishes the fact that globalized robustness is an optimization goal in its own right which is not readily attained by nominal or strict robust optimization.

As the overall result from these experiments, we can conclude that the globalized robust counterpart with any of the two quadratic uncertainty reserves is practically never too conservative to perform well in nominal quality, and at the same time it is practically always conservative enough to protect the solution even from the worst-case deviations. For the linear uncertainty reserves, it resulted that they were chosen as very conservative such that they virtually made no difference in solution quality compared to the strict robust counterpart. Still, it might be a good heuristic to choose a linear uncertainty reserve with a small factor α with respect to achieving smaller computation times as in the strict robust counterpart while obtaining virtually the same robustness guarantees. From the computational point of view, we already saw that globalized robust solutions are not harder to obtain than strict robust solutions, in many cases it is even easier. Altogether, globalized robustness appears to be a worthwile tool to implement a suitable balance between protection against uncertainty and deterioriation in nominal cost at no increase in computational complexity.

In the following, we repeat the same experiments on two more sets of instances to see in how far these results for the minimum-spanning-tree problem apply to other problem classes as well.

5.3 Shortest Paths

Our second benchmark problem is the shortest-path problem (SPP). The instances considered here are derived from the data of Ben Stabler's library of transportation networks (see Stabler (2018)). The latter consists of 23 street networks, most of them based on real-world maps. From each such network (excluding the mostly trivial *Braess* network), we generated 10 randomized problem instances, interpreting the original network topology as a directed graph $G = (V, A)$ and using the *free-flow times* as the arc costs.

We chose 5 origin-destination pairs at random, and out of these we defined 5 instances with deterministic arc cost deviations and 5 more instances with random arc cost deviations. The two types of edge cost deviations and the four types of uncertainty reserves were the same as for the minimum-spanning-tree problem. For the globalized Γ-robust problem, we chose $\Gamma = \{0, \ldots, 20\}$ while for the strict robust problem we considered the cases $\Gamma = 10$ and $\Gamma = 20$.

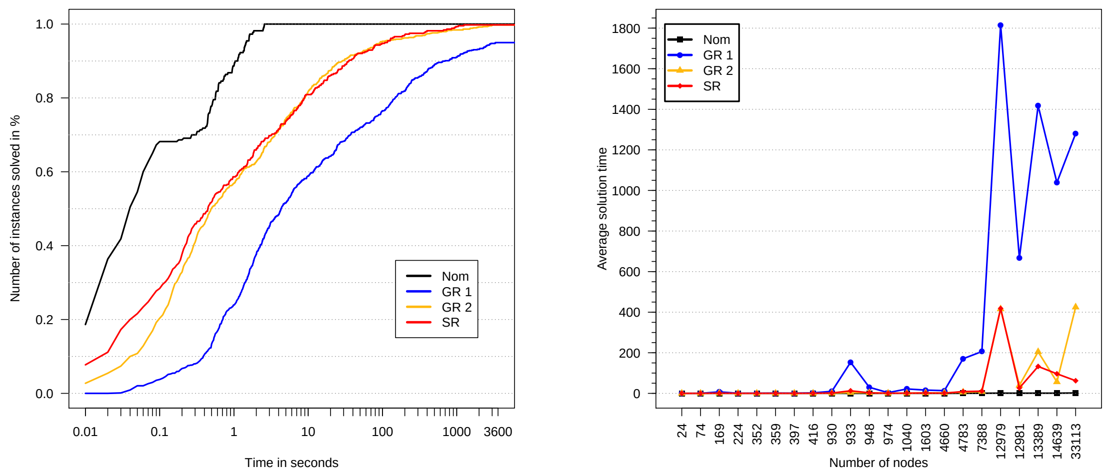
Table 1 in the appendix lists the networks used for this experiment together with their respective size, ordered by the number of nodes, again including minimal and maximal computation times for the different model types. As for MSTP, the nominal instances of the problem are trivial to solve, while the robustified versions can be much harder.

As we have considered five different origin-destination pairs for each network together with two different types of deviations each, the total number of uncertain instances is 220. Each one is solved in the nominal version, in the globalized Γ-robust version with four different uncertainty reserves combined with our two possible model formulations and in the strict robust version with two different Γ-values. Altogether, this makes 2420 problems to be solved.

5.3.1 Comparison of Computation Times

As before, we compare the computations times of Nom, GR 1, GR 2, SR on all the instances. Again, we use MIP formulations to solve all these models.

The computational performance for the SPP instances up to the time limit of 3600 seconds is shown in Figure 4.



(a) Performance profile

(b) Averages per instance size

Figure 4: Computational performance of the different problems and formulations for SPP

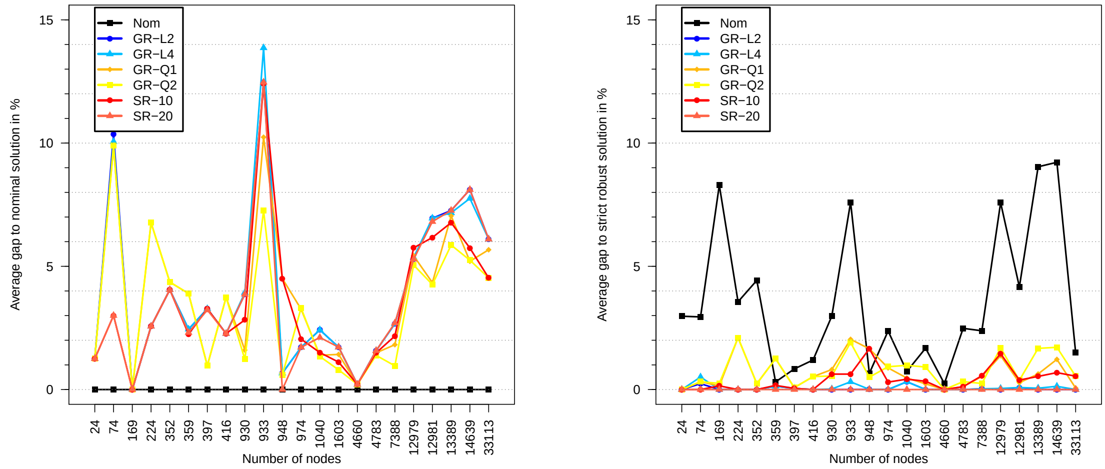
We see in Figure 4a that the nominal problem is solved within 1 second for all the instances. Concerning the globalized robust counterpart, GR 2 again significantly outperforms GR 1, solving 30% more instances in under 1 second and 5% instances more overall. We can also confirm the previous finding that GR 2 takes about the same time to solve as SR on average, with a slight edge of the latter on the easier instances and the reverse on the more difficult instances.

Figure 4b indicates that the average computation times of the different problems grow moderately with instance size. Up to a few thousand nodes in the graph, they remain relatively low; from this magnitude on, they differ more pointedly. While Nom stays easy to solve, GR 1 becomes significantly harder. The latter is vastly outperformed by GR 2, which mostly takes about the same time to solve as SR. Concerning the influence of the type of uncertainty on the computation times, we found the reverse effect compared to the minimum-spanning-tree problem: the instances with fixed deviations were significantly harder to solve than those with fixed deviations. The computations times of the former are by a factor of ten higher for all robust counterparts, where GR 1 already has relatively high solution times for the instances with fixed deviations. A reason for this discrepancy we could observe is that in the case of the fixed deviations, the optimal value of the linear relaxation of the shortest-path instances is already very close to the integer optimal value, while this does not apply to the random deviations. The trend to smaller computational complexity with a higher uncertainty reserve is even more pointed for the shortest-path instances in comparison to the minimum-spanning-tree instances.

5.3.2 Protection against Uncertainties

The performance of the different robust approaches when it comes to the protection of the solutions against data uncertainties was much more uniform for SPP than it was for MSTP. The possible advantage of the globalized robust solutions depended more on individual instances, with their quality being significantly better on larger instances, as we will see in the following.

In Figure 5 we evaluate the costs produced by all robust SPP models, both in the nominal and the worst-case objective function.



(a) Gaps to the nominal optima

(b) Gaps to the worst-case optima for $\Gamma = 20$

Figure 5: Comparison of all SPP models evaluated in the nominal and worst-case objective function respectively

Figure 5a shows that in general the objectives of the robust models are much closer together overall than for the MSTP instances. This is in part explained by the fact that between 60 and 70% of all robust solutions of a given model attain the optimal nominal value. We see from the figure that on the smaller instances (with $\leq$ 416 nodes), the globalized robust counterparts even attain worse nominal values than the strict robust counterparts, most notably for the *EMA* network with 74 nodes, where the former is about 7 percentage points worse (and up to 57 percentage points in one particular instance of that size). On the other hand, the globalized robust counterparts mostly outperform the strict robust counterparts on the larger instances – by up to 7 percentage points on average for the *ChicagoSketch* network with 933 nodes. Overall, this leads to a slight advantage of about 3 percentage points for the least conservative globalized robust counterpart compared to the two strict robust counterparts. We remark that the instances with fixed deviations had nominal values within 1% of the nominal optimum for all considered models, similar to before, such that the differences almost entirely stem from the results on the instances with random deviations.

The quality of all obtained SPP solutions evaluated in worst-case cost is shown in Figure 5b. The findings on the MSTP instances are mostly confirmed here, as the globalized robust solutions still generally yield a high level of protection against cost deviations as compared to the nominal solutions. Even the least conservative globalized robust solutions are always within about 2% of the strict robust optima on average, in many cases they are even way closer to them.

Lastly, we compare the attained solutions of the different models in the globalized objective function for the two uncertainty reserves $f_{\text{quad},0.2}$ and $f_{\text{lin},0.4}$. Here, the nominal solutions are again far off while the strict robust solutions are closer to the globalized robust ones yet still much more conservative by several percentage points.

With the results on the MSTP instances largely confirmed by those on the SPP instance, both polynomial-time-solvable combinatorial problems in the nominal case, we finally consider one more set of instances where the nominal problem is already NP-hard.

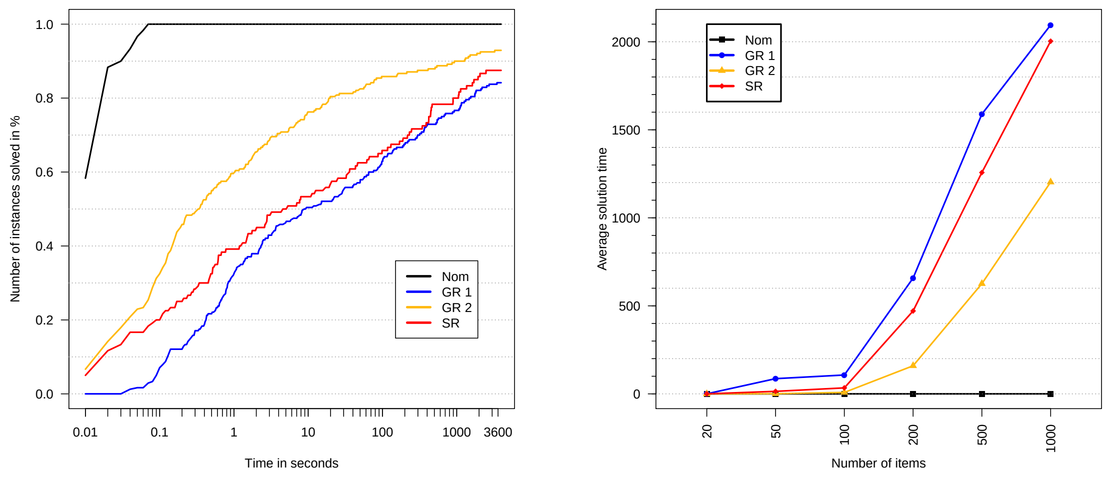
5.4 Knapsacks

Our final set of benchmarks are random knapsack problems (KP) instances. We generated them according to the paradigm of strongly correlated instances where all items have similar ratios between benefit and weight. This approach generally leads to harder and thus computationally more interesting instances (see Pisinger (2005)). We created instances in 6 different sizes, 5 of each size, with the number of items n ranging from 20 to 1000. The types of cost deviations, the uncertainty reserves for the globalized robust counterparts and the Γ-values for the strict robust counterparts in the different instances were chosen analogously to the setting for MSTP and SPP.

Table 3 in the appendix shows the minimal and maximal computation times of all knapsacks instances of a given size for all four model types under consideration. While all nominal instances were solved in under 1 second, the robust instances were generally hard to solve – from 200 items on, every model had at least one unsolved instance.

With 30 basic instances and two different types of deviations, we had 60 instances in total. For each one, we solved the nominal version, the globalized robust counterparts with four different types of uncertainty reserves in two different formulations and the strict robust counterparts with two different values of Γ, which makes 760 overall. As before, we solve the different models using MIP formulations and study both their computational behaviour as well as the resulting solution quality.

Figure 6 shows the computations times of all knapsack instances, both as a performance profile and as arithmetic means for each number of items.



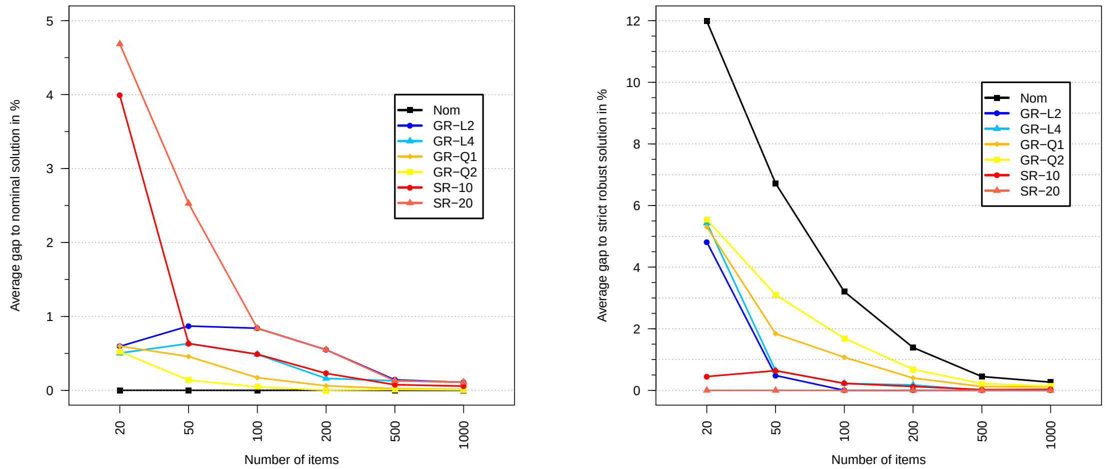
(a) Performance diagram

(b) Averages per instance size

Figure 6: Computation times of the different problems and formulations for the KP instances

The performance profile in Figure 6a confirms the dominance of GR 2 over GR 1 for the globalized robust counterpart. The latter is even clearly easier to solve here than SR. From Figure 6b, we see that the computation time grows exponentially with the instance size (as expected), with the instances up to 100 items solvable within 200 seconds – the larger ones take significantly longer to solve. It is remarkable that the globalized robust counterpart using the improved formulation can be solved within about half the time needed for the strict robust counterpart in most cases.

Figure 7 gives the quality of the solutions obtained by the different models as measured in the nominal objective function and the worst-case objective function respectively.



(a) Gaps to the nominal optima

(b) Gaps to the worst-case optima for $\Gamma = 20$

Figure 7: Comparison of all KP models evaluated in the nominal and worst-case objective function respectively

As Figure 7a shows, GR-Q2 loses almost nothing in nominal optimality, while SR-10 and SR-20 trail by several percentage points in solution quality. This mostly stems from the results on the smaller knapsack instances, where the results differ more pointedly. The differences shrink notably with an increasing number of items in a given instance.

Lastly, we examine the quality of the obtained solutions in the worst-case objective functions for $\Gamma = 20$ in Figure 7b. With the globalized robust solutions being visibly better than the nominal solutions and not too far off the strict robust solutions, the results for KP underline those already observed for SPP and MSTP.

Altogether, our results on the different benchmark sets lead us to the conclusion that the globalized robust counterpart comes at no additional computational complexity compared to the strict robust counterpart and at the same time allows the planner to come to a very fine-tuned trade-off between protection against deviations in the input data and nominal solution quality. These features make it a valuable complement to existing tools for decision-making under uncertainty.

6 Conclusions

In this work, we have introduced a discrete analogue of the globalized robustness concept that was originally introduced for continuous robust optimization. First, we have discussed how the general concept can be modelled for the Γ-robustness scheme. Then we have presented equivalent compact reformulations and algorithmically tractable robust counterparts for mixed-integer linear optimization problems. In fact, a globally robustified MIP remains an MIP – only with an increased number of variables and constraints. In the binary case, it can be further reformulated such that the problem remains fixed-parameter tractable in Γ.

In our experimental results on classical combinatorial optimization problems, namely the minimum-spanning-tree, the shortest-path and the knapsack problem, we have shown that the robust counterparts indeed remain algorithmically tractable, as the solution time usually increases mildly when compared to solving the nominal instances. The globalized robustness concept introduced here not only protects solutions against scenarios from within a given uncertainty set, but incorporates a controllable protection against realizations of the uncertainty outside this set as well. In summary, it is a concept that is readily usable in theory and in practice for combinatorial as well as for mixed-integer linear problems.

Appendix

In this appendix, we list all the nominal instances that we used in our computational study. Each table shows the names of the nominal instances in the first column, followed by information about the size of the respective instance as well as the nominal and robust solutions times according to the models we compared. Table 1 is for the minimum-spanning-tree problem (MSTP), Table 2 for the shortest-path problem (SPP) and Table 3 for the knapsack problem (KP). The MSTP instances are based on the networks from SNDlib (see Orlowski et al. (2007)), the SPP instances were derived from the transportation networks from Ben Stabler's library (see Stabler (2018)), and the KP instances were generated randomly according to Pisinger's model for strongly correlated instances (see Pisinger (2005)).

			Solution time (in s)							
			Nom		GR 1		GR 2		SR	
Network	$\|V\|$	$\|E\|$	min	max	min	max	min	max	min	max
dfn-bwin	10	45	< 1	< 1	< 1	-	< 1	202	< 1	-
di-yuan	11	42	< 1	< 1	< 1	2	< 1	< 1	< 1	< 1
dfn-gwin	11	47	< 1	< 1	< 1	14	< 1	< 1	< 1	< 1
pdh	11	34	< 1	< 1	< 1	3	< 1	< 1	< 1	< 1
abilene	12	15	< 1	< 1	< 1	< 1	< 1	< 1	< 1	< 1
polska	12	18	< 1	< 1	< 1	< 1	< 1	< 1	< 1	< 1
nobel-us	14	21	< 1	< 1	< 1	< 1	< 1	< 1	< 1	< 1
atlanta	15	22	< 1	< 1	< 1	< 1	< 1	< 1	< 1	< 1
newyork	16	49	< 1	< 1	< 1	23	< 1	4	< 1	2
nobel-germany	17	26	< 1	< 1	< 1	1	< 1	< 1	< 1	< 1
geant	22	36	< 1	< 1	< 1	6	< 1	2	< 1	< 1
ta1	24	55	< 1	< 1	< 1	2	< 1	< 1	< 1	< 1
france	25	45	< 1	< 1	< 1	-	< 1	-	< 1	-
janos-us	26	84	< 1	< 1	< 1	2	< 1	1	< 1	< 1
norway	27	51	< 1	< 1	< 1	7	< 1	2	< 1	1
sun	27	102	< 1	< 1	< 1	5	< 1	2	< 1	< 1
nobel-eu	28	41	< 1	< 1	< 1	13	< 1	4	< 1	1
india35	35	80	< 1	< 1	< 1	557	< 1	77	< 1	2448
cost266	37	57	< 1	< 1	< 1	22	< 1	10	< 1	3
janos-us-ca	39	122	< 1	< 1	< 1	7	< 1	3	< 1	1
giul39	39	172	< 1	< 1	< 1	534	< 1	217	1	94
pioro40	40	89	< 1	< 1	< 1	81	< 1	41	< 1	44
germany50	50	88	< 1	< 1	< 1	-	< 1	-	48	-
zib54	54	81	< 1	< 1	< 1	7	< 1	6	< 1	1
ta2	65	108	< 1	< 1	< 1	82	< 1	31	< 1	6
brain	161	332	< 1	< 1	< 1	1	< 1	3	< 1	< 1

Table 1: Sizes of the underlying networks of the MSTP instances and min/max solution times under the different models; “-” means “not solved within the time limit”.

			Min / max solution time (in s)							
			Nom		GR 1		GR 2		SR	
Network	$\|V\|$	$\|A\|$	min	max	min	max	min	max	min	max
SiouxFalls	24	76	< 1	< 1	< 1	< 1	< 1	< 1	< 1	< 1
EMA	74	258	< 1	< 1	< 1	< 1	< 1	< 1	< 1	< 1
STC	169	624	< 1	< 1	< 1	77	< 1	3	< 1	4
f-c	224	523	< 1	< 1	< 1	1	< 1	< 1	< 1	< 1
b-p-c	352	749	< 1	< 1	< 1	1	< 1	< 1	< 1	< 1
b-t	359	766	< 1	< 1	< 1	2	< 1	< 1	< 1	< 1
b-m-c	397	871	< 1	< 1	< 1	2	< 1	< 1	< 1	< 1
Anaheim	416	914	< 1	< 1	< 1	15	< 1	1	< 1	< 1
Barcelona	930	2522	< 1	< 1	< 1	76	< 1	3	< 1	9
ChicagoSketch	933	2950	< 1	< 1	1	2786	< 1	66	< 1	185
W-A	948	2535	< 1	< 1	1	362	< 1	21	< 1	16
b-m-p-f-c	974	2184	< 1	< 1	1	6	< 1	1	< 1	1
Winnipeg	1040	2836	< 1	< 1	1	230	< 1	4	< 1	6
Terrassa-Asym	1603	3264	< 1	< 1	1	159	< 1	14	< 1	16
Hessen-Asym	4660	6674	< 1	< 1	1	48	< 1	18	< 1	2
Goldcoast	4783	11140	< 1	< 1	6	1205	< 1	37	< 1	62
Austin	7388	18956	< 1	< 1	12	1306	2	35	< 1	41
CR	12979	39018	< 1	1	53	-	4	-	2	-
berlin-center	12981	28370	< 1	1	15	-	3	557	2	117
Philadelphia	13389	40003	< 1	1	26	-	2	2707	3	972
Birmingham	14639	33937	< 1	1	44	-	3	624	3	930
Sydney	33113	75379	1	2	70	-	7	3001	3	397

Table 2: Sizes of the underlying networks of the SPP instances and min/max solution times under the different models; "-" means "not solved within the time limit". Note that several network names are abbreviated due to space constraints. Namely, *STC* is *SymmetricaTestCase*, *f-c* is *friedrichshain-center*, *b-p-c* is *berlin-prenzlauerberg-center*, *b-t* is *berlin-tiergarten*, *W-A* is *Winnipeg-Asym*, *b-m-p-f-c* is *berlin-prenzlauerberg-center*, and *CR* is *ChicagoRegional*.

		Min / max solution time (in s)							
		Nom		GR 1		GR 2		SR	
Network	n	min	max	min	max	min	max	min	max
Knapsack-20	20	< 1	< 1	< 1	< 1	< 1	< 1	< 1	< 1
Knapsack-50	50	< 1	< 1	< 1	2195	< 1	18	< 1	231
Knapsack-100	100	< 1	< 1	< 1	1071	< 1	97	< 1	247
Knapsack-200	200	< 1	< 1	< 1	-	< 1	-	< 1	-
Knapsack-500	500	< 1	< 1	2	-	< 1	-	< 1	-
Knapsack-1000	1000	< 1	< 1	22	-	< 1	-	46	-

Table 3: Numbers of items of the KP instances and min/max solution times under the different models; "-" means "not solved within the time limit".

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