# Relating Single-Scenario Facets to the Convex Hull of the Extensive Form of a Stochastic Single-Node Flow Polytope

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We study when a facet-defining inequality for a deterministic, single-scenario subproblem is also facet-defining for the extensive form of a two-stage stochastic mixed-integer linear program (SMIP). To answer this question, we introduce a novel stochastic variant of the well-known single-node flow (SNF) polytope, and present necessary and sufficient conditions for single-scenario facet-defining inequalities to be facet-defining for the extensive form. We further demonstrate that our stochastic SNF polytope is a relaxation of a broad subclass of SMIPs, illustrating its generality.

**Keywords:** stochastic mixed-integer programming, facets, polyhedral combinatorics

## 1. Introduction and Background

Stochastic mixed-integer programs (SMIPs) with recourse are a powerful modeling tool for optimization under uncertainty in which some of the decisions are constrained to take on integer values. Such models have found application in diverse areas, including healthcare, supply chain management, finance, natural resource management, production planning, and more [21]. A general linear two-stage SMIP given in its *extensive form* [4] can be expressed as

$$\min\ c^{\mathrm{T}}x + \sum_{\omega\in\Omega} \Pr(\omega)(q^{\omega})^{\mathrm{T}}y^{\omega} \tag{1a}$$

$$\text{s.t. } Ax \geq b \tag{1b}$$

$$T^{\omega}x + W^{\omega}y^{\omega} \geq r^{\omega} \qquad \forall\, \omega\in\Omega \tag{1c}$$

$$x \in \mathcal{X} \tag{1d}$$

$$y^{\omega} \in \mathcal{Y} \qquad \forall\, \omega\in\Omega \tag{1e}$$

where $\Omega$ is a finite set of scenarios, and each scenario $\omega \in \Omega$ occurs with probability $\Pr(\omega)$; $A$, $(T^{\omega})_{\omega\in\Omega}$, $(W^{\omega})_{\omega\in\Omega}$ are matrices of conformable dimension, and $c$, $(q^{\omega})_{\omega\in\Omega}$ and $(r^{\omega})_{\omega\in\Omega}$ are vectors of conformable dimension. The variables $x$ correspond to a first-stage "here-and-now" decision, which must be made before the uncertain parameters are known. Once $x$ is fixed, a random scenario $\omega \in \Omega$ is realized, at which point a recourse decision $y^{\omega}$ is chosen. The set $\mathcal{X}$ (resp. $\mathcal{Y}$) enforces integrality on a subset of the $x$ (resp. $y^{\omega}$) variables. Let $Z$ denote the feasible region of (1), and for each $\omega \in \Omega$ let $Z^{\omega}$ denote the set of vectors $(x, y^{\omega})$ satisfying (1b)–(1e), with the constraints (1c) and (1e) removed for all scenarios $\omega' \neq \omega$. We refer to $\mathrm{conv}(Z^{\omega})$ as the single-scenario polyhedron, where conv(·) denotes the convex hull. We refer the reader to [4, 9] for a general background on SMIPs.

Although there exist many sophisticated techniques for solving (1) via decomposition, the direct solution of the extensive form using mixed-integer programming solvers remains a popular solution approach for SMIPs (see, e.g., [2, 5, 8, 14, 23] for recent examples). Although this approach does not generally scale well as $|\Omega|$ increases, its advantages include the fact that it is particularly easy to implement [22], and it requires virtually no structural assumptions (e.g., pure-binary first stage variables, fixed recourse, etc.) on the underlying SMIP. As such, direct solution of (1) is still incorporated as an option in various SMIP modeling software tools [15, 19, 22].

It is well-known that strong valid inequalities for the convex hull of the feasible region are a crucial ingredient for the efficient solution of mixed-integer programs, both stochastic and otherwise. Consequently, knowledge of strong valid inequalities for the set $\text{conv}(Z)$ are of particular interest for the solution of (1). For many problems, the facial structure of the single-scenario polyhedron $\text{conv}(Z^\omega)$ is well studied. That is, there are often known classes of valid inequalities for $\text{conv}(Z^\omega)$ of the form

$$\alpha^{\mathrm{T}} x + \beta^{\mathrm{T}} y^\omega \geq \tau, \tag{F}$$

where $\alpha$ and $\beta$ are vectors of conformable length, and $\tau \in \mathbb{R}$. It is trivial to show that if (F) is valid for $\text{conv}(Z^\omega)$ then it is valid for $\text{conv}(Z)$, but far less is known about the strength of (F) for $\text{conv}(Z)$ (see, e.g., [7] for an example of such a result in the context of lot sizing). Understanding when valid inequalities for $\text{conv}(Z^\omega)$ are strong for $\text{conv}(Z)$ could provide a potentially large class of strong valid inequalities for $\text{conv}(Z)$, using only information known from the deterministic setting. In this work, we look for conditions under which single-scenario inequalities of the form (F) are facet-defining for $\text{conv}(Z)$, the convex hull of the feasible region of the extensive form. Of particular interest is answering the question of when facet-defining inequalities for $\text{conv}(Z^\omega)$ are also facet-defining for $\text{conv}(Z)$.

Because answering this question in the full generality of the SMIP (1) is quite difficult, we focus here on a particular structured polytope. Specifically, we answer these questions for a stochastic variant of the well-known single-node flow (SNF) polytope $S$, given by the convex hull of the set [13, 20]:

$$X := \left\{ \begin{array}{l} x \in \{0,1\}^n \\ y \in \mathbb{R}^n_+ \end{array} \;\middle|\; \begin{array}{l} \sum_{i \in N^+} y_i - \sum_{i \in N^-} y_i \geq d \\ y_i \leq u_i x_i \;\forall\, i \in N \end{array} \right\}, \tag{2}$$

where $N = \{1, \ldots, n\}$, $(N^+, N^-)$ is a partition of $N$, $d \in \mathbb{R}$ and $u_i \in \mathbb{R}_+$ for all $i \in N$. The SNF polytope models a single node within a network, with some set of arcs $N^-$ entering the node, and another set $N^+$ leaving (Figure 1). The node itself has a fixed demand value $d$, and each arc $i$ has a capacity $u_i$. Each arc $i$ can be switched on or off by the binary variable $x_i$, and the amount of flow on that arc is controlled by the variable $y_i$. If arc $i$ is turned off ($x_i = 0$), it may not carry any flow.

The SNF polytope has received significant attention in the literature, and there are many known classes of valid inequalities for $S$ [1, 6, 10, 13, 18, 20, 24]. This widespread attention is due to the

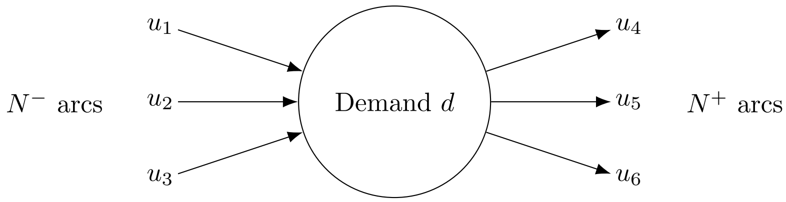


Figure 1: A single node modeled by the single-node flow (SNF) polytope $S$. Note that the "demand" $d$ may be either positive or negative.

fact that, in spite of its specific appearance, the SNF polytope can serve as a general relaxation for a very broad class of mixed-integer programs [12, 20]. The valid inequalities derived for the SNF polytope have proven very successful for the solution of general mixed-integer linear programs—in particular, the flow cover inequalities [13] and generalized flow cover inequalities [20].

In the sequel, we refer to $S$ as the *single-scenario* SNF polytope, to distinguish it from the stochastic version we introduce. We will demonstrate that, like its deterministic counterpart, the stochastic version of the SNF polytope is more general than it appears. In particular, valid inequalities for the stochastic SNF polytope correspond to valid inequalities for a broad subclass of two-stage SMIPs. This construction, given in Section 3, generalizes well-known results for the single-scenario polytope $S$ [12, 20].

The main contributions of this paper are as follows:

1. We introduce a stochastic variant of the single-scenario SNF polytope $S$, and illustrate that this stochastic variant can be viewed as a relaxation of a broad subclass of two-stage SMIPs with binary first-stage variables.
2. We provide necessary and sufficient conditions for valid inequalities for the single-scenario SNF polytope to be facet-defining for the stochastic SNF polytope.
3. We prove a series of corollaries that both illustrate the utility of our main theorem and extend a result for the single-scenario SNF polytope. In particular, we provide mild conditions under which *every* facet-defining inequality for the single-scenario SNF polytope is facet-defining for the stochastic SNF polytope.

## 2. The Stochastic Single-Node Flow Polytope

We define a stochastic (two-stage) extension of the single-node flow polytope $S$. Specifically, given a finite set of scenarios $\Omega$, we define $S(\Omega)$ to be the convex hull of the set

$$X(\Omega) := \left\{ \begin{array}{l} x \in \{0,1\}^n \\ y^\omega \in \mathbb{R}^n_+ \ \forall\, \omega \in \Omega \end{array} \;\middle|\; \begin{array}{ll} \sum_{i \in N^+} y_i^\omega - \sum_{i \in N^-} y_i^\omega \geq d^\omega & \forall\, \omega \in \Omega \\ y_i^\omega \leq u_i^\omega x_i & \forall\, i \in N,\ \omega \in \Omega \end{array} \right\},$$

where the parameters $u$ and $d$ are uncertain. Note that when $|\Omega| = 1$, $S(\Omega)$ reduces to the single-scenario SNF flow polytope $S$. We show in Section 3 that the polytope $S(\Omega)$ is more general than it appears. Although we assume for clarity that the second-stage variables $y^\omega$ are continuous in the sequel, we note that our main result (Theorem 1) also holds in the case when the second-stage variables are restricted to be integer (see Remark 1).

We consider two polytopes associated with $S(\Omega)$. For each $\omega \in \Omega$, let $X^\omega$ denote the set (2) with data $d^\omega$ and $u_i^\omega$ for $i \in N$, and let $S^\omega := \text{conv}(X^\omega)$ denote the single-scenario SNF polytope. Let $P^\omega$ denote the projection of $S(\Omega)$ onto the $(x, y^\omega)$ coordinates. It is easy to verify that $P^\omega \subseteq S^\omega$ for all $\omega \in \Omega$.

The following two propositions establish some basic properties of $S(\Omega)$, both of which generalize the corresponding results for the single-scenario polytope $S$.

**Proposition 1.** $S(\Omega)$ is full-dimensional if and only if all of the following hold:

$$u_i^\omega > 0 \text{ for all } i \in N, \omega \in \Omega \tag{A1}$$

$$\sum_{i \in N^+ \setminus \{j\}} u_i^\omega \geq d^\omega \text{ for all } j \in N^+,\ \omega \in \Omega. \tag{A2}$$

$$\sum_{i \in N^+} u_i^\omega > d^\omega \text{ for all } \omega \in \Omega. \tag{A3}$$

*Proof.* To show necessity, suppose $S(\Omega)$ is full-dimensional, i.e. $\dim(S(\Omega)) = n(|\Omega| + 1)$. First, suppose that $u_i^\omega \leq 0$ for some $i \in N$, $\omega \in \Omega$. If $u_i^\omega < 0$, then $S(\Omega) = \varnothing$, a contradiction. Otherwise, $u_i^\omega = 0$, which implies $y_i^\omega = 0$ for any point in $S(\Omega)$, and thus $S(\Omega)$ lies in a subspace of dimension $n(|\Omega| + 1) - 1$, a contradiction. Second, suppose $\sum_{i \in N^+ \setminus \{j\}} u_i^\omega < d^\omega$ for some $j \in N^+$ and $\omega \in \Omega$. Then $x_j = 1$ for any point in $S(\Omega)$, again a contradiction. Third, we take $N^+ = \varnothing$ (if $N^+ \neq \varnothing$, then (A1) and (A2) imply (A3), completing the proof), and suppose $d^\omega \geq 0$ for some $\omega \in \Omega$. If $d^\omega > 0$ then $S(\Omega) = \varnothing$, a contradiction. Otherwise, $d^\omega = 0$, and every point in $S(\Omega)$ satisfies $y_i^\omega = 0$ for all $i \in N$, again a contradiction.

Sufficiency is proven constructively using the $n(|\Omega| + 1) + 1$ affinely independent points shown in Figure 2. For this construction, we define $\varepsilon_i^\omega = \min\{u_i^\omega, d^\omega + \sum_{j \in N^-} u_j^\omega\}$ for all $i \in N^-$ and $\omega \in \Omega$. By (A1) and (A3), we have that $\varepsilon_i^\omega > 0$ for all $i \in N^-$ and $\omega \in \Omega$. It is simple to verify that these points lie in $X(\Omega) \subseteq S(\Omega)$ and are affinely independent using (A1) through (A3). □

Note that if $N^+$ is non-empty, then (A1) and (A2) imply (A3). Otherwise, we interpret (A3) to mean $d^\omega < 0$ for all $\omega \in \Omega$. We assume that (A1) through (A3) hold throughout the remainder of the paper. We note that this assumption is without loss of generality—the proof of Proposition 1 shows that if (A1) or (A2) does not hold, then we may eliminate variables to construct a full-dimensional instance. Otherwise, if (A3) does not hold, then $S(\Omega)$ is empty.

Before proceeding, we recall the definition of a facet-defining inequality, which will be used extensively for the remainder of the paper. Let $P \subset R^d$ be a polyhedron with $\dim(P) = q$. The inequality $a^{\mathrm{T}} z \leq b$ is *facet-defining* for $P$ if $a^{\mathrm{T}} z \leq b$ for all $z \in P$, there exists a point $z' \in P$ such that $a^{\mathrm{T}} z' < b$, and there exists $q$ affinely independent points in $P$ satisfying $a^{\mathrm{T}} z = b$. We note that if $q = d$, then $a^{\mathrm{T}} z \leq b$ for all $z \in P$ implies that there exists $z' \in P$ such that $a^{\mathrm{T}} z' < b$.

**Proposition 2.** The following inequalities are facet-defining for $S(\Omega)$: $x_i \leq 1$ (for all $i \in N$), $y_i^\omega \geq 0$ (for all $i \in N^-$, $\omega \in \Omega$), and $y_i^\omega \leq u_i^\omega x_i$ (for all $i \in N^+$, $\omega \in \Omega$).

*Proof (sketch).* To see that $x_i \leq 1$ is facet-defining, fix $i \in N$. If $i \in N^+$, the required affinely independent points are given by the points in Figure 2 with the point indexed by $i$ removed. If $i \in N^-$, the required points are obtained by first removing the point indexed by $i$. Next, for any remaining point with $x_i = 0$, set instead $x_i = 1$. The resulting points are still affinely independent. For the remaining inequalities, fix $i \in N$ and $\omega \in \Omega$. The required affinely independent points are given in Figure 2 with the point indexed by $i$ and $\omega$ removed. □

Lemma 1 characterizes how the facets of a polyhedron change under projection.

**Lemma 1 (Corollary 3.6 in [3]).** Let $Q = \{(u, z) \in \mathbb{R}^p \times \mathbb{R}^q \mid Au + Bz \leq b\}$ be a non-empty polyhedron, and let $(A^=, B^=, b^=)$ denote the equality subsystem of $Q$ (i.e., the set of inequalities satisfied with equality by all points in $Q$). Let $F = \{(u, z) \in Q \mid \eta^{\mathrm{T}} u + \gamma^{\mathrm{T}} z = \gamma_0\}$ be a facet of $Q$. Then $\mathrm{proj}_z(F)$ is a facet of a $\mathrm{proj}_z(Q)$ if and only if $\eta$ can be expressed as a linear combination of the rows of $A^=$. In particular, if $\eta = 0$, then $\mathrm{proj}_z(F)$ is always a facet of $\mathrm{proj}_z(Q)$.

**Proposition 3.** If for some $\omega \in \Omega$ the inequality (F) is facet-defining for $S(\Omega)$, then it is facet-defining for $P^\omega$. Moreover, if (F) is facet-defining for $S(\Omega)$ and $\beta = 0$, then (F) is facet-defining for $P^\omega$ for all $\omega \in \Omega$.

*Proof.* Follows from Lemma 1 with $Q = S(\Omega)$, $z = (x, y^\omega)$, $u = (y^1, \ldots, y^{\omega-1}, y^{\omega+1}, \ldots, y^{|\Omega|})$, $\gamma = (\alpha, \beta)$, $\gamma_0 = \tau$, and $\eta = 0$. □

As suggested by Proposition 3, the projected polytopes $P^\omega$ will play an important role in our analysis. In particular, our main result in Section 4 and its corollaries require constructing points which lie in the projected polytopes $P^\omega$. To that end, Proposition 4 characterizes the relationship between these projections $P^\omega$ and the single-scenario polytopes $S^\omega := \mathrm{conv}(X^\omega)$. In particular, we demonstrate that $P^\omega$ is equal to the convex hull of $X^\omega$ subject to the addition of $n$ knapsack constraints on the binary $x$ variables.

**Proposition 4.** For any scenario $\omega \in \Omega$,

$$P^\omega = \mathrm{conv}\left\{(x, y^\omega) \in X^\omega \;\middle|\; \sum_{j \in N^+} u_j^\ell x_j \geq \max\{d^\ell, 0\} \text{ for all } \ell \in \Omega\right\}.$$

In particular, if $d^\omega \leq 0$ for all $\omega \in \Omega$, then $P^\omega = S^\omega$ for all $\omega \in \Omega$.

*Proof.* "$\subseteq$" Fix $(x, z^\omega) \in \mathrm{proj}_{(x,y^\omega)}(X(\Omega))$. It is immediate that $(x, z^\omega) \in X^\omega$. Furthermore, for all scenarios $\ell \neq \omega$, there exists a vector $z^\ell \in \mathbb{R}^n$ such that the point $(x, z^1, \ldots, z^\omega, \ldots, z^{|\Omega|}) \in X(\Omega)$.

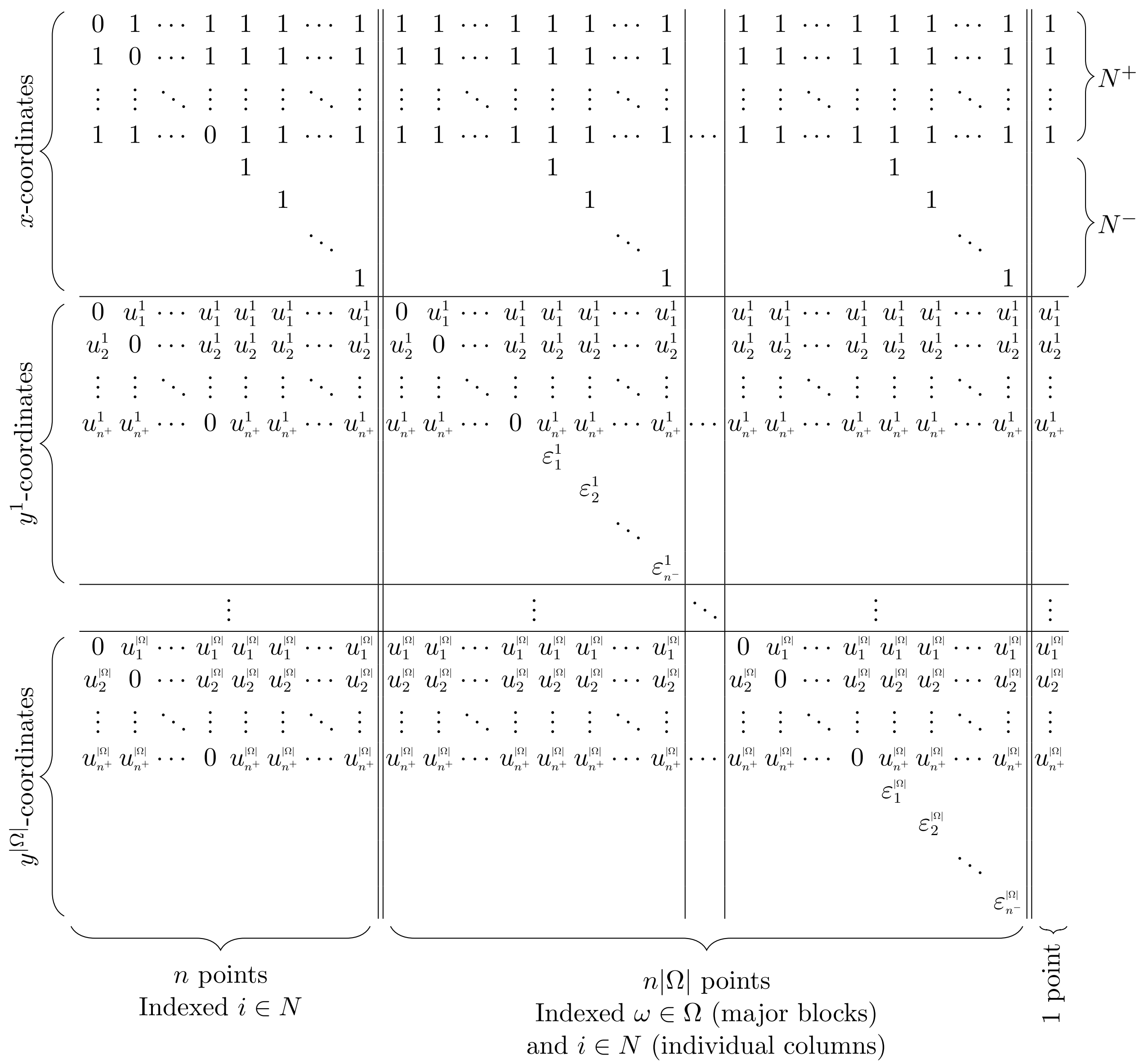


Figure 2: Set of $n(|\Omega|+1)+1$ affinely independent points that lie in the stochastic SNF polytope $S(\Omega)$, used to prove Proposition 1 and Proposition 2. Each point is a column in the above matrix. Blank entries are assumed to be zero, and we use the notation $n^+ = |N^+|$, $n^- = |N^-|$.

In particular, for any $\ell \in \Omega$, the point $(x, z^\ell) \in X(\Omega)$, and hence $\sum_{j \in N^+} u_j^\ell x_j \geq \sum_{j \in N^+} z_j^\ell \geq \sum_{j \in N^+} z_j^\ell - \sum_{j \in N^-} z_j^\ell \geq d^\ell$. Moreover, by (A1), $\sum_{j \in N^+} u_j^\ell x_j \geq 0$. Hence $\sum_{j \in N^+} u_j^\ell x_j \geq \max\{d^\ell, 0\}$.

"$\supseteq$" Fix $(x, z^\omega) \in X^\omega$ such that $\sum_{j \in N^+} u_j^\ell x_j \geq \max\{d^\ell, 0\}$ for all $\ell \in \Omega$. For all scenarios $\ell \neq \omega$, define $z^\ell \in \mathbb{R}^n$ by $z_j^\ell = u_j^\ell x_j$ for $j \in N^+$, and 0 otherwise. We wish to show that $(x, z^1, \ldots, z^\omega, \ldots, z^{|\Omega|}) \in X(\Omega)$. By definition of $X(\Omega)$ and $X^\ell$, it suffices to show that $(x, z^\ell) \in X^\ell$

for all $\ell \neq \omega$. Clearly, $(x, z^\ell)$ satisfies $0 \leq z_j^\ell \leq u_j^\ell x_j$ for all $j \in N$. The flow constraint is also satisfied: $\sum_{j\in N^+} z_j^\ell - \sum_{j\in N^-} z_j^\ell = \sum_{j\in N^+} u_j^\ell x_j \geq \max\{d^\ell, 0\} \geq d^\ell$.

Finally, if $d^\omega \leq 0$ for all $\omega \in \Omega$, then $\max\{d^\omega, 0\} = 0$ for all $\omega \in \Omega$, and the constraint $\sum_{j\in N^+} u_j^\omega x_j \geq 0$ is redundant (recall that $u_j^\omega > 0$ for all $j$ and $\omega$ by (A1)). We conclude that $P^\omega = S^\omega$. □

Example 1 shows that the converse of Proposition 3 is not true—that is, an inequality of the form (F) which is facet-defining for the projection $P^\omega$ is not necessarily facet-defining for $S(\Omega)$. It also illustrates the utility of Proposition 4.

**Example 1.** Consider an instance of the stochastic SNF polytope $S(\Omega)$ with $N = N^+ = \{1, 2, 3, 4\}$ and $N^- = \varnothing$. Let there be two scenarios ($\Omega = \{1, 2\}$), with data given by $u^1 = (1,\ 1,\ 1,\ 1)$, $u^2 = (1.1,\ 1.1,\ 1.1,\ 0.9)$, and $d^1 = d^2 = 3$. The single-scenario polytope $S^1$ is defined by the inequalities $x_i \leq 1$ for all $i \in N$, $y_i^1 \leq u_i^1 x_i$ for all $i \in N$ and $\sum_{i\in N} y_i^1 \geq d^1$. The polytope $S^2$ is defined by the inequalities $x_i \leq 1$ for all $i \in N$, $y_i^2 \leq u_i^2 x_i$ for all $i \in N$, $\sum_{i\in N} y_i^2 \geq d^2$, the cover inequality $\sum_{i\in N} x_i \geq 3$, and the flow cover inequalities:

$$5x_2 + 5x_3 + 4x_4 + 5y_1^2 \geq 14, \qquad 5x_1 + 5x_3 + 4x_4 + 5y_2^2 \geq 14,$$

$$5x_1 + 5x_2 + 4x_4 + 5y_3^2 \geq 14, \qquad 4x_1 + 4x_2 + 4x_3 + 5y_4^2 \geq 12.$$

By applying Proposition 4, we can verify that $P^1 = S^1$ and $P^2 = S^2$. However, the inequality $\sum_{i\in N} x_i \geq 3$, which is facet-defining for $P^2 = S^2$, is not facet-defining for $S(\Omega)$. To see this, note that $\sum_{i\in N} x_i \geq 3$ can be expressed as a linear combination (sum) of the facet-defining (for $S(\Omega)$) inequalities $\sum_{i\in N} y_i^1 \geq 3$ and $y_i^1 \leq x_i$ for all $i \in N$.

## 3. Generality of the Stochastic SNF Polytope

It is well known that the single-scenario single-node flow polytope $S$ is more general than it appears—under mild assumptions, any mixed-integer constraint with variable upper bounds on some variables can be relaxed into a single-node flow polytope [12, Sec. II.2.4], [20]. We now illustrate that, under slightly different assumptions, a similar result holds for the stochastic SNF polytope $S(\Omega)$.

Consider a mixed-integer set of the form

$$Q := \left\{ \begin{array}{l|l} & \sum_{i\in I_1}(\pi_i^\omega v_i^\omega + \theta_i^\omega x_i) + \sum_{i\in I_2} \pi_i^\omega v_i^\omega + \sum_{i\in I_3} \theta_i^\omega x_i \geq d^\omega \ \forall\, \omega \in \Omega \\ x_i \in \{0,1\} \ \forall\, i \in I_1 \cup I_3 & \\ v^\omega \in \mathcal{S} \ \forall\, \omega \in \Omega & 0 \leq v_i^\omega \leq m_i^\omega x_i \ \forall\, i \in I_1, \omega \in \Omega \\ & 0 \leq v_i^\omega \leq m_i^\omega \ \forall\, i \in I_2, \omega \in \Omega \end{array} \right\},$$

where $I_j$ for $j = 1, 2, 3$ are disjoint index sets with $n_j = |I_j|$, and $\mathcal{S} \subseteq \mathbb{R}_+^{n_1+n_2}$ imposes integrality restrictions on a subset of the variables (this allows for binary variables by taking $m_i^\omega = 1$). The set $Q$ can be viewed as a single row of a two-stage SMIP with binary first-stage variables, where the first-stage variables enforce variable upper bounds on a subset of second-stage variables.

Next, we make two assumptions. First, for any fixed $i \in I_1 \cup I_2$ (resp. $i \in I_1 \cup I_3$), the coefficients $\pi_i^\omega$ (resp. $\theta_i^\omega$) have the same sign for all $\omega \in \Omega$. Second, $\pi_i^\omega \theta_i^\omega \geq 0$ for all $i \in I_1$ and $\omega \in \Omega$ (cf. [12, Sec. II.2.4]). Define the index sets $I_1^+ := \{i \in I_1 \mid \pi_i^\omega > 0 \text{ or } \pi_i^\omega = 0 \text{ and } \theta_i^\omega > 0\}$, $I_2^+ := \{i \in I_2 \mid \pi_i^\omega > 0\}$, $I_3^+ := \{i \in I_3 \mid \theta_i^\omega > 0\}$, and $I_j^- := I_j \setminus I_j^+$ for $j = 1, 2, 3$. By the first of these assumptions, the sets $I_j^+$ and $I_j^-$ do not depend on the scenario $\omega$. Define

$$y_i^\omega := \begin{cases} \pm(\pi_i^\omega v_i^\omega + \theta_i^\omega x_i), & i \in I_1^\pm, \\ \pm\pi_i^\omega v_i^\omega, & i \in I_2^\pm, \\ \pm\theta_i^\omega x_i, & i \in I_3^\pm \end{cases} \quad \text{and} \quad u_i^\omega := \begin{cases} \pm(\pi_i^\omega m_i^\omega + \theta_i^\omega), & i \in I_1^\pm, \\ \pm\pi_i^\omega m_i^\omega, & i \in I_2^\pm, \\ \pm\theta_i^\omega, & i \in I_3^\pm \end{cases} \tag{3}$$

where the sign $+$ is chosen if $i \in I_1^+$, etc. Note that the second assumption ensures that $y_i^\omega \geq 0$ for all $i$ and $\omega$. Then the set $Q$ can be equivalently represented by the constraints

$$\sum_{i \in N^+} y_i^\omega - \sum_{i \in N^-} y_i^\omega \geq d^\omega \; \forall \, \omega \in \Omega$$
$$0 \leq y_i^\omega \leq u_i^\omega x_i \forall \, i \in N, \omega \in \Omega$$

where $N^+ = I_1^+ \cup I_2^+ \cup I_3^+$, $N^- = I_1^- \cup I_2^- \cup I_3^-$, and $N = N^+ \cup N^-$, subject to the additional constraints $x_i = 1$ for all $i \in I_2$, $v^\omega \in \mathcal{S}$ for all $\omega \in \Omega$, and (3). A concrete illustration of such a relaxation (Example 2) is given in Section 4.

## 4. Main Result and Corollaries

Theorem 1, our main result, provides necessary and sufficient conditions for an inequality of the form (F) to be facet-defining to the stochastic single-node flow polytope $S(\Omega)$.

**Theorem 1.** Suppose that, for some scenario $\omega \in \Omega$, the inequality (F) is valid for $S^\omega$, and both of the following conditions hold:

1. For all $i \in N$ and $\ell \in \Omega \setminus \{\omega\}$, there exists a point $(x, y^1, \ldots, y^{|\Omega|}) \in X(\Omega)$ that satisfies (F) with equality such that $x_i = 1$ and $\sum_{j \in N^+} u_j^\ell x_j > d^\ell$.
2. The inequality (F) is facet-defining for $P^\omega$.

Then (F) is facet-defining for the stochastic SNF polytope $S(\Omega)$.

Moreover, if (F) is not a scalar multiple of $y_i^\omega \geq 0$ for any $i \in N^-$ nor a scalar multiple of $y_i^\omega \leq u_i^\omega x_i$ for any $i \in N^+$, then Conditions 1 and 2 above are also necessary.

*Proof.* The proof is constructive and is given in Appendix A. □

**Remark 1.** Theorem 1 extends to the case where an arbitrary subset of the second-stage variables are restricted to be integers, provided that the problem data $(u^\omega, d^\omega)_{\omega \in \Omega}$ are integral. The latter assumption results in only a minor loss of generality and holds true even if the problem data are rational (which can be made integral by an appropriate scaling). In this case, the proof of Theorem 1 goes through by noting that the condition $\sum_{j \in N^+} u_j^\ell x_j > d^\ell$ is equivalent to $\sum_{j \in N^+} u_j^\ell x_j \geq d^\ell + 1$, and using construction given in Appendix A.

Theorem 1 can be extended to the case when the inequality (F) satisfies $\beta \equiv 0$—i.e., the only non-zero coefficients in the inequality are on the binary $x$ variables.

**Corollary 1.** An inequality of the form $\alpha^{\mathrm{T}} x \geq \tau$ is facet-defining for $S(\Omega)$ if and only if both of the following conditions hold:

1. For all $i \in N$ and $\omega \in \Omega$, there exists a point $(x, y^1, \ldots, y^{|\Omega|}) \in X(\Omega)$ that satisfies $\alpha^{\mathrm{T}} x = \tau$, $x_i = 1$ and $\sum_{j \in N^+} u_j^\omega x_j > d^\omega$.
2. The inequality is facet-defining for $\mathrm{proj}_x(S(\Omega))$.

*Proof.* The proof of sufficiency is similar to the constructive proof of Theorem 1, except that the set of point given by (P1) now contains $n|\Omega|$ points (one for each $i \in N$ and $\omega \in \Omega$), and the set of points given by (P2) contains only $n$ points, because $\mathrm{proj}_x(S(\Omega)) \subset \mathbb{R}^n$ (rather than $P^\omega \subset \mathbb{R}^{2n}$ in Theorem 1). The proof of necessity of Condition 1 is identical to that of Theorem 1. The necessity of Condition 2 follows from Lemma 1 with $Q = S(\Omega)$, $z = x$, $u = (y^1, \ldots, y^{|\Omega|})$, $\gamma = \alpha$, $\gamma_0 = \tau$, and $\eta = 0$. $\square$

Although Condition 2 of Theorem 1 and Corollary 1 may be difficult to verify in general, Corollary 2 and Corollary 3 illustrate important cases in which Condition 2 can be shown to hold. Corollary 2 shows that if all of the demands $d^\omega$ are strictly negative, then every inequality that is facet-defining for the well-studied SNF polytopes $S^\omega$ is also facet-defining for the stochastic SNF polytope $S(\Omega)$.

**Corollary 2.** Suppose that $d^\omega < 0$ for all $\omega \in \Omega$. Then for any $\omega \in \Omega$, any inequality (F) that is facet-defining for $S^\omega$ is facet-defining for $S(\Omega)$.

*Proof.* Fix $\omega \in \Omega$. It suffices to show that Conditions 1 and 2 of Theorem 1 hold. Because $d^\omega < 0$ for all $\omega \in \Omega$, Condition 1 is equivalent to showing that, for all $i \in N$, there exists a point $(\hat{x}, \hat{y}^1, \ldots, \hat{y}^{|\Omega|}) \in X(\Omega)$ that satisfies (F) with equality such that $\hat{x}_i = 1$. Fix $i \in N$. Because (F) is facet-defining for $P^\omega$, there exists a point $(\hat{x}, \hat{y}^\omega) \in \mathrm{proj}_{(x,y^\omega)}(X(\Omega))$ that satisfies (F) with equality such that $\hat{x}_i = 1$ (this follows because the inequality $x_i \geq 0$ can never be facet-defining for $\mathrm{proj}_{(x,y^\omega)}(X(\Omega))$—any point which satisfies $x_i = 0$ must also satisfy $y_i^\omega = 0$, and thus the face defined by $x_i \geq 0$ is not of sufficiently high dimension to be facet-defining). Because $(\hat{x}, \hat{y}^\omega) \in \mathrm{proj}_{(x,y^\omega)}(X(\Omega))$, for all scenarios $\ell \neq \omega$ there exists $\hat{y}^\ell \in \mathbb{R}^n$ such that $(\hat{x}, \hat{y}^1, \ldots, \hat{y}^\omega, \ldots, \hat{y}^{|\Omega|}) \in X(\Omega)$. Moreover, this point satisfies (F) with equality, as well as $\hat{x}_i = 1$. Hence, Condition 1 of Theorem 1 is satisfied. Moreover, by Proposition 4, $S^\omega = P^\omega$ for all $\omega \in \Omega$, and thus Condition 2 of Theorem 1 is satisfied. We conclude that (F) is facet-defining for $S(\Omega)$. $\square$

We note that the conditions of Corollary 2 are satisfied in many applications.

**Example 2 (Stochastic Facility Location).** Define the stochastic capacitated facility location problem as follows (see [17] for a review): consider facilities $i = 1, \ldots, n$, and customers $j =$

$1,\ldots,m$. Let $b_j^\omega > 0$ denote the demand by customer $j$ under scenario $\omega \in \Omega$, and let $c_i^\omega > 0$ denote the capacity of facility $i$ under scenario $\omega \in \Omega$, where as before, $|\Omega| < \infty$. Let the variable $y_{ij}^\omega$ denote the amount of customer $j$'s demand fulfilled by facility $i$ under scenario $\omega$. Let the variable $x_i \in \{0,1\}$ denote whether facility $i$ is opened (a decision made before the facility capacities and customer demands are known). The feasible region of the stochastic (capacitated) facility location problem is defined by the constraints

$$\sum_{i=1}^{n} y_{ij}^\omega \leq b_j^\omega \qquad \text{for all } j = 1,\ldots,m \text{ and for all } \omega \in \Omega, \tag{4a}$$

$$\sum_{j=1}^{m} y_{ij}^\omega \leq c_i^\omega x_i \qquad \text{for all } i = 1,\ldots,n \text{ and for all } \omega \in \Omega, \tag{4b}$$

$$y_{ij}^\omega \geq 0 \qquad \text{for all } i = 1,\ldots,n \text{ and } j = 1,\ldots,m, \text{ and for all } \omega \in \Omega, \tag{4c}$$

$$x_i \in \{0,1\} \qquad \text{for all } i = 1,\ldots,n. \tag{4d}$$

Models related to (4) arise in the literature (e.g., [11, 16, 17]). If we consider only a single customer $j$, then (4) can be relaxed into a stochastic SNF polytope $S(\Omega)$, with $N = N^+ = \{1,\ldots,n\}$ ($N^- = \emptyset$), $d^\omega = b_j^\omega$ for all $\omega \in \Omega$ and $u_i^\omega = c_i^\omega$ for all $i \in N$ and $\omega \in \Omega$. By multiplying the demands by $-1$ and interchanging the sets $N^+$ and $N^-$, then applying Corollary 2, it follows that for all scenarios $\omega \in \Omega$, any facet-defining inequality for the ($\leq$-constrained) single-scenario, single-customer SNF polytope is facet-defining for the stochastic SNF polytope.

When only the demands $d^\omega$ are uncertain, the stochastic SNF polytope may be viewed as representing a single node within a larger network facing demand uncertainty (for example, as a single customer in the stochastic facility location problem above). In this setting, Corollary 3 tells us that the facet-defining inequalities corresponding to the most restrictive (i.e., highest demand) scenario are also facet-defining for $S(\Omega)$.

**Corollary 3.** Consider a stochastic SNF polytope $S(\Omega)$ that satisfies $u_i^\omega = u_i$ for all $\omega \in \Omega$ and $i \in N$, and let $\omega_0 \in \arg\max\{d^\omega \mid \omega \in \Omega\}$. Then every facet-defining inequality for $S^{\omega_0}$ is also facet-defining for $S(\Omega)$.

*Proof.* Suppose that (F) is facet-defining for $S^{\omega_0}$. If (F) is a scalar multiple of the inequality $y_i^{\omega_0} \geq 0$ (for some $i \in N^-$) or $y_i^{\omega_0} \leq u_i^{\omega_0} x_i$ (for some $i \in N^+$), then by Proposition 2, we are done. Hence, for the remainder of the proof we assume that (F) is not a scalar multiple of the inequalities $y_i^{\omega_0} \geq 0$ for any $i \in N^-$ or $y_i^{\omega_0} \leq u_i^{\omega_0} x_i$ for any $i \in N^+$.

We next claim that $S^{\omega_0} = P^{\omega_0}$. That $P^{\omega_0} \subseteq S^{\omega_0}$ is trivial. To establish $S^{\omega_0} \subseteq P^{\omega_0}$, it suffices to show that $X^{\omega_0} \subseteq P^{\omega_0}$, because every point in $S^{\omega_0}$ can be written as a finite convex combination of points in $X^{\omega_0}$, and the set $P^{\omega_0}$ is convex. To that end, fix $(\bar{x}, \bar{z}^{\omega_0}) \in X^{\omega_0} \subseteq S^{\omega_0}$. Then

$$\sum_{j \in N^+} u_j \bar{x}_j \geq \sum_{j \in N^+} \bar{z}_j^{\omega_0} \geq \sum_{j \in N^+} \bar{z}_j^{\omega_0} - \sum_{j \in N^-} \bar{z}_j^{\omega_0} \geq d^{\omega_0} \geq d^\omega,$$

where the last inequality holds for any scenario $\omega \in \Omega$. By Proposition 4, the point $(\bar{x}, \bar{z}^{\omega_0})$ lies in

$P^{\omega_0}$, and thus $S^{\omega_0} \subseteq P^{\omega_0}$, which implies $S^{\omega_0} = P^{\omega_0}$. Because $S^{\omega_0} = P^{\omega_0}$, Condition 2 of Theorem 1 is satisfied. To show that Condition 1 is also satisfied, it suffices to show that for all $i \in N$ there exists a point $(\hat{x}, \hat{y}^1, \ldots, \hat{y}^{|\Omega|}) \in X(\Omega)$ (depending on $i$) that satisfies (F) with equality such that $\hat{x}_i = 1$ and $\sum_{j \in N^+} u_j \hat{x}_j > d^{\omega_0}$. Fix $i \in N$. If $i \in N^+$, then, because (F) is facet-defining for $P^{\omega_0}$ and is not a scalar multiple of $y^{\omega_0} \leq u_i x_i$, there exists a point $(\hat{x}, \hat{y}^{\omega_0}) \in \mathrm{proj}_{(x,y^{\omega_0})}(X(\Omega))$ that satisfies (F) with equality, and $\hat{y}^{\omega_0} < u_i \hat{x}_i$ (in particular, $\hat{x}_i = 1$). Thus

$$\sum_{j \in N^+} u_j \hat{x}_j > \sum_{j \in N^+} \hat{y}_j^{\omega_0} \geq \sum_{j \in N^+} \hat{y}^{\omega_0} - \sum_{j \in N^-} \hat{y}_j^{\omega_0} \geq d^{\omega_0}.$$

Because $(\hat{x}, \hat{y}^{\omega_0}) \in \mathrm{proj}_{(x,y^{\omega_0})}(X(\Omega))$, for all scenarios $\ell \neq \omega_0$, there exist $\hat{y}^\ell \in \mathbb{R}^n$ such that $(\hat{x}, \hat{y}^1, \ldots, \hat{y}^{\omega_0}, \ldots, \hat{y}^{|\Omega|}) \in X(\Omega)$. Moreover, this point satisfies (F) with equality, $\hat{x}_i = 1$, and $\sum_{j \in N^+} u_j \hat{x}_j > d^{\omega_0}$, and thus satisfies Condition 1 of Theorem 1. A similar argument follows if $i \in N^-$ (*cf.* the proof of necessity in Theorem 1), and we conclude by Theorem 1 that (F) is facet-defining for $S(\Omega)$. □

Finally, we show that Theorem 1 has implications for the single-scenario (i.e., deterministic) setting as well. In particular, Corollary 1 generalizes a theorem of Padberg et al. [13], which relates the single-scenario SNF polytope $S$ with the so-called "associated knapsack polytope." In fact, the associated knapsack polytope is simply the set $\mathrm{proj}_x(S(\Omega))$ in the case when $|\Omega| = 1$.

**Corollary 4.** (Theorem 2 of [13]) Suppose $N^- = \varnothing$, and that the inequality $\alpha^{\mathrm{T}} x \geq \tau$ is facet-defining for $\mathrm{proj}_x(S)$. Then $\alpha^{\mathrm{T}} x \geq \tau$ is facet-defining for $S$ if and only if for all $i \in N$, there exists a point $(x^{(i)}, y^{(i)}) \in X$ that satisfies $\alpha^{\mathrm{T}} x^{(i)} = \tau$, $x_i^{(i)} = 1$, and $y_i^{(i)} < u_i$.

*Proof.* Apply Corollary 1 with $|\Omega| = 1$ and $N^- = \varnothing$. □

## 5. Conclusion and Future Work

We have introduced a stochastic programming variant of the single-node flow polytope, and provided conditions under which facet-defining inequalities for only a single scenario are also facet-defining for the convex hull of the stochastic SNF polytope itself. Future directions include extending these results to other polyhedra, in order to gain a better understanding of the facial structure of the convex hull of the extensive form for more general SMIPs. The results of Section 3 also illustrate the potential utility of deriving valid inequalities for the stochastic SNF polytope which incorporate non-zero coefficients for multiple scenarios, which could be used in the solution of the extensive form of more general two-stage SMIPs.

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## A. Proof of Main Result

The following two lemmas will be used to prove Theorem 1—their proofs are straightforward and omitted for brevity.

**Lemma 2.** Suppose $\{x^{(i)} \in \mathbb{R}^M \mid i = 1, \ldots, n\}$ is a set of $n$ affinely independent points, and let $\lambda_1, \ldots, \lambda_n$ be scalars such that $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_1 \neq 0$. Define $\hat{x} = \sum_{i=1}^n \lambda_i x^{(i)}$. Then $\hat{x}, x^{(2)}, \ldots, x^{(n)}$ are affinely independent.

**Lemma 3.** Suppose $\{(f^{(i)}, g^{(i)}) \in \mathbb{R}^{M_1+M_2} \mid i = 1, \ldots, N\}$ is a set of $N$ points such that $f^{(1)}, \ldots, f^{(N)}$ are affinely independent. Suppose $h^{(1)}, \ldots, h^{(m)}$ are $m$ linearly independent points in $\mathbb{R}^{M_2}$. Then for any $s \in \{1, \ldots, N\}$, the $N + m$ points

$$\begin{pmatrix} f^{(1)} \\ g^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} f^{(N)} \\ g^{(N)} \end{pmatrix}, \quad \begin{pmatrix} f^{(s)} \\ g^{(s)} + h^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} f^{(s)} \\ g^{(s)} + h^{(m)} \end{pmatrix}$$

are affinely independent.

In the proof of Theorem 1, we use the following notation. Let $\vec{e}_{i\omega}$ denote the standard basis vector in $\mathbb{R}^{n(|\Omega|-1)}$, with $y^\omega = e_i$ and $y^\ell = 0$ for all scenarios $\ell \neq \omega$, where $e_i$ is the $i^{\text{th}}$ standard basis vector in $\mathbb{R}^n$.

*Proof of Theorem 1.*

**Sufficiency** It is trivial to show that (F) being valid for $S^\omega$ implies that (F) is valid for $S(\Omega)$. We now construct $n(|\Omega|+1)$ affinely independent points in $S(\Omega)$ that satisfy (F) with equality. We construct these points in two separate sets, Set 1 containing $n(|\Omega|-1)$ points, and Set 2 containing $2n$ points. In our construction, we use $\ell \in \Omega \setminus \{\omega\}$ to index points in $X(\Omega)$, and use $r \in \Omega$ to index second-stage variables of points in $X(\Omega)$.

**Set 1:** Fix $(i,\ell) \in N \times \Omega \setminus \{\omega\}$, let $(x^{(i,\ell)}, y^{1,(i,\ell)}, \dots, y^{|\Omega|,(i,\ell)}) \in X(\Omega)$ denote the point whose existence is asserted by Condition 1 of the theorem, and define the positive constant

$$\varepsilon_{i\ell} = \min\left\{u_i^\ell, \sum_{j\in N^+} u_j^\ell x_j^{(i,\ell)} - d^\ell\right\} > 0$$

(recall that $u_i^\ell > 0$ by (A1)). For all $r \in \Omega$, define the vector $z^{r,(i,\ell)} \in \mathbb{R}^n$ by $z^{r,(i,\ell)} = y^{r,(i,\ell)}$ for $r \neq \ell$, and

$$z_j^{\ell,(i,\ell)} = \begin{cases} u_i^\ell - \varepsilon_{i\ell}, & \text{if } i \in N^+ \text{ and } j = i, \\ \varepsilon_{i\ell}, & \text{if } i \in N^- \text{ and } j = i, \\ u_j^\ell x_j^{(i,\ell)}, & \text{if } j \in N^+ \text{ and } j \neq i, \\ 0, & \text{otherwise (i.e., } j \in N^- \text{ and } j \neq i). \end{cases}$$

For brevity, let $z^{(i,\ell)} := (z^{1,(i,\ell)}, \dots, z^{|\Omega|,(i,\ell)}) \in \mathbb{R}^{n|\Omega|}$.

We now show that the point $(x^{(i,\ell)}, z^{(i,\ell)})$ satisfies (F) with equality and lies in $X(\Omega)$. The former is true because $\ell \neq \omega$ implies $z^{\omega,(i,\ell)} = y^{\omega,(i,\ell)}$, and $(x^{(i,\ell)}, y^{\omega,(i,\ell)})$ satisfies (F) with equality by Condition 1. Because $x^{(i,\ell)}$ is integral, to show $(x^{(i,\ell)}, z^{(i,\ell)})$ lies in $X(\Omega)$, it suffices to show that $(x^{(i,\ell)}, z^{r,(i,\ell)}) \in X^r$ for all scenarios $r \in \Omega$. If $r \neq \ell$, then $z^{r,(i,\ell)} = y^{r,(i,\ell)}$, and thus $(x^{(i,\ell)}, z^{r,(i,\ell)}) \in X^r$ by Condition 1.

Otherwise, if $r = \ell$, we show that $(x^{(i,\ell)}, z^{\ell,(i,\ell)})$ satisfies both the variable upper bound constraints and the flow constraint:

- *Variable upper bound constraints*: For $j \neq i$, $z_j^{\ell,(i,\ell)}$ is either 0 or $u_j^\ell x_j^{(i,\ell)}$. In either case, $z_j^{\ell,(i,\ell)} \leq u_j^\ell x_j^{(i,\ell)}$. For $j = i$, Condition 1 gives that $x_i^{(i,\ell)} = 1$. The coordinate $z_i^{\ell,(i,\ell)}$ is either $\varepsilon_{i\ell}$ or $u_i^\ell - \varepsilon_{i\ell}$, both of which lie between 0 and $u_i^\ell$.
- *Flow constraint*: If $i \in N^+$, then

$$\sum_{j\in N^+} z_j^{\ell,(i,\ell)} = -\varepsilon_{i\ell} + \sum_{j\in N^+} u_j^\ell x_j^{(i,\ell)}, \text{ and } \sum_{j\in N^-} z_j^{\ell,(i,\ell)} = 0.$$

Otherwise, if $i \in N^-$, then

$$\sum_{j\in N^+} z_j^{\ell,(i,\ell)} = \sum_{j\in N^+} u_j^\ell x_j^{(i,\ell)}, \text{ and } \sum_{j\in N^-} z_j^{\ell,(i,\ell)} = \varepsilon_{i\ell}.$$

In either case, it follows from the definition of $\varepsilon_{i\ell}$ that

$$-\varepsilon_{i\ell} + \sum_{j\in N^+} u_j^\ell x_j^{(i,\ell)} = \sum_{j\in N^+} z_j^{\ell,(i,\ell)} - \sum_{j\in N^-} z_j^{\ell,(i,\ell)} \geq d^\ell.$$

We conclude that the point $(x^{(i,\ell)}, z^{(i,\ell)})$ lies in $X(\Omega)$ and satisfies (F) with equality.

Next, note that if $i \in N^+$, we may add $\varepsilon_{i\ell}$ to the $i^{\text{th}}$ entry of $z^{\ell,(i,\ell)}$, and the vector $(x^{(i,\ell)}, z^{(i,\ell)})$ will remain in $X(\Omega)$. Similarly, if $i \in N^-$, we may subtract $\varepsilon_{i\ell}$ from the $i^{\text{th}}$ entry of $z^{\ell,(i,\ell)}$ and the vector $(x^{(i,\ell)}, z^{(i,\ell)})$ will remain in $X(\Omega)$. Hence, if $i \in N^+$, the vector

$$\tfrac{1}{n(|\Omega|-1)} \begin{pmatrix} x^{(i,\ell)} \\ z^{(i,\ell)} + \varepsilon_{i\ell}\vec{e}_{i\ell} \end{pmatrix} + \tfrac{1}{n(|\Omega|-1)} \sum_{\substack{(j,r)\in N\times\Omega\setminus\{\omega\} \\ (j,r)\neq(i,\ell)}} \begin{pmatrix} x^{(j,r)} \\ z^{(j,r)} \end{pmatrix} \tag{5}$$

lies in $S(\Omega)$ as a convex combination of points in $X(\Omega)$, and satisfies (F) with equality, as a convex combination of points that satisfy (F) with equality. Similarly, if $i \in N^-$, the point

$$\tfrac{1}{n(|\Omega|-1)} \begin{pmatrix} x^{(i,\ell)} \\ z^{(i,\ell)} - \varepsilon_{i\ell}\vec{e}_{i\ell} \end{pmatrix} + \tfrac{1}{n(|\Omega|-1)} \sum_{\substack{(j,r)\in N\times\Omega\setminus\{\omega\} \\ (j,r)\neq(i,\ell)}} \begin{pmatrix} x^{(j,r)} \\ z^{(j,r)} \end{pmatrix} \tag{6}$$

lies in $S(\Omega)$ and satisfies (F) with equality. Define the midpoint of the vectors $(x^{(i,\ell)}, z^{(i,\ell)})$ as

$$\begin{pmatrix} \hat{x} \\ \hat{z} \end{pmatrix} := \tfrac{1}{n(|\Omega|-1)} \sum_{(i,\ell)\in N\times\Omega\setminus\{\omega\}} \begin{pmatrix} x^{(i,\ell)} \\ z^{(i,\ell)} \end{pmatrix}. \tag{7}$$

Set 1 is given by the $n(|\Omega|-1)$ points

$$\left\{ \begin{pmatrix} \hat{x} \\ \hat{z} + \frac{1}{N(|\Omega|-1)}\varepsilon_{i\ell}\vec{e}_{i\ell} \end{pmatrix} : i \in N^+, \ell \in \Omega\setminus\{\omega\} \right\} \bigcup \left\{ \begin{pmatrix} \hat{x} \\ \hat{z} - \frac{1}{n(|\Omega|-1)}\varepsilon_{i\ell}\vec{e}_{i\ell} \end{pmatrix} : i \in N^-, \ell \in \Omega\setminus\{\omega\} \right\}. \tag{P1}$$

We emphasize that the points (P1) lie in $S(\Omega)$ and satisfy (F) with equality.

**Set 2:** By Condition 2, there exist $2n$ affinely independent points $\{(\bar{x}^{(q)}, \bar{y}^{\omega,(q)})\}_{q=1}^{2n}$ which lie in $\text{proj}_{(x,y^\omega)}(X(\Omega)) \subseteq \mathbb{R}^{2n}$ and satisfy (F) with equality. Hence, the affine hull of these points equals $\{(x,y) \in \mathbb{R}^{2n} \mid \alpha^{\mathrm{T}}x + \beta^{\mathrm{T}}y = \tau\}$. In particular, if we let $\hat{z}^\omega$ denote the scenario $\omega$ component of the vector $\hat{z}$ in (7), then the point $(\hat{x}, \hat{z}^\omega)$ can be expressed as an affine combination of the points

$\{(\bar{x}^{(q)}, \bar{y}^{\omega,(q)})\}$, and thus, by Lemma 2, the $2n$ points

$$\begin{pmatrix} \hat{x} \\ \hat{z}^{\omega} \end{pmatrix}, \begin{pmatrix} \bar{x}^{(2)} \\ \bar{y}^{\omega,(2)} \end{pmatrix}, \ldots, \begin{pmatrix} \bar{x}^{(2n)} \\ \bar{y}^{\omega,(2n)} \end{pmatrix}$$

are affinely independent. Moreover, for all $q = 2, \ldots, 2n$, $(\bar{x}^{(q)}, \bar{y}^{\omega,(q)}) \in \text{proj}_{(x,y^{\omega})}(X(\Omega))$ implies that, for all $\ell \in \Omega \setminus \{\omega\}$ there exists $\bar{y}^{\ell,(q)} \in \mathbb{R}^n$ such that $(\bar{x}^{(q)}, \bar{y}^{1,(q)}, \ldots, \bar{y}^{\omega,(q)}, \ldots, \bar{y}^{|\Omega|,(q)}) \in X(\Omega)$. Hence, the $2n$ points

$$\begin{pmatrix} \hat{x} \\ \hat{z}^{1} \\ \vdots \\ \hat{z}^{|\Omega|} \end{pmatrix}, \begin{pmatrix} \bar{x}^{(2)} \\ \bar{y}^{1,(2)} \\ \vdots \\ \bar{y}^{|\Omega|,(2)} \end{pmatrix}, \ldots, \begin{pmatrix} \bar{x}^{(2n)} \\ \bar{y}^{1,(2n)} \\ \vdots \\ \bar{y}^{|\Omega|,(2n)} \end{pmatrix} \tag{P2}$$

lie in $S(\Omega)$ and satisfy (F) with equality. The points (P2) form Set 2.

The $n(|\Omega| - 1)$ points (P1) and the $2n$ points (P2) all lie in $S(\Omega)$ and satisfy (F) with equality. To show that they are affinely independent, we apply Lemma 3 as follows. Take $N = M_1 = 2n$, $m = M_2 = n(|\Omega| - 1)$, $f^{(1)} = (\hat{x}, \hat{z}^{\omega})$, and $f^{(q)} = (\bar{x}^{(q)}, \bar{y}^{\omega,(q)})$ for $q = 2, \ldots, 2n$ (note that the points $f^{(q)}$ are affinely independent for $q = 1, \ldots, 2n$). Take the point $g^{(1)} = (\hat{z}^1, \ldots, \hat{z}^{\omega-1}, \hat{z}^{\omega+1}, \ldots, \hat{z}^{|\Omega|}) \in \mathbb{R}^{n(|\Omega|-1)}$ and $g^{(q)} = (\bar{y}^{1,(q)}, \ldots, \bar{y}^{\omega-1,(q)}, \bar{y}^{\omega+1,(q)}, \ldots, \bar{y}^{|\Omega|,(q)}) \in \mathbb{R}^{n(|\Omega|-1)}$ for $q = 2, \ldots, 2n$. The point $(f^{(s)}, g^{(s)})$ in Lemma 3 is given by $(\hat{x}, \hat{z})$, and the points $h^{(j)}$ are given by $\pm \frac{1}{n(|\Omega|-1)} \varepsilon_{i\ell} \vec{e}_{i\ell}$ (*cf.* (P1)), which are non-zero scalar multiples of distinct standard basis vectors, and thus linearly independent. Hence, the $n(|\Omega|+1)$ points (P1) and (P2) are affinely independent, and we conclude that (F) is facet-defining for $S(\Omega)$, completing the proof of sufficiency.

**Necessity:** Suppose that (F) is facet-defining for $S(\Omega)$. Condition 2 follows from Proposition 3. To show that Condition 1 holds, fix $i \in N$ and $\ell \in \Omega \setminus \{\omega\}$. Suppose first that $i \in N^+$. Because (F) is not a scalar multiple of the inequality $y_i^{\ell} \leq u_i^{\ell} x_i$, there exists a point $(\hat{x}, \hat{y}^1, \ldots, \hat{y}^{|\Omega|}) \in X(\Omega)$ that satisfies (F) with equality such that $\hat{y}_i^{\ell} < u_i^{\ell} \hat{x}_i$ (in particular, $\hat{x}_i = 1$). Hence

$$\sum_{j \in N^+} u_j^{\ell} \hat{x}_j > \sum_{j \in N^+} \hat{y}_j^{\ell} \geq \sum_{j \in N^+} \hat{y}_j^{\ell} - \sum_{j \in N^-} \hat{y}_j^{\ell} \geq d^{\ell},$$

as desired. If, on the other hand, $i \in N^-$, then, because (F) is not a scalar multiple of $y_i^{\ell} \geq 0$, there exists a point $(\hat{x}, \hat{y}^1, \ldots, \hat{y}^{|\Omega|}) \in X(\Omega)$ that satsifies (F) with equality such that $\hat{y}_i^{\ell} > 0$ (in particular, $\hat{x}_i = 1$). Hence

$$\sum_{j \in N^+} u_j^{\ell} \hat{x}_j \geq \sum_{j \in N^+} \hat{y}_j^{\ell} > \sum_{j \in N^+} \hat{y}_j^{\ell} - \sum_{j \in N^-} \hat{y}_j^{\ell} \geq d^{\ell},$$

concluding the proof of necessity. □