New facets and facet-generating procedures for the orientation model for vertex coloring problems

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Abstract

In this work, we study the *orientation model* for vertex coloring problems with the aim of finding partial descriptions of the associated polytopes. We present new families of valid inequalities, most of them supported by paths of the input graph. We develop facet-generating procedures for the associated polytopes, which we denominate *path-lifting procedures*. Given a path between two vertices and any two valid inequalities, the path-lifting procedure generates an infinite family of valid inequalities. We study conditions ensuring that the obtained inequalities define facets. Finally, we present a large family of valid inequalities, which is obtained by iteratively applying the path-lifting procedure. We conjecture that this family may be sufficient to completely describe the associated polytope when G is a path.

KEYWORDS: vertex coloring, orientation model, facet-generating procedures.

1. Introduction

A vertex coloring of a graph $G = (V, E)$ with a given set of colors C is an assignment $c : V \to C$ such that $c(v) \neq c(w)$ for each edge $vw \in E$. In the last decades, *integer linear programming* (ILP) has been successfully applied to graph coloring problems, by resorting to several formulations for the classical version of the problem. Some of these include the *standard formulation* [10, 11], the *orientation model* [1], the *representatives formulation* [2, 3], the *distance model* [4], and the *maximal independent sets formulation* [9].

The *orientation model* for vertex coloring problems [1] uses an integer variable $x_v \in \{0, \ldots, |C| - 1\}$ for each vertex $v \in V$ to indicate the color assigned to v. In addition, a binary variable y_{vw} is used for each edge $vw \in E$, $v < w$,

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to indicate whether $x_v < x_w$ or not in the represented coloring. With these definitions C-colorings of a graph $G = (V, E)$ can be described by the following constraints:

$$\begin{aligned} x_v - x_w &\geq 1 - |C| y_{vw} && \forall vw \in E, v < w, && (1) \\ x_w - x_v &\geq 1 - |C|(1 - y_{vw}) && \forall vw \in E, v < w, && (2) \\ x_v &\in \{0, |C| - 1\} && \forall v \in V, && (3) \\ y_{vw} &\in \{0, 1\} && \forall vw \in E. && (4) \end{aligned}$$

Constraints (1) and (2) assert that $|x_v - x_w| \geq 1$ for each edge $vw \in E$. If $y_{vw} = 0$ then (1) implies $x_v > x_w$ (and (2) does not impose any restriction), whereas if $y_{vw} = 1$ then (2) implies $x_w > x_v$ (and (1) does not impose any restriction). In order to find an optimal coloring this formulation can be extended by using a variable z as an upper bound for variables x_v and minimizing this new variable. However, in this work we focus our attention to the non-extended version of this formulation.

We call $P_{\text{col}}(G, C)$ to the convex hull of points $(x, y) \in \mathbb{R}^{|V|+|E|}$ satisfying constraints (1)-(4). If $\mathcal{G}$ is a graph family, then $P_{\text{col}}(\mathcal{G}, C)$ denotes the corresponding family of polytopes. We may omit the set C whenever it is clear from the context. We call $\mathcal{P}_{col}$ to the family of polytopes $P_{\text{col}}(G)$ with G being a general graph.

An interesting property of $\mathcal{P}_{col}$ is the strong geometric symmetry presented by these polytopes. As it was first observed in [6] and afterwards explored in [7] and [8], every point in a polytope $P_{\text{col}}(G, C)$ has a symmetrical point inside $P_{\text{col}}(G, C)$ with respect to the central point $p = (x^p, y^p)$, with $x_v^p = \frac{|C|-1}{2}$ and $y_e^p = \frac{1}{2}$ for each $v \in V$ and $e \in E$. This means that for each point $(x, y) \in P_{\text{col}}(G, C)$, the symmetrical point $\text{sym}(x, y) = 2p - (x, y)$ also belongs to $P_{\text{col}}(G, C)$. Due to this symmetry, for each face of $P_{\text{col}}(G, C)$ there exists a *parallel* face with the same dimension. Moreover, this parallel face is given by a simple formula.

Theorem 1.1 ([7]). *If $b \leq a^T(x, y)$ is a valid (resp. facet-inducing) inequality for $P_{col}(G, C)$, then $a^T(x, y) \leq 2a^T p - b$ is also a valid (resp. facet-inducing) inequality for $P_{col}(G, C)$.*

It is interesting to note that the symmetric version of constraint (1) is (2).

Although variable y_{vw} is defined only when $v < w$, we also define $y_{wv} = 1 - y_{vw}$ as a notational convenience. Note that $y_{wv} = 1$ if $x_w < x_v$ in the represented coloring, i.e., y_{wv} has the opposite meaning of y_{vw}. In practice, variable y_{vw} should be replaced by $1 - y_{wv}$ whenever $v > w$. Similarly, for an inequality $\pi x + \mu y \leq \alpha$, where π_v and μ_e are the coefficients of x_v and y_e, respectively, for $v \in V$ and $e \in E$, we define

$$\vec{\mu}_{vw} = \begin{cases} \mu_{vw} & \text{if } v < w \\ -\mu_{wv} & \text{if } w < v. \end{cases}$$

We will use this notation (mainly in Section 2) while proving that certain inequalities define facets of $P_{\text{col}}(G)$. Also, for a feasible solution $\hat{z} = (\hat{x}, \hat{y})$ we define $\hat{z}(c) := \hat{x}(c) := \{v \in V : \hat{x}_v = c\}$ as the *class* of color c induced by $\hat{z}$. In the following sections, we will refer to the following result from [8], where a generalization of $\mathcal{P}_{col}$ is studied.

Theorem 1.2 ([8]). *Given a vertex $u \in V$ and a clique $K \subseteq N(u)$, the* clique inequality

$$\sum_{w \in K} y_{wu} \leq x_u \tag{5}$$

is valid for $P_{col}(G)$. If K is a maximal clique in $G[N(u)]$ and $|C| \geq \chi(G) + 3$, then (5) defines a facet of $P_{col}(G)$.

2. New facet-defining inequalities for $\mathcal{P}_{col}$

We present in this section several families of facet-inducing inequalities for $P_{\text{col}}(G)$, mainly based on cliques and paths of the input graph G.

2.1. Reinforced orientation inequalities

As a first result, we present a strengthening of constraints (1) and (2), which also generalizes the *double-covering clique* inequalities introduced in [8].

Definition 2.1. *Let $uv \in E$. Let $K^u \subseteq N(u) \setminus N(v)$, $K^v \subseteq N(v) \setminus N(u)$ and $K^{uv} \subseteq N(u) \cap N(v)$ be three cliques such that $K^u \cup K^{uv}$ and $K^v \cup K^{uv}$ are also cliques (see Figure 1). Define $Q = K^u \cup K^v \cup K^{uv}$. The* reinforced orientation inequality *(ROI) is defined as*

$$x_u - x_v \geq 1 - (|C| - |Q|)y_{uv} - \sum_{w \in K^u} y_{uw} - \sum_{w \in K^v} y_{wv} - \sum_{w \in K^{uv}} (y_{uw} - y_{vw}). \tag{6}$$

Proposition 2.1. *The reinforced orientation inequalities are valid for $P_{col}(G)$.*

We omit the proof of Proposition 2.1 since it is a direct consequence of the results of Section 3 (where it is also discussed). The following theorem characterizes the cases in which (6) defines a facet of $P_{\text{col}}(G)$, when $|C| \geq \chi(G) + 2$.

Theorem 2.2. *If $|C| \geq \chi(G) + 2$, then the reinforced orientation inequality (6) defines a facet of $P_{col}(G)$ if and only if every vertex in $(N(u) \cap N(v)) \setminus Q$ has a non-neighbour in Q.*

Proof. Let F be the face of $P_{\text{col}}(G)$ defined by (6). Note that there are two types of feasible solutions $(\hat{x}, \hat{y}) \in F$:

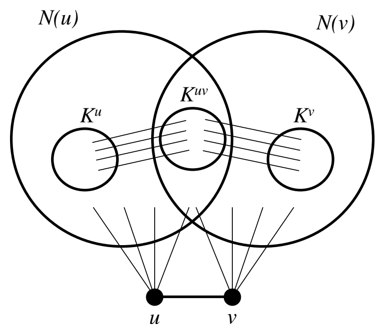

Figure 1: Structural scheme for the *reinforced orientation inequalities*.

Type (i): solutions satisfying $\hat{x}_v < \hat{x}_u$ and such that:

- for each color $c \in (\hat{x}_v, \hat{x}_u)$ there exists $w \in K^{uv}$ with $\hat{x}_w = c$,
- each $z \in K^u$ (resp. $z \in K^v$) has $\hat{x}_z < \hat{x}_v$ (resp. $\hat{x}_u < \hat{x}_z$).

Type (ii): solutions satisfying $\hat{x}_u < \hat{x}_v$ and such that for each color $c < \hat{x}_u$ (resp. $c > \hat{x}_v$) there is a vertex $w \in (K^u \cup K^{uv})$ (resp. $z \in (K^v \cup K^{uv})$) with $\hat{x}_w = c$ (resp. $\hat{x}_z = c$).

Let $\pi \in \mathbb{R}^n, \mu \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$. We shall prove that if $\pi x + \mu y = \alpha$ for each $(x, y) \in F$, then $\pi x + \mu y = \alpha$ is a multiple of the coefficient vector of (6), thus implying that (6) defines a facet of $P_{\text{col}}(G)$.

Claim 1: $\boldsymbol{\pi_w = 0}$, for all $\boldsymbol{w \notin \{u, v\}}$. Let $\hat{z} = (\hat{x}, \hat{y}) \in F$ be a type (ii) solution with $\hat{x}_u = 0$, $\hat{x}_v = |C| - 1$ and such that there is a color c with $|c - \hat{x}_w| = 1$ and $\hat{z}(c) = \emptyset$ (such a solution exists since $|C| \geq \chi(G) + 2$). Let $\bar{z}$ be the solution that coincides with $\hat{z}$, with the exception of vertex w, which is assigned color c. As $\hat{z}, \bar{z} \in F$ and the only difference is the value for x_w, then $\pi_w = 0$.
Claim 2: $\boldsymbol{\vec{\mu}_{wq} = 0}$, for each $\boldsymbol{wq \in E}$ with $\boldsymbol{w, q \notin \{u, v\}}$. Let $\hat{z} = (\hat{x}, \hat{y}) \in F$ be a type (ii) solution with $\hat{x}_u = 0$, $\hat{x}_v = |C| - 1$ and such that there exists a color c with $\hat{z}(c) = \{w\}$ and $\hat{z}(c+1) = \{q\}$ (as $|C| \geq \chi(G) + 2$, such a solution exists). Let $\bar{z}$ be a solution that coincides with $\hat{z}$, with the exception of vertices w and q, which swap colors w.r.t. $\hat{z}$. Since $\hat{z}, \bar{z} \in F$ and these solutions only differ in the variables x_w, x_q and y_{wq}, then by Claim 1 we have $\vec{\mu}_{wq} = 0$.
Claim 3: $\boldsymbol{\vec{\mu}_{uq} = 0}$ (resp. $\boldsymbol{\vec{\mu}_{vq} = 0}$), for each $\boldsymbol{q \in N(u) \setminus (K^u \cup N(v))}$ (resp. $\boldsymbol{q \in N(v) \setminus (K^v \cup N(u))}$). Let $q \in N(u) \setminus (K^u \cup N(v))$ and $\hat{z} = (\hat{x}, \hat{y}) \in F$ be a type (i) solution with $\hat{x}_v = \hat{x}_q = c$, $\hat{x}_u = c + 1$ and $\hat{z}(c+2) = \emptyset$ (as $|C| \geq \chi(G) + 2$, we can construct such a solution by setting $\hat{z}(c) = \{v, q\}$). Let $\bar{z}$ be constructed from $\hat{z}$ by setting $\bar{x}_q = c + 2$ and leaving the remaining variables unchanged. As $\hat{z}, \bar{z} \in F$ and they only differ in the values of x_q, y_{uq} and maybe some variables y_{wq} with $w \neq u, v$, then by Claim 1 and Claim 2 we have $\vec{\mu}_{uq} = 0$. The proof for $\vec{\mu}_{vq} = 0$ is analog.

Claim 4: $\vec{\mu}_{uq} = \vec{\mu}_{vq} = 0$, for all $q \in (N(u) \cap N(v)) \setminus K^{uv}$. By hypothesis there is a vertex $w \in Q$ such that $wq \notin E$. If $w \in K^{uv}$, take $\hat{z} = (\hat{x}, \hat{y}) \in F$ of type (i) with $\hat{x}_w = \hat{x}_q = c$, $\hat{x}_v = c - 1$, $\hat{x}_u = c + 1$ and such as $\hat{z}(c+2) = \emptyset$ (as $|C| \geq \chi(G) + 2$, we can set $\hat{z}(c) = \{w, q\}$). Let $\bar{z}$ be constructed from $\hat{z}$ by setting $\bar{x}_q = c + 2$ and leaving the remaining variables unchanged. As $\hat{z}, \bar{z} \in F$ and they only differ in the values of x_q, y_{uq} and maybe some variables $y_{w'q}$ with $w' \neq u, v$, then by Claim 1 and Claim 2 we have $\vec{\mu}_{uq} = 0$. Analogously, we can see that $\vec{\mu}_{vq} = 0$. Assume now that $w \in K^u$ and take $\hat{z} = (\hat{x}, \hat{y}) \in F$ of type (ii) with $\hat{x}_w = \hat{x}_q = 0$, $\hat{x}_u = 1$, $\hat{x}_v = |C| - 1$ and $\hat{z}(2) = \emptyset$ (assuming w and q are the only vertices in their color class we can see that the latter solution exists). Let $\bar{z}$ be constructed from $\hat{z}$ by setting $\bar{x}_q = 2$ and leaving the remaining variables unchanged. As $\hat{z}, \bar{z} \in F$ and they only differ in the values of x_q, y_{uq} and maybe some variables $y_{w'q}$ with $w' \neq u, v$, then by Claim 1 and Claim 2 we have $\vec{\mu}_{uq} = 0$. We shall prove now that $\vec{\mu}_{vq} = 0$. Let $\hat{z} = (\hat{x}, \hat{y}) \in F$ be of type (i) with $\hat{x}_q = c$, $\hat{x}_v = c + 1$, $\hat{x}_u = c + 2$ and $\hat{z}(c+3) = \emptyset$ (this solution can be obtained by reordering the colors of an optimal coloring in such a way that the color $c + 3$ is not assigned to any vertex). Let $\bar{z}$ be constructed from $\hat{z}$ by setting $\bar{x}_q = c + 3$ and leaving the remaining variables unchanged. As $\hat{z}, \bar{z} \in F$ and these solutions differ in the values of x_q, y_{uq}, y_{vq} and maybe some variables $y_{w'q}$ with $w' \neq u, v$, then by Claim 1 and Claim 2 we have $\vec{\mu}_{uq} + \vec{\mu}_{vq} = 0$, hence $\vec{\mu}_{vq} = 0$. The case $w \in K^v$ is analog.

Claim 5: $\pi_v = -\pi_u$. Let $\hat{z} = (\hat{x}, \hat{y}) \in F$ of type (i) with $\hat{x}_v = c$, $\hat{z}(c+1) = \{x_u\}$ and $\hat{z}(c+2) = \emptyset$ (since $|C| \geq \chi(G) + 2$, we can reorder the colors of an optimal coloring to get such a solution). Let $\bar{z}$ be constructed from $\hat{z}$ by setting $\bar{x}_v = c + 1$ and $\bar{x}_u = c + 2$. As $\hat{z}, \bar{z} \in F$ and they only differ in the values of x_u and x_v, then we get $c\pi_u + (c+1)\pi_v = (c+1)\pi_u + (c+2)\pi_v$. Hence $\pi_u + \pi_v = 0$.

Claim 6: $\vec{\mu}_{uv} = (|C| - |Q|)\pi_u$. Let $\hat{z} = (\hat{x}, \hat{y}) \in F$ be of type (ii) in which every vertex $w \in K^u \cup K^{uv}$ has $\hat{x}_w < \hat{x}_u$ and every vertex $z \in K^v$ has $\hat{x}_z > \hat{x}_v$ (i.e., $\hat{x}_u = |K^u| + |K^{uv}|$ and $\hat{x}_v = |C| - 1 - |K^v|$), and $\hat{z}(\hat{x}_u) = \{u\}$ and $\hat{z}(\hat{x}_u + 1) = \emptyset$. Let $\bar{z}$ equal to $\hat{z}$ but where u is moved to color $|K^u| + |K^{uv}| + 1$ and v to color $|K^u| + |K^{uv}|$; we can see that $\bar{z}$ is a type (i) solution in F. As $\hat{z}, \bar{z} \in F$ and they only differ in the values of x_u, x_v, y_{uv} and maybe some variables y_e with $\vec{\mu}_e = 0$, then we have

$$\vec{\mu}_{uv} + (|K^u| + |K^{uv}|)\pi_u + (|C| - 1 - |K^v|)\pi_v = (|K^u| + |K^{uv}| + 1)\pi_u + (|K^u| + |K^{uv}|)\pi_v,$$

and therefore

$$\begin{aligned} \vec{\mu}_{uv} &= \pi_u + (|K^u| + |K^{uv}| - |C| + 1 + |K^v|)\pi_v \\ &= \pi_u + (|Q| - |C|)\pi_v + \pi_v \\ &= (|Q| - |C|)\pi_v && \text{(by Claim 5)} \\ &= (|C| - |Q|)\pi_u. && \text{(by Claim 5)} \end{aligned}$$

Claim 7: $\vec{\mu}_{uw} = \pi_u$, for each $w \in K^u \cup K^{uv}$. Let $\hat{z} = (\hat{x}, \hat{y}) \in F$ be of type (ii) with $\hat{x}_w = 0$ and $\hat{x}_u = 1$ (we shall assume $\hat{x}_v = |C| - 1$ although it is not necessary). Let $\bar{z}$ equal to $\hat{z}$ but swapping color classes 0 and 1. Note that $\hat{z}$ is also of type (ii). As $\hat{z}, \bar{z} \in F$ and they only differ on the values of y_{uw}, x_u, x_w and maybe some variables with null coefficients, then we have $\vec{\mu}_{uw} + \pi_w = \pi_u$ and since $\pi_w = 0$, then $\vec{\mu}_{uw} = \pi_u$.

Claim 8: $\boldsymbol{\vec{\mu}_{wv} = -\pi_v}$, for each $\boldsymbol{w \in K^v}$. Let $\hat{z} = (\hat{x}, \hat{y}) \in F$ be of type (ii) with $\hat{x}_w = |C| - 2$ and $\hat{x}_v = |C| - 1$ (we shall assume $\hat{x}_u = 0$ although it is not necessary). Let $\bar{z}$ equal to $\hat{z}$ but swapping color classes $|C| - 1$ and $|C| - 2$. Note that $\hat{z}$ is also of type (ii). As $\hat{z}, \bar{z} \in F$ and they only differ in the values of y_{wv}, x_v, x_w and maybe some variables with null coefficients, then we have $\vec{\mu}_{wv} + (|C| - 1)\pi_v + (|C| - 2)\pi_w = (|C| - 2)\pi_v + (|C| - 1)\pi_w$ and since $\pi_w = 0$, then $\vec{\mu}_{wv} = -\pi_v$.

Claim 9: $\boldsymbol{\vec{\mu}_{vw} = \pi_v}$, for each $\boldsymbol{w \in K^{vw}}$. Let $\hat{z} = (\hat{x}, \hat{y}) \in F$ be of type (ii) with $\hat{x}_v = |C| - 2$ and $\hat{x}_w = |C| - 1$. Again, we get $\bar{z}$ from $\hat{z}$ swapping color classes $|C| - 1$ and $|C| - 2$. Note that $\hat{z}$ is also of type (ii). As $\hat{z}, \bar{z} \in F$ and they only differ on y_{vw}, x_v, x_w and some other variables with null coefficients, then we get $\vec{\mu}_{vw} + (|C| - 2)\pi_v + (|C| - 1)\pi_w = (|C| - 1)\pi_v + (|C| - 2)\pi_w$ and since $\pi_w = 0$, then $\vec{\mu}_{vw} = \pi_v$.

Claims 1 to 9 prove that $\pi x + \mu y = \alpha$ is actually a multiple of (6) thus this inequality defines a facet of $P_{\text{col}}(G)$.

We now prove that the condition asking every vertex in $(N(u) \cap N(v)) \setminus K^{uv}$ to have a non-neighbor in Q is necessary for (6) to define a facet. Assume there is a vertex $k \in (N(u) \cap N(v)) \setminus K^{uv}$ being neighbor of every vertex in Q. In this case, we claim that every solution in F satisfies also the equality $y_{uk} + y_{kv} = 1 + y_{uv}$, thus F is not a facet of $P_{\text{col}}(G)$. The claim is true since in every type (i) solution (i.e., when $y_{uv} = 0$), vertex k should have a color lesser than the color of v or greater than the color of u, and in every type (ii) solution, where $y_{uv} = 1$, vertex k should receive a color greater than the color assigned to u and lesser than the color assigned to v. □

2.2. Path-based valid inequalities

Cliques and *holes* are usually essential structures in the context of graph coloring, constantly appearing associated to many facet-defining inequalities, for most known formulations. Nevertheless, for the *orientation model* there is another (very simple) structure that seems to have a strong presence within the facets of $\mathcal{P}_{col}$, namely paths. In the following, we say that the length $|P|$ of a path P is the number of edges (not vertices) of P.

2.2.1. Path inequalities

If the colors assigned to the vertices of a path P follow an increasing order along the path, then the difference between the colors of the first and the last vertex of P is at least the length of the path.

Definition 2.2. *Given a path $P = \{v_0, v_1, \dots, v_k\}$, with $k = |C|$, the* path inequality *is defined as*

$$\sum_{i=0}^{k-1} y_{v_i v_{i+1}} \leq k - 1. \tag{7}$$

Proposition 2.3. *The path inequalities are valid for $P_{col}(G)$.*

Proof. If $\sum_{i=0}^{k-1} y_{v_i v_{i+1}} \geq k$, then $x_{v_k} - x_{v_0} \geq k = |C|$, which is not possible as $x_{v_k}, x_{v_0} \in [0, |C| - 1]$, thus (7) is valid. □

Inequality (7) is valid also when $k > |C|$, but in that case (7) is the sum of the path inequality associated with the first $|C|$ edges and the inequalities $y_e \leq 1$ for the remaining edges. Next we give sufficient conditions for (7) to define a facet of $P_{\text{col}}(G)$. For the sake of clarity, we first give the following definition.

Definition 2.3. *Let P be a simple path of a graph $G = (V, E)$. For two vertices $u, v \in P$, we say that a* (P, u, v)-lining *on G is a coloring $c : V \to C$ such that $c(u) = c(v)$ and $c(w_1) \neq c(w_2)$, for every $w_1, w_2 \in P \setminus \{u\}$. We say that G is* (P, u, v)-lineable *if there exists a (P, u, v)-lining on G.*

Theorem 2.4. *Given a graph $G = (V, E)$ and a path $P = \{v_0, v_1, \dots, v_k\}$, with $k = |P| = |C|$, if $l = \lceil \frac{k+1}{2} \rceil$ and*

i. $|C| \geq \chi(G \setminus P) + l + 3$,

ii. P has no chords (i.e., $v_i v_j \notin E$, for all $i, j \in [0, k]$, with $|i - j| > 1$), and

iii. for each $i \in [0, k-1]$, there exists a (P, v_{j_1}, v_{j_2})-lining of G with $j_1 \leq i < j_2$,

then the path inequality *(7) associated to P defines a facet of $P_{col}(G)$.*

Proof. Let F be the face of $P_{\text{col}}(G)$ defined by (7) and let $(x, y) \in F$ be a feasible solution. Since (7) is satisfied with equality, then there is exactly one edge $v_i v_{i+1}$ of P for which $y_{v_i v_{i+1}} = 1$. Hence, for some $t \in [1, k]$, we have:

- $x_{v_0} < x_{v_1} < \dots < x_{v_{t-1}}$,
- $x_{v_{t-1}} > x_{v_t}$, and
- $x_{v_t} < x_{v_{t+1}} < \dots < x_{v_k}$.

Note that as P satisfies condition ii, its vertices can be colored as above by using just l colors (e.g., by incrementally coloring the first l vertices of P with these l colors and replicating this coloring for the remaining vertices with the same l colors, or maybe $l - 1$ given the parity of $|P|$). By doing this, condition i implies that the rest of the graph can be colored while leaving 3 unused colors. This fact allows us to construct particular solutions in F in order to prove some of the following claims.

Let $\pi \in \mathbb{R}^n, \mu \in \mathbb{R}^m$, and $\alpha \in \mathbb{R}$. We shall prove that if $\pi x + \mu y = \alpha$ for each $(x, y) \in F$, then $\pi x + \mu y = \alpha$ is a multiple (7), thus implying that (7) defines a facet of $P_{\text{col}}(G)$.

Claim 1: $\boldsymbol{\pi_u = 0}$, for all $\boldsymbol{u \in V \setminus P}$. Let $\hat{z} = (\hat{x}, \hat{y}) \in F$ be a solution in which the vertices of P are colored with the first l colors (i.e., colors $[0, l-1]$) as explained above, and the vertices of $V \setminus P$ only use colors greater than l, being $l+1$ the color assigned to u. Let $\bar{z}$ be constructed from $\hat{z}$ by assignint color l to the vertex u. Since $\hat{z}, \bar{z} \in F$ and these solutions only differ in the value of x_v, then $\pi_v = 0$.
Claim 2: $\boldsymbol{\pi_{v_i} = 0}$, for all $\boldsymbol{v_i \in P}$. Let $\hat{z} = (\hat{x}, \hat{y}) \in F$ be a solution in which the vertices of $V \setminus P$ only use colors greater than l and the vertices of P are colored with the first $l+1$ colors (i.e., colors $[0, l]$), being $c \in [0, l-1]$ the color assigned to v_i and $\hat{z}(c+1) = \emptyset$. Note that this solution may be obtained from solution $\hat{z}$ from Claim 1, by assigning color $t+1$ to every vertex receiving color t in $\hat{z}$, for every $t \in [c+1, l)$, thus leaving the color class $c+1$ empty. Let $\bar{z}$ be constructed from $\hat{z}$ by assigning color $c+1$ to the vertex v_i. Since $\hat{z}, \bar{z} \in F$ and they only differ in the value of x_{v_i}, then $\pi_{v_i} = 0$.
Claim 3: $\boldsymbol{\vec{\mu}_{uw} = 0}$, for each $\boldsymbol{u, w \in V \setminus P}$. Let $\hat{z} = (\hat{x}, \hat{y}) \in F$ be a solution in which the vertices of P are colored with the first l colors (i.e., colors $[0, l-1]$), vertices u and w use colors l and $l+1$, respectively, and the rest of the vertices of $V \setminus P$ only use colors greater than $l+1$. Let $\bar{z}$ be constructed from $\hat{z}$ by swapping the colors of u and w. Since $\hat{z}, \bar{z} \in F$ and they only differ in the values of x_u, x_w, and y_{uw}, then by Claim 1, $\vec{\mu}_{uw} = 0$.
Claim 4: $\boldsymbol{\vec{\mu}_{uv_i} = 0}$, for each $\boldsymbol{u \in V \setminus P, v_i \in P}$. Let $\hat{z} = (\hat{x}, \hat{y}) \in F$ a solution in which vertices from $V \setminus (P \cup \{u\})$ only use colors greater than $l+2$, vertices from $P \setminus \{v_i\}$ only use colors in the interval $[0, l+2]$ not using three consecutive colors c, $c+1$ y $c+2$, being c and $c+1$ the colors assigned to u and to v_i, respectively. Let $\bar{z}$ equal to $\hat{z}$ but in which u takes color $c+2$. Since $\hat{z}, \bar{z} \in F$ and they only differ in the values of x_u and y_{uv_i}, then by Claim 1, $\vec{\mu}_{uv_i} = 0$.
Claim 5: $\boldsymbol{\vec{\mu}_{v_{i-1}v_i} = \vec{\mu}_{v_i v_{i+1}}}$, for each $\boldsymbol{i \in [1, k-1]}$. By condition iii, there exists a (P, v_{j_1}, v_{j_2})-lining of G with $j_1 \leq i < j_2$. Then, we can construct a solution $\hat{z} = (\hat{x}, \hat{y}) \in F$ in which $\hat{y}_{v_i v_{i+1}} = 0$ by simply ordering the color classes given by the lining in such a way that $\hat{x}_{v_{i+1}} < \hat{x}_{v_i}$. To this end, we can perform the following operations:

1. assign vertices $v_0, \ldots, v_{j_1-1}$ to colors $0, \ldots, j_1 - 1$,
2. assign vertices $v_{j_1}, \ldots, v_i$ to colors $(j_1 + j_2 - (i+1)), \ldots, j_2 - 1$,
3. assign vertices $v_{i+1}, \ldots, v_{j_2}$ to colors $j_1, \ldots, (j_1 + j_2 - (i+1))$ and
4. assign vertices $v_{j_2+1}, \ldots, v_k$ to colors $j_2, \ldots, k-1$.

Figure 2 illustrates this assignment; labels reference the index of the vertices in the path and the horizontal alignment indicates the assigned color (increasing from left to right). We can see that the only edge e from P with $\hat{y}_e = 0$ is $v_i v_{i+1}$. Analogously for v_{i-1}, we can construct a solution $\bar{z} \in F$ in which the only edge e from P with $\bar{y}_e = 0$ is $v_{i-1} v_i$ (yet for $i = 1$, by using $j_1 = 0$ y $j_2 > 1$). Note that $\hat{z}$ and $\bar{z}$ may differ in several variables, but by the previous claims, all these variables have null coefficients in $\pi x + \mu y$ except $y_{v_{i-1} v_i}$ and $y_{v_i v_{i+1}}$. Hence, as

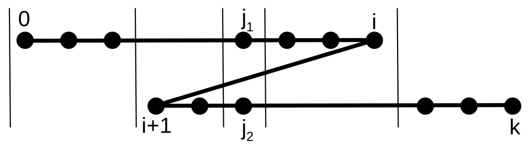

Figure 2: Solution constructed in Claim 5.

both solutions belong to F, we deduce $\vec{\mu}_{v_{i-1}v_i} = \vec{\mu}_{v_i v_{i+1}}$.

The conjunction of the above claims shows that $\pi x + \mu y = \alpha$ is a multiple of (7), thus implying that (7) defines a facet of $P_{\text{col}}(G)$.

□

2.2.2. Weighted path inequalities

Path inequalities, from Section 2.2.1, are valid only when the length of the path is greater than or equal to $|C|$. However, when $|P| < |C|$ we can deduce a bound for the color assigned to vertices of P whenever the path is colored with a monotonic order. We exploit these properties in the following valid inequalities.

Definition 2.4. *Given a path $P = \{v_0, v_1, \ldots, v_k\}$, with $1 \leq k < |C|$, the* weighted path inequalities *(WPI) are defined as*

$$x_{v_0} \geq k - \sum_{j=0}^{k-1} (k-j) y_{v_j v_{j+1}}, \tag{8}$$

$$x_{v_0} \leq (|C| - 1) - k + \sum_{j=0}^{k-1} (k-j) y_{v_{j+1} v_j}. \tag{9}$$

Proposition 2.5. *The weighted path inequalities are valid for $P_{col}(G)$.*

Note that (9) is the symmetrical version of (8). The *weighted path inequalities* are a generalization of the *Drei-Kanten-Pfad* inequalities introduced in [6]. In Section 2.2.3 we present a generalization of the WPI, thus proving the validity of (8) and (9). However, we present the particular case of WPI since this family requires weaker conditions in order to define facets of $\mathcal{P}_{col}$.

Theorem 2.6. *Assume that $|C| \geq \chi(G \setminus P) + k + 2$ and that P is chordless. Then, the WPI (8) and (9) define facets of $P_{col}(G)$ if and only if each vertex in $N(v_0) \setminus P$ has a non-neighbor in P.*

We omit the proof of the above result as it is long and it resorts to the same techniques than our previous proofs. The proof of this result can be found in the e-companion to this work.

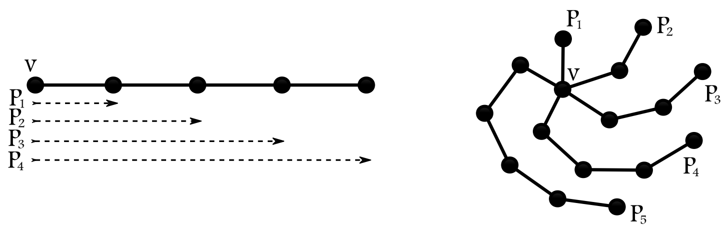

Figure 3: *Spiral inequalities* (two examples with and without path intersections).

It worth noting that $|C| \geq \chi(G \setminus P) + k + 2$ is not in general a necessary condition for (8) to be facet-inducing and that we found many examples in practice in which a WPI defines a facet and this condition does not hold. Unfortunately, it seems to be quite difficult to tighten this condition for the general case. In [5] we give an extensive list of necessary conditions for (8) to be facet-inducing.

2.2.3. Spiral inequalities

Following a similar idea as in the WPI, we present now a generalization of this family, which considers several paths starting in a given vertex. For the sake of clarity we introduce the following notation: given a path $P = \{v_0, v_1, \ldots, v_k\}$, we define

$$\overrightarrow{Y}(P) = \sum_{i=0}^{k-1} y_{v_i v_{i+1}}, \qquad \overleftarrow{Y}(P) = \sum_{i=0}^{k-1} y_{v_{i+1} v_i}.$$

Note that $\overrightarrow{Y}(P) = 0$ (resp. $\overleftarrow{Y}(P) = 0$) means that P is assigned colors in a decreasing (resp. increasing) order.

Definition 2.5. *Let $v \in V$ and $k \in [1, |C| - 1]$. For each $i = 1, \ldots, k$, let $P_i = \{w_0^i, w_1^i, \ldots, w_i^i\}$ be a path starting from v, i.e., $w_0^i = v$ (see Figure 3). With these definitions, the* spiral inequalities *(SI) are defined as*

$$x_v \geq k - \sum_{i=1}^{k} \overrightarrow{Y}(P_i), \tag{10}$$

$$x_v \leq (|C| - 1 - k) + \sum_{i=1}^{k} \overleftarrow{Y}(P_i). \tag{11}$$

Proposition 2.7. *The spiral inequalities are valid for $P_{col}(G)$.*

Proof. Due to Theorem 1.1, we may just prove that (11) is valid for $P_{\text{col}}(G)$. Let $(\hat{x}, \hat{y})$ be a feasible solution. If $\hat{x}_v \leq |C| - 1 - k$ then there is nothing to be proved, as the sum in the right hand side is always non-negative. Otherwise, if $\hat{x}_v = |C| - 1$, then $\hat{y}_{w_1^i w_0^i} = 1$, for all $i = 1, \ldots, k$ (as $w_0^i = v$), thus the inequality holds. Assume then that $\hat{x}_v = |C| - 1 - t$, with $1 \leq t < k$ and let us analyze the value of the sum

$$\sum_{j=1}^{t+1} \hat{y}_{w_j^i w_{j-1}^i}, \tag{12}$$

for each of the paths P_i with $i > t$ (note that w^i_{t+1} exists as $i > t$). If there exists some $j \in [1, t]$ with $\hat{y}_{w^i_j w^i_{j-1}} = 1$, then (12) is greater or equal than 1. Assume then such j does not exist and so the edges of P_i involved in (12) are a simple path (thus being a cycle would imply the existance of such j by transitivity). This means that $\hat{x}_{w^i_0} < \hat{x}_{w^i_1} < \cdots < \hat{x}_{w^i_t}$ and since $\hat{x}_{w^i_0} = \hat{x}_v = |C| - 1 - t$, then $\hat{x}_{w^i_t} = |C| - 1$, thus implying $\hat{y}_{w^i_{t+1} w^i_t} = 1$. In any case, (12) is greater than or equal to 1. Therefore, as there are $k - t$ paths P_i with $i > t$, then the inequality (11) holds. □

Figure 3 shows two examples of supporting graph structures for *spiral inequalities*. In first example, all paths are subpaths of just one path on the graph. In the second example, all paths are pairwise edge- and vertex-disjoint. These are extreme examples (with respect to path intersections), but since there is no condition on the intersection of the paths, any set of (possibly intersecting) paths gives rise to a *spiral inequality*. In particular, the inequality obtained in the case of full path intersection (left image in Figure 3) is a *weighted path inequality* (from Section 2.2.2), thus implying that some *spiral inequalities* define facets of $P_{\text{col}}(G)$. Nevertheless, we next describe other sufficient conditions for this fact for another case, namely the full disjoint case on the right of Figure 3.

Theorem 2.8. *Let $\mathcal{P} = \{v\} \cup \bigcup_{i=1}^k P_i$, where v and $P_1, \ldots, P_k$ are the vertex and the paths associated with two inequalities (10) and (11), respectively. Assume that $|C| \geq \chi(G \setminus \mathcal{P}) + k + 2$, that $P_i \cap P_j = \{v\}$ if $i \neq j$ and that there are no edges in $G[\mathcal{P}]$ but the edges of $P_1, \ldots, P_k$. Then, inequalities (10) and (11) define facets of $P_{col}(G)$ if and only if every vertex in $N(v) \backslash \mathcal{P}$ has a non-neighbor in $\mathcal{P}$.*

We omit the proof of the above result as it is long and it resorts to the same techniques than our previous proofs. The proof of this result can be found in the e-companion to this work.

Proposition 2.9. *The separation problem associated to the* spiral inequalities (10) *and* (11) *can be solved in polynomial time.*

Proof. Given a fractional solution $\hat{z} = (\hat{x}, \hat{y})$, let $H(\hat{z})$ be a digraph with vertex set V and arcs $A = \{uw : uw \in E \text{ or } wu \in E\}$ with an arc weight $\omega_{uw} = \hat{y}_{uw}$ for each $uw \in A$ (in case $w < u$ then $\omega_{uw} = 1 - \hat{y}_{wu}$). Given a vertex $v \in V$ and a value $k \in \mathbb{Z}_+$, it is easy to see that the value of

$$\sum_{i=1}^{k} \overrightarrow{Y}(P_i) \tag{13}$$

with P_i being a path with i edges starting from v, is minimized when each P_i is an i-edge directed path of minimum weight in $H(\hat{z})$ starting from v, for each $i = 1, \ldots, k$. Therefore, given v and k, the separation problem over (10) consists in verifying whether the minimum value for (13) is less than $k - \hat{x}_v$ or not. So, to finish this proof we give a polynomial algorithm to find a directed path of minimum weight with fixed length t in a digraph (note that in our context $t \le k < |C| \le n$). Finding the minimum-weighted directed paths of fixed length between any pair of vertices on a digraph H may be easily solved by means of dynamic programming via the following recursive definition:

$$D_1[v,w] = \begin{cases} \omega_{vw}, & \text{if } vw \in E \\ +\infty, & \text{if } vw \notin E \end{cases}$$
$$D_{i+1}[v,w] = \min_{\substack{u \in V \\ u \neq v,w}} (D_i[v,u] + D_1[u,w]), \text{ for } i = 1,\ldots,t-1.$$

In this definition, $D_i[v,w]$ stores the weight of a minimum-weighted path of length i from v to w. This path may not be a simple path, but this is not a requirement for the *spiral inequalities*.

□

2.2.4. Double and triple spiral inequalities

In this section, we introduce two families of facet-inducing inequalities for $\mathcal{P}_{col}$, which we derive from facets found in experiments with small instances. Both families share an interesting structure, which motivates the procedures presented in Section 3, so we shall use these families as an introduction for the ideas explored afterwards. We omit the proofs of validity of these families as they are a direct consequence of the results in Section 3.

Given a path P between vertices u and v, it is easy to see that if a solution has $\overrightarrow{Y}(P) = 0$, i.e., all edges of P are "oriented" from v to u, then $x_u - x_v \ge |P|$, and so we can say that

$$x_u - x_v \ge |P| - (|P| + |C| - 1)\overrightarrow{Y}(P), \tag{14}$$

is valid for $P_{\text{col}}(G)$. Note that when $|P| = 1$, then (14) equals the model constraint (1).

By taking k_u paths starting from u of incremental lenghts and applying the *spiral inequality* (11), we get an upper bound for x_u. Analogously for v and similar k_v paths, through the application of (10), we get a lower bound for x_v. These bounds can be used along with (14) to obtain the following family of inequalities.

Definition 2.6. *Let $u, v \in V$ and a path P from u to v. Let $k_u, k_v \in \mathbb{Z}_+$ with $k_u + k_v \le |C| + |P| - 1$. For each $i = 1,\ldots,k_u$, let P_i^u be a path starting on u with $|P_i^u| = i$ and for each $i = 1,\ldots,k_v$, let P_i^v be a path starting on v with $|P_i^v| = i$ (see Figure 4). With these definitions, the* double spiral inequality *(2SI) is defined as*

$$x_u - x_v \ge |P| - (|P| + |C| - 1 - k_u - k_v)\overrightarrow{Y}(P) - \sum_{i=1}^{k_u} \overrightarrow{Y}(P_i^u) - \sum_{i=1}^{k_v} \overleftarrow{Y}(P_i^v). \tag{15}$$

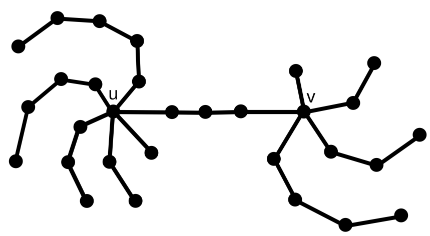

Figure 4: *Double spiral inequalities* (with no path intersections).

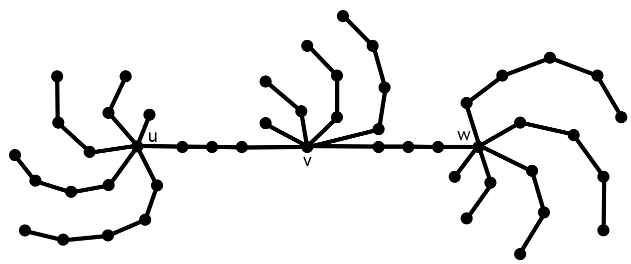

Figure 5: *Triple spiral inequalities* (with no path intersections).

Proposition 2.10. *The double spiral inequalities are valid for* $P_{col}(G)$*.*

We go futher, now taking a path from v to another vertex w. Lets call this path P_{vw} and the previous path $P_{uv} := P$. We can now take a *spiral inequality* (11) associated with w and, following the same idea as above, we can combine it with (15) and P_{vw} to obtain a new valid inequality (as ilustrated in Figure 5).

Definition 2.7. *Let* $u, v, w \in V$ *with a path* P_{uv} *from* u *to* v *and a path* P_{vw} *from* v *to* w*. Let* $k_u, k_v, k_w \in \mathbb{Z}_+$ *such that* $k_u + k_v \leq |C| + |P_{uv}| - 1$ *y* $k_u + |P_{vw}| > k_w + |P_{uv}|$*. For each* $i = 1, \ldots, k_u$ *(resp.* $j = 1, \ldots, k_v$ *and* $t = 1, \ldots, k_w$*) let* P_i^u *(resp.* P_j^v *y* P_t^w*) a path from* u *such that* $|P_i^u| = i$ *(resp. from* v *such that* $|P_j^v| = j$ *and from* w *such that* $|P_t^w| = t$*). With these definitions, the* triple spiral inequality *(3SI) is defined as*

$$x_u - x_v + x_w \geq |P_{uv}| + k_w + 1 - (|P_{uv}| + |C| - 1 - k_u - k_v)\overrightarrow{Y}(P_{uv}) - \overleftarrow{Y}(P_{vw}) - \sum_{i=1}^{k_u} \overrightarrow{Y}(P_i^u) - \sum_{i=1}^{k_v} \overleftarrow{Y}(P_i^v) - \sum_{i=1}^{k_w} \overrightarrow{Y}(P_i^w). \quad (16)$$

Figure 5 illustrates the structure of this inequality.

Proposition 2.11. *The triple spiral inequalities are valid for* $P_{col}(G)$*.*

Both the the *double* and *triple spiral inequalities* arise from the same idea, i.e., to combine the variables of a path P with two valid inequalities (each associated to one of the extreme vertices of P). Although we just have proofs for the validity of these inequalities (see Section 3), we have found that both families define facets in the polytopes as sociated with small instances. In the following section we formalize these ideas in procedures that employ general valid inequalities. We then explore the facet-inducing properties of inequalities associated with such structures.

3. ***Path lifting*** procedures

Both *double* and *triple spiral inequalities* consider a path P and two valid inequalities each of them associated to each endpoint of P. In this section we explore such a structure, by presenting a facet-generating procedure, which we call *path θ-lifting* procedure. Starting from two (almost) generic valid inequalities and a path between two distinguished vertices, this procedure generates new valid (sometimes facet-inducing) inequalities by combining these elements. We next give a simple example to introduce the idea of *path lifting* and we afterwards formalize the procedure.

Given two vertices $v, w \in V$, assume that the following inequalities are valid for $P_{\text{col}}(G)$:

$$x_w + \mu y \geq \alpha \tag{17}$$

$$-x_v + \mu' y \geq \alpha'. \tag{18}$$

The sum of them

$$x_w - x_v + (\mu + \mu')y \geq \alpha + \alpha' \tag{19}$$

is also valid. Assume now that there exists a path P from v to w. We know that a feasible solution satisfying $\overleftarrow{Y}(P) = 0$ must have $x_w - x_v \geq |P|$. This remark allows us to deduce that the following inequality

$$x_w - x_v + (\mu + \mu')y \geq |P| - (|P| - \alpha - \alpha')\overleftarrow{Y}(P), \tag{20}$$

is also valid, whenever $\mu, \mu' \geq \mathbf{0}$ and $|P| \geq \alpha + \alpha'$. Note that when $\overleftarrow{Y}(P) \geq 1$, then (20) is dominated by, or equal to, (19), as $|P| - \alpha - \alpha' \geq 0$. It is worth noting that when (17) and (18) are *spiral inequalities*, then (20) is the *double spiral inequality* (15) from Section 2.2.4.

Similarly, we can use the mentioned path to create the following inequality instead of (20),

$$x_w - x_v + (\mu + \mu')y \geq \alpha + \alpha' + 1 - \overleftarrow{Y}(P), \tag{21}$$

which is also valid, whenever $\mu, \mu' \geq \mathbf{0}$ and $|P| \geq \alpha + \alpha' + 1$ (again, when $\overleftarrow{Y}(P) \geq 1$, then (21) is dominated by (19)). We should remark that, *a priori*,

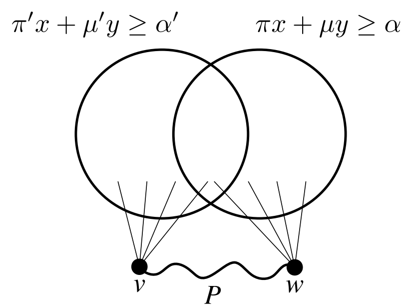

Figure 6: General scheme of the structures used by *path lifting* procedure.

there is no domination relation between (20) and (21). It is worth noting in this case that when (17) is a *spiral inequality* and (18) is a *double spiral inequality*, then (21) is the *triple spiral inequality* (16) from Section 2.2.4.

Inequalities (20) and (21) differ only in their left-hand sides. In particular we can rewrite both left-hand sides as

$$\alpha + \alpha' + \theta + \theta \overleftarrow{Y}(P)$$

with $\theta = |P| - \alpha - \alpha'$ for (20) and $\theta = 1$ for (21), suggesting that a generalization can be made for these procedures. We explore this idea in the next subsection, where we present a procedure that generates infinite families of valid inequalities.

3.1. The path θ-lifting procedure

The procedures skteched above take as input inequalities involving color variables only for vertices v and w (i.e., x_v and x_w). We now present a procedure that takes as input general valid inequalities involving also variables x_u, for $u \notin \{v, w\}$ in addition to x_v and x_w. Figure 6 may help the reader by presenting the general scheme of the structures used by *path lifting* procedures.

Theorem 3.1. *Let $v, w \in V$. Let $\pi^1 x + \mu^1 y \geq \alpha_1$ be a valid inequality for $P_{col}(G)$ and let $\mu^2 \geq \boldsymbol{0}$ and $\alpha_2 \in \mathbb{R}$ such that*

$$\pi^1 x - x_v + (\mu^1 + \mu^2) y \geq (\alpha_1 + \alpha_2) \tag{22}$$

is valid for $P_{col}(G)$. Analogously, let $\pi^3 x + \mu^3 y \geq \alpha_3$ be a valid inequality for $P_{col}(G)$ and let $\mu^4 \geq \boldsymbol{0}$ and $\alpha_4 \in \mathbb{R}$ such that

$$\pi^3 x + x_w + (\mu^3 + \mu^4) y \geq (\alpha_3 + \alpha_4) \tag{23}$$

is valid for $P_{col}(G)$. Finally, let P be a path from v to w and let $\theta \in \mathbb{R}_+$. If $|P| \geq \theta + \alpha_2 + \alpha_4$, then the inequality

$$\pi x + \mu y + (x_w - x_v) \geq \alpha + \theta - \theta \overleftarrow{Y}(P) \tag{24}$$

with $\mu = \mu^1 + \mu^2 + \mu^3 + \mu^4, \pi = \pi^1 + \pi^3$ *and* $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$*, is valid for* $P_{col}(G)$.

Proof. If $\overleftarrow{Y}(P) = 0$ then $x_w - x_v \geq |P|$, and since $\pi^1 x + \mu^1 y \geq \alpha_1$ and $\pi^3 x + \mu^3 y \geq \alpha_3$, then

$$\begin{aligned}
\pi x + \mu y + (x_w - x_v) &= (\pi^1 + \pi^3)x + (\mu^1 + \mu^2 + \mu^3 + \mu^4)y + (x_w - x_v) \\
&= (\pi^1 x + \mu^1 y) + (\pi^3 x + \mu^3 y) + (\mu^2 + \mu^4)y + (x_w - x_v) \\
&\geq \alpha_1 + \alpha_3 + (\mu^2 + \mu^4)y + |P| \qquad (*) \\
&\geq \alpha_1 + \alpha_3 + |P| \\
&\geq \alpha + \theta,
\end{aligned}$$

hence, the inequality holds. On the other side, if $\overleftarrow{Y}(P) = k > 0$, then

$$\begin{aligned}
\alpha + \theta - \theta \overleftarrow{Y}(P) &= \alpha + \theta - \theta k \\
&\leq \alpha + \theta - \theta \\
&= (\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4) \\
&\leq (\pi^1 x - x_v + (\mu^1 + \mu^2)y) + (\pi^3 x + x_w + (\mu^3 + \mu^4)y) \\
&= \pi x + \mu y + (x_w - x_v)
\end{aligned}$$

thus proving the validity of the inequality. □

The conditions $\mu^2, \mu^4 \geq \mathbf{0}$ and $|P| \geq \theta + \alpha_2 + \alpha_4$ in Theorem 3.1 are not strictly necessary. In fact, they could be replaced by the weaker condition $|P| \geq \theta + \alpha_2 + \alpha_4 - (\bar{\mu}^2 + \bar{\mu}^4)\mathbf{1}$, where $\bar{\mu}_e = \min\{\mu_e, 0\}$, since the latter suffices to perform the step marked with $(*)$ in the proof, for which the prior conditions were required. However, for the sake of clarity, we omit such strengthening on the statement of this result.

We refer to this procedure as the *path θ-lifting* (or simply *path lifting*) procedure. As $\theta \in \mathbb{R}_+$, this procedure generates infinite families of valid inequalities for $P_{\text{col}}(G)$ which define a non-poliedral convex containing $P_{\text{col}}(G)$. Obviously, just a finite set of these inequalities will define facets of $P_{\text{col}}(G)$, however, *a priori* there is no "dominating value" for θ, i.e., a value such that the obtained inequality is stronger than the inequality for any other values of θ. This is due to the fact that the left hand side of the inequality does not depend on θ while the right hand side is $\alpha + \theta(1 - \overleftarrow{Y}(P))$, and $1 - \overleftarrow{Y}(P)$ may be positive or negative, depending on the point being analyzed.

Remark 3.1.

1. *The* reinforced orientation inequality (6) *associated to two vertices u and v can be obtained via a* path θ-lifting *procedure by using one* clique inequality*, its symmetric* clique inequality*, the path given by the edge uv and $\theta = 1 - \alpha$.*

2. *The* double spiral inequality (15) *associated to two vertices u and v can be obtained via a* path θ-lifting *procedure by using two* spiral inequalities*, the corresponding path P between u and v and $\theta = |P| - \alpha$.*
3. *The* triple spiral inequality (16) *can be obtained via a* path θ-lifting *procedure by using a* double spiral inequality*, a* spiral inequality*, the corresponding path P joining the associated vertices and $\theta = 1$.*

Theorem 2.2 and Remark 3.1 (1.) imply that the *path lifting* procedure generates facets of $P_{\text{col}}(G)$. In fact, this shows that the obtained inequality may define a facet even when the valid inequalities used in the procedure are not facet-inducing. Indeed, clique inequalities only define facets for maximal cliques, but this is not a requirement for the ROI (generated by the procedure) to be facet-inducing. This gives an idea of the potential of the *path lifting* procedure, concerning the facet-inducing properties of the generated inequalities. In Section 3.2 we explore these topics and give sufficient conditions for the path lifing procedures to generate facet-inducing inequalities.

It is worth to note that *reinforced orientation inequalities* (in the context of path lifting generated inequalities) use paths of length 1. However, it is easy to see that the conditions of the procedure stand also when using longer paths, thus generating valid inequalities generalizing ROI (though probably with stronger conditions about its facet-inducing properties).

3.2. Facets from path lifting *procedures*

Although many facets can be generated by path lifting procedures (e.g, 2SI, 3SI and ROI), it nevertheless seems to be very difficult to fully characterize the cases in which the generated inequality defines a facet of $P_{\text{col}}(G)$. In this section, we focus on the particular case where the path P used in the procedure is composed by a single edge. We should note that, since $|P| = 1$ we may fix θ as big as possible (i.e., $\theta = 1 - \alpha_2 - \alpha_4$), in order to obtain the strongest inequality. As we mentioned before, this is not always clear when $|P| > 1$. By using this single-edge path, we give next a partial characterization for some cases in which the *path θ-lifting* procedure generates a facet-inducing inequality for $P_{\text{col}}(G)$. First we give some useful definitions.

For a graph $G = (V, E)$, we define the *support vertex set* of an inequality $\pi x + \mu y \geq \alpha$ as $V(\pi, \mu) := \{u \in V : \pi_u \neq 0\} \cup \{u \in V : \mu_{uv} \neq 0, \text{ for some } uv \in E\}$. We also define $V(\pi) := \{u \in V : \pi_u \neq 0\}$ and $E(\mu) = \{uv \in E : \mu_{uv} \neq 0\}$. Finally, we say that an edge $vw \in E$ is a *cut edge* of G if it is the only path from v to w.

Theorem 3.2. *Let $vw \in E$ be a cut edge of a graph G and let the inequalities $\pi^1 x + \mu^1 y \geq \alpha_1$, $\pi^3 x + \mu^3 y \geq \alpha_3$,*

$$\pi^1 x - x_v + (\mu^1 + \mu^2) y \geq \alpha_1 + \alpha_2, \text{ and} \tag{25}$$

$$\pi^3 x + x_w + (\mu^3 + \mu^4) y \geq \alpha_3 + \alpha_4 \tag{26}$$

be valid for $P_{col}(G)$, with $\mu^2, \mu^4 \geq \mathbf{0}$, $\alpha_4 < -\alpha_2$, and such that the support vertex sets of (25) and (26) have no intersection. Let F_1 and F_2 be the faces of $P_{col}(G)$ defined by inequalities (25) and (26), respectively. Let $\theta = 1 - \alpha_2 - \alpha_4$ and let $z^a = (x^a, y^a)$ and $z^b = (x^b, y^b)$ be two points in $F_1 \cap F_2$ with $x^a_v = x^b_v = -\alpha_2$, $x^a_w = x^b_w = \alpha_4$ and such that

- *for all $u \in N(w)$, $x^a_u \neq \alpha_4 + \theta$ and if $\mu^3_{uw} \neq 0$ or $\mu^4_{uw} \neq 0$, then either $x^a_u < \alpha_4$ or $x^a_u > \alpha_4 + \theta$, and*
- *for all $u \in N(v)$, $x^b_u \neq -\alpha_2 - \theta$ and if $\mu^1_{uv} \neq 0$ or $\mu^2_{uv} \neq 0$, then either $x^b_u < -\alpha_2 - \theta$ or $x^b_u > -\alpha_2$.*

If (25) *and (26) define facets of $P_{col}(G)$ then the inequality obtained by* path θ-lifting *over (25) and (26) with $P = \{v, w\}$, i.e.,*

$$(\pi^1 + \pi^3)x + x_w - x_v + \mu y \geq \alpha + \theta - \theta y_{wv} \tag{27}$$

with $\mu = \mu^1 + \mu^2 + \mu^3 + \mu^4$ and $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, defines a facet of $P_{col}(G)$.

Proof. Inequality (27) is valid for $P_{\mathrm{col}}(G)$ since $|P| = 1 \geq \theta + \alpha_2 + \alpha_4$, so it is left to prove that it defines a facet of $P_{\mathrm{col}}(G)$.

Since $\mu^2 \geq \mathbf{0}$, then $\pi^1 x + (\mu^1 + \mu^2)y \geq \alpha_1$ for every solution in F_1, thus $-x_v \leq \alpha_2$ for these points, as (25) is an equality for F_1. Analogously, since $\mu^4 \geq \mathbf{0}$, then $\pi^3 x + (\mu^3 + \mu^4)y \geq \alpha_3$ for every solution in F_2 thus $x_w \leq \alpha_4$ for these points. Hence, as $\alpha_4 < -\alpha_2$, then every solution in $F_1 \cap F_2$ satisfies $x_w < x_v$ and so $y_{wv} = 1$. On the other side, if $y_{wv} = 1$ then (27) is the sum of (25) and (26), and so (27) is an equality whenever the latter two are equalities. Therefore, being F the face of $P_{\mathrm{col}}(G)$ defined by (27), we have that $F_1 \cap F_2 \subseteq F$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be the two disjoint subgraphs of G separated by the cut edge wv, thus $G = (V_1 \cup V_2, E_1 \cup E_2 \cup \{wv\})$ and call $d := \dim(P_{\mathrm{col}}(G)) + 1$. Since F_1 is a facet of $P_{\mathrm{col}}(G)$, there exist affinely independent points $z^1, \ldots, z^d \in F_1$. For $i = 1, \ldots, d$, let $\bar{z}^i = (\bar{x}^i, \bar{y}^i)$ be a point equal to z^i but with $\bar{y}^i_e = 0$ for all $e \in (E_2 \cup \{wv\})$ and $\bar{x}^i_u = 0$ for all $u \in V_2$. We can see that the set $\{\bar{z}_i : i = 1, \ldots, d\}$ must contain a subset of $d_1 := d - |E_2 \cup \{wv\}| - |V_2|$ affinely independent points; w.l.o.g., we may call such a subset $D_1 = \{\bar{z}_i : i = 1, \ldots, d_1\}$. In the same way, as F_2 is a facet of $P_{\mathrm{col}}(G)$, we can construct $|E_2| + |V_2|$ affinely independent points $\hat{z}^i = (\hat{x}^i, \hat{y}^i)$, with $\hat{y}^i_e = 0$ for each $e \notin E_2$ and $\hat{x}^i_u = 0$ for each $u \notin V_2$, by using a set of d affinely independent points from F_2; w.l.o.g., we call this subset $D_2 = \{\hat{z}_i : i = 1, \ldots, d_2\}$, with $d_2 = |E_2| + |V_2|$. Consider now the following set of points:

$$D = \{\bar{z}^i + \hat{z}^1 + e_{wv} : i = 1, \ldots, d_1\} \cup \{\bar{z}^1 + \hat{z}^i + e_{wv} : i = 2, \ldots, d_2\},$$

where e_{wv} is the vector having a 1 in the coordinate associated to y_{wv} and 0 in the remaining coordinates. Since G_1 and G_2 are disjoint subgraphs of G, we can see that the points in D are solutions of $P_{\mathrm{col}}(G)$, as variables from $\bar{z}_i$ and

$\hat{z}_i$ represent parts of solutions in F_1 and F_2 respectively and, as we saw above, every solution in $F_1 \cap F_2$ satisfies $y_{wv} = 1$. Therefore, $D \subseteq F_1 \cap F_2 \subseteq F$. On the other hand, we can see that that D is a set of $d_1 + d_2 - 1 = d - 2$ affinely independent points. Additionally, this implies that $\dim(F_1 \cap F_2) = d - 1$, as this polytope has dimension at most $\dim(P_{\text{col}}(G)) - 2$.

Consider now the solution z^a from hypothesis in which $x^a_v = -\alpha_2$ and $x^a_w = \alpha_4$, thus $y^a_{wv} = 1$. Since $z^a \in F_1 \cap F_2 \subseteq F$, then $(\pi^1 + \pi^3)x^a + x^a_w - x^a_v + \mu y^a = \alpha$. We construct a new point $\hat{z}^a$ from z^a by assigning $\hat{x}^a_w = x^a_w + \theta$. Note that the conditions on the hypothesis ensure that this is a feasible solution. As the difference between the colors of v and w in z^a was $x^a_v - x^a_w = -\alpha_2 - \alpha_4 = \theta - 1$, then the new solution has $\hat{y}^a_{wv} = 0$. Also, since the difference between z^a and $\hat{z}^a$ is given by the values of variables x^a_w, y^a_{wv} and maybe some other variables with null coefficient in (27), we have

$$\begin{aligned}(\pi^1 + \pi^3)\hat{x}^a + \hat{x}^a_w - \hat{x}^a_v + \mu\hat{y}^a &= (\pi^1 + \pi^3)x^a + (x^a_w + \theta) - x^a_v + \mu y^a \\ &= \alpha + \theta \\ &= \alpha + \theta + \theta\hat{y}^a_{wv},\end{aligned}$$

and so $\hat{z}^a \in F$. Since the only changes from z^a to $\hat{z}^a$ were in variables with null coefficients in (25), then $\hat{z}^a \in F_1$ and also it is easy to see that $\hat{z}^a \notin F_2$, as $\hat{z}^a$ does not satisfies (26) by equality (because $\hat{x}^a_w > \alpha_4$). Recall that $D \subseteq F_1 \cap F_2$, therefore the $d-2$ (affine independent) points de D are contained in the hyperplane defined by (26) and since $\hat{z}^a$ is not there, then $D \cup \{\hat{z}^a\}$ is a set of $d-1$ affine independent points.

Consider now solution z^b from hypothesis in which $x^b_v = -\alpha_2$, $x^b_w = \alpha_4$, thus $y^a_{wv} = 1$. As before, since $z^b \in F_1 \cap F_2$, then $(\pi^1 + \pi^3)x^b + x^b_w - x^b_v + \mu y^b = \alpha$. Analogously, we construct $\hat{z}^b$ from z^b by assigning $\hat{x}^b_v = x^b_v - \theta$. Again, as the difference between the color of v and w in z^b was $x^b_v - x^b_w = -\alpha_2 - \alpha_4 = \theta - 1$, then the new solution has $\hat{y}^b_{wv} = 0$. Also, since the difference between z^b and $\hat{z}^b$ is given by the values of variables x^b_v, y^b_{wv} and maybe some other variables with null coefficient in (27), we have

$$\begin{aligned}(\pi^1 + \pi^3)\hat{x}^b + \hat{x}^b_w - \hat{x}^b_v + \mu\hat{y}^b &= (\pi^1 + \pi^3)x^b + x^b_w - (x^b_v - \theta) + \mu y^b \\ &= \alpha + \theta \\ &= \alpha + \theta - \theta\hat{y}^b_{wv},\end{aligned}$$

and so $\hat{z}^b \in F$. Since the only changes from z^b to $\hat{z}^b$ were in variables with null coefficients in (26), then $\hat{z}^b \in F_2$ and also it is easy to see that $\hat{z}^b \notin F_1$, as $\hat{z}^b$ does not satisfies (25) by equality (because $\hat{x}^b_v < -\alpha_2$). Recall that $D \cup \{\hat{z}^a\} \subseteq F_1$, therefore the $d-1$ (affine independent) points of D are contained in the hyperplane defined by (25) and since $\hat{z}^b$ is not there, then $D \cup \{\hat{z}^a, \hat{z}^b\}$ is a set of d affinely independent points, all of them belonging to F, thus proving that F is a facet of $P_{\text{col}}(G)$. □

4. *Alternating spiral path inequalities*

The path lifting procedure can be applied to any valid inequality and also, it can be successively iterated by re-applying the procedure to the obtained inequality along with another path to some other inequality. In this section, we describe an infinite family of valid inequalities which arises from the iterative application of *path θ-lifting* using *spiral inequalities*, alternating between (11) and (10). For the sake of simplicity, for a path P and an integer $j \in \mathbb{Z}$, we introduce the notation

$$Y_j(P) = \begin{cases} \overrightarrow{Y}(P) & \text{if } j \text{ is even,} \\ \overleftarrow{Y}(P) & \text{if } j \text{ is odd.} \end{cases}$$

Therefore, the *spiral inequalities* (10) and (11) can be rewritten, respectively,

$$x_{v_0} + \sum_{i=1}^{k_0} Y_0(P_i^0) \geq k_0, \text{ and} \tag{28}$$

$$-x_{v_1} + \sum_{i=1}^{k_1} Y_1(P_i^1) \geq 1 - |C| + k_1 \tag{29}$$

for two vertices v_0 and v_1, and the corresponding paths. Given now $\theta_0 \in \mathbb{R}_+$ and a path $P_{v_0v_1}$ from v_0 to v_1, with $|P_{v_0v_1}| + |C| \geq \theta_0 + 1 + k_0 + k_1$, by applying a *path lifting* with (28) and (29) we get:

$$x_{v_0} - x_{v_1} + \sum_{j=0}^{1} \sum_{i=1}^{k_j} Y_j(P_i^j) \geq (1 - |C| + k_0 + k_1) + \theta_0 - \theta_0 \overrightarrow{Y}(P_{v_0v_1}), \tag{30}$$

since here $\alpha_1 = 0$, $\alpha_2 = 1 - |C| + k_1$, $\alpha_3 = 0$ and $\alpha_4 = k_0$. Now take a vertex v_2 and a path $P_{v_1v_2}$ from v_1 to v_2. Take also $\theta_1 \in \mathbb{R}_+$ and a *spiral inequality* (10) associated to v_2 and k_2 proper paths $P_1^2, \ldots, P_{k_2}^2$. Applying a *path lifting* with θ_1 to such inequality along with (30) we get:

$$x_{v_0} - x_{v_1} + x_{v_2} + \sum_{j=0}^{2} \sum_{i=1}^{k_j} Y_j(P_i^j) \geq (1 - |C| + k_0 + k_1 + k_2) + \sum_{j=0}^{1} (\theta_j - \theta_j Y_j(P_{v_jv_{j+1}})). \tag{31}$$

thus here $\alpha_1 = k_0$, $\alpha_2 = \theta_0 + 1 - |C| + k_1$, $\alpha_3 = 0$ and $\alpha_4 = k_2$. We know that if $|P_{v_1v_2}| + |C| \geq \theta_1 + \theta_0 + 1 + k_1 + k_2$, then (31) is valid for $P_{\text{col}}(G)$. Going one step further, take a vertex v_3 and a path $P_{v_2v_3}$ from v_2 to $v_3$and take a *spiral inequality* (11) associated to v_3 y to k_3 proper paths $P_1^3, \ldots, P_{k_3}^3$. Finally, applying a *path lifting* to such inequality along with (31) and $\theta_2 \in \mathbb{R}_+$ we get:

$$x_{v_0} - x_{v_1} + x_{v_2} - x_{v_3} + \sum_{j=0}^{3} \sum_{i=1}^{k_j} Y_j(P_i^j) \geq \left[2(1 - |C|) + \sum_{j=0}^{3} k_j \right] + \sum_{j=0}^{2} (\theta_j - \theta_j Y_j(P_{v_jv_{j+1}})) \tag{32}$$

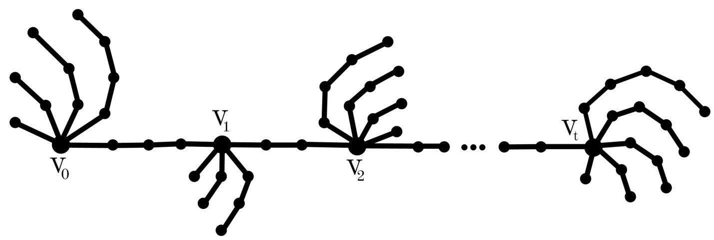

Figure 7: Example structure for the ASPI inequalities.

thus here we have $\alpha_1 = 0$, $\alpha_2 = 1 - |C| + k_3$, $\alpha_3 = 1 - |C| + k_0 + k_1 + \theta_0$ and $\alpha_4 = k_2 + \theta_1$. As before, (32) is valid as long as $|P_{v_2 v_3}| + |C| \geq \theta_2 + \theta_1 + 1 + k_2 + k_3$.

Clearly, this procedure may be continued by concatenating more paths and inequalities in the process (as it is illustrated on Figure 7). We next formalize the depicted family of valid inequalities for $P_{\text{col}}(G)$, which alternates *spiral inequalities* (11) and (10) with path lifting procedures.

Definition 4.1. *Let $v_0, \ldots, v_t$ be a sequence of vertices and t paths $P_{v_j v_{j+1}}$ from v_j to v_{j+1} for $j = 0, \ldots, t-1$. Let $P_1^j, \ldots, P_{k_j}^j$, be k_j paths starting on v_j with $|P_i^j| = i$, for $i = 1, \ldots, k_j$ and $j = 0, \ldots, t$ (see Figure 7). Given $\theta_0, \ldots, \theta_{t-1} \in \mathbb{R}_+$, if $|P_{v_0 v_1}| + |C| \geq \theta_0 + k_0 + k_1 + 1$ and $|P_{v_j v_{j+1}}| + |C| \geq \theta_j + \theta_{j-1} + k_j + k_{j+1} + 1$ for all $j = 1, \ldots, t-1$, then the* alternating spiral path inequality (ASPI) *is defined as*

$$\sum_{j=0}^{t} (-1)^j x_{v_j} + \sum_{j=0}^{t} \sum_{i=1}^{k_j} Y_j(P_i^j) \geq \sum_{j=0}^{t} \lambda_j + \sum_{j=0}^{t-1} (\theta_j - \theta_j Y_j(P_{v_j v_{j+1}})) \tag{33}$$

where $\lambda_j = k_j$ if j is even and $\lambda_j = 1 - |C| + k_j$ if j is odd.

Proposition 4.1. *The alternating spiral path inequalities are valid for $P_{col}(G)$.*

Proof. The proof is given by induction on t. If $t = 0$, then (33) is just a *spiral inequality* of type (10). If $t = 1$, then (33) is (30), which is obtained by a *path lifting* under valid assumptions (as $|P_{v_0 v_1}| \geq \theta_0 + 1 - |C| + k_0 + k_1 = \theta_0 + \alpha_2 + \alpha_4$), hence is valid for $P_{\text{col}}(G)$.
Assume then $t \geq 2$. By inductive hypothesis, we know that ASPI inequalities for the first $t-1$ and $t-2$ vertices from the sequence are valid. These inequalities are, respectively,

$$\sum_{j=0}^{t-1} (-1)^j x_{v_j} + \sum_{j=0}^{t-1} \sum_{i=1}^{k_j} Y_j(P_i^j) \geq \sum_{j=0}^{t-1} \lambda_j + \sum_{j=0}^{t-2} (\theta_j - \theta_j Y_j(P_{v_j v_{j+1}})). \tag{34}$$

$$\sum_{j=0}^{t-2} (-1)^j x_{v_j} + \sum_{j=0}^{t-2} \sum_{i=1}^{k_j} Y_j(P_i^j) \geq \sum_{j=0}^{t-2} \lambda_j + \sum_{j=0}^{t-3} (\theta_j - \theta_j Y_j(P_{v_j v_{j+1}})), \tag{35}$$

We shall see now that (33) is generated by a *path lifting* with θ_{t-1} using (35) and a *spiral inequality* associated to vertex v_t (which may be (10) or (11) given the parity of t). It is easy to see that the difference between (34) and (35) is

$$(-1)^{t-1}x_{v_{t-1}} + \sum_{i=1}^{k_{t-1}} Y_{t-1}(P_i^{t-1}) \geq \lambda_{t-1} + \theta_{t-2} - \theta_{t-2}Y_{t-2}(P_{v_{t-2}v_{t-1}})$$

If t is odd, we may use (35) as $\pi^3 x + \mu^3 y \geq \alpha_3$ and (34) as (23), and so $\mu^4 \geq \mathbf{0}$ and $\alpha_4 = \lambda_{t-1} + \theta_{t-2} = k_{t-1} + \theta_{t-2}$. Then, by using the following *spiral inequality*

$$-x_{v_t} + \sum_{i=1}^{k_t} \overleftarrow{Y}(P_i^t) \geq 1 - |C| + k_t$$

in the role of (22) we have $\mu^2 \geq \mathbf{0}$ and $\alpha_2 = 1 - |C| + k_t = \lambda_t$. The obtained inequality from the *path lifting* is (33), and since $|P_{v_{t-1}v_t}| \geq \theta_{t-1} + \theta_{t-2} + 1 - |C| + k_t + k_{t-1} = \theta_{t-1} + \alpha_2 + \alpha_4$, it is valid for $P_{\text{col}}(G)$. In the case that t is even, by using (35) as $\pi^1 x + \mu^1 y \geq \alpha_1$ and (34) as (22), we have $\mu^2 \geq \mathbf{0}$ and $\alpha_2 = \lambda_{t-1} + \theta_{t-2} = 1 - |C| + k_{t-1} + \theta_{t-2}$, and so using the following *spiral inequality*

$$x_{v_t} + \sum_{i=1}^{k_t} \overrightarrow{Y}(P_i^t) \geq k_t$$

in the role of (23) we have $\mu^4 \geq \mathbf{0}$ and $\alpha_4 = k_t = \lambda_t$. Again, the obtained inequality from the *path lifting* is (33), and since $|P_{v_{t-1}v_t}| \geq \theta_{t-1} + \theta_{t-2} + 1 - |C| + k_t + k_{t-1} = \theta_{t-1} + \alpha_2 + \alpha_4$, it is valid for $P_{\text{col}}(G)$. □

Note that the ASPI inequalities are a generalization of *double* and *triple spiral inequalities* (using $t = 2$ and $t = 3$, respectively, and the proper values for the corresponding θ). Also, being the product of successive applications of *path lifting* procedures, Theorem 3.2 suggests that under suitable hypothesis, some of these inequalities may define facets of $P_{\text{col}}(G)$. We omit the statement of these hypothesis as it would amount to replicating the conditions for each applied *path lifting*.

We conclude this section by mentioning that in our experimentation with G being a path up to 7 vertices, we found facets of $P_{\text{col}}(G)$ from the ASPI family with $t = 2, 3, 4$. Moreover, every facet obtained for the analyzed small instances (G being a path) seem to come from *path liftings*, and more specifically from the ASPI family. We strongly believe that this family is enough to characterize $P_{\text{col}}(G)$ when G is a path (and maybe even when G is a tree).

5. Final remarks

In this work we have introduced several families of facet-inducing inequalities and a facet-generating procedure for the polytope associated with the orientation model for the classical vertex coloring problem. Although these results are of a theoretical nature, these families may be useful within a cutting plane environment in order to solve instances of vertex coloring and variations in practice. Although the orientation model is not the best available formulation for solving the classical vertex coloring problem, its variables and constraints appear as substructures of many other formulations (as, e.g., interval coloring and bandwidth allocation), hence these results could be applicable in these settings as well.

From a theoretical point of view, it would be interesting to further develop the ideas present in the path-lifting procedure studied in this work. In particular, the existence of a real-valued input parameter allows the procedure to generate an infinite number of valid inequalities, and detecting which values of this parameter generate facet-inducing inequalities is not a straightforward task. Providing insight on this issue may be an interesting theoretical endeavor and might enhance the applicability of this procedure in practice.

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