OPTIMAL CONTROL OF DIFFERENTIAL INCLUSIONS

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In memory of Oleg Vladimirovich Vasiliev, a colleague and dear friend

Abstract. This paper is devoted to optimal control of dynamical systems governed by differential inclusions in both frameworks of Lipschitz continuous and discontinuous velocity mappings. The latter framework mostly concerns a new class of optimal control problems described by various versions of the so-called sweeping/Moreau processes that are very challenging mathematically and highly important in applications to mechanics, engineering, economics, robotics, etc. Our approach is based on developing the method of discrete approximations for optimal control problems of such differential inclusions that addresses both numerical and qualitative aspects of optimal control. In this way we establish necessary optimality conditions for optimal solutions to differential inclusions and discuss their various applications. Deriving necessary optimality conditions strongly involves advanced tools of first-order and second-order variational analysis and generalized differentiation.

Key words. Optimal control, differential inclusions, variational analysis, sweeping processes, discrete approximations, generalized differentiation

AMS subject classifications. 49J52, 49J53, 49K24, 49M25, 90C30

1 Introduction

Classical optimal control theory deals with dynamical systems of the type

$$\dot{x}(t) = f\big(t, x(t), u(t)\big), \quad u(t) \in U \quad \text{a.e.} \quad t \in [a, b], \tag{1.1}$$

in the class of measurable controls $u(\cdot)$, where $f\colon [a, b] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is a vector function that is continuously differentiable in x, and where U is a compact set. The main result there is the *Pontryagin maximum principle* (PMP), which provides necessary optimality conditions for strong local minimizers via the maximization of a certain Hamiltonian function; see [44] and further developments in [10, 21, 33, 49, 51] with the references therein, where the reader can also find extensions of the classical PMP to various hereditary systems, nonsmooth problems, partial differential equations of parabolic and hyperbolic types, and other controlled dynamical systems.

More recently optimal control theory has been extended to dynamical systems without explicit control parameterizations by considering *differential inclusions* of the type

$$\dot{x}(t) \in F\big(t, x(t)\big) \quad \text{a.e.} \quad t \in [a, b] \tag{1.2}$$

in the class of absolutely continuous trajectories $x(\cdot)$, where $F\colon [a, b] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping/multifunction acting in finite-dimensional spaces. We refer the reader to the books [10, 33, 38, 51] with the vast bibliographies and commentaries therein for various results on optimization problems for differential inclusions obtained under Lipschitzian assumptions on the set-valued velocity mapping $F(t, \cdot)$ in (1.2). Observe that optimization problems for Lipschitzian differential inclusions are intrinsically nonsmooth, and thus their study requires appropriate tools of generalized differentiation. Necessary optimality conditions for such problems were obtained in extended Euler-Lagrange and Hamiltonian forms (including maximization conditions of the Pontryagin type); see the aforementioned monographs for more details.

It is worth mentioning that the differential inclusion framework (1.2) covers (via measurable selection theorems) not only the standard optimal control setting (1.2) with constant control sets U (which may evolve in time), but also much more challenging situations where control sets depend on state variables

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$U = U(t,x)$. The latter setting corresponds to the representation $F(t,x) = f(t,x,U(t,x))$ in (1.2) while reflecting a certain *feedback control* effect that is crucial, in particular, for engineering design. Observe also that the differential inclusion formalism arises not only in describing the parameterized control systems of type (1.1) with $U = U(t,x)$, but in other numerous applications to economic, mechanic, and behavioral science models that do not involve any control parametrization.

In this paper we mainly discuss a constructive approach to the study and solving of optimization problems for differential inclusions that is based on the *method of discrete approximations*. This approach clearly has a computational flavor to justify the possibility of the numerical solution of infinite-dimensional optimization problems by optimizing their finite-dimensional discrete-time counterparts. But our main goal here is to derive necessary optimality conditions for the original infinite-dimensional control problems by reducing them to finite-dimensional ones and employing optimality conditions in mathematical programming. The idea of this approach goes back to Euler [20] who used it to obtain a necessary optimality condition ("Euler equation") for a specific ("simplest") problem of the calculus of variations on minimizing a particular integral functional depending on the velocity variable.

The development of this idea in problems with dynamic constraints of type (1.2), or even of the standard optimal control type (1.1) with smooth dynamics, is significantly more challenging. The reader is referred to the author's books [33, 38] with the extensive bibliographies and commentaries therein for the implementation of this approach in various classes of dynamical systems: ordinary differential equations and inclusions, delay-differential and neutral-type inclusions, partial differential equations and inclusions of the parabolic type, etc. In what follows we start our discussions with some results obtained in this direction for optimization problems governed by *Lipschitzian differential inclusions* of type (1.2).

Quite recently, other types of optimization problems for *discontinuous* differential inclusions have been formulated and investigated from the viewpoint of deriving necessary optimality conditions. The dynamics of such systems is governed by various versions of the *sweeping process*. The original *uncontrolled* version of the sweeping process was introduced by Moreau in the 1970s motivated by applications to problems of elastoplasticity; see [43] and the book [31] for more details. Similar processes were independently considered by Krasnosel'skii and Pokrovskii for dynamical systems with hysteresis; see their book [28]. Later on models of the sweeping type appeared in other areas of applied science and practical modeling; see, e.g., [1] for electric circuits, [30, 50] for traffic equilibria, etc. We also refer the reader to the survey paper [16] with the extensive bibliographies therein for the mathematical theory of uncontrolled sweeping processes as a part of nonlinear analysis.

Recall that the basic sweeping process of Moreau is described by the differential inclusion

$$\dot{x}(t) \in -N\big(x(t); C(t)\big) \ \text{ a.e. } \ t \in [0,T], \quad x(0) := x_0 \in C(0), \tag{1.3}$$

on the fixed time interval $[0,T]$, where $N(x;\Omega) = N_\Omega(x)$ signifies the normal cone [46] to a convex set $\Omega \subset \mathbb{R}^n$ at x in the standard sense of convex analysis

$$N(x;\Omega) := \begin{cases} \big\{v \in \mathbb{R}^n \big|\ \langle v, u - x\rangle \le 0 \ \text{ for all } \ u \in \Omega\big\} & \text{if } \ x \in \Omega, \\ \emptyset & \text{otherwise,} \end{cases} \tag{1.4}$$

and where the moving convex set $C(t)$ continuously depends on time.

Denoting $F(x) := -N(x; C(t))$ in (1.3), we observe that this (always unbounded) velocity mapping is discontinuous and hence is never Lipschitzian, i.e., fails to satisfy the crucial assumption of the aforementioned control theory for differential inclusions. There is a more striking thing to say on comparison of (1.3) with Lipschitzian differential inclusions: the Cauchy problem in (1.3) admits a *unique* solution due to the well-known maximal monotonicity of the normal cone mapping $x \mapsto N(x; C(t))$ in convex analysis [46]. This excludes, in contrast to the Lipschitzian theory for (1.2), the possibility of optimization of the sweeping differential inclusion as given in (1.3) with a fixed moving set $C(t)$.

In [11] we suggested for the first time in the literature to insert *control actions* into the *moving sets*

$$C(t) := C\big(u(t)\big) \ \text{ for all } \ t \in [0,T], \tag{1.5}$$

which makes it possible to change and optimize the *shape* of the right-hand side in (1.3) in order to achieve a desired performance of the controlled sweeping process with respect to a prescribed cost functional. This novel and practically motivated approach led us to a new class of control systems that is essentially different from those considered before in control theory. Besides the discontinuity and the changeable shape of the velocity mapping in (1.3), we unavoidably have the *pointwise mixed control-state constraints*

$$x(t) \in C\big(u(t)\big) \text{ for a.e. } t \in [0,T],$$

which are intrinsic in (1.3) and (1.5) due to the normal cone construction (1.4). Such constraints are among the most difficult even for standard optimal control of smooth systems (1.1) while being investigated therein only under restrictive regularity assumptions.

Other classes of dynamic optimization problems for controlled sweeping processes correspond to the appearance of control actions in additive *external perturbations* of the type

$$\dot{x}(t) \in g\big(x(t), w(t)\big) - N\big(x(t); C(t)\big) \text{ for a.e. } t \in [0,T], \quad x(0) := x_0 \in C(0), \tag{1.6}$$

where control functions $w(\cdot)$ may be applied either together with controls $u(\cdot)$ in the moving sets (1.5), or in the absence of them. Problems of type (1.6) also exhibit new phenomena in control theory and require the development and implementation of advanced tools of variational analysis.

Our recent results discussed in what follows show that the machinery of well-posed discrete approximations married to powerful tools of first-order and second-order generalized differentiation lead us to deriving new necessary optimality conditions for local minimizers in optimal control problems of both types as in (1.3), (1.5) and in (1.6) expressed entirely in terms of the given problem data. Some of the obtained necessary optimality conditions contain conventional Hamiltonian maximization of the PMP type. On the other hand, we show that in problems with controlled moving sets (1.5) the conventional PMP formalism *fails*, while we are able to establish a new one in terms of a novel Hamiltonian function.

The rest of the paper is organized as follows. In Section 2 we present and discuss some basic robust constructions of first-order and second-order *generalized differentiation* in variational analysis that are appropriate to study differential inclusions while being widely used in all the subsequent sections.

Section 3 is devoted to optimization problems for dynamical systems governed by *Lipschitzian* differential inclusions. We discuss here the well-posedness of discrete approximations and their applications to deriving necessary optimality conditions of the extended Euler-Lagrange type accompanied by the maximization condition for a broad class of intermediate (including strong) local minimizers.

In Section 4 we consider some classes of *sweeping control problems* of type (1.6) with controls in *additive perturbations*. First we investigate problems with smooth controls $w(\cdot)$ and $u(\cdot)$ in perturbations and in (possibly nonconvex) moving sets, respectively, and then study optimization problems with constrained discontinuous controls only in perturbations. In the first case the method of discrete approximation leads us to deriving extended Euler-Lagrange conditions of a new type, while for the sweeping control systems of the second kind we derive optimality conditions extending the maximum principle.

Section 5 deals with optimal control problems for sweeping processes with control functions acting in *parameterized moving sets*. Employing discrete approximations, we derive necessary optimality conditions in appropriate Euler-Lagrange and Hamiltonian forms, where the new Hamiltonian function is introduced to establish a novel version of the maximum principle. It is observed that the conventional form of the maximum principle fails to provide necessary optimality conditions for such control systems.

The concluding Section 6 is devoted to *applications* of the obtained necessary optimality conditions to some practical models with smooth and nonsmooth dynamics. We discuss here recent applications to corridor and planar versions of the crowd motion model and related models of traffic equilibria, to hysteresis systems and elastoplasticity problems, and to typical control models arising in robotics.

Throughout the paper we use the standard notation of variational analysis, generalized differentiation and control theory; see, e.g., [39, 47, 51]. We specified them in the places where they appear for the first time in the paper. Among other symbols, recall that A^* signifies for the transposed/adjoint matrix to

A and that $I\!N := \{1, 2, \ldots\}$. We also mention that $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ indicates that F may be a set-valued mapping, in contrast to the usual notation $F\colon \mathbb{R}^n \to \mathbb{R}^m$ for single-valued ones.

2 Tools of Generalized Differentiation

In this section we provide a brief overview of those constructions of generalized differentiation for nonsmooth functions, nonconvex sets, and set-valued mappings that are used in the paper. These constructions have been initiated by the author in [32] for the first order and in [34] for the second order while now being major in variational analysis and its applications to optimization, control theory, and numerous applications; see, e.g., the books [33, 37, 38, 39, 47, 51] and the references therein for more details.

We start with extended-real-valued functions $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$, which is a standard and convenient framework in convex and variational analysis. Given $\bar{x} \in \operatorname{dom}\varphi := \{x \in \mathbb{R}^m \mid \varphi(x) < \infty\}$, the (first-order) *subdifferential* (or the set of subgradients) of φ at $\bar{x}$ is defined by

$$\partial\varphi(\bar{x}) := \Big\{ v \in \mathbb{R}^n \Big| \begin{array}{l} \exists\, x_k \to \bar{x}, \exists\, v_k \to v \ \text{ such that } \varphi(x_k) \to \varphi(\bar{x}) \ \text{ and} \\ \displaystyle\liminf_{x \to x_k} \frac{\varphi(x) - \varphi(x_k) - \langle v_k, x - x_k\rangle}{\|x - x_k\|} \ge 0 \ \text{ as } \ k \to \infty \end{array} \Big\}. \tag{2.1}$$

This construction reduces to the gradient $\partial\varphi(\bar{x}) = \{\nabla\varphi(\bar{x})\}$ for smooth functions and to the classical subdifferential of convex analysis if φ is convex. The subgradient set (2.1) is nonempty for any function φ that is locally Lipschitzian around $\bar{x}$ while may be nonconvex even for simple Lipschitzian functions; e.g., $\partial\varphi(0) = \{-1, 1\}$ for $\varphi(x) := -|x|$ on $\mathbb{R}$. Nevertheless the subdifferential (2.1) and associated constructions for sets and set-valued mappings enjoy comprehensive calculus rules, which are based on variational/extremal principles of variational analysis.

Given a set $\Omega \subset \mathbb{R}^n$, consider its indicator function $\delta_\Omega(x)$, which equals 0 for $x \in \Omega$ and ∞ otherwise, and define the *normal cone* to Ω at $\bar{x}$ by

$$N_\Omega(\bar{x}) = N(\bar{x}; \Omega) := \partial\delta_\Omega(\bar{x}) \ \text{ for } \ x \in \Omega \ \text{ and } \ N(\bar{x}; \Omega) := \emptyset \ \text{ for } \ \bar{x} \notin \Omega. \tag{2.2}$$

Considering then a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with the graph $\operatorname{gph} F := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in F(x))\}$, the *coderivative* of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is defined by

$$D^*F(\bar{x}, \bar{y})(u) := \big\{ v \in \mathbb{R}^n \big| \ (v, -u) \in N\big((\bar{x}, \bar{y}); \operatorname{gph} F\big)\big\} \ \text{ for all } \ u \in \mathbb{R}^m, \tag{2.3}$$

while we drop $\bar{y}$ in (2.3) when F is single-valued. In the case where F is smooth ($\mathcal{C}^1$) around $\bar{x}$ we have

$$D^*F(\bar{x})(u) = \big\{\nabla F(\bar{x})^* u\big\} \ \text{ for all } \ u \in \mathbb{R}^m$$

via the transpose Jacobian matrix, but in general the coderivative (2.3) is a positively homogeneous set-valued mapping enjoying full calculus rules and providing complete characterizations (call "Mordukhovich criteria" in [47]) of the major well-posedness properties in nonlinear analysis related to Lipschitzian stability, metric regularity, and linear openness/covering of multifunctions; see [35] and then [37, 39, 47] for different proofs and numerous applications. In particular, these characterizations play a crucial role in deriving necessary optimality conditions for Lipschitzian differential inclusions.

Now we turn to second-order generalized differential constructions for extended-real-valued functions $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ by employing the dual "derivative-of-derivative" approach. Given $\bar{x} \in \operatorname{dom}\varphi$, pick a subgradient $\bar{v} \in \partial\varphi(\bar{x})$ and define the *second-order subdifferential* (or *generalized Hessian*) $\partial^2\varphi(\bar{x}, \bar{v})\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ of φ at $\bar{x}$ relative to $\bar{v}$ as the coderivative of the first-order subgradient mapping by

$$\partial^2\varphi(\bar{x}, \bar{v})(u) := (D^*\partial\varphi)(\bar{x}, \bar{v})(u), \quad u \in \mathbb{R}^n, \tag{2.4}$$

where the indication of $\bar{v} = \nabla\varphi(\bar{x})$ is dropped when φ is differentiable at $\bar{x}$. If φ is $\mathcal{C}^2$-smooth around $\bar{x}$, then (2.4) reduces to the classical (symmetric) Hessian matrix of φ at $\bar{x}$:

$$\partial^2\varphi(\bar{x})(u) = \big\{\nabla^2\varphi(\bar{x})u\big\} \ \text{ for all } \ u \in \mathbb{R}^n.$$

Second-order subdifferential constructions of type (2.4) naturally appear in the study of the sweeping processes defined via the normal cone mappings as in (1.3) and its nonconvex extensions. This is due to the description of adjoint systems in first-order optimality conditions for differential inclusions via coderivatives. In fact, in modeling of a large class of sweeping processes with control-dependent moving sets we use the parameterized normal cone mapping $\mathcal{N}\colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ given by

$$\mathcal{N}(x,w) := N\big(x; S(w)\big) \;\text{ for }\; x \in S(w) := \big\{x \in \mathbb{R}^n \big|\; \theta(x,w) \in \Theta\big\}, \quad w \in \mathbb{R}^m. \tag{2.5}$$

To proceed in more detail, consider a function $\varphi\colon \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ of two variables and define the *partial second-order subdifferential* of φ with respect to x at $(\bar{x},\bar{w})$ relative to $\bar{v}$ by

$$\partial_x^2\varphi(\bar{x},\bar{w},\bar{v})(u) := \big(D^*\partial_x\varphi\big)(\bar{x},\bar{w},\bar{v})(u) \;\text{ for all }\; u \in \mathbb{R}^n \tag{2.6}$$

via the coderivative (2.3) of the first-order partial subdifferential mapping

$$\partial_x\varphi(x,w) := \partial\varphi_w(x) \;\text{ with }\; \varphi_w(x) := \varphi(x,w).$$

Observe that $\mathcal{N}(x,w) = \partial_x\varphi(x,w)$ with $\varphi(x,w) := (\delta_\Theta \circ \theta)(x,w)$, where θ and Θ are taken from (2.5). We clearly have the following coderivative representation for the normal cone mapping (2.5):

$$D^*\mathcal{N}(\bar{x},\bar{w},\bar{v})(u) = \partial_x^2\varphi(\bar{x},\bar{w},\bar{v})(u) \;\text{ whenever }\; \bar{v} \in \mathcal{N}(\bar{x},\bar{w}) \;\text{ and }\; u \in \mathbb{R}^n.$$

Further elaborations of this formula require developing chain rules for the partial second-order subdifferential (2.4) for the composite function φ therein. The following *generalized second-order chain rule* taken from [42, Theorem 3.1] is efficient in our applications to controlled sweeping processes: Let $\theta\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^d$ be $\mathcal{C}^2$-smooth around $(\bar{x},\bar{w})$ with the surjective partial Jacobian operator $\nabla_x\theta(\bar{x},\bar{w})$. Then for each $\bar{v} \in \mathcal{N}(\bar{x},\bar{w})$ there exists a unique vector $\bar{q} \in N_\Theta(\theta(\bar{x},\bar{w}))$ such that $\nabla_x\theta(\bar{x},\bar{w})^*\bar{q} = \bar{v}$ and the coderivative (2.3) of the normal cone mapping (2.5) is calculated by

$$D^*\mathcal{N}(\bar{x},\bar{w},\bar{v})(u) = \begin{bmatrix} \nabla^2_{xx}\langle\bar{q},\theta\rangle(\bar{x},\bar{w}) \\ \nabla^2_{xw}\langle\bar{q},\theta\rangle(\bar{x},\bar{w}) \end{bmatrix} u + \nabla\theta(\bar{x},\bar{w})^* D^*N_\Theta\big(\theta(\bar{x},\bar{w}),\bar{q}\big)\big(\nabla_x\theta(\bar{x},\bar{w})u\big), \quad u \in \mathbb{R}^n. \tag{2.7}$$

As we see, the second-order chain rule (2.7) reduces the calculation of $D^*\mathcal{N}$ to that of D^*N_Θ. Constructive computations of the latter for various classes of sets Θ, which are overwhelmingly encountered in optimization, control and their applications, can be found in [24, 39, 41, 42] and the references therein.

3 Optimization of Lipschitzian Differential Inclusions

Here we consider the following Bolza-type optimal control problem for differential inclusions under the (Lipschitzian) assumptions listed below in Theorem 3.2:

$$\text{minimize } \; J[x] := \varphi\big(x(0),x(T)\big) + \int_0^T \ell\big(x(t),\dot{x}(t)\big)dt \tag{3.1}$$

over absolutely continuous trajectories $x\colon [0,T] \to \mathbb{R}^n$ of the autonomous differential inclusion

$$\dot{x}(t) \in F\big(x(t)\big) \;\text{ a.e. }\; t \in [0,T] \tag{3.2}$$

subject to the geometric endpoint constraints

$$\big(x(0),x(T)\big) \in \Omega. \tag{3.3}$$

We refer the reader to [38, 51] for more general versions of this problem for nonautonomous differential inclusions without convexity assumptions on $F(x)$ and $\ell(x,\cdot)$ and Lipschitzian assumptions on the terminal and running costs φ and ℓ. We choose here the model in (3.1)–(3.3) for simplicity and better comparison with controlled sweeping processes. The crucial assumption in the necessary optimality conditions of Theorem 3.2 is the *Lipschitzian* dependence of F on the state variable x.

As mentioned in Section 1, our approach to deriving necessary optimality conditions for local minimizers of the above problem (3.1)–(3.3) is based on the *method of discrete approximations*. The main issues of this approach are as follows:

• Construct a family of discrete approximations involving a finite-difference replacement of the derivative in (3.2) and a consistent perturbation of the endpoint constraints in (3.3). Then approximate any feasible trajectory of (3.2) by feasible trajectories of discrete systems in a topology implying the a.e. convergence of the discrete derivatives that are piecewise constantly extended on the continuous-time interval $[0,T]$. In this step we address not only qualitative aspects of well-posedness but also numerical ones with estimating error bounds, convergence rates, etc. Achieving it leads us to the $W^{1,2}$-norm approximation of a given local minimizer for the continuous-time problem (3.1)–(3.3) by a sequence of optimal solutions to the discrete-time problems that are piecewise linearly extended to the whole interval $[0,T]$. In [38] it was done for a class of the so-called "intermediate local minimizers" introduced in [36]. This class includes strong local minimizers while occupying an intermediate position between the latter and weak local minimizers in dynamic optimization; see Definition 3.1 and subsequent discussions.

• Each discrete-time problem that approximates the original one can be reduced to a nondynamic problem of mathematical programming in finite dimensions with increasingly many geometric constraints of the graphical type. We employ the powerful tools of generalized differentiation discussed in Section 2 for deriving necessary optimality conditions in the approximating discrete-time problems. It can be done without any Lipschitzian and convexity assumptions by applying the well-developed generalized differential calculus for them. Note that dealing with the graphical structure of the geometric constraints requires that the used generalized differential constructions should be subtle and small enough to handle graphical sets. In particular, the convexified normal cone by Clarke [10] cannot be employed for these purposes since applying it to graphical sets often gives us the whole space or its subspace of maximal dimension; see [37, 39, 47] for more details. On the other hand, our constructions discussed in Section 2 satisfy all the required properties and thus can be successfully implemented.

• Finally, we derive necessary optimality conditions for local minimizers of (3.1)–(3.3) by passing to the limit from those for discrete approximations. This part is the most challenging while requiring the clarification and justification of an appropriate convergence of dual arcs. For the case of Lipschitzian differential inclusions it is done by using the coderivative/Mordukhovich criterion for the Lipschitz continuity of set-valued mappings mentioned in Section 2.

The necessary optimality conditions formulated below concerns the following notion of local minimizers for (3.1)–(3.3) introduced in [36].

Definition 3.1 *Let $\bar{x}(\cdot)$ be a feasible solution to problem* (3.1)–(3.3). *We say that $\bar{x}(\cdot)$ is an* INTERMEDIATE LOCAL MINIMIZER *of rank $p \in [1,\infty)$ for this problem if there are $\varepsilon > 0$ and $\alpha \ge 0$ such that $J[\bar{x}] \le J[x]$ for any feasible solution to* (3.1)–(3.3) *satisfying the constraints*

$$\|x(t)-\bar{x}(t)\| < \varepsilon \ \text{ for all } \ t \in [0,T] \ \text{ and } \ \alpha \int_0^T \|\dot{x}(t)-\dot{\bar{x}}(t)\|^p dt < \varepsilon. \tag{3.4}$$

The localization in (3.4) means in fact that a neighborhood of $\bar{x}(\cdot)$ in the space $W^{1,p}\big([0,T];\mathbb{R}^n\big)$ is considered. If $\alpha = 0$ in (3.4), we get the classical *strong* local minimum corresponding to a neighborhood of $\bar{x}$ in the norm topology of $\mathcal{C}([0,T];\mathbb{R}^n)$. If (3.4) is replaced by the more restrictive requirement

$$\|\dot{x}(t)-\dot{\bar{x}}(t)\| < \varepsilon \ \text{ a.e. } \ t \in [0,T],$$

we get the classical *weak* local minimum in the framework of Definition 3.1, which corresponds to considering a neighborhood of $\bar{x}(\cdot)$ in the norm topology of $W^{1,\infty}([0,T];\mathbb{R}^n)$. The reader is referred to [36, 51] for various examples showing that the intermediate notion of Definition 3.1 is properly different from both strong and weak local minimizers of (3.1)–(3.3) for convex autonomous differential inclusions.

Here are the aforementioned necessary optimality conditions for Lipschitzian differential inclusions obtained by using discrete approximations. Note that the assumptions imposed in the theorem ensure that we can consider the case of $p = 2$ without loss of generality and thus refer to $\bar{x}(\cdot)$ as to an intermediate local minimizer for problem (3.1)–(3.3).

Theorem 3.2 *Let $\bar{x}(\cdot)$ be an intermediate local minimizer for problem* (3.1)–(3.3) *under the assumptions that F is locally Lipschitzian, convex-valued, and bounded around $\bar{x}(\cdot)$, that the running cost ℓ is locally Lipschitzian in both variables and convex with respect to velocities, that the terminal cost φ is locally Lipschitzian while the constraint set Ω is locally closed around $(\bar{x}(0), \bar{x}(T))$. Then there exist a number $\lambda \geq 0$ and an absolutely continuous adjoint arc $p\colon [0,T] \to \mathbb{R}^n$, not equal to zero simultaneously, satisfying the extended Euler-Lagrange inclusion*

$$\dot{p}(t) \in \text{co}\Big\{u \in \mathbb{R}^n \Big|\ \begin{array}{l}\big(u, p(t)\big) \in \lambda\partial\ell\big(\bar{x}(t), \dot{\bar{x}}(t)\big)\\ +N\big((\bar{x}(t), \dot{\bar{x}}(t)); \text{gph}\, F\big)\end{array}\Big\} \quad a.e. \quad t \in [0,T],$$

the Weierstrass-Pontryagin maximum condition

$$\big\langle p(t), \dot{\bar{x}}(t)\big\rangle - \lambda\vartheta\big(\bar{x}(t), \dot{\bar{x}}(t)\big) = \max_{v \in F(\bar{x}(t))} \Big\{\big\langle p(t), v\big\rangle - \lambda\ell\big(\bar{x}(t), v\big)\Big\} \quad a.e. \quad t \in [0,T], \tag{3.5}$$

and the transversality inclusion at both endpoints

$$\big(p(0), -p(T)\big) \in \lambda\partial\varphi\big(\bar{x}(0), \bar{x}(T)\big) + N\big((\bar{x}(0), \bar{x}(T)); \Omega\big).$$

4 Sweeping Processes with Controlled Perturbations

In this section we turn to controlled sweeping processes and start with the system

$$\dot{x}(t) \in g\big(x(t), w(t)\big) - N\big(x(t); C(t)\big) \ \text{ for a.e. } \ t \in [0,T], \quad x(0) := x_0 \in C(0), \tag{4.1}$$

where control functions $w(\cdot)$ are acting in the additive perturbations. When the moving set $C(t)$ in (4.1) is given a priori, optimal control problems of the Bolza type were studied in [9, 19, 48] from the viewpoints of the existence of optimal solutions and relaxation stability.

In [5, 6] we considered the perturbed sweeping process in (4.1), where—along with the controls $w(\cdot)$ in perturbations—the other type of controls $u(\cdot)$ were applied to the moving set $C(t)$ given by

$$C(t) := C + u(t) \ \text{ with } \ C := \big\{x \in \mathbb{R}^n \big|\ \langle x_i^*, x\rangle \leq 0 \ \text{ for all } \ i = 1, \ldots, m\big\} \tag{4.2}$$

and the fixed vectors x_i^* generating the convex polyhedron C in (4.2). The optimal control problem formulated and investigated in [5, 6] was as follows:

$$\text{minimize } \ J[x,u,w] := \varphi\big(x(T)\big) + \int_0^T \ell\big(x(t), u(t), w(t), \dot{x}(t), \dot{u}(t), \dot{w}(t)\big)\, dt \tag{4.3}$$

over control pairs $u(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^n)$, $w(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^d)$ and the corresponding trajectories $x(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^n)$ of the controlled sweeping system in (4.1), (4.2). The main attention in [5, 6] was paid to the construction of well-posed discrete approximations of (4.1)–(4.3) and using the discrete approximation approach to derive necessary conditions for local optimal solutions to this problem. The obtained results were then applied in [6] to solving some optimal control problems for the corridor version of the crowd motion model of traffic equilibria; see Section 6 for more details on this model.

One of the strongest motivations for our subsequent paper [7] was to formulate and investigate a class of sweeping control system, which is suitable for applications to the much more realistic planar crowd motion model the dynamic of which was described in [30, 50] as a sweeping process over a nonpolyhedral moving set. To accomplish this goal, we considered in [7] the controlled sweeping process given by (4.1) with the *nonconvex* (and hence nonpolyhedral) moving set

$$C(t) := C + u(t) = \bigcap_{i=1}^{m} C_i + u(t), \quad C_i := \big\{x \in \mathbb{R}^n \big|\ \xi_i(x) \geq 0\big\} \ \text{ for all } \ i = 1, \ldots, m, \tag{4.4}$$

defined via some convex $\mathcal{C}^2$-smooth functions $\xi_i \colon \mathbb{R}^n \to \mathbb{R}$. Due to the nonconvexity of the set $C(t)$ in (4.4), we replaced therein the normal cone of convex analysis (1.4) by the nonconvex one from (2.2). The optimal control problem formulated in [7] reads as follows: minimize the cost functional (4.3) over control pairs $u(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$, $v(\cdot) \in W^{1,2}([0,T];\mathbb{R}^d)$ and the corresponding trajectories $x(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$ of (4.1) with the controlled moving set (4.4). Besides the dynamic constraints (4.1), we imposed the pointwise constraints on the u-controls as

$$0 < r_1 \leq \|u(t)\| \leq r_2 \ \text{ for all } \ t \in [0,T]$$

with the given constraint bounds r_1, r_2. Note that (4.1) yields the mixed state-control constraints

$$\xi_i\big(x(t) - u(t)\big) \geq 0 \ \text{ for all } \ t \in [0,T] \ \text{ and } \ i = 1, \ldots, m.$$

The method of discrete approximations combined with the machinery of first-order and second-order generalized differentiation from Section 2 led us in [7] to deriving constructive necessary optimality conditions for intermediate local minimizers in the above problem.

Let us now discuss yet another setting of sweeping optimal control, where control actions $w(\cdot)$ entering the additive perturbations in (4.1) are *constrained* and *discontinuous* in contrast to [5]–[7]. On the other hand, the set C in the sweeping dynamics considered in [14] does not depend on control and time variables. The precise formulation of such problems is as follows:

$$\text{minimize} \quad J[x,u] := \varphi\big(x(T)\big) \tag{4.5}$$

over feasible pairs $(x(\cdot), u(\cdot))$ of measurable controls $u(t)$ and absolutely continuous trajectories $x(t)$ on $[0,T]$ satisfying the perturbed controlled sweeping process of type (4.1) written as

$$\dot{x}(t) \in g\big(x(t), u(t)\big) - N\big(x(t); C\big) \ \text{ a.e. } \ t \in [0,T],\ x(0) := x_0 \in C \subset \mathbb{R}^n, \tag{4.6}$$

with the conventional notation for control functions ($u(t)$ instead of $w(t)$ as in (4.1), since the set C is uncontrolled now) subject to the pointwise control constraints given by

$$u(t) \in U \subset \mathbb{R}^d \ \text{ a.e. } \ t \in [0,T]. \tag{4.7}$$

The set C in (4.6) is a convex polyhedron defined by

$$C := \bigcap_{i=1}^{m} C_i \ \text{ with } \ C_i := \big\{x \in \mathbb{R}^n \big|\ \langle x_i^*, x\rangle \leq c_i\big\}. \tag{4.8}$$

Developing an advanced version of the method of discrete approximations, we recently obtained in [14] a collection of new necessary optimality conditions for (4.5)–(4.8) that includes the maximization condition of the PMP type. Let us first describe the class of local minimizers studied in [14]. We say that a feasible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ for (4.5)–(4.8) is a $W^{1,2} \times L^2$*-local minimizer* for this problem if there exists a number $\varepsilon > 0$ such that $J[\bar{x}, \bar{u}] \leq J[x,u]$ whenever a feasible pair $(x(\cdot), u(\cdot))$ satisfies

$$\int_0^T \Big(\|\dot{x}(t) - \dot{\bar{x}}(t)\|^2 + \|u(t) - \bar{u}(t)\|^2\Big)\, dt < \varepsilon.$$

For the reader's convenience and brevity, we present now the major result of [14] under the following simplified assumptions in comparison with those imposed in [14]. The pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ therein is a fixed

$W^{1,2} \times L^2$-local minimizer under consideration.

(A1) The cost function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ in (4.3) is continuous differentiable around $\bar{x}(T)$.
(A2) The perturbation mapping $g\colon \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ in (4.6) is continuous differentiable around $(\bar{x}(\cdot), \bar{u}(\cdot))$ and satisfies the sublinear growth condition

$$\|g(x,u)\| \le \alpha\big(1 + \|x\|\big) \text{ for all } u \in U \text{ with some } \alpha > 0.$$

(A3) The control set U is compact and convex in $\mathbb{R}^d$.
(A4) The image set $g(x, U)$ is convex in $\mathbb{R}^n$.
(A5) The vertices x_i^* of (4.8) satisfy the linear independence constraint qualification

$$\Big[\sum_{i \in I(\bar{x})} \alpha_i x_i^* = 0,\ \alpha_i \in \mathbb{R}\Big] \Longrightarrow \big[\alpha_i = 0 \text{ for all } i \in I(\bar{x})\big\}$$

along the trajectory $\bar{x} = \bar{x}(t)$ as $t \in [0,T]$, where $I(\bar{x}) := \{i \in \{1,\ldots,m\} \mid \langle x_i^*, \bar{x}\rangle = c_i\}$.

Note that the convexity assumptions imposed in (A3) and (A4) can be removed in the necessary optimality conditions presented below for the case of *strong* minimizers by considering a certain relaxation procedure as in [14]. We refer the reader to [18, 19, 48] for various relaxation results (of the Bogolyubov-Young type) for non-Lipschitzian differential inclusions.

Theorem 4.1 *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a $W^{1,2} \times L^2$-local minimizer for problem (4.5)–(4.8) under the assumptions in (A1)–(A5), where $\bar{u}(\cdot)$ is of bounded variation (BV) with a right continuous representative on $[0,T]$. Then there exist a multiplier $\lambda \ge 0$, a measure $\gamma = (\gamma_1,\ldots,\gamma_n) \in C^*([0,T];\mathbb{R}^n)$ as well as adjoint arcs $p(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$ and $q(\cdot) \in BV([0,T];\mathbb{R}^n)$ such that $\lambda + \|q(t)\|_{L^\infty} + \|p(T)\| > 0$ and the following conditions are satisfied:*

• Primal velocity representation:

$$-\dot{\bar{x}}(t) = \sum_{i=1}^m \eta_i(t) x_i^* - g\big(\bar{x}(t), \bar{u}(t)\big) \ \text{ for a.e. } t \in [0,T], \tag{4.9}$$

where $\eta^i(\cdot) \in L^2([0,T];\mathbb{R}_+)$ being uniquely determined by (4.9) and well defined at $t = T$.

• Adjoint system:

$$\dot{p}(t) = -\nabla_x g\big(\bar{x}(t), \bar{u}(t)\big)^* q(t) \ \text{ for a.e. } t \in [0,T],$$

where the dual arcs $q(\cdot)$ and $p(\cdot)$ are precisely connected by the equation

$$q(t) = p(t) - \int_{(t,T]} d\gamma(\tau)$$

that holds for all $t \in [0,T]$ except at most a countable subset.

• Maximization condition*:*

$$\big\langle \psi(t), \bar{u}(t)\big\rangle = \max\big\{\big\langle \psi(t), u\big\rangle \big|\ u \in U\big\} \ \text{ with } \psi(t) := \nabla_u g\big(\bar{x}(t), \bar{u}(t)\big)^* q(t) \ \text{ for a.e. } t \in [0,T].$$

• Complementarity conditions:

$$\big\langle x_i^*, \bar{x}(t)\big\rangle < c_i \Longrightarrow \eta_i(t) = 0 \ \text{ and } \ \eta_i(t) > 0 \Longrightarrow \big\langle x_i^*, q(t)\big\rangle = c_i$$

for a.e. $t \in [0,T]$ including $t = T$ and for all $i = 1,\ldots,m$.

• Right endpoint transversality conditions*:*

$$-p(T) = \lambda \nabla\varphi\big(\bar{x}(T)\big) + \sum_{i \in I(\bar{x}(T))} \eta_i(T) x_i^* \ \text{ with } \sum_{i \in I(\bar{x}(T))} \eta_i(T) x_i^* \in N\big(\bar{x}(T); C\big).$$

• Measure nonatomicity condition: *If $t \in [0,T)$ and $\langle x_i^*, \bar{x}(t)\rangle < c_i$ for all $i = 1,\ldots,m$, then there is a neighborhood V_t of t in $[0,T]$ such that $\gamma(V) = 0$ for all the Borel subsets V of V_t.*

Note that necessary optimality conditions in sweeping control theory containing the maximization of the corresponding Hamiltonian were first obtained in [4] for (global) optimal solutions to a sweeping process of another type with an uncontrolled strictly smooth and convex set $C(t) \equiv C$ having nonempty interior and control functions that linearly enter an adjacent ordinary differential equation. Further results in this direction were derived in the case of the sweeping control system (4.6), where measurable controls $u(t)$ enter the additive smooth term g while the uncontrolled moving set $C(t)$ is compact, convex or mildly nonconvex, and possesses a $\mathcal{C}^3$-smooth boundary for each $t \in [0,T]$ along with some additional assumptions. The very recent paper [17] also deals with a sweeping control system of type (4.6) and establishes necessary optimality conditions for global minimizers involving the maximization of the standard Hamiltonian function provided that the convex and compact set $C(t) \equiv C$ of nonempty interior given by $C := \{x \in \mathbb{R}^n \mid \psi(x) \le 0\}$ via a $\mathcal{C}^2$-smooth function ψ under other assumptions, which are partly differ from [3]. Certain penalty-type approximation methods developed in [3], [4], and [17] are different from each other, significantly based on the *smoothness* of *uncontrolled moving sets* while being sharply distinct from the method of discrete approximations used in our approach.

5 Sweeping Processes with Controlled Moving Sets

In this section we concentrate on a challenging class of controlled sweeping processes with control functions acting in moving sets. Such control problems were introduced and studied in [11] for the case where the set $C(u)$ in (1.5) was defined by a half-space in $\mathbb{R}^n$. A more general and involved case of the polyhedral description of $C(u)$ was fully investigated in [13]. The following optimal control problem was considered therein: minimize the cost functional (4.3) over the collection of absolutely continuous controls $u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot))$, $w(\cdot) = (w_1(\cdot), \ldots, w_m(\cdot))$ and the corresponding absolutely continuous trajectories $x(\cdot)$ satisfying the sweeping differential inclusion (1.3) with the controlled moving set

$$C(t) := \big\{x \in \mathbb{R}^n \big| \ \langle u_i(t), x\rangle \le w_i(t), \ \ i = 1, \ldots, m\big\} \tag{5.1}$$

subject to the control constraints

$$\|u_i(t)\| = 1 \ \text{ for all } \ t \in [0,T], \ \ i = 1, \ldots, m, \tag{5.2}$$

As follows from (1.4) and (5.1), we automatically have also the pointwise state-control constraints

$$x(t) \in C\big(u(t), w(t)\big) \ \text{ for all } \ t \in [0,T] \tag{5.3}$$

Using the method of discrete approximations and advanced tools of variational analysis, we derived in [11, 13] necessary optimality conditions of the extended Euler-Lagrange type for polyhedral sweeping control problems. In this section we present more general results in this direction taken from [27, 40] and obtained without any polyhedrality assumptions. Besides the conditions of the Euler-Lagrange type, the novel optimality conditions of the extended Hamiltonian type are also given therein with discovering that the conventional PMP formalism fails for such control systems.

Here we address the following sweeping control problem:

$$\text{minimize } J[x,u] := \varphi\big(x(T)\big) + \int_0^T \ell\big(x(t), u(t), \dot{x}(t), \dot{u}(t)\big)dt \tag{5.4}$$

over absolutely continuous control functions $u\colon [0,T] \to \mathbb{R}^m$ and the corresponding absolutely continuous trajectories $x\colon [0,T] \to \mathbb{R}^n$ of the sweeping differential inclusion

$$\dot{x}(t) \in g\big(x(t)\big) - N\big(h(x(t)); C(u(t))\big) \ \text{ a.e. } \ t \in [0,T], \quad x(0) = x_0 \in C\big(u(0)\big) \tag{5.5}$$

with the controlled moving set defined by the inverse images

$$C(u) := \big\{x \in \mathbb{R}^n \big| \ \theta(x,u) \in \Theta\big\}, \quad u \in \mathbb{R}^m, \tag{5.6}$$

where $g\colon \mathbb{R}^n \to \mathbb{R}^n$ and $h\colon \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable mappings, and where $\theta\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^d$ is a twice continuously differentiable mapping around the references points with its partial Jacobian matrix $\nabla_x\theta$ of full rank at the point in question. The set Θ in (5.6) is locally closed and is not assumed to be convex. Hence the set $C(u)$ is generally nonconvex as well, while the normal cone in (5.5) is understood in the sense of (2.2). The moving set description (5.6) surely covers the polyhedral case (5.1).

In addition to the standing assumptions on the given data of (5.4)–(5.6) discussed above, suppose that there exist a number $r > 0$ and a mapping $\theta\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ locally Lipschitz continuous and uniformly bounded on bounded sets such that for all $\bar{v} \in N(\theta_{\bar{u}}(\bar{x}); \Theta)$ and $x \in \theta_u^{-1}(\Theta)$ with $u := \bar{u} + \theta(x-\bar{x}, x, \bar{x}, \bar{u})$ there exists a vector $v \in N(\theta_u(x); \Theta)$ satisfying $\|v - \bar{v}\| \le r\|x - \bar{x}\|$. This assumption is technical. It automatically holds not only in the polyhedral setting of (5.1), but also in some nonconvex settings; see [27] for more details and examples.

We consider the two types of local minimizers for problem (5.4)–(5.6) as formulated below.

Definition 5.1 *Fix a feasible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ to problem (5.4)–(5.6) and say that:*

(i) *$(\bar{x}(\cdot), \bar{u}(\cdot))$ is a* LOCAL $W^{1,2} \times W^{1,2}$-MINIMIZER *for this problem if $\bar{x}(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^n)$, $\bar{u}(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^m)$, and we have*

$$J[\bar{x}, \bar{u}] \le J[x, u] \;\; \text{for} \;\; x(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^n) \;\; \text{and} \;\; u(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^m),$$

which are sufficiently close to $(\bar{x}(\cdot), \bar{u}(\cdot))$ in the norm topology of the corresponding $W^{1,2}$ spaces.

(ii) *Let the integrand $\ell(\cdot)$ in (5.4) do not depend on $\dot{u}$. Then $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a* LOCAL $W^{1,2} \times \mathcal{C}$-MINIMIZER *in (5.4)–(5.6) if $\bar{x}(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^n)$, $\bar{u}(\cdot) \in \mathcal{C}([0,T]; \mathbb{R}^m)$, and we have*

$$J[\bar{x}, \bar{u}] \le J[x, u] \;\; \text{for all} \;\; x(\cdot) \in W^{1,2}([0,T]; \mathbb{R}^n) \;\; \text{and} \;\; u(\cdot) \in \mathcal{C}([0,T]; \mathbb{R}^m), \tag{5.7}$$

which are sufficiently close to $(\bar{x}(\cdot), \bar{u}(\cdot))$ in the norm topology of the corresponding spaces in (5.7).

The following major result was proved in [27, Theorem 4.3] by using the method of discrete approximations combined with generalized second-order calculus rule from (2.7). For simplicity we present this result in the case where $g(x) := 0$ and $h(x) := x$ for all $x \in \mathbb{R}^n$.

Theorem 5.2 *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be one of the local minimizers for problem (5.4)–(5.6) taken from Definition 5.1. The following assertions hold:*

(i) *If $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a local $W^{1,2} \times W^{1,2}$-minimizer, then there exist a multiplier $\lambda \ge 0$, an adjoint arc $p(\cdot) = (p^x, p^u) \in W^{1,2}([0,T]; \mathbb{R}^n \times \mathbb{R}^m)$, a vector measure $\gamma \in C^*([0,T]; \mathbb{R}^d)$, as well as pairs $(w^x(\cdot), w^u(\cdot)) \in L^2([0,T]; \mathbb{R}^n \times \mathbb{R}^m)$ and $(v^x(\cdot), v^u(\cdot)) \in L^\infty([0,T]; \mathbb{R}^n \times \mathbb{R}^m)$ with*

$$\big(w^x(t), w^u(t), v^x(t), v^u(t)\big) \in \operatorname{co}\partial\ell\big(\bar{x}(t), \bar{u}(t), \dot{\bar{x}}(t), \dot{\bar{u}}(t)\big) \;\; a.e. \;\; t \in [0,T] \tag{5.8}$$

satisfying the following necessary optimality conditions:

- PRIMAL-DUAL DYNAMIC RELATIONSHIPS*:*

$$\dot{p}(t) = \lambda w(t) + \left[\begin{array}{c} \nabla^2_{xx}\langle \eta(t), \theta\rangle\big(\bar{x}(t), \bar{u}(t)\big) \\ \nabla^2_{xw}\langle \eta(t), \theta\rangle\big(\bar{x}(t), \bar{u}(t)\big) \end{array}\right]\big(-\lambda v^x(t) + q^x(t)\big) \;\; a.e. \;\; t \in [0,T], \tag{5.9}$$

$$q^u(t) = \lambda v^u(t) \;\; a.e. \;\; t \in [0,T], \tag{5.10}$$

where $\eta(\cdot) \in L^2([0,T]; \mathbb{R}^s)$ is a uniquely defined vector function determined by the representation

$$\dot{\bar{x}}(t) = -\nabla_x\theta\big(\bar{x}(t), \bar{u}(t)\big)^*\eta(t) \;\; a.e. \;\; t \in [0,T] \tag{5.11}$$

with $\eta(t) \in N(\theta(\bar{x}(t), \bar{u}(t)); \Theta)$, and where $q\colon [0,T] \to \mathbb{R}^n \times \mathbb{R}^m$ is a function of bounded variation on $[0,T]$ with its left-continuous representative given, for all $t \in [0,T]$ except at most a countable subset, by

$$q(t) = p(t) - \int_{[t,T]} \nabla\theta\big(\bar{x}(\tau), \bar{u}(\tau)\big)^* d\gamma(\tau). \tag{5.12}$$

• Measured coderivative condition: *Considering the t-dependent outer limit*

$$\operatorname*{Lim\,sup}_{|B|\to 0}\frac{\gamma(B)}{|B|}(t):=\Big\{y\in\mathbb{R}^s\Big|\ \exists\ \text{sequence}\ \ B_k\subset[0,1]\ \ \text{with}\ \ t\in I\!B_k,\ |B_k|\to 0,\ \frac{\gamma(B_k)}{|B_k|}\to y\Big\} \tag{5.13}$$

over Borel subsets $B\subset[0,1]$ *with the Lebesgue measure* $|B|$, *for a.e.* $t\in[0,T]$ *we have*

$$D^*N_\Theta\big(\theta(\bar{x}(t),\bar{u}(t)),\eta(t)\big)\big(\nabla_x\theta(\bar{x}(t),\bar{u}(t))(q^x(t)-\lambda v^x(t))\big)\cap\operatorname*{Lim\,sup}_{|B|\to 0}\frac{\gamma(B)}{|B|}(t)\neq\emptyset. \tag{5.14}$$

• Transversality condition *at the right endpoint:*

$$-\big(p^x(T),p^u(T)\big)\in\lambda\big(\partial\varphi(\bar{x}(T)),0\big)+\nabla\theta\big(\bar{x}(T),\bar{u}(T)\big)N_\Theta\big((\bar{x}(T),\bar{u}(T)\big). \tag{5.15}$$

• Measure nonatomicity condition: *Whenever* $t\in[0,T)$ *with* $\theta(\bar{x}(t),\bar{u}(t))\in\operatorname{int}\Theta$ *there is a neighborhood* V_t *of* t *in* $[0,T]$ *such that* $\gamma(V)=0$ *for any Borel subset* V *of* V_t.

• Nontriviality condition:

$$\lambda+\sup_{t\in[0,T]}\|p(t)\|+\|\gamma\|\neq 0\ \ \text{with}\ \ \|\gamma\|:=\sup_{\|x\|_{C([0,T]}=1}\int_{[0,T]}x(s)d\gamma. \tag{5.16}$$

(ii) *If* $(\bar{x}(\cdot),\bar{u}(\cdot))$ *is a local* $W^{1,2}\times\mathcal{C}$*-minimizer, then all the conditions* (5.9)–(5.16) *in* (i) *hold with the replacement of the quadruple* $(w^x(\cdot),w^u(\cdot),v^x(\cdot),v^u(\cdot))$ *in* (5.8) *by the triple* $(w^x(\cdot),w^u(\cdot),v^x(\cdot))\in L^2([0,T];\mathbb{R}^n)\times L^2([0,T];\mathbb{R}^m)\times L^\infty([0,T];\mathbb{R}^n)$ *satisfying the inclusion*

$$\big(w^x(t),w^u(t),v^x(t)\big)\in\operatorname{co}\partial\ell\big(\bar{x}(t),\bar{u}(t),\dot{\bar{x}}(t)\big)\ \ \text{a.e.}\ \ t\in[0,T].$$

It is worth mentioning that the extended Euler-Lagrange optimality conditions of Theorem 5.2, as well as the previous results from [11] and [13] obtained for polyhedral sweeping control problems with movings sets defined in (5.1) and (5.2), do not contain a maximization condition like in the Pontryagin maximum principle for standard control problems governed by the ODE systems (1.1) with smooth dynamics and also like for Lipschitzian differential inclusions as given in (3.5) of Theorem 3.2.

Next we present necessary optimality conditions in the novel *Hamiltonian form*, which is complemented to Theorem 5.2 and does contain a maximization condition of the new type appeared in our papers [27, 40] as the first version of the maximum principle for sweeping process with controlled moving sets. To proceed, consider problem (5.4)–(5.6) with $\Theta=\mathbb{R}^d_-$ being a nonpositive orthant in $\mathbb{R}^d$. This result is based on the generalized second-order chain rule given in Section 2 and the precise calculation of the second-order construction $D^*N_{\mathbb{R}^d_-}$ taken from [41]. When $\Theta=\mathbb{R}^d_-$ in (5.6), define the active index set

$$I(x,u):=\big\{i\in\{1,\ldots,d\}\big|\ \theta_i(x,u)=0\big\}$$

and observe that under the standing surjectivity assumption on the partial Jacobian operator $\nabla_x\theta$ for each $v\in -N(x;C(u))$ there exists a unique collection $\{\alpha_i\}_{i\in I(x,u)}$ with $\alpha_i\le 0$ and $v=\sum_{i\in I(x,u)}\alpha_i[\nabla_x\theta(x,u)]_i$. Given $\nu\in\mathbb{R}^d$, define further the vector $[\nu,v]\in\mathbb{R}^n$ by

$$[\nu,v]:=\sum_{i\in I(x,u)}\nu_i\alpha_i\big[\nabla_x\theta(x,u)\big]_i$$

and introduce the *new Hamiltonian function* by

$$H_\nu(x,u,p):=\sup\big\{\langle[\nu,v],p\rangle\big|\ v\in -N\big(x;C(u)\big)\big\},\quad (x,u,p)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n. \tag{5.17}$$

The aforementioned novel form of the maximum principle that is fulfilled for the class of sweeping control problems under consideration is as follows.

Theorem 5.3 *Consider the optimal control problem* (5.4)–(5.6) *in the frameworks of Theorem* 5.2 *with* $\Theta = \mathbb{R}^d_-$*. Then, in addition to all the conditions in assertions* (i) *and* (ii) *of Theorem* 5.2*, we have the maximization condition*

$$\big\langle \big[\nu(t), \dot{\bar{x}}(t)\big], q^x(t) - \lambda v^x(t)\big\rangle = H_{\nu(t)}\big(\bar{x}(t), \bar{u}(t), q^x(t) - \lambda v^x(t)\big) = 0 \quad a.e. \quad t \in [0,T].$$

holds with a measurable vector function $\nu\colon [0,T] \to \mathbb{R}^d$ *satisfying the inclusion*

$$\nu(t) \in D^* N_{\mathbb{R}^d_-}\big(\theta(\bar{x}(t), \bar{u}(t)), \mu(t)\big)\big(\nabla_x \theta(\bar{x}(t), \bar{u}(t))(q^x(t) - \lambda v^x(t))\big) \cap \operatorname*{Lim\,sup}_{|B| \to 0} \frac{\gamma(B)}{|B|}(t)$$

for a.e. $t \in [0,T]$*, where* Lim sup *is defined in* (5.13).

Furthermore, it is shown in [27] that a conventional form of the maximum principle with replacing the new Hamiltonian function (5.17) by the standard one

$$H(x,u,p) := \sup\big\{\langle p, v\rangle \big|\, v \in -N\big(x; C(u)\big)\big\}, \quad (x,u,p) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n.$$

fails as a necessary optimality condition even for global minimizers of (5.4)–(5.6).

6 Selected Applications

In the final section of the paper we discuss selected applications of the necessary optimality conditions for sweeping control systems presented above as well as some directions of future research. The original sweeping control applications in [11, 13] concerned some models from *elastoplasticity*. In particular, the reader can find in [13] the results for the *quasistatic elastoplasticity models with hardening* the sweeping dynamics of which is described in [22], while the control version was developed and optimized in [13].

The applications in [6, 8] addressed the corridor and planar versions of the *crowd motion model*. The original developments on the crowd motion model concern local interactions between participants in order to describe the dynamics of pedestrian traffic. Nowadays this model is successfully used to study more general classes of problems in socioeconomics, mechanics, operations research, etc. The microscopic form of the crowd motion model is based on the following two postulates. Firstly, each individual has a *spontaneous* velocity that he/she intends to implement in the absence of other participants. However, in reality the *actual* velocity must be considered. The latter one is incorporated via a projection of the spontaneous velocity into the set of admissible velocities, i.e., those which do not violate certain *nonoverlapping constraints*. A mathematical description of the *uncontrolled* microscopic crowd motion model was given in [30, 50] as a *sweeping process*, and then it was used in these and other papers for numerical simulations and various applications.

Let us discuss here some developments in [8] dealing with a practical *control formulation* of the microscopic planar crowd motion model. In contrast to the corridor model considered in [6] by using the necessary optimality conditions obtained in [5] for the polyhedral control description of moving sets, the major overlapping condition in the planar crowd motion model is not polyhedral anymore while being represented in the following form of (4.4):

$$\big\{x \in \mathbb{R}^{2n} \big|\, D_{ij}(x) \ge 0 \ \text{ for all } \ i \ne j\big\},$$

where $D_{ij}(x) := \|x_i - x_j\| - 2R$ is the signed distance between the disks i and j of the same radius R identified with $n \ge 2$ participants on the plane. The corresponding optimal control problem formulated and investigated in [8] is described via the sweeping dynamics as follows: minimize the cost functional of type (4.3) over the constrained controlled sweeping process

$$\begin{cases} \dot{x}(t) \in g\big(x(t), w(t)\big) - N\big(x(t); C(t)\big) \ \text{ for a.e. } \ t \in [0,T], \\ C(t) := C + \bar{u}(t), \ \|\bar{u}(t)\| = r \in [r_1, r_2] \ \text{ on } \ [0,T], \ x(0) = x_0 \in C(0), \end{cases}$$

where the initial data and constraints are given by

$$g\big(x(t),w(t)\big) := \big(s_1w_1(t)\cos\theta_1(t), s_1w_1(t)\sin\theta_1(t),\ldots,s_nw_n(t)\cos\theta_n(t),s_nw_n(t)\sin\theta_n(t)\big),$$

$$\bar u_{i+1}(t)=\bar u_i(t):=\left(\frac{r}{\sqrt{2n}},\frac{r}{\sqrt{2n}}\right),\quad i=1,\ldots,n-1,$$

$$C:=\big\{x=(x_1,\ldots,x_n)\in\mathbb{R}^{2n}\big|\ \theta_{ij}(x)\ge 0\ \text{ for all }\ i\ne j\ \text{ as }\ i,j=1,\ldots,n\big\}$$

with the functions $\theta_{ij}(x):=D_{ij}(x)=\|x_i-x_j\|-2R$, and with

$$x(t)-\bar u(t)\in C\ \text{ for all }\ t\in[0,T].$$

This model belongs to optimal control theory for sweeping processes governed by *prox-regular moving sets*, which was partly discussed in Section 4. Applying the necessary optimality conditions for such problems developed in [7] allowed us to obtain in [8] a complete solution to this model in the case of lower numbers of participants and also to establish efficient relationships to determine optimal parameters in the general crowd model setting with finitely many participants. On the other hand, further algorithmic developments are needed in the case of many participants in crowd motion modeling.

The necessary optimality conditions for the sweeping optimal control problem presented in Theorem 4.1, which is based on [14], were applied in [15] to two practical models written therein in the form of the constrained controlled sweeping process (4.5)–(4.8). The first model is an optimal control version of the *mobile robot model with obstacles* the dynamics of which was described as a sweeping process in [23]. The second one is a continuous-time, deterministic, and optimal control version of the *pedestrian traffic flow model through a doorway* for which a stochastic, discrete-time, and simulation (uncontrolled) counterpart was originated in [29]. The application of Theorem 4.1 led us in [15] to complete calculations of optimal solutions for both models in several important settings, but many unsolved issues still remain in further numerical implementations and applications.

The obtained necessary optimality conditions for the sweeping control problem (5.4)–(5.6) and its specifications presented in Section 5 also admit various applications to practical models. We refer the reader to [27] for some applications to *nonpolyhedral* models of *elastoplasticity* and *hysteresis*. The necessary optimality conditions given in Theorems 5.2 and 5.3 are used therein for complete calculations of optimal solutions in the controlled hysteresis model the dynamics of which dynamics is described in the sweeping form (5.5) in [2]. Subsequent applications in this direction, including hysteresis models that arise in problems of *contact and nonsmooth mechanics* [1, 45], require further elaborations of the results obtained in [27]. Among other future developments we mention *rate-independent* systems arising in hysteresis and related areas. Some of such (uncontrolled) models are formulated in [1, 2, 4, 26, 28] with sweeping process descriptions of their dynamics.

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