Characterizations of explicitly quasiconvex vector functions w.r.t. polyhedral cones*

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Abstract

The aim of this paper is to present new characterizations of explicitly cone-quasiconvex vector functions with respect to a polyhedral cone of a finite-dimensional Euclidean space. These characterizations are given in terms of classical explicit quasiconvexity of certain real-valued functions, defined by composing the vector-valued function with appropriate scalarization functions, namely the extreme directions of the polar cone or some nonlinear scalarization functions, currently used in vector optimization.

Keywords: Generalized convex vector functions; polyhedral cones; extreme directions; nonlinear scalarization function

Mathematics Subject Classification: 26B25; 46A40; 90C29

1 Introduction

The role of generalized convexity in scalar and vector optimization is well recognized (see, e.g., the books by Avriel, Diewert, Schaible and Zang [2], Cambini and Martein [8], Crouzeix [10], Göpfert, Riahi, Tammer and Zălinescu [13], Jahn [15], and Luc [18]).

Among various classes of generalized convex real-valued functions, the explicitly quasiconvex ones (i.e., quasiconvex functions that are also semistrictly quasiconvex, cf. Definition 2.1) are of special interest for scalar optimization, since they preserve some local-global extremality properties of convex functions (see, e.g., Ponstein [21],

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Bagdasar and Popovici [3], and the references therein). It should be noticed that the notions of semistrict and explicit quasiconvexity appeared in the early literature, around fifty years ago, under different names (the semistrictly quasiconvex functions were called "strictly quasiconvex" by Ponstein [21], while Stoer and Witzgall [27] used the terms "pseudoconvexity" and "strong quasiconvexity" instead of semistrict quasiconvexity and explicit quasiconvexity, respectively).

As usual in multicriteria optimization problems, the objective function is defined on a nonempty (convex) subset D of a linear space X and takes values in a finite-dimensional Euclidean space $\mathbb{R}^m$. Such a vector-valued function, $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$, is called componentwise explicitly quasiconvex if its scalar components $f_1, \ldots, f_m : D \to \mathbb{R}$ are explicitly quasiconvex in classical sense. Multicriteria optimization problems involving componentwise explicitly quasiconvex objective functions have been intensively studied in the literature. For instance, Luc and Schaible [20] and, quite recently, Bagdasar and Popovici [4], have obtained local-global properties; the connectedness/contractibility of the efficient sets has been studied in a series of papers, initiated by the early work of Schaible [26] and followed by Benoist [5], Benoist and Popovici [7], Chew, Choo and Schaible [9], Daniilidis, Hadjisavvas and Schaible [11]; as shown by Popovici [23], the multicriteria optimization problems with componentwise explicitly quasiconvex objective functions have the property of Pareto reducibility (i.e. the weakly efficient solution set is the union of the sets of efficient solutions of all subproblems obtained from the original one by selecting certain criteria).

In more general vector optimization problems, the objective function takes values in a real linear space Y, partially ordered by a convex cone K. As shown by Popovici [24], one can define a concept of explicit K-quasiconvexity for such vector-valued functions, which recovers the componentwise explicit quasiconvexity in the particular case when $Y = \mathbb{R}^m$ and $K = \mathbb{R}^m_+$. The principal aim of this paper is to show that when K is a nontrivial solid polyhedral cone in $Y = \mathbb{R}^m$, the explicitly K-quasiconvex functions can be characterized in terms of usual explicit quasiconvexity of certain real-valued functions, obtained from the vector-valued function by composition with certain scalarization functionals, namely the extreme directions of the polar cone K^+ (Theorem 4.1) and the well-known Gerstewitz (Tammer) nonlinear scalarization functions (Theorem 4.4). Our results follow a series of works devoted to the characterization of K-quasiconvex vector-valued functions (see, e.g., Benoist, Borwein and Popovici [6], Luc [18], and the references therein) and the characterization of K-convex vector-valued functions (see, e.g., La Torre, Popovici and Rocca [17], and Popovici [22]).

The paper is organized as follows. In Section 2 we recall some basic notions of generalized convexity used in the sequel for real-valued and vector-valued functions, respectively. Section 3 is devoted to a geometrical analysis of the polyhedral cones and their polars. Finally, our new characterizations of explicitly K-quasiconvex vector-valued functions are given in Section 4, along with an illustrative example.

2 Preliminaries

Throughout this paper X denotes a real topological linear space, while $D \subseteq X$ is assumed to be a nonempty convex set.

2.1 Generalized convex scalar functions

We start by recalling four classical concepts of generalized convexity for real-valued functions.

Definition 2.1 *A function $\varphi : D \to \mathbb{R}$ is called:*

- *convex, if for all $x, x' \in D$ and $t \in]0,1[$ we have*

$$\varphi((1-t)x + tx') \le (1-t)\varphi(x) + t\varphi(x').$$

- *quasiconvex, if for all $x, x' \in D$ and $t \in]0,1[$ we have*

$$\varphi((1-t)x + tx') \le \max\{\varphi(x), \varphi(x')\}.$$

- *semistrictly quasiconvex, if for any $x, x' \in D$ such that $\varphi(x) \ne \varphi(x')$ and all $t \in]0,1[$ we have*

$$\varphi((1-t)x + tx') < \max\{\varphi(x), \varphi(x')\}.$$

- *explicitly quasiconvex, if φ is both quasiconvex and semistrictly quasiconvex.*

The next result follows by Definition 2.1 and a characterization of explicit quasiconvexity given by Popovici [24, Rem. 3.1].

Proposition 2.2 *For any $\varphi : D \to \mathbb{R}$ the following characterizations hold:*

1° *φ is convex if and only if for any $\lambda, \lambda' \in \mathbb{R}$ and $x, x' \in D$ such that $\varphi(x) \le \lambda$ and $\varphi(x') \le \lambda'$, and for all $t \in]0,1[$, we have*

$$\varphi((1-t)x + tx') \le (1-t)\lambda + t\lambda'.$$

2° *φ is quasiconvex if and only if for any $\lambda \in \mathbb{R}$ and $x, x' \in D$ such that $\varphi(x) \le \lambda$ and $\varphi(x') \le \lambda$, and for all $t \in]0,1[$, we have*

$$\varphi((1-t)x + tx') \le \lambda.$$

3° *φ is semistrictly quasiconvex if and only if for any $\lambda \in \mathbb{R}$ and $x, x' \in D$ such that $\varphi(x) = \lambda$ and $\varphi(x') < \lambda$, and for all $t \in]0,1[$, we have*

$$\varphi((1-t)x + tx') < \lambda.$$

4° *φ is explicitly quasiconvex if and only if for any $\lambda \in \mathbb{R}$ and $x, x' \in D$ such that $\varphi(x) \le \lambda$ and $\varphi(x') < \lambda$, and for all $t \in]0,1[$, we have*

$$\varphi((1-t)x + tx') < \lambda.$$

Remark 2.3 *The following properties hold (see, e.g., Avriel* et al. *[2]):*
a) All convex functions are explicitly quasiconvex.
b) Quasiconvexity and semistrict quasiconvexity do not imply each other.

c) Any semistrictly quasiconvex function, which is lower semicontinuous, is in fact explicitly quasiconvex.

Remark 2.4 *In view of Proposition 2.2 it is easy to check that whenever a function $\varphi : D \to \mathbb{R}$ is convex (quasiconvex, semistrictly quasiconvex, explicitly quasiconvex), then for any numbers $a \in \mathbb{R}_+$ and $b \in \mathbb{R}$ the function*

$$a\varphi + b : D \to \mathbb{R}$$

is also convex (quasiconvex, semistrictly quasiconvex, explicitly quasiconvex, respectively). Moreover, if several functions $\varphi_1 : D \to \mathbb{R}, \ldots, \varphi_n : D \to \mathbb{R}$ ($n \in \mathbb{N}$, $n \geq 2$) are convex (quasiconvex, explicitly quasiconvex), then the real-valued function

$$x \in D \;\longmapsto\; \max\{\varphi_1(x), \ldots, \varphi_n(x)\}$$

is convex (quasiconvex, explicitly quasiconvex, respectively). However, the maximum of several semistrictly quasiconvex functions is not necessarily semistrictly quasiconvex, as for instance when $\varphi_1, \ldots, \varphi_n : D = \mathbb{R} \to \mathbb{R}$ are defined by

$$\varphi_i(x) := \begin{cases} 0 & \text{if } \; x \in \mathbb{R} \setminus \{i\} \\ 1 & \text{if } \; x = i. \end{cases}$$

Notice that these functions are not lower semicontinuous.

2.2 Generalized convex vector functions

In this subsection we recall three concepts of generalized convexity for vector functions that have found many interesting applications in vector optimization (see, e.g., Bagdasar and Popovici [4], Luc [18], Popovici [23, 24], and the references therein).

Consider a finite-dimensional real Euclidean space $\mathbb{R}^m$ ($m \geq 2$), endowed with a convex cone $C \subseteq \mathbb{R}^m$, i.e.,

$$0 \in C = \mathbb{R}_+ \cdot C := \{tx \mid t \in \mathbb{R}_+,\ x \in C\} = C + C := \{y + y' \mid y, y' \in C\},$$

where 0 stands for the null vector and $\mathbb{R}_+ := [0, +\infty[$. Assume that C is solid and nontrivial, i.e., $\emptyset \neq \operatorname{int} C \neq \mathbb{R}^m$.

Definition 2.5 *A function $f : D \to \mathbb{R}^m$ is called*

- *C-convex if for all $x, x' \in D$ and $t \in]0, 1[$ we have*

$$f((1-t)x + tx') \in (1-t)f(x) + tf(x') - C.$$

- *C-quasiconvex if for any $y \in \mathbb{R}^m$ and $x, x' \in D$ such that $f(x) \in y - C$ and $f(x') \in y - C$, and for all $t \in]0, 1[$, we have*

$$f((1-t)x + tx') \in y - C.$$

- *explicitly C-quasiconvex if and only if for any $y \in \mathbb{R}^m$ and $x, x' \in D$ such that $f(x) \in y - C$ and $f(x') \in y - \operatorname{int} C$, and for all $t \in]0, 1[$, we have*

$$f((1-t)x + tx') \in y - \operatorname{int} C.$$

The following characterizations of generalized convex vector functions are well-known (see, e.g., Luc [18] and Popovici [24]).

Proposition 2.6 *For any function $f : D \to \mathbb{R}^m$ the following hold:*

1° *f is C-convex if and only if its epigraph*

$$\mathrm{epi}_C(f) := \{(x, y) \in D \times \mathbb{R}^m \mid y \in f(x) + C\}$$

is convex.

2° *f is C-quasiconvex if and only if for all $y \in \mathbb{R}^m$ the lower level set*

$$f^{-1}[y - C] = \{x \in D \mid f(x) \in y - C\}$$

is convex.

3° *f is explicitly C-quasiconvex if and only if for any points $y \in \mathbb{R}^m$, $x \in f^{-1}[y - C]$ and $x' \in f^{-1}[y - \mathrm{int}\, C]$, and for all $t \in]0, 1[$, we have*

$$(1 - t)x + tx' \in f^{-1}[y - \mathrm{int}\, C].$$

Remark 2.7 *It was shown by Popovici [24, Props. 3.3 and 3.4] that:*
a) C-convexity implies C-quasiconvexity and explicit C-quasiconvexity.
b) If C is closed, then explicit C-quasiconvexity implies C-quasiconvexity.

Remark 2.8 *In the particular case when $C = \mathbb{R}^m_+$ is the standard ordering cone, it is important to notice that:*
a) A vector function $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$ is $\mathbb{R}^m_+$-convex (resp. $\mathbb{R}^m_+$-quasiconvex, explicitly $\mathbb{R}^m_+$-quasiconvex) if and only if it is componentwise convex (quasiconvex, explicitly quasiconvex), i.e., all scalar components $f_1, \ldots, f_m : D \to \mathbb{R}$ are convex (quasiconvex, explicitly quasiconvex). More precisely, these characterizations of $\mathbb{R}^m_+$-convexity and $\mathbb{R}^m_+$-quasiconvexity directly follow from Definitions 2.1 and 2.5, while the characterization of explicit $\mathbb{R}^m_+$-quasiconvexity has been established by Popovici [24, Th. 3.1].
b) As shown by Bagdasar and Popovici [4, Th. 3.17], any componentwise explicitly quasiconvex function $f : D \to \mathbb{R}^m$ satisfies the property that for all $x, x' \in D$ such that $f(x) \neq f(x')$ and $t \in]0, 1[$, we have

$$f((1 - t)x + tx') \in \max\{f(x), f(x')\} - (\mathbb{R}^m_+ \setminus \{0\}),$$

where $\max\{u, v\} := (\max\{u_1, v_1\}, \ldots, \max\{u_m, v_m\})$ for all $u = (u_1, \ldots, u_m)$, $v = (v_1, \ldots, v_m) \in \mathbb{R}^m$. Actually, Luc and Schaible [20, p. 152] introduced an original concept of generalized convexity, also called "explicit quasiconvexity," by combining the above property with the componentwise quasiconvexity. However, their concept does not imply componentwise explicit quasiconvexity.

3 Properties of polyhedral cones and their polars

As usual we define the polar cone of any nonempty set $S \subseteq \mathbb{R}^m$ by

$$S^+ := \{x \in \mathbb{R}^m \mid \forall\, y \in S:\ \langle x, y\rangle \geq 0\},$$

where $\langle\cdot,\cdot\rangle$ stands for the usual inner product. Notice that S^+ is actually a closed convex cone. A cone $K \subseteq \mathbb{R}^m$ is said to be polyhedral if there exists a nonempty finite set $S \subseteq \mathbb{R}^m \setminus \{0\}$ such that $K = S^+$. It is well-known that the polar cone of any polyhedral cone is also polyhedral (see, e.g., Aliprantis and Tourky [1]).

In what follows we consider a polyhedral cone

$$K := \{u^1, \dots, u^p\}^+ = \{v \in \mathbb{R}^m \mid\ \langle v, u^i\rangle \geq 0,\ \forall\, i \in I_p\},$$

where $p \in \mathbb{N}$, $I_p := \{1, \dots, p\}$ and $u^1, \dots, u^p \in \mathbb{R}^m$ satisfy the conditions

$$0 \notin \operatorname{conv}\{u^1, \dots, u^p\}, \tag{3.1}$$

$$u^i \notin \textstyle\sum_{j\in I_p\setminus\{i\}} \mathbb{R}_+ \cdot u^j,\ \forall\, i \in I_p. \tag{3.2}$$

It is understood that the latter sum of sets is empty for $p = 1$.

For all $i \in I_p$, we define the set

$$E_i := \{v \in \mathbb{R}^m \mid\ \langle v, u^i\rangle = 0\} \cap K,$$

which is known to be an exposed face of K, whose relative interior $\operatorname{ri} E_i$ is nonempty, since $0 \in E_i = \operatorname{argmin}_{v\in K}\langle v, u^i\rangle$ (see, e.g., Rockafellar [25]). Also, we denote the set of all extreme directions of the polar cone K^+ by

$$\operatorname{extd} K^+ := \{w \in K^+ \setminus \{0\} \mid \forall\, w^1, w^2 \in K^+ : w = w^1 + w^2 \Rightarrow w^1, w^2 \in \mathbb{R}_+ \cdot w\}.$$

Proposition 3.1 *The following properties hold:*

1° $\operatorname{int} K = \{v \in \mathbb{R}^m \mid \langle v, u^i\rangle > 0, \forall\, i \in I_p\}$.

2° K *is nontrivial and solid.*

3° $K^+ = \mathbb{R}_+ \cdot \operatorname{conv}\{u^1, \dots, u^p\} = \sum_{i\in I_p} \mathbb{R}_+ \cdot u^i$.

4° K^+ *is pointed.*

5° $\operatorname{extd} K^+ = \mathbb{R}^*_+ \cdot \{u^1, \dots, u^p\}$.

6° $\operatorname{ri} E_i = \{v \in \mathbb{R}^m \mid \langle v, u^i\rangle = 0, \langle v, u^j\rangle > 0, \forall j \in I_p\setminus\{i\}\}$ *for all* $i \in I_p$.

Proof: Assertions 1°, 2° and 3° are based on classical arguments in Convex Analysis (see, e.g., Zălinescu [30] and our paper [14, Prop. 4.6]). Assertion 4° directly follows from the first equality in 3°, which actually shows that the closed cone K^+ is based (see, e.g., Göpfert *et al.* [13, p. 3]).

Assertion 5° trivially holds for $p = 1$. Assuming that $p \geq 2$, consider any $v \in \operatorname{extd} K^+$. Then $v \in K^+ \setminus \{0\}$, hence there exist $\alpha_1, \dots, \alpha_p \in \mathbb{R}_+$ with $\alpha_j > 0$ for some $j \in I_p$, such that $v = \sum_{i\in I_p} \alpha_i u^i$, in view of 3°. Thus we have $v = v' + v''$, where $v' := \alpha_j u^j \in K^+ \setminus \{0\}$ and $v'' := \sum_{i\in I_p\setminus\{j\}} \alpha_i u^i \in K^+$ by 3° and (3.1). Since $v \in \operatorname{extd} K^+$, we infer that $v', v'' \in \mathbb{R}_+ \cdot v$ and, more precisely, $v' \in \mathbb{R}^*_+ \cdot v$, hence

$v \in \mathbb{R}^*_+ \cdot v' = \mathbb{R}^*_+ \cdot u^j \subseteq \mathbb{R}^*_+ \cdot \{u^1, \dots, u^p\}$. Thus, the inclusion "$\subseteq$" in 5° holds true. In order to prove the converse inclusion, consider a point $w \in \mathbb{R}^*_+ \cdot \{u^1, \dots, u^p\}$, i.e., $w = \beta u^k$ with $\beta \in \mathbb{R}^*_+$ and $k \in I_p$. Notice that $w \neq 0$ by (3.1). Assume that $w = w' + w''$ for some $w', w'' \in K^+$. Then, by 3° there exist $\beta'_1, \dots, \beta'_p \in \mathbb{R}_+$ and $\beta''_1, \dots, \beta''_p \in \mathbb{R}_+$ such that $w' = \sum_{i \in I_p} \beta'_i u^i$ and $w'' = \sum_{i \in I_p} \beta''_i u^i$, hence $\beta u^k = \sum_{i \in I_p} (\beta'_i + \beta''_i) u^i$, i.e., $(\beta - \beta'_k - \beta''_k) u^k = \sum_{i \in I_p \setminus \{k\}} (\beta'_i + \beta''_i) u^i$. By assumption (3.2) we infer that $\beta - \beta'_k - \beta''_k \leq 0$, hence $0 = (\beta'_k + \beta''_k - \beta) u^k + \sum_{i \in I_p \setminus \{k\}} (\beta'_i + \beta''_i) u^i$ with $\beta'_k + \beta''_k - \beta \geq 0$ and $\beta'_i + \beta''_i \geq 0$ for all $i \in I_p \setminus \{k\}$. Then, by using (3.1) it is easy to deduce that $\beta'_k + \beta''_k - \beta = 0$ and $\beta'_i + \beta''_i = 0$, i.e., $\beta'_i = \beta''_i = 0$, for all $i \in I_p \setminus \{k\}$. Hence we have $w' = \beta'_k u^k \in \mathbb{R}_+ \cdot u^k = \mathbb{R}_+ \cdot w$ and $w'' = \beta''_k u^k \in \mathbb{R}_+ \cdot u^k = \mathbb{R}_+ \cdot w$, which allows us to conclude that $w \in \operatorname{extd} K^+$. Thus the inclusion "$\supseteq$" in 5° is also true.

Finally, let us prove 6°. To this aim, let $i \in I_p$ and let $H \subseteq I_p$ be the largest index set for which the face E_i admits the representation

$$E_i = \{v \in \mathbb{R}^m \mid \langle v, u^h \rangle = 0, \forall\, h \in H\} \cap K.$$

Notice that $i \in H$. According to Luc [19, Th. 2.3.3 and Cor. 2.3.5], we have

$$\operatorname{ri} E_i = \{v \in \mathbb{R}^m \mid \langle v, u^h \rangle = 0, \forall\, h \in H,\ \langle v, u^j \rangle > 0, \forall\, j \in I_p \setminus H\}.$$

Therefore, in order to prove 6° we just have to show that $H = \{i\}$.

Assume by the contrary that there is some $k \in H \setminus \{i\}$. Since

$$E_k := \{v \in \mathbb{R}^m \mid \langle v, u^k \rangle = 0\} \cap K$$

and $k \in H$, we have $E_i \subseteq E_k$. By polar calculus rules, it follows that $E_k^+ \subseteq E_i^+$, which means that $\mathbb{R} \cdot u^k + K^+ \subseteq \mathbb{R} \cdot u^i + K^+$ (see, e.g., Jahn [15, Lem. 1.24]). Actually, by decomposing $\mathbb{R}$ into $(-\mathbb{R}_+) \cup \mathbb{R}_+$ and taking into account that $\mathbb{R}_+ \cdot u^k \subseteq K^+$ and $\mathbb{R}_+ \cdot u^i \subseteq K^+$ by 3°, we deduce that $-\mathbb{R}_+ \cdot u^k + K^+ \subseteq -\mathbb{R}_+ \cdot u^i + K^+$. In particular, this shows that $-u^k \in -\mathbb{R}_+ \cdot u^i + K^+$, i.e., there is some $\alpha \geq 0$ such that

$$-u^k \in -\alpha u^i + K^+. \tag{3.3}$$

We distinguish two cases.

Case 1: If $\alpha = 0$, then (3.3) yields $-u^k \in K^+$. By 3° and 4°, we infer that $u^k = 0$, which contradicts the assumption (3.1).

Case 2: If $\alpha > 0$, then (3.3) shows that $u^i \in \frac{1}{\alpha} u^k + \frac{1}{\alpha} u$ for some $u \in K^+$. Since $u^i \in \operatorname{extd} K^+$ by 5°, we infer that $u^k, u \in \mathbb{R}_+ \cdot u^i$, which contradicts the assumption (3.2). □

Lemma 3.2 *Let $i \in I_p$. For any point $v^i \in \operatorname{ri} E_i$, we have*

$$\mathbb{R} \cdot v^i + K = \{v \in \mathbb{R}^m \mid \langle v, u^i \rangle \geq 0\}; \tag{3.4}$$

$$\mathbb{R} \cdot v^i + \operatorname{int} K = \{v \in \mathbb{R}^m \mid \langle v, u^i \rangle > 0\}. \tag{3.5}$$

Proof: Consider an arbitrary point $v^i \in \operatorname{ri} E_i$.

For any $v \in \mathbb{R} \cdot v^i + K$ we have $v = \alpha v^i + x$ for some $\alpha \in \mathbb{R}$ and $x \in K$, hence $\langle v, u^i \rangle = \alpha \langle v^i, u^i \rangle + \langle x, u^i \rangle = \langle x, u^i \rangle \geq 0$, by Proposition 3.1 (6°) and the definition of K. Thus the inclusion "$\subseteq$" in (3.4) holds.

In order to prove the converse inclusion, we distinguish two cases.

Case 1: If $p = 1$, then we have $\{v \in \mathbb{R}^m \mid \langle v, u^i \rangle \geq 0\} = K \subseteq \mathbb{R} \cdot v^i + K$, hence the desired inclusion holds.

Case 2: If $p > 1$, then let $v^0 \in \mathbb{R}^m$ be any point such that $\langle v^0, u^i \rangle \geq 0$. For proving that $v^0 \in \mathbb{R} \cdot v^i + K$, consider the real number

$$\alpha_0 := \min_{j \in I_p \setminus \{i\}} \frac{\langle v^0, u^j \rangle}{\langle v^i, u^j \rangle},$$

which is well-defined by Proposition 3.1 (6°), and let

$$x^0 := v^0 - \alpha_0 v^i.$$

Then, in view of Proposition 3.1 (6°), we can further deduce that $\langle x^0, u^i \rangle = \langle v^0, u^i \rangle - \alpha_0 \langle v^i, u^i \rangle = \langle v^0, u^i \rangle \geq 0$ and $\langle x^0, u^j \rangle = \langle v^0, u^j \rangle - \alpha_0 \langle v^i, u^j \rangle \geq \langle v^0, u^j \rangle - \frac{\langle v^0, u^j \rangle}{\langle v^i, u^j \rangle} \langle v^i, u^j \rangle = 0$ for all $j \in I_p \setminus \{i\}$. It follows that $x^0 \in K$, hence $v^0 = \alpha_0 v^i + x^0 \in \mathbb{R} \cdot v^i + K$. Thus the inclusion "$\supseteq$" in (3.4) holds.

For proving (3.5), recall that $\operatorname{int} K \neq \emptyset$, by Proposition 3.1 (2°). Hence

$$\mathbb{R} \cdot v^i + \operatorname{int} K = \operatorname{int}(\mathbb{R} \cdot v^i + K),$$

according to Tanaka and Kuroiwa [29, Th. 2.2]. Moreover, since $u^i \neq 0$ by (3.1), we also have

$$\{v \in \mathbb{R}^m \mid \langle v, u^i \rangle > 0\} = \operatorname{int}\{v \in \mathbb{R}^m \mid \langle v, u^i \rangle \geq 0\}.$$

The above relations show that (3.5) directly follows from (3.4). □

We end this section by giving an explicit representation of the nonlinear scalarization function in the sense of Gerstewitz (Tammer) [12], associated to the polyhedral cone K and two points $e \in \operatorname{int} K$ and $d \in \mathbb{R}^m$, namely the real-valued function $\sigma_{K,e,d} : \mathbb{R}^m \to \mathbb{R}$ defined by

$$\sigma_{K,e,d}(y) \quad := \quad \min\{s \in \mathbb{R} \mid y \in d + se - K\}, \quad \forall\, y \in \mathbb{R}^m.$$

These kind of functions have found many applications in vector- and set-optimization (see, e.g., Göpfert *et al.* [13], Khan, Tammer and Zălinescu [16], and Luc [18]), being known also as the smallest strictly monotonic functions in the sense of Luc [18], or under other names in mathematical economics (see, e.g., Tammer and Zălinescu [28] and the references therein).

Lemma 3.3 *For any $e \in \operatorname{int} K$ and $d \in \mathbb{R}^m$, we have*

$$\sigma_{K,e,d}(y) \quad := \quad \max_{i \in I_p} \frac{\langle y - d, u^i \rangle}{\langle e, u^i \rangle}, \quad \forall\, y \in \mathbb{R}^m. \tag{3.6}$$

Proof: Let $y \in \mathbb{R}^m$. By definition of K, we have

$$\begin{aligned}\sigma_{K,e,d}(y) &:= \min\{s \in \mathbb{R} \mid d + se - y \in K\} \\ &= \min\{s \in \mathbb{R} \mid \langle d + se - y, u^i\rangle \geq 0,\ \forall\, i \in I_p\} \\ &= \min\{s \in \mathbb{R} \mid s\langle e, u^i\rangle \geq \langle y - d, u^i\rangle,\ \forall\, i \in I_p\}.\end{aligned}$$

Taking into account that $e \in \operatorname{int} K$, i.e., $\langle e, u^i\rangle > 0$ for all $i \in I_p$, we infer

$$\begin{aligned}\sigma_{K,e,d}(y) &= \min\{s \in \mathbb{R} \mid s \geq \langle y - d, u^i\rangle / \langle e, u^i\rangle,\ \forall\, i \in I_p\} \\ &= \max\{\langle y - d, u^i\rangle / \langle e, u^i\rangle \mid i \in I_p\},\end{aligned}$$

i.e., (3.6) holds. □

4 Characterizations of explicit cone-quasiconvexity

4.1 Characterization of explicit cone-quasiconvexity by means of the polar cone's extreme directions

Theorem 4.1 *For any function $f : D \to \mathbb{R}^m$ the following assertions are equivalent:*

1° *f is explicitly K-quasiconvex.*

2° *For every $i \in I_p$, the real-valued function*

$$x \in D \longmapsto \langle f(x), u^i\rangle$$

is explicitly quasiconvex.

3° *For all $u \in \operatorname{extd} K^+$, the real-valued function*

$$x \in D \longmapsto \langle f(x), u\rangle$$

is explicitly quasiconvex.

Proof: In order to prove the implication 1° $\Longrightarrow$ 2°, assume that 1° holds and let $i \in I_p$. For proving the explicit quasiconvexity of the scalar function $\langle f(\cdot), u^i\rangle$ we will use the characterization given in Proposition 2.2 (4°). Let $x, x' \in D$ and $\lambda \in \mathbb{R}$ be such that

$$\langle f(x), u^i\rangle \leq \lambda \ \text{ and } \ \langle f(x'), u^i\rangle < \lambda \tag{4.1}$$

and let $t \in]0, 1[$. We just have to show that

$$\langle f((1-t)x + x'), u^i\rangle < \lambda. \tag{4.2}$$

Since $u^i \neq 0$ by (3.1), there exists $v \in \mathbb{R}^m$ such that

$$\langle v, u^i\rangle = \lambda, \tag{4.3}$$

hence (4.1) can be rewritten as

$$\langle v - f(x), u^i \rangle \geq 0 \ \text{ and } \ \langle v - f(x'), u^i \rangle > 0. \tag{4.4}$$

Now, choose a point $v^i \in \operatorname{ri} E_i$. Then, Lemma 3.2 together with (4.4) ensure the existence of two numbers $\alpha, \alpha' \in \mathbb{R}$, such that

$$v - f(x) \in \alpha v^i + K \ \text{ and } \ v - f(x') \in \alpha' v^i + \operatorname{int} K.$$

Consider the number

$$\alpha_0 := \min\{\alpha, \alpha'\}.$$

Since $v^i \in K$ and $\alpha - \alpha_0, \alpha' - \alpha_0 \in \mathbb{R}_+$, we have $(\alpha - \alpha_0)v^i, (\alpha' - \alpha_0)v^i \in K$, i.e., $\alpha v^i, \alpha' v^i \in \alpha_0 v^i + K$. It follows that $v - f(x) \in \alpha_0 v^i + K + K = \alpha_0 v^i + K$ and $v - f(x') \in \alpha_0 v^i + K + \operatorname{int} K = \alpha_0 v^i + \operatorname{int} K$, hence

$$f(x) \in (v - \alpha_0 v^i) - K \ \text{ and } \ f(x') \in (v - \alpha_0 v^i) - \operatorname{int} K.$$

By 1° we deduce that

$$f((1-t)x + tx') \in (v - \alpha_0 v^i) - \operatorname{int} K,$$

i.e., $v - \alpha_0 v^i - f((1-t)x + tx') \in \operatorname{int} K$. By Proposition 3.1 (1°), we infer that $\langle v - \alpha_0 v^i - f((1-t)x + tx'), u^i \rangle > 0$, i.e.,

$$\langle f((1-t)x + tx'), u^i \rangle < \langle v, u^i \rangle - \alpha_0 \langle v^i, u^i \rangle,$$

which in view of Proposition 3.1 (6°) and (4.3) reduces to the desired relation (4.2).

For proving the implication 2° $\Longrightarrow$ 1°, assume that 2° holds and consider some points $x, x' \in D$ and $y \in \mathbb{R}^m$ such that

$$f(x) \in y - K \ \text{ and } \ f(x') \in y - \operatorname{int} K, \tag{4.5}$$

and an arbitrary $t \in]0,1[$. We have to prove that

$$f((1-t)x + x') \in y - \operatorname{int} K,$$

which in view of Proposition 3.1 (1°) reduces to show that, for any $i \in I_p$,

$$\langle y - f((1-t)x + tx'), u^i \rangle > 0. \tag{4.6}$$

Indeed, in view of the definition of K and Proposition 3.1 (1°), it follows from (4.5) that $\langle y - f(x), u^i \rangle \geq 0$ and $\langle y - f(x'), u^i \rangle > 0$, i.e.,

$$\langle f(x), u^i \rangle \leq \langle y, u^i \rangle \ \text{ and } \ \langle f(x'), u^i \rangle < \langle y, u^i \rangle. \tag{4.7}$$

Since by assumption 2° the scalar function $\langle f(\cdot), u^i \rangle$ is explicitly quasiconvex, we deduce from (4.7) and Proposition 2.2 (4°) that

$$\langle f((1-t)x + tx'), u^i \rangle < \langle y, u^i \rangle,$$

which yields the desired relation (4.6).

Finally, observe that the equivalence $2^\circ \iff 3^\circ$ holds by Proposition 3.1 (5°), since the explicit quasiconvexity is stable under positive scalar multiplication, in view of Remark 2.4. □

From the previous theorem we can derive a characterization of the explicit cone-quasiconvexity w.r.t. the polyhedral cone K in terms of the explicit cone-quasiconvexity w.r.t. the standard cone $\mathbb{R}^p_+$. To this aim, we make use of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^p$, defined by

$$T(v) := (\langle v, u^1 \rangle, \ldots, \langle v, u^p \rangle), \ \forall\, v \in \mathbb{R}^m.$$

Corollary 4.2 *For any function $f : D \to \mathbb{R}^m$ the following assertions are equivalent:*

1° *f is explicitly K-quasiconvex.*

2° *$T \circ f : D \to \mathbb{R}^p$ is explicitly $\mathbb{R}^p_+$-quasiconvex.*

Proof: According to Theorem 4.1, f is explicitly K-quasiconvex if, and only if, for every $i \in I_p$ the real-valued function

$$x \in D \longmapsto \langle f(x), u^i \rangle = (T \circ f)_i(x)$$

is explicitly quasiconvex, i.e., $T \circ f$ is componentwise explicitly quasiconvex. Thus, the desired conclusion immediately follows by Remark 2.8 (a), applied to $T \circ f$ in the role of f. □

Example 4.3 *In the Euclidean space $\mathbb{R}^3$ consider the polyhedral cone $K := \{u^1, u^2, u^3, u^4\}^+$ (i.e., $m = 3$ and $p = 4$), where*

$$u^1 = (1, 0, 1),\ u^2 = (0, 1, 1),\ u^3 = (-1, 0, 1),\ u^4 = (0, -1, 1).$$

It can be easily checked that conditions (3.1) *and* (3.2) *hold. In this case, Theorem 4.1 shows that a vector function $f = (f_1, f_2, f_3) : D \to \mathbb{R}^3$ is explicitly K-quasiconvex if and only if the real-valued functions $f_1 + f_3$, $f_2 + f_3$, $-f_1 + f_3$ and $-f_2 + f_3$ are explicitly quasiconvex on D in the classical sense. Corollary 4.2 also shows that f is explicitly K-quasiconvex if and only if $T \circ f$ is explicitly $\mathbb{R}^4$-quasiconvex, where $T : \mathbb{R}^3 \to \mathbb{R}^4$ is the linear operator defined for all $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ by*

$$T(v) = (v_1 + v_3, v_2 + v_3, -v_1 + v_3, -v_2 + v_3).$$

4.2 Characterization of explicit cone-quasiconvexity by means of nonlinear scalarization functions

Theorem 4.4 *Let $e \in \operatorname{int} K$. For any function $f : D \to \mathbb{R}^m$ the following assertions are equivalent:*

1° *f is explicitly K-quasiconvex.*

2° *For any $d \in \mathbb{R}^m$, the composite function $\sigma_{K,e,d} \circ f : D \to \mathbb{R}$, i.e., the real-valued function*

$$x \in D \longmapsto \max_{i \in I_p} \frac{\langle f(x) - d, u^i \rangle}{\langle e, u^i \rangle}$$

is explicitly quasiconvex.

Proof: According to Lemma 3.3, for any $d \in \mathbb{R}^m$ and $x \in D$ we have

$$(\sigma_{K,e,d} \circ f)(x) = \max_{i \in I_p} \frac{\langle f(x) - d, u^i \rangle}{\langle e, u^i \rangle} = \max_{i \in I_p} \frac{\langle f(x), u^i \rangle - \langle d, u^i \rangle}{\langle e, u^i \rangle}. \tag{4.8}$$

In order to prove the implication 1° $\Longrightarrow$ 2°, assume that 1° holds and let $d \in \mathbb{R}^m$. Then, by Theorem 4.1, all functions $\langle f(\cdot), u^i \rangle$ with $i \in I_p$ are explicitly quasiconvex. Taking into account that $\langle e, u^i \rangle > 0$ for all $i \in I_p$ (since $e \in \operatorname{int} K$), we infer by Remark 2.4 and (4.8) that $\sigma_{K,e,d} \circ f : X \to \mathbb{R}$ is explicitly quasiconvex.

Conversely, for proving the implication 2° $\Longrightarrow$ 1°, assume that 2° holds and suppose by the contrary that f is not explicitly K-quasiconvex. Then, by Theorem 4.1, it would exists $i_0 \in I_p$ for which $\langle f(\cdot), u^{i_0} \rangle$ is not explicitly quasiconvex. By Proposition 2.2 (4°), we infer the existence of $\eta \in \mathbb{R}$ and $x^1, x^2, x^3 \in D$ with $x^3 := (1-t)x^1 + tx^2$ for some $t \in \,]0, 1[$, such that

$$\langle f(x^1), u^{i_0} \rangle \leq \eta, \quad \langle f(x^2), u^{i_0} \rangle < \eta \quad \text{and} \quad \langle f(x^3), u^{i_0} \rangle \geq \eta. \tag{4.9}$$

Now, choose a point $v \in \operatorname{ri} E_{i_0}$. In view of Proposition 3.1 (6°) we have

$$\langle v, u^{i_0} \rangle = 0 \;\text{ and }\; \langle v, u^j \rangle > 0, \; \forall\, j \in I_p \setminus \{i_0\}. \tag{4.10}$$

Moreover, since $e \in \operatorname{int} K$, we have

$$\langle e, u^i \rangle > 0, \; \forall\, i \in I_p. \tag{4.11}$$

Thus we can define the number

$$\alpha := \max_{\substack{j \in I_p \setminus \{i_0\} \\ k \in \{1,2,3\}}} \frac{\langle e, u^j \rangle}{\langle v, u^j \rangle} \cdot \left(\frac{\langle f(x^k), u^j \rangle}{\langle e, u^j \rangle} - \frac{\langle f(x^k), u^{i_0} \rangle}{\langle e, u^{i_0} \rangle} \right).$$

In view of (4.10) and (4.11), it is easy to check that

$$\alpha \cdot \left(\frac{\langle v, u^i \rangle}{\langle e, u^i \rangle} - \frac{\langle v, u^{i_0} \rangle}{\langle e, u^{i_0} \rangle} \right) \geq \frac{\langle f(x^k), u^i \rangle}{\langle e, u^i \rangle} - \frac{\langle f(x^k), u^{i_0} \rangle}{\langle e, u^{i_0} \rangle}, \; \forall\, i \in I_p. \tag{4.12}$$

By applying (4.8) for

$$d := \alpha v,$$

and by using (4.12), we deduce that

$$\begin{aligned}(\sigma_{K,e,\alpha v} \circ f)(x^k) &= \sigma_{K,e,\alpha v}(f(x^k)) \\ &= \max_{i \in I_p} \left(\frac{\langle f(x^k), u^i \rangle}{\langle e, u^i \rangle} - \frac{\alpha \langle v, u^i \rangle}{\langle e, u^i \rangle} \right) \\ &= \frac{\langle f(x^k), u^{i_0} \rangle}{\langle e, u^{i_0} \rangle} - \frac{\alpha \langle v, u^{i_0} \rangle}{\langle e, u^{i_0} \rangle} \\ &= \frac{\langle f(x^k), u^{i_0} \rangle}{\langle e, u^{i_0} \rangle}, \ \forall\, k \in \{1, 2, 3\}.\end{aligned}$$

Therefore (4.9) can be rewritten as

$$\begin{aligned}(\sigma_{K,e,\alpha v} \circ f)(x^1) &\leq \frac{\eta}{\langle e, u^{i_0} \rangle}, \\ (\sigma_{K,e,\alpha v} \circ f)(x^2) &< \frac{\eta}{\langle e, u^{i_0} \rangle}, \\ (\sigma_{K,e,\alpha v} \circ f)(x^3) &\geq \frac{\eta}{\langle e, u^{i_0} \rangle},\end{aligned}$$

which contradicts the explicit quasiconvexity of the function $\sigma_{K,e,\alpha v} \circ f$, in view of Proposition 2.2 (4°). Thus, the implication 2° $\Longrightarrow$ 1° holds true. □

Corollary 4.5 *Assume that* $e \in \operatorname{int} K$ *and* $\langle e, u^i \rangle = 1$ *for all* $i \in I_p$*. Then, for any function* $f : D \to \mathbb{R}^m$ *the following assertions are equivalent:*

1° *f is explicitly K-quasiconvex.*

2° *For any $c \in \mathbb{R}^m$, the real-valued function*

$$x \in D \ \longmapsto \ \max_{i \in I_p} \langle f(x) + c, u^i \rangle$$

is explicitly quasiconvex.

Proof: Follows by Theorem 4.4 by letting $d = -c$. □

Remark 4.6 *The assumption of Corollary 4.5 actually means that* $u^1, \ldots, u^p$ *are the extreme points of*

$$B \ := \ \{u \in K^+ \mid \langle e, u \rangle = 1\},$$

which is known to be a compact convex base of K^+ *whenever* $e \in \operatorname{int} K$ *(see, e.g., Göpfert* et al. *[13, Lemma 2.2.17]).*

Example 4.7 *Consider the particular case when* $K = \mathbb{R}^m_+$ *is the standard ordering cone (hence* $K^+ = K$ *and* $p = m$*). Then, letting* $e = (1, \ldots, 1)$ *and* $u^1 = (1, 0, \ldots, 0), \ldots, u^p = (0, \ldots, 0, 1)$ *in Corollary 4.5, we conclude that a vector function* $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$ *is explicitly* $\mathbb{R}^m_+$*-quasiconvex, i.e., componentwise explicitly quasiconvex, according to Remark 2.8 (a), if and only if for all* *constants* $c_1, \dots, c_m \in \mathbb{R}$ *the real-valued function*

$$x \in D \longmapsto \max\{f_1(x) + c_1, \dots, f_m(x) + c_m\}$$

is explicitly quasiconvex.

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References

[1] C. D. Aliprantis and R. Tourky, *Cones and Duality*, American Mathematical Society, 2007.

[2] M. Avriel, W. E. Diewert, S. Schaible, I. Zang, *Generalized Concavity*, Mathematical Concepts and Methods in Science and Engineering **36**, Plenum Press, New York, 1988.

[3] O. Bagdasar and N. Popovici, *Local maximum points of explicitly quasiconvex functions*, Optim. Lett. **9** (2015), 769–777.

[4] O. Bagdasar and N. Popovici, *Local maximizers of generalized convex vector-valued functions*, J. Nonlinear Convex Anal. **18** (2017), 2229–2250.

[5] J. Benoist, *Connectedness of the efficient set for strictly quasiconcave sets*, J. Optim. Theory Appl. **96** (1998), 627–654.

[6] J. Benoist, J. M. Borwein and N. Popovici, *A characterization of quasiconvex vector-valued functions*, Proc. Amer. Math. Soc. **131** (2003), 1109–1113.

[7] J. Benoist and N. Popovici, *Contractibility of the efficient frontier of three-dimensional simply-shaded sets*, J. Optim. Theory Appl. **111** (2001), 81–116.

[8] A. Cambini and L. Martein, *Generalized Convexity and Optimization. Theory and applications*, Springer-Verlag, Berlin Heidelberg, 2009.

[9] K. P. Chew, E. U. Choo and S. Schaible, *Connectedness of the efficient set in three-criteria quasiconcave programming*, Cahiers du C.E.R.O. **27** (1985), 213–220.

[10] J.-P. Crouzeix, *Generalized Convexity and Generalized Monotonicity*, Monografías del Instituto de Matemática y Ciencias Afines, IMCA, Lima, 2000.

[11] A. Daniilidis, N. Hadjisavvas and S. Schaible, *Connectedness of the efficient set for three-objective quasiconcave maximization problems*, J. Optim Theory Appl. **93** (1997), 517–524.

[12] C. Gerstewitz, *Nichtkonvexe Dualität in der Vektoroptimierung*, Wiss. Zeitschrift Tech. Hochsch. Leuna-Merseburg **25** (1983), 357–364.

[13] A. Göpfert, H. Riahi, C. Tammer and C. Zălinescu, *Variational Methods in Partially Ordered Spaces*, Springer, New York, 2003.

[14] C. Günther and N. Popovici, *New algorithms for discrete vector optimization based on the Graef-Younes method and cone-monotone sorting functions*, Optimization **67** (2018), 975–1003.

[15] J. Jahn, *Vector Optimization - Theory, Applications, and Extensions*, 2nd ed., Springer, Berlin, Heidelberg, 2011.

[16] A. A. Khan, C. Tammer and C. Zălinescu, *Set-valued Optimization: An Introduction with Applications*, Springer, Berlin, 2015.

[17] D. La Torre, N. Popovici and M. Rocca, *Scalar characterizations of weakly cone-convex and weakly cone-quasiconvex functions*, Nonlinear Anal. **72** (2010), 1909–1915.

[18] D. T. Luc, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems **319**, Springer, Berlin, 1989.

[19] D. T. Luc, *Multiobjective Linear Programming. An Introduction*, Springer, Cham, 2016.

[20] D. T. Luc and S. Schaible, *Efficiency and generalized concavity*, J. Optim. Theory Appl. **94** (1997), 147–153.

[21] J. Ponstein, *Seven kinds of convexity*, SIAM Review **9** (1967), 115–119.

[22] N. Popovici, *A characterization of cone-convex functions*, Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity **1** (2003), 123–131.

[23] N. Popovici, *Pareto reducible multicriteria optimization problems*, Optimization **54** (2005), 253–263.

[24] N. Popovici, *Explicitly quasiconvex set-valued optimization*, J. Global Optim. **38** (2007), 103–118.

[25] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[26] S. Schaible, *Bicriteria quasiconcave programs*, Cahiers du C.E.R.O. **25** (1983), 93–101.

[27] J. Stoer and C. Witzgall, *Convexity and Optimization in Finite Dimensions. I*, Die Grundlehren der mathematischen Wissenschaften **163**, Springer-Verlag, Berlin, 1970.

[28] C. Tammer and C. Zălinescu, *Lipschitz properties of the scalarization function and applications*, Optimization **59** (2010), 305–319.

[29] T. Tanaka and D. Kuroiwa, *The convexity of A and B assures* $\operatorname{int} A + B = \operatorname{int}(A + B)$, Appl. Math. Lett. **6** (1993), 83–86.

[30] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, River Edge, 2002.