A BILEVEL APPROACH FOR IDENTIFYING THE WORST CONTINGENCIES FOR NONCONVEX ALTERNATING CURRENT POWER SYSTEMS *

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Abstract. We address the bilevel optimization problem of identifying the most critical attacks to an alternating current (AC) power flow network. The upper-level binary maximization problem consists in choosing an attack that is treated as a parameter in the lower-level defender minimization problem. Instances of the lower-level global minimization problem by themselves are NP-hard due to the nonconvex AC power flow constraints, and bilevel solution approaches commonly apply a convex relaxation or approximation to allow for tractable bilevel reformulations at the cost of underestimating some power system vulnerabilities. Our main contribution is to provide an alternative branch-and-bound algorithm whose upper bounding mechanism (in a maximization context) is based on a reformulation that avoids relaxation of the AC power flow constraints in the lower-level defender problem. Lower bounding is provided with semidefinite programming (SDP) relaxed solutions to the lower-level problem. We establish finite termination with guarantees of either a globally optimal solution to the original bilevel problem, or a globally optimal solution to the SDP-relaxed bilevel problem which is included in a vetted list of upper-level attack solutions, at least one of which is a globally optimal solution to the bilevel problem. We demonstrate through computational experiments applied to IEEE case instances both the relevance of our contribution, and the effectiveness of our contributed algorithm for identifying power system vulnerabilities whose value is underestimated when using standard convex relaxations of the lower level problem. We conclude with a discussion of future extensions and improvements.

Key words. Optimal power flow, nonconvex robust optimization, network contingency identification, nonlinear mixed-integer programming

AMS subject classifications. 90C06, 90C11, 90C22, 90C30, 90C47, 90C56, 90C57, 90C90

1. Introduction.

We present solution methods for application to the problem of identifying the most critical attacks to an alternating current (AC) power system, which we model as a Stackelberg game, where the attacker (i.e., leader) aims to compromise the functionality of a small subset of components whose failure or malfunction results in a system disruption that cannot be adequately remedied with available defensive measures by a defender (i.e., follower). We allow for an "attack" to be of either malevolent or natural origin, such as from a "perfect storm" of naturally occurring component failures. One example of where a small number of component failures had significant consequence was during the large-scale blackout in the northeastern United States and neighboring parts of Canada in the summer of 2003 [48, 37].

The optimization model associated with this problem of power system vulnerability analysis (PSVA) naturally has the form of a bilevel problem, a specific type of bilevel optimization problem (see, e.g., [2, 19, 40]) where the upper-level (attacker) problem seeks to maximize the same objective that the lower level (defender) seeks to minimize. For purpose of brevity, we refer to the problem of interest as *the PSVA bilevel problem*. Formulation and solution techniques for the PSVA bilevel problem are well-studied [43, 2, 9, 3, 53, 16, 44, 51], and some noteworthy variants include: unit-commitment [45, 13]; probabilistic line failure [46] trilevel defender-attacker-defender [1, 18, 27], defender line-switching capability [53] heuristics, metaheuris-

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tics [10, 3, 29].

Even well-structured bilevel optimization problems (such as linear bilevel programs) are known to be NP-hard, and the PSVA bilevel problem is furthermore complicated by the nonlinearity and nonconvexity of the PSVA lower-level defender problem due to its AC power flow equations. Naturally, many solution approaches either relax or linearize the lower-level problem of the PSVA bilevel problem to yield a modified structure more favorable to single-level reformulations. But in doing so, accuracy of the underlying model is compromised, and the resulting PSVA bilevel model solutions may misidentify attacks as harmless that are in fact not. For this reason, one avenue of PSVA bilevel research is to develop and test solution approaches that avoid relaxing or approximating the AC power flow equations of the PSVA lower-level problem. Similarly to [10, 3, 29], we focus on the aspect of PSVA bilevel research in which solution approaches preserve the nonlinear, nonconvex AC power equation structure. We shift focus from heuristic/metaheuristic approaches considered in [10, 3, 29] and instead develop a single-level reformulation that is distinct from the reformulation approaches based on KKT conditions [2, 39] or Lagrangian duality [2, 51], and that is suitable for the nonconvex structure of the lower-level defender problem.

We focus on the case where attacks consist of the deactivation of up to K transmission lines (including transformers). For the purpose of evaluating power network security, solving $K > 1$ instances of the bilevel problem are of the most practical value; the $K = 0$ instance trivializes to a feasibility problem for the baseline power network, and power networks are assumed to be secure against the removal of any one line as allowed by $K = 1$.

The PSVA bilevel problem may be formulated equivalently as a single-level maximization problem whose objective is the optimal value function of the lower-level defender problem. The *optimal value function* of the lower-level problem is a function of the upper-level attack variable whose output is the optimal value of the lower-level problem for the input attack. Thus, earlier efforts involved developing equivalent single-level reformulations that are tractable to known solution method.

The paper [2] describes two single-level reformulation approaches: (i) replacement of the lower-level problem with either its Karush-Kuhn-Tucker (KKT) conditions; and (ii) replacement of the lower-level problem with its Lagrangian dual problem. The KKT conditions are necessary under constraint qualification, and sufficient under convexity of the lower level problem. Thus, under these two assumptions, the KKT condition-based single-level reformulation is equivalent to the original bilevel optimization problem. Under the same two assumptions, The Lagrangian dual has zero duality gap with the primal lower-level problem, likewise yielding an equivalent single-level reformulation.

Due to the two assumptions (i.e., constraint qualifications and convexity) for the KKT condition-based or Lagrangian dual-based single-level reformulations to be equivalent to the original bilevel optimization problem, solution approaches typically rely on relaxations or approximations of the lower-level problem. Various approaches based on these ideas have been well-studied and well-developed (e.g., [43, 2, 39, 54, 25, 7, 20, 51]). Some of these relaxations or approximations are linear, while other relaxations preserve some nonlinearity while yielding convex lower-level problem relaxations. Well-studied convex relaxations of the AC power flow equations include the semidefinite programming (SDP) relaxation [4, 28, 32, 26], the quadratic constraint (QC) relaxation [15], and the second-order cone (SOC) relaxation [24]. Thus, in various ways, these approaches obtain reformulations that are solvable with well-developed solver technology, but at the cost of diminishing the accuracy of the lower-level defender model.

The solution approaches based on convex relaxations of the lower-level defender problem can be expected to yield "false negatives" in terms of of identifying power system vulnerabilities. That is, the relaxed lower-level defender problem can have optimal value zero for a given attack when in fact the optimal value for the non-relaxed lower-level defender problem is nonzero and perhaps even of substantial value. Furthermore, solution approaches based on approximations for lower-level defender models can be expected to yield both "false negatives" and "false positives," which include the identification of attacks as being substantial that, in fact, are not.

In contrast, our goal is to explore severity of the inaccuracies caused by these relaxations and/or approximation to the lower-level defender problem. We ask to what degree can power system vulnerabilities be missed or inaccurately assessed? Toward this end, we develop and analyze an alternative bilevel formulation and solution approach that preserves the original structure to the AC power flow equations in the lower-level defender problem.

In this paper we present a new solution approach to solving the PSVA bilevel problem for identifying the most severe attacks to a power system for which optimal defender response is assumed. The key contributions of this paper are (i) to show that AC power-flow equations can be used within a rigorous global optimization approach to analyze grid contingencies, and (ii) to provide an implementable and effective branch-and-bound framework for solving the PSVA bilevel problem with lower-level AC power-flow equations. Our branch-and-bound framework is based on two tailored subproblems for computing lower and upper bounds. In particular, we develop a single-level reformulation of the PSVA bilevel problem obtained by reformulating the max-min problem as a min-max problem whose objective provides a valid upper bound for the original PSVA bilevel problem. We obtain lower bounds through the use of an SDP relaxation of the lower-level defender subproblem. In addition, we provide an analysis of the solutions produced upon finite termination of the branch-and-bound framework, including conditions under which its solutions are either globally optimal, or are contained within a finite list of candidate solutions. We demonstrate the effectiveness of our approach in computational experiments, applying our branch-and-bound approach to the identification of the most severe attacks in IEEE power network instances. We compare our approach to the SOC relaxation and Lagrangian dual single-level reformulation approach of [51]. Finally, we use the resulting comparison to provide insights toward understanding the limitations of using a convex-relaxed lower-level defender subproblem in formulating and evaluating solutions to the bilevel problem.

This paper is organized as follows. In Section 2, we present the parameterized lower-level defender problem as a feasibility problem for satisfying the AC power flow constraints, with system infeasibilities penalized by absolute slack values, and binary-valued parameters corresponding to attack states. We embed the lower-level problem in the bilevel problem, and also in the related bilevel minmax problem. We derive key properties of these problems and their relationship, and we describe how to apply the SDP and SOC convex relaxations to the AC power flow constraints. In Section 3, we present our branch-and-bound algorithm, and we prove properties of the generated solutions at termination. Section 4 describes our numerical experiments for (i) demonstrating the advantages of using the AC power-flow equations directly in the PSVA bilevel problem, and (ii) testing the developed branch-and-bound approach and reporting the numerical results for the new method and a conic-based method on different power-grid instances. In Section 5, we summarize our conclusions and describe future work

2. Problem Formulation.

To begin, we specify the lower-level AC optimal power flow (ACOPF) with details of the power flow physics. We follow the development of Chapter 3 in [57].

2.1. Nonconvex lower-level ACOPF problem.

We consider an AC power flow system consisting of buses, indexed by i or j with index set N, that are linked by lines (including transformers) indexed by l from set L. When the terminal buses of a line are to be specified, we denote the line as $(i, j) = l \in L$, which is connected from bus i to bus j. Distinction is made between lines that are *active* indexed from the set $L^0 \subseteq L$, and lines that are *inactive* indexed from the set $L^1 \subseteq L$. Hence, $L^0 \cup L^1 = L$ and $L^0 \cap L^1 = \emptyset$. Lines may become inactive due to an intentional attack on the network or merely due to unexpected component failures.

For a given resistance r_l and reactance χ_l of line $l \in L$, the complex-valued line impedance is defined by $z_l := (r_l + \mathfrak{i}\chi_l)$, where the imaginary unit is denoted by $\mathfrak{i} := \sqrt{-1}$. We denote complex current flows by ι_l^f and ι_l^t, where the superscripts f and t indicate the forward and backward direction of flows, respectively. Given complex bus voltages $v_i \in \mathbb{C}$, $i \in N$, the line $l \in L$ current flows are determined by

$$\begin{bmatrix} \iota_l^f \\ \iota_l^t \end{bmatrix} := \begin{bmatrix} Y_l^{ff} & Y_l^{ft} \\ Y_l^{tf} & Y_l^{tt} \end{bmatrix} \begin{bmatrix} v_i \\ v_j \end{bmatrix}, \tag{2.1a}$$

where the line admittance entries Y_l are

$$Y_l := \begin{bmatrix} Y_l^{ff} & Y_l^{ft} \\ Y_l^{tf} & Y_l^{tt} \end{bmatrix} := \begin{bmatrix} (z^{-1} + \mathfrak{i}\frac{b_l}{2})\frac{1}{\tau_l^2} & -z_l^{-1}\frac{1}{\tau_l e^{-\mathfrak{i}\psi_l}} \\ -z_l^{-1}\frac{1}{\tau_l e^{\mathfrak{i}\psi_l}} & z_l^{-1} + \mathfrak{i}\frac{b_l}{2} \end{bmatrix} \tag{2.1b}$$

given the charging susceptance b_l, the tap ratio τ_l, and the phase angle shift ψ_l. At each line $(i, j) = l \in L$, complex power flows from bus i and to bus j are given respectively by

$$s_l^f := v_i(\iota_l^f)^* = v_i v_i^*(Y_l^{ff})^* + v_i v_j^*(Y_l^{ft})^*, \quad s_l^t := v_j(\iota_l^t)^* = v_j v_i^*(Y_l^{tf})^* + v_j v_j^*(Y_l^{tt})^*, \tag{2.1c}$$

where $(\cdot)^*$ applied to a complex-valued argument returns its complex conjugate. The shunt power flow associated with bus $i \in N$ is given by

$$s_i^{sh} := |v_i|^2(Y_i^{shR} - \mathfrak{i}Y_i^{shI}), \tag{2.1d}$$

where Y_i^{shR} and Y_i^{shI} are bus $i \in N$ shunt admittance parameters. Active and reactive power components are defined as follows:

$$p_l^f := \Re\{s_l^f\}, \quad q_l^f := \Im\{s_l^f\}, \quad l \in L \tag{2.1e}$$

$$p_l^t := \Re\{s_l^t\}, \quad q_l^t := \Im\{s_l^t\}, \quad l \in L \tag{2.1f}$$

$$p_i^{sh} := \Re\{s_i^{sh}\}, \quad q_i^{sh} := \Im\{s_i^{sh}\}, \quad i \in N. \tag{2.1g}$$

The real and imaginary voltage components are denoted $v_i^R := \Re\{v_i\}$ and $v_i^I := \Im\{v_i\}$ for each $i \in N$. Collecting $v^R := \left[v_i^R\right]_{i \in N}$ and $v^I := \left[v_i^I\right]_{i \in N}$, the active and reactive power quantities (2.1e), (2.1f), and (2.1g) may be written in the form

$$p_l^f = \left\langle P_l^f, W \right\rangle, \quad q_l^f = \left\langle Q_l^f, W \right\rangle, \quad p_l^t = \left\langle P_l^t, W \right\rangle, \quad q_l^t = \left\langle Q_l^t, W \right\rangle, \quad l \in L, \tag{2.2a}$$

$$p_i^{sh} = \left\langle P_i^{sh}, W \right\rangle, \quad q_i^{sh} = \left\langle Q_i^{sh}, W \right\rangle, \quad i \in N, \tag{2.2b}$$

where P_l^f, Q_l^f, P_l^t, Q_l^t, $l \in L$, and P_i^{sh}, Q_i^{sh}, $i \in N$ are constant sparse $2|N| \times 2|N|$ symmetric real-valued matrices, $W \in \mathbb{R}^{2|N| \times 2|N|} \in \mathcal{W}^{AC}$ with

$$\mathcal{W}^{AC} := \left\{ \begin{bmatrix} v^R \\ v^I \end{bmatrix} \begin{bmatrix} v^R \\ v^I \end{bmatrix}^T : v^R, v^I \in \mathbb{R}^{|N|} \right\}, \tag{2.2c}$$

and $\langle \cdot, \cdot \rangle$ is the Frobenius inner product.

The power system operator (PSO) has direct control of the bus voltage settings $v_i \in \mathcal{C}$, $i \in N$, and through control of these settings, indirect control of the bus $i \in N$ shunt power flows via (2.2b) and the active line $l \in L^0$ flows via (2.2a). The PSO also has direct control over active and reactive power generation at bus $i \in N$, which are denoted by $p^G := (p_i^G)_{i \in N}$ and $q^G := (q_i^G)_{i \in N}$, respectively. The bus voltage settings $v_i \in \mathbb{C}$, $i \in N$, the line power flow quantities (2.2a), and the power generation are subject to the following physical constraints.

1. Voltage magnitude bounds:

$$(V_i^{min})^2 \leq \left\langle V_i^M, W \right\rangle \leq (V_i^{max})^2, \quad i \in N, \tag{2.2d}$$

   where V_i^M is the constant sparse $2|N| \times 2|N|$ symmetric matrix for which

$$|v_i|^2 = \left\langle V_i^M, W \right\rangle, \quad i \in N.$$

2. Active and reactive power generation bounds:

$$P_i^{min} \leq p_i^G \leq P_i^{max}, \quad Q_i^{min} \leq q_i^G \leq Q_i^{max}, \quad i \in N, \tag{2.2e}$$

3. Thermal line flow limits:

$$(p_l^f)^2 + (q_l^f)^2 \leq (s_l^{max})^2, \quad (p_l^t)^2 + (q_l^t)^2 \leq (s_l^{max})^2, \quad l \in L^0 \tag{2.2f}$$

   where $s_l^{max} \in \mathbb{R}_+ \cup \{\infty\}$ is the upper bound on the absolute value line l flow in either the 'from' or 'to' direction. Constraint (2.2f) is only relevant for active lines $l \in L^0$.

Using necessary adjustments of v_i, p_i^G, q_i^G, $i \in N$, as allowed by the physical constraints (2.2d)–(2.2f), the PSO is tasked with maintaining, for each bus $i \in N$, the balance between constant active P_i^D and reactive Q_i^D power demands and the bus i-specific power injection quantities due to 1) power generation p_i^G, q_i^G; 2) via (2.2b), shunt power flows p_i^{sh}, q_i^{sh}; 3) via (2.2a), line power flows p_l^f, q_l^f, $l \in L_i^f \cap L^0$ where L_i^f is the set of lines with the origin of bus i; and p_l^t, q_l^t, $l \in L_i^t \cap L^0$, where L_i^t is the set of lines with the destination of bus i. Furthermore, the resulting active and reactive power flow balance constraint equations at each bus $i \in N$ are given by

$$\left\langle P_i^{sh}, W \right\rangle + \sum_{l \in L_i^f \cap L^0} p_l^f + \sum_{l \in L_i^t \cap L^0} p_l^t = p_i^G - P_i^D \tag{2.2g}$$

$$\left\langle Q_i^{sh}, W \right\rangle + \sum_{l \in L_i^f \cap L^0} q_l^f + \sum_{l \in L_i^t \cap L^0} q_l^t = q_i^G - Q_i^D. \tag{2.2h}$$

We pose a model to accommodate change in the active line index set L^0 due to an attack or other disruption to the power system. In such a situation, it may not be possible for the PSO to enforce all of the power system constraints (2.2) by adjustment of the voltage settings v and power generations p^G, and q^G. That is, load shedding, excessive power generation, or other violations of system power constraints might be unavoidable. Consequently, through the introduction of slacks associated with various types of power quantities, we model the lower-level problem below in (2.3) to minimize these violations.

In order to model the mutable nature of the active line index set L^0, we replace the use of L^0 in (2.2) with the use of binary valued parameters $x_l \in \{0,1\}$, $l \in L$ in (2.3). Lines for which $x_l = 0$ behave like active lines, while lines for which $x_l = 1$ behave like inactive lines. Consequently, with the use of $x := (x_l)_{l\in L}$, the use of the full index set L instead of L^0 in the following model (2.3) is deliberate. In (2.3), $x := (x_l)_{l\in L}$ is treated as a parameter, but in subsequent models for the attacker problem, it becomes a decision variable. The PSO lower-level problem is given by

$$\begin{aligned}
\phi^{\mathcal{W}}(x) := \min_{W,p,q,d} \quad & \sum_{i\in N} \left[|d^N_{i,p}| + |d^N_{i,q}|\right] \\
& + \sum_{l\in L} (1-x_l) \left[|d^f_{l,p}| + |d^t_{l,p}| + |d^f_{l,q}| + |d^t_{l,q}| + d^f_{l,s} + d^t_{l,s}\right] \\
& + \sum_{l\in L} x_l \left[|p^f_l| + |p^t_l| + |q^f_l| + |q^t_l|\right] && \text{(2.3a)} \\
\text{s.t.} \quad & W \in \mathcal{W} && \text{(2.3b)} \\
& P^{min}_i \le p^G_i \le P^{max}_i, \quad Q^{min}_i \le q^G_i \le Q^{max}_i, \quad i \in N && \text{(2.3c)} \\
& (V^{min}_i)^2 \le \left\langle V^M_i, W \right\rangle \le (V^{max}_i)^2, \quad i \in N, && \text{(2.3d)} \\
& (p^f_l)^2 + (q^f_l)^2 \le (s^{max}_l + d^f_{l,s})^2, \quad l \in L && \text{(2.3e)} \\
& (p^t_l)^2 + (q^t_l)^2 \le (s^{max}_l + d^t_{l,s})^2, \quad l \in L && \text{(2.3f)} \\
& d^f_{l,s}, d^t_{l,s} \ge 0, \quad l \in L && \text{(2.3g)} \\
& \langle P^{sh}_i, W\rangle + \sum_{l\in L^f_i} p^f_l + \sum_{l \in L^t_i} p^t_l - p^G_i + P^D_i = d^N_{i,p}, \quad i \in N, && \text{(2.3h)} \\
& \langle Q^{sh}_i, W\rangle + \sum_{l\in L^f_i} q^f_l + \sum_{l \in L^t_i} q^t_l - q^G_i + Q^D_i = d^N_{i,q}, \quad i \in N, && \text{(2.3i)} \\
& p^f_l = \left\langle P^f_l, W \right\rangle + d^f_{l,p}, \quad p^t_l = \left\langle P^t_l, W \right\rangle + d^t_{l,p}, \quad l \in L && \text{(2.3j)} \\
& q^f_l = \left\langle Q^f_l, W \right\rangle + d^f_{l,q}, \quad q^t_l = \left\langle Q^t_l, W \right\rangle + d^t_{l,q}, \quad l \in L, && \text{(2.3k)}
\end{aligned}$$

where W, p, q, d are decision variables. For $\mathcal{W} = \mathcal{W}^{AC}$, entries of W are bilinear terms in the entries of v^R and v^I, resulting in a nonconvex nonlinear program due to the quadratic rank-1 constraint, as specified in (2.2c). The thermal limit constraints (2.3e) and (2.3f) are convex nonlinear, which can also be reformulated as second-order cone (SOC) constraints. All other constraints are linear.

The power flow balance constraints as originally given in (2.2g)–(2.2h) are softened to (2.3h)–(2.3i) with the use of active and reactive bus power slacks $d^N_{i,p}$ and $d^N_{i,q}$, $i \in N$, whose absolute values are penalized in the objective function; and, via (2.3j)–(2.3k), line flow power slacks $d^f_{l,p}, d^t_{l,p}, d^f_{l,q}, d^t_{l,q}$, $l \in L$, whose absolute values are penalized only for active lines l with $x_l = 0$. Constraints (2.3j)–(2.3k) quantify the line power flow slacks as the discrepancies between the *actual* line power flows over active lines as given by the right-hand sides of (2.2a) and the *target* power flows $p_l^f, p_l^t, q_l^f, q_l^t$, $l \in L$ applied to the satisfaction of (2.3h)–(2.3i).

As required to guarantee the feasibility of problem (2.3), the thermal line limit constraints (2.2f) are also softened in (2.3e) and (2.3f) with the use of the nonnegative slacks (2.3g). As with the line flow power slacks, these thermal line limit slacks are only penalized over lines l that are active ($x_l = 0$). For lines $l \in L$ that are flagged as inactive with $x_l = 1$, the softened line flow constraints (2.3j)–(2.3k) and the softened thermal line limit constraints (2.3e)–(2.3g) become irrelevant since their corresponding slack variable terms in the objective (2.3a) have coefficient zero. Rather, any nonzero targeted power flow $p_l^f, p_l^t, q_l^f, q_l^t$ over lines $l \in L$ that are flagged as inactive with $x_l = 1$ are penalized in absolute value. Throughout the paper, we also assume the following conditions in order to prevent problem (2.3) from being trivially infeasible:

A1 The physical constraints (2.2d), (2.2e), and (2.2f) are consistent, i.e., $V_i^{min} \leq V_i^{max}$, $P_i^{min} \leq P_i^{max}$, $Q_i^{min} \leq Q_i^{max}$, and $s_l^{max} \geq 0$;

A2 $\mathcal{W} \supseteq \mathcal{W}^{AC}$.

We now show the following properties of the lower-level problem (2.3).

Proposition 2.1. *If A1 and A2 hold, then for any realization of* $\mathcal{W}$*,*

1. *problem* (2.3) *is always feasible for all* $x \in [0,1]^{|L|}$*;*
2. *for any given* $x \in \{0,1\}^{|L|}$, $\phi^{\mathcal{W}}(x) = 0$ *if and only if there exists a feasible solution* (W^0, p^0, q^0, d^0) *such that*
   (a) $d_{l,p}^f = d_{l,p}^t = d_{l,q}^f = d_{l,q}^t = d_{l,s}^f = d_{l,s}^t = 0$ *for all* $l \in L$ *with* $x_l = 0$ *and*
   (b) $p_l^f = p_l^t = q_l^f = q_l^t = 0$ *for all* $l \in L$ *with* $x_l = 1$*,*

   which in turn holds if and only if (W^0, p^0, q^0) *satisfies the constraints* (2.2)*;*
3. $x \mapsto \phi^{\mathcal{W}}(x)$ *is concave (and thus continuous) over* $x \in [0,1]^{|L|}$*.*

Proof. For the first claim, we fix $W = W^0$, $p^G = (p^G)^0$, and $q^G = (q^G)^0$. Due to A1 and A2, we can assign line flow discrepancies $d_p^f = (d_p^f)^0, d_p^t = (d_p^t)^0, d_q^f = (d_q^f)^0, d_q^t = (d_q^t)^0$ and target line flows $p^f = (p^f)^0, p^t = (p^t)^0, q^f = (q^f)^0, q^t = (q^t)^0$ to satisfy the power balance constraints (2.3h)–(2.3i) and the discrepancy-defining constraints (2.3j)–(2.3k). That leaves only the thermal line limit constraints (2.3e) and (2.3f), which can be satisfied with sufficiently large slack values $d_s^f = (d_s^f)^0$ and $d_s^t = (d_s^t)^0$. Thus, the existence of a feasible solution (W^0, p^0, q^0, d^0) satisfying all constraints to problem (2.3) has been demonstrated.

The second claim is obvious from the definition of the objective function (2.3a) and the natural correspond between constraints (2.3c)–(2.3k) and the power system constraints (2.2).

For the third claim, we note that $\phi^{\mathcal{W}}(x)$ is just the infimum of an arbitrary collection of affine functions in x with coefficients parameterized by all feasible values of the decision variables in (2.3). For this reason, $\phi^{\mathcal{W}}(x)$ is a concave function. (See, e.g., [42, Lemma 2.58].) ∎

We make the following remarks.

Remark 2.2. The first and second properties of Proposition 2.1 cover the baseline situation with no attack $x = 0$, in which $\phi^{\mathcal{W}}(0) = 0$ with $\mathcal{W} = \mathcal{W}^{AC}$ implies the existence of PSO settings that satisfy the AC power system requirements (2.2) under normal (i.e., noncontingent) operating conditions.

Remark 2.3. One may possibly consider certain variations to the objective in the lower-level problem (2.3), while preserving the properties of Proposition 2.1. For example, one may add weighted expressions to the current objective function (2.3a) that model the cost of power generation. These additional expressions need to be weighted in such a way so that the original expressions in (2.3a) behave like exact penalty terms for the soft constraints whose violation they penalize. Though not known a priori, such a weighting would exist under the satisfaction of a constraint qualification. For now, we consider only lower-level subproblem objective functions focusing on system feasibility having the form (2.3a).

2.2. Convex relaxations of the lower-level subproblem. In what follows, we shall make reference to two well-known convex relaxations of $\mathcal{W}^{AC}$: SDP and SOC relaxations. We use the SDP relaxation in order to provide lower bounds for our maximization branch-and-bound approach, while the SOC is introduced for the later comparison with the single-level reformulation based on the Lagrangian dual approach as in [51].

The SDP relaxation (e.g., [4, 28]) of $\phi^{AC}(x)$ is realized with the use of $\mathcal{W} = \mathcal{W}^{SDP} \supset \mathcal{W}^{AC}$ where

$$\mathcal{W}^{SDP} := \left\{ W \in \mathbb{R}^{2|N| \times 2|N|} \ : \ W \succeq 0 \right\} = \mathcal{S}_+^{2|N|}. \tag{2.4}$$

The SOC relaxation of the power flow equations (2.1) is applied to the entries of the complex voltage products given in the rank-1 matrix vv^* for each line as follows. For each line $(i,j) = l \in L$ with *from* bus i and *to* bus j, denote the complex voltage products between buses i and j by

$$w_{ii} = |v_i|^2, \ w_{jj} = |v_j|^2, \ w_{ij} = w_{ji}^* = v_i v_j^*. \tag{2.5}$$

Collectively, denote $w_l := [w_{ii}, w_{jj}, w_{ij}]$ for each $(i,j) = l \in L$. In terms of W matrix entries, the real and imaginary components of the complex voltage products as defined in (2.5) are

$$w_{ij}^R := \Re\{w_{ij}\} = W_{i,j} + W_{|N|+i,|N|+j}, \quad w_{ij}^I := \Im\{w_{ij}\} = W_{j,|N|+i} - W_{i,|N|+j}. \tag{2.6}$$

The SOC relaxation is then applied to the constraint $W \in \mathcal{W}^{AC}$ by replacing $W \in \mathcal{W}^{AC}$ with the following SOC constraints for each line $l \in L$:

$$\left\| \begin{matrix} w_{ii} - w_{jj} \\ 2w_{ij}^R \\ 2w_{ij}^I \end{matrix} \right\| \le w_{ii} + w_{jj}, \quad (i,j) = l \in L. \tag{2.7}$$

(Note that $w_{ii} = w_{ii}^R$ for each $i \in N$.) Denote for brevity $\hat{\imath} := i + |N|$ and $\hat{\jmath} := j + |N|$. In terms of W, SOC constraint (2.7) is written $W \in \mathcal{W}^{SOC} = \bigcap_{l \in L} \mathcal{W}_l^{SOC}$, where each $\mathcal{W}_l^{SOC}$, $l \in L$, is defined by

$$\mathcal{W}_l^{SOC} := \left\{ W \ : \ \left\| \begin{matrix} W_{i,i} + W_{\hat{\imath},\hat{\imath}} - W_{j,j} - W_{\hat{\jmath},\hat{\jmath}} \\ 2(W_{i,j} + W_{\hat{\imath},\hat{\jmath}}) \\ 2(W_{j,\hat{\imath}} - W_{i,\hat{\jmath}}) \end{matrix} \right\| \le W_{i,i} + W_{\hat{\imath},\hat{\imath}} + W_{j,j} + W_{\hat{\jmath},\hat{\jmath}} \right\} \tag{2.8}$$

From this point on, for brevity, we denote $\phi^{AC} := \phi^{\mathcal{W}^{AC}}$, $\phi^{SDP} := \phi^{\mathcal{W}^{SDP}}$, and $\phi^{SOC} := \phi^{\mathcal{W}^{SOC}}$.

2.3. PSVA bilevel problem formulation. With the lower-level subproblem (2.3), we formulate the PSVA bilevel optimization problem as follows:

$$\Phi^{\mathcal{W}} := \max_x \left\{ \phi^{\mathcal{W}}(x) \quad \text{s.t.} \quad \sum_{l \in L} x_l \le K, \quad x_l \in \{0,1\}, \ l \in L \right\}, \tag{2.9}$$

as a discrete optimization problem with nonsmooth objective value function $x \mapsto \phi^{\mathcal{W}}(x)$.

Remark 2.4. By Proposition 2.1, $\phi^{\mathcal{W}}$ is concave (and thus continuous) in $x \in [0,1]^{|L|}$ even for a nonconvex realization of $\mathcal{W}$. As such, even for $\mathcal{W} = \mathcal{W}^{AC}$, problem (2.9) has the structure of a convex mixed-integer nonlinear program (MINLP). In theory, we may solve MINLPs using such methodologies as the generalized extended cutting plane (ECP) or outer approximation (OA) approaches [50, 22]. However, in practice, the use of such approaches require the reliable evaluation of (at least quantifiably approximate) values and (at least quantifiably approximate) subgradients to the lower-level optimal value function, and this in turn requires solving the corresponding subproblem to global optimality. Global optimization of the lower-level subproblem with the nonconvex AC power flow constraints is known to be strongly NP-hard [8].

Such a convexity structure of $\phi^{\mathcal{W}}$, $\mathcal{W} = \mathcal{W}^{AC}$, is not necessary to establish the global optimality properties of solutions generated in our contributed approach. Nevertheless, such objective function structures are generally desirable within a branch-and-bound context for informing meaningful branching rules that are more than mere guesses within what is otherwise a blind combinatorially enumerative process.

When $\mathcal{W}$ is a convex relaxation of $\mathcal{W}^{AC}$ for which a constraint qualification holds, the bilevel problem (2.9) can be reformulated into an equivalent single-level maximization problem by replacing the function $\phi^{\mathcal{W}}(x)$ with the Lagrangian dual problem to problem (2.3). Hence, a mixed-integer convex programming algorithm can be applied directly to the resulting single-level reformulation. Such an approach was applied in the recent paper [51] where in our present notation, $\mathcal{W} = \mathcal{W}^{SOC} \supset \mathcal{W}^{AC}$ realizes the well-known SOC relaxation of $\mathcal{W}^{AC}$. One may then apply a branch-and-bound approach to the single-level maximization problem, with upper bounds computed as solutions to the single-level problem relaxation due to relaxing the integrality constraints on x. Lower bounds are due to the verification of feasible solutions to the bilevel problem.

The lower-level problem (2.3) with $\mathcal{W} = \mathcal{W}^{AC}$ is nonconvex and may have a nonzero gap with its Lagrangian dual, so the Lagrangian dual-based single-level reformulation will *not* be equivalent to the nonrelaxed PSVA bilevel problem. Thus, within a maximizing branch-and-bound context, the node upper bounding procedure is not yet evident. We address this in the next section.

3. Algorithms and Methods.

In this section, we present an implementable and effective branch-and-bound framework for solving the PSVA bilevel problem with lower-level AC power-flow equations. Our branch-and-bound framework is based on two tailored subproblems for computing lower and upper bounds.

Let $\mathcal{T}$ be a branch-and-bound tree that consists of a set of tree nodes $\mathcal{N}$. At each tree node $\mathcal{N} \in \mathcal{T}$, the algorithm may fix some of the binary variables x_l to either 0 or 1 as part of the branching process. We denote by $L_0^{\mathcal{N}}$ the set of indices of x that are fixed to 0 at node $\mathcal{N}$, and we denote by $L_1^{\mathcal{N}}$ the set of indices of x components that are fixed to 1 at node $\mathcal{N}$. The set of remaining line indices for which the x components are not fixed is denoted Let $L_*^{\mathcal{N}} := L \backslash \{L_0^{\mathcal{N}} \cup L_1^{\mathcal{N}}\}$. The upper bound associated with each node is $\Phi_{UB}^{\mathcal{N}}$, and the incumbent lower bound associated with the best-known feasible solution is Φ_{LB}. We now define $X^{\mathcal{N}}$ for a given node $\mathcal{N}$ as

$$X^{\mathcal{N}} := \left\{ x \in \{0,1\}^{|L|} : \sum_{l \in L} x_l \leq K,\ x_l = 0\ \forall l \in L_0^{\mathcal{N}},\ x_l = 1\ \forall l \in L_1^{\mathcal{N}} \right\},$$

Before proceeding, we define the following.

1. The solution $x^{\mathcal{N}}$ associated with node $\mathcal{N}$ is the unique element of $X^{\mathcal{N}}$ such that $x_l = 0$ for all $l \in L_*^{\mathcal{N}}$.
2. The active node tree $\mathcal{T}$ is a collection of nodes $\mathcal{N}$ that have not been fathomed yet; the collection of nodes that have been fathomed due to optimality is denoted $\mathcal{F}^{OPT}$, and the collection of nodes that have been fathomed due to bound is denoted $\mathcal{F}^{BD}$.

We denote the node $\mathcal{N}$ specific instances of problems (2.9) by

$$\Phi^{\mathcal{W},\mathcal{N}} := \max_{x} \left\{ \phi^{\mathcal{W}}(x) \quad \text{s.t.} \quad x \in X^{\mathcal{N}} \right\}. \tag{3.1}$$

As we noted in subsection 2.3, we do not readily have a tractable single-level reformulation for the bilevel problem (2.9), and so it is with its node $\mathcal{N}$ refinements (3.1). In addition, even the continuous relaxations of (3.1) is not tractable for providing the upper bounds.

We develop a tractable single-level reformulation of (3.1) by replacing the max-min problem with a min-max problem, whose objective provides a valid upper bound for the node subproblem (3.1). The resulting upper bounding min-max problem is given by

$$\begin{aligned} \Psi^{\mathcal{W},\mathcal{N}} := \min_{W,p,q,d} & \sum_{i\in N} \left[d_{i,p}^{N} + d_{i,q}^{N}\right] \\ & + \max_{x} \left\{ \begin{array}{l} \sum_{l\in L}(1-x_l)\left[|d_{l,p}^{f}| + |d_{l,p}^{t}| + |d_{l,q}^{f}| + |d_{l,q}^{t}| + d_{l,s}^{f} + d_{l,s}^{t}\right] \\ \quad + \sum_{l\in L} x_l \left[|p_l^f| + |p_l^t| + |q_l^f| + |q_l^t|\right] \\ \text{s.t.} \quad x \in X^{\mathcal{N}} \end{array} \right\} \\ \text{s.t. } & (2.3b) - (2.3k). \end{aligned} \tag{3.2}$$

We note the following relationships.

PROPOSITION 3.1. *Let A1-A2 hold. For each* $\mathcal{N} \in \mathcal{T} \cup \mathcal{F}^{OPT} \cup \mathcal{F}^{BD}$*, we have*

$$\phi^{\mathcal{W}}(x^{\mathcal{N}}) \le \Phi^{\mathcal{W},\mathcal{N}} \le \Psi^{\mathcal{W},\mathcal{N}}. \tag{3.3}$$

Furthermore, if $K = |L_1^{\mathcal{N}}|$ *or* $|L_*^{\mathcal{N}}| = 0$*, then*

$$\phi^{\mathcal{W}}(x^{\mathcal{N}}) = \Phi^{\mathcal{W},\mathcal{N}} = \Psi^{\mathcal{W},\mathcal{N}}. \tag{3.4}$$

Proof. The first claim follows readily from the definition of $\Phi^{\mathcal{W},\mathcal{N}}$ given that $x^{\mathcal{N}} \in X^{\mathcal{N}}$ (first inequality) and from elementary minimax theory [41, Lemma 36.1] (second inequality). To see the second claim, if $K = |L_1^{\mathcal{N}}|$ or $|L_*^{\mathcal{N}}| = 0$, then the set $X^{\mathcal{N}} = \{x^{\mathcal{N}}\}$ is singleton, and so problems (3.1) and (3.2) are evidently equivalent to instances of problem (2.3) with $x = x^{\mathcal{N}}$. ∎

For purpose of notational brevity, we denote, for each $\mathcal{N} \in \mathcal{T}$, $\Phi^{AC,\mathcal{N}} := \Phi^{\mathcal{W}^{AC},\mathcal{N}}$, $\Phi^{SDP,\mathcal{N}} := \Phi^{\mathcal{W}^{SDP},\mathcal{N}}$ and $\Psi^{AC,\mathcal{N}} := \Psi^{\mathcal{W}^{AC},\mathcal{N}}$, $\Psi^{SDP,\mathcal{N}} := \Psi^{\mathcal{W}^{SDP},\mathcal{N}}$.

The node $\mathcal{N}$ upper bounding problems (3.2) are still not readily solvable as given, but they may be easily reformulated into equivalent single-level reformulations.

PROPOSITION 3.2. *The node* $\mathcal{N}$*-specific attacker-defender problem with defender as leader defined in* (3.2) *may be equivalently written as follows.*

$$\Psi^{\mathcal{W},\mathcal{N}} = \min_{W,p,q,d,u} (K - |L_1^{\mathcal{N}}|)u^k + \sum_{l\in L_*^{\mathcal{N}}} u_l$$

$$+ \sum_{l \in L_0^{\mathcal{N}} \cup L_*^{\mathcal{N}}} \left[|d_{l,p}^f| + |d_{l,p}^t| + |d_{l,q}^f| + |d_{l,q}^t| + d_{l,s}^f + d_{l,s}^t \right]$$

$$+ \sum_{l \in L_1^{\mathcal{N}}} \left[|p_l^f| + |p_l^t| + |q_l^f| + |q_l^t| \right] + \sum_{i \in N} \left[d_{i,p}^N + d_{i,q}^N \right]$$

$$s.t. (2.3b)–(2.3k)$$

(3.5a)
$$\left(|p_l^f| + |p_l^t| + |q_l^f| + |q_l^t| \right) - \left(|d_{l,p}^f| + |d_{l,p}^t| + |d_{l,q}^f| + |d_{l,q}^t| + d_{l,s}^f + d_{l,s}^t \right) \le u_l + u^K, \quad l \in L_*^{\mathcal{N}}$$

$$u^K \ge 0, \quad u_l \ge 0, \quad l \in L_*^{\mathcal{N}}.$$

Proof. The inner maximization problem is bounded and feasible in x for all values of (W, p, q, d). and its constraint matrix has a simple structure that is easily verified to be totally unimodular [36, III.2]. As a result, the integrality restriction of the inner maximization problem can be relaxed while keeping the same optimal value. Strong duality readily holds for the continuously relaxed inner maximization problem. In summary, the above reasoning is written mathematically as

$$\max_x \left\{ \begin{array}{l} \sum_{l \in L} (1 - x_l) \left[|d_{l,p}^f| + |d_{l,p}^t| + |d_{l,q}^f| + |d_{l,q}^t| + d_{l,s}^f + d_{l,s}^t \right] \\ \qquad + \sum_{l \in L} x_l \left[|p_l^f| + |p_l^t| + |q_l^f| + |q_l^t| \right] \\ \text{s.t.} \quad x \in X^{\mathcal{N}} \end{array} \right\}$$

$$= \max_x \left\{ \begin{array}{l} \sum_{l \in L} (1 - x_l) \left[|d_{l,p}^f| + |d_{l,p}^t| + |d_{l,q}^f| + |d_{l,q}^t| + d_{l,s}^f + d_{l,s}^t \right] \\ \qquad + \sum_{l \in L} x_l \left[|p_l^f| + |p_l^t| + |q_l^f| + |q_l^t| \right] \\ \text{s.t.} \qquad x_l = 0, \;\; l \in L_0^{\mathcal{N}}, \qquad x_l = 1, \;\; l \in L_1^{\mathcal{N}} \\ \qquad\qquad 0 \le x_l \le 1, \;\; l \in L_*^{\mathcal{N}}, \qquad \sum_{l \in L} x_l \le K. \end{array} \right\}$$

$$= \min_u (K - |L_1^{\mathcal{N}}|) u^k + \sum_{l \in L_*^{\mathcal{N}}} u_l$$

$$+ \sum_{l \in L_0^{\mathcal{N}} \cup L_*^{\mathcal{N}}} \left[|d_{l,p}^f| + |d_{l,p}^t| + |d_{l,q}^f| + |d_{l,q}^t| + d_{l,s}^f + d_{l,s}^t \right]$$

$$+ \sum_{l \in L_1^{\mathcal{N}}} \left[|p_l^f| + |p_l^t| + |q_l^f| + |q_l^t| \right] + \sum_{i \in N} \left[d_{i,p}^N + d_{i,q}^N \right]$$

$$s.t. \left(|p_l^f| + |p_l^t| + |q_l^f| + |q_l^t| \right) - \left(|d_{l,p}^f| + |d_{l,p}^t| + |d_{l,q}^f| + |d_{l,q}^t| + d_{l,s}^f + d_{l,s}^t \right) \le u_l + u^K, \quad l \in L_*^{\mathcal{N}}$$

$$u^K \ge 0, \quad u_l \ge 0, \quad l \in L_*^{\mathcal{N}}.$$

which results in the claimed single-level reformulation. ☐

We introduce more notation.

1. Given $x \in X^{\mathcal{N}}$, the global optimal value to (2.3) was denoted by $\phi^{AC}(x)$; but we also allow for the use of locally optimal values denoted $\bar{\phi}^{AC}(x)$ and thus $\phi^{AC}(x) \le \bar{\phi}^{AC}(x)$.
2. We denote $\bar{\Psi}^{AC,\mathcal{N}}$ for discerning the possible local optimality of computed solutions to (3.2), so that $\Psi^{AC,\mathcal{N}} \le \bar{\Psi}^{AC,\mathcal{N}}$.
3. In solving problem (3.2) for either globally or merely locally optimal value, we denote the dual values associated with the constraints (3.5a) by $\xi_l^{\mathcal{N}}$, $l \in L_*^{\mathcal{N}}$.

Due to the use of problem (3.5) solutions for computing upper bounds $\Phi_{UB}^{\mathcal{N}}$, the branching index selection rule used in Algorithm 3.1 cannot be based on x component values. Instead, the branching index is selected based on $\xi^{\mathcal{N}}$ component values, in particular selecting an index corresponding the the maximal $\xi^{\mathcal{N}}$ component value. This particular branching rule is motivated by the role of each ξ_l, $l \in L_*^{\mathcal{N}}$, as the dual value associated with the corresponding $x_l \le 1$ bound in the inner maximization problem (3.2). In general, nonzero values for ξ_l mean that the corresponding bound

Algorithm 3.1 Branch-and-bound applied to (2.9) with $\mathcal{W} = \mathcal{W}^{AC}$

Inputs: optimality tolerance $\epsilon \geq 0$
Initialize $\mathcal{T} \leftarrow \emptyset$, $\mathcal{F}^{OPT} \leftarrow \emptyset$, $\mathcal{F}^{BD} \leftarrow \emptyset$
Create a root node $\mathcal{N}$ such that $L_0^{\mathcal{N}} \leftarrow \emptyset$, $L_1^{\mathcal{N}} \leftarrow \emptyset$, and $L_*^{\mathcal{N}} \leftarrow L$
Set $\Phi_{UB}^{\mathcal{N}} \leftarrow \infty$, compute $\Phi_{LB}^{\mathcal{N}} \leftarrow \phi^{SDP}(x^{\mathcal{N}})$
Set $\Phi_{LB} \leftarrow \Phi_{LB}^{\mathcal{N}}$ and $x_{LB} \leftarrow x^{\mathcal{N}}$
Set $\mathcal{T} \leftarrow \{\mathcal{N}\}$
while $\mathcal{T} \neq \emptyset$ **do**
  Select a node $\mathcal{N} \in \arg\max_{\mathcal{N} \in \mathcal{T}} \Phi_{UB}^{\mathcal{N}}$
  Set $\Phi_{UB} \leftarrow \max_{\mathcal{N} \in \mathcal{T}} \Phi_{UB}^{\mathcal{N}}$ and $\mathcal{T} \leftarrow \mathcal{T} \setminus \{\mathcal{N}\}$
  if $\Phi_{UB} < \Phi_{LB}$ **then**
    $\mathcal{F}^{BD} \leftarrow \mathcal{F}^{BD} \cup \{\mathcal{N}\} \cup \mathcal{T}$, $\mathcal{T} \leftarrow \emptyset$ {Fathom remaining active nodes by bound}

    return $\mathcal{T}, \mathcal{F}^{OPT}, \mathcal{F}^{BD}$ { **terminate** }
  else
    Solve the node $\mathcal{N}$ subproblem (3.2) with $\mathcal{W} = \mathcal{W}^{AC}$
      for a locally maximal value $\bar{\Psi}^{AC,\mathcal{N}}$ and multipliers $\xi_l^{\mathcal{N}}$, $l \in L_*^{\mathcal{N}}$
    Update $\Phi_{UB}^{\mathcal{N}} \leftarrow \min\{\Phi_{UB}^{\mathcal{N}}, \bar{\Psi}^{AC,\mathcal{N}}\}$
    if $|L_*^{\mathcal{N}}| = 0$ OR $K = |L_1^{\mathcal{N}}|$ OR $\Phi_{UB}^{\mathcal{N}} - \Phi_{LB}^{\mathcal{N}} \leq \epsilon$ **then**
      $\mathcal{F}^{OPT} \leftarrow \mathcal{F}^{OPT} \cup \{\mathcal{N}\}$ {Fathom by optimality}
    else if $\Phi_{UB}^{\mathcal{N}} \leq \Phi_{LB}$ **then**
      $\mathcal{F}^{BD} \leftarrow \mathcal{F}^{BD} \cup \{\mathcal{N}\}$ {Fathom due to bound}
    else
      Select $l^* \in \arg\max_{l \in L_*^{\mathcal{N}}} \xi_l^{\mathcal{N}}$
      Create two nodes $\mathcal{N}_0$ and $\mathcal{N}_1$ such that
        1) $L_0^{\mathcal{N}_0} \leftarrow L_0^{\mathcal{N}} \cup \{l^*\}$, $L_1^{\mathcal{N}_0} \leftarrow L_1^{\mathcal{N}}$, , $\Phi_{LB}^{\mathcal{N}_0} \leftarrow \Phi_{LB}^{\mathcal{N}}$, $\Phi_{UB}^{\mathcal{N}_0} \leftarrow \Phi_{UB}^{\mathcal{N}}$
        2) $L_0^{\mathcal{N}_1} \leftarrow L_0^{\mathcal{N}}$, $L_1^{\mathcal{N}_1} \leftarrow L_1^{\mathcal{N}} \cup \{l^*\}$, $\Phi_{UB}^{\mathcal{N}_1} \leftarrow \Phi_{UB}^{\mathcal{N}}$
      Compute $\Phi_{LB}^{\mathcal{N}_1} \leftarrow \phi^{SDP}(x^{\mathcal{N}_1})$
      if $\Phi_{LB}^{\mathcal{N}_1} > \Phi_{LB}$ **then**
        $\Phi_{LB} \leftarrow \Phi_{LB}^{\mathcal{N}_1}$ and $x_{LB} \leftarrow x^{\mathcal{N}_1}$
      end if
      Set $\mathcal{T} \leftarrow \mathcal{T} \cup \{\mathcal{N}_0, \mathcal{N}_1\}$
    end if
  end if
end while
return $\mathcal{T}, \mathcal{F}^{OPT}, \mathcal{F}^{BD}$

$x_l \leq 1$ is binding and thus favored as a choice in at least one optimal attack for the inner maximization problem (3.2). This particular branching index selection is by no means forced, and additional study into branching index selection rules and analogs to strong and pseudocost branching (e.g., [5]) for similar algorithms is needed.

The finite termination of Algorithm 3.1 is combinatorially evident.

THEOREM 3.3. *Let A1–A2 hold, and let Algorithm 3.1 be applied to problem (2.9) with $\mathcal{W} = \mathcal{W}^{AC}$ with optimality tolerance $\epsilon > 0$. Then Algorithm 3.1 terminates after processing a finite number of nodes and the following hold.*

1. *The incumbent solution x_{LB} is ϵ-optimal for problem (2.9) with $\mathcal{W} = \mathcal{W}^{SDP}$;*
2. *If $\Phi_{UB}^{\mathcal{N}} - \Phi_{LB} \leq \epsilon$ for all $\mathcal{N} \in \mathcal{F}^{OPT}$, then x_{LB} is furthermore ϵ-optimal for problem (2.9) with $\mathcal{W} = \mathcal{W}^{AC}$.*

3. *Otherwise, there is at least one solution* $x^{\mathcal{N}}$, $\mathcal{N} \in \mathcal{F}^{OPT}$, *for which* $\Phi_{LB} \leq \Phi_{UB}^{\mathcal{N}}$, *and these solutions are candidate* ϵ*-optimal solutions for problem* (2.9) *with* $\mathcal{W} = \mathcal{W}^{AC}$.

Proof. By A1–A2, the initial evaluation of Φ_{LB} in line 5 is finite and hence the subsequent part of the algorithm is nontrivial. The terminating after processing a finite number of tree nodes is evident from the combinatorial association of each node with one of the finite number of possible ways to partition the line index set $L = \{L_0^{\mathcal{N}}, L_1^{\mathcal{N}}, L_*^{\mathcal{N}}\}$.

For the first claim, the ϵ-optimality of x_{LB} for problem (2.9) with $\mathcal{W} = \mathcal{W}^{SDP}$ follows since, by construction we have the following three cases for each $\mathcal{N} \in \mathcal{F}^{OPT} \cup \mathcal{F}^{BD}$.

1. $\Phi_{UB}^{\mathcal{N}} \leq \Phi_{LB}$, and since so $\Phi^{SDP,\mathcal{N}} \leq \Phi_{UB}^{\mathcal{N}}$, we have $\Phi^{SDP,\mathcal{N}} \leq \Phi_{LB}$.
2. $\Phi_{UB}^{\mathcal{N}} > \Phi_{LB}$ and $|L_1^{\mathcal{N}}| = K$ or $|L_*^{\mathcal{N}}| = 0$. By Proposition 3.1 part 2, $\Phi^{SDP,\mathcal{N}} = \Phi_{LB}^{\mathcal{N}}$, and so $\Phi^{SDP,\mathcal{N}} \leq \Phi_{LB}$.
3. $\Phi_{UB}^{\mathcal{N}} > \Phi_{LB}$ and $\Phi_{UB}^{\mathcal{N}} - \Phi_{LB}^{\mathcal{N}} \leq \epsilon$. Thus, $0 < \Phi_{UB}^{\mathcal{N}} - \Phi_{LB} \leq \epsilon$. Furthermore, since $\Phi^{SDP,\mathcal{N}} \leq \Phi_{UB}^{\mathcal{N}}$, we have $\Phi^{SDP,\mathcal{N}} \leq \Phi_{LB} + \epsilon$

For the second claim if $\Phi_{UB}^{\mathcal{N}} - \Phi_{LB} \leq \epsilon$ for all $\mathcal{N} \in \mathcal{F}^{OPT}$, then global ϵ-optimality of x_{LB} with respect to (2.9), $\mathcal{W} = \mathcal{W}^{AC}$ is also established. Otherwise, for the third claim, Algorithm 3.1 there is a nonempty list of solutions $x^{\mathcal{N}}$ with $\mathcal{N} \in \mathcal{F}^{OPT}$ and $\Phi_{UB}^{\mathcal{N}} \geq \Phi_{LB}$ since Φ_{LB} corresponds to one of the $\Phi_{LB}^{\mathcal{N}}$ by construction, and at least one of these solutions is globally optimal for the bilevel problem (2.9), $\mathcal{W} = \mathcal{W}^{AC}$. ∎

The ability to verify the global optimality $\bar{\Psi}^{AC,\mathcal{N}} = \Psi^{AC,\mathcal{N}}$ is sufficient for resolving the uncertainty about which of the said candidate solutions are globally optimal.

4. Computational Experiments.

In this section, we present the results of two sets of computational experiments on the test instances IEEE 30-, 57-, 118-, and 300-bus test systems [38]. (We additionally applied experiments with larger Pegase 1354-, and 2869-bus system instances, and we defer discussion of these experiments to the conclusion.)

Significant parameters associated with each test instance are summarized in Table 1. We formulate the optimization models using our Julia Package MaximinOPF.jl [17] built on top of PowerModels.jl [14] and JuMP [21] modeling interface. We use Julia version 1.4 [6].

TABLE 1
Data associated with each of the test problems.

Number of Components by Type		IEEE Case				Pegase Case	
Type	Index Set	30	57	118	300	1354	2869
Buses	N	30	57	118	300	1354	2869
Generators	G	6	7	54	69	260	510
Loads (fixed)		20	42	99	201	673	1491
Shunts		2	3	14	29	1082	2197
Branches	L	41	80	186	411	1991	4582
Transformers		0	17	9	107	234	496

4.1. First set of experiments for comparing reliability of lower level model relaxations.

In the first set of experiments, we simply solve instances of the PSVA min level defender problem (2.3) over $\mathcal{W} \in \{\mathcal{W}^{AC}, \mathcal{W}^{SDP}, \mathcal{W}^{SOC}\}$ for the smaller IEEE Case 30 and IEEE Case 57 instances over all enumerated line attacks $x \in \{0,1\}^{|L|}$ under attack budget $K = 3$. For the $\mathcal{W} = \mathcal{W}^{AC}$ instance, we use the open source interior point solver Ipopt [49] with HSL linear solver ma57 [23]. For the $\mathcal{W} = \mathcal{W}^{SDP}$ and $\mathcal{W}^{SOC}$ tests, we use the commercially available Mosek [35]. Computations for the first set of experiments are carried out on a workstation with dual socket Intel Xeon Gold 6140 CPUs, 512 GB RAM, and a total of 36 physical cores.

We denote specific solutions x with superscripts to indicate the inactive line indices l ($x_l = 1$). For example, using IEEE Case 30 with $|L| = 41$ lines, then the specific x solution with $x_{10} = 1$, $x_{40} = 1$ and all other $x_l = 0$ would be denoted $x^{[10,40]}$. When the use of x notation is not required, we simply describe the attack with attacked lines bracketed, for example, $[10, 40]$.

TABLE 2
Tabulating enumerated contingencies with significant discrepancies between relaxations

IEEE Case 30, $K = 1$				IEEE Case 30, $K = 3$			
lines cut	ϕ^{AC}	ϕ^{SDP}	ϕ^{SOC}	lines cut	ϕ^{AC}	ϕ^{SDP}	ϕ^{SOC}
[16]	0.013	0.009	0	[5, 6, 7]	0.117	0.108	0
[25]	0.003	0.003	0	[6, 7, 8]	0.127	0.118	0
[36]	0.07	0.07	0	[11, 15, 16]	0.162	0.162	0
				[14, 15, 16]	0.162	0.162	0
IEEE Case 57, $K = 1$				IEEE Case 57, $K = 1$			
lines cut	ϕ^{AC}	ϕ^{SDP}	ϕ^{SOC}	lines cut	ϕ^{AC}	ϕ^{SDP}	ϕ^{SOC}
[29]	0.002	0	0	[35]	0.003	0	0
[36]	0.001	0	0	[37]	0.019	0.003	0
[38]	0.022	0.003	0	[39]	0.089	0.057	0
[51]	0.006	0	0	[52]	0.014	0.001	0
[55]	0.025	0.007	0	[60]	0.104	0	0
[61]	0.006	0	0	[65]	0.025	0	0
[66]	0	0	0	[79]	0.009	0	0
IEEE Case 57, $K = 2$				IEEE Case 57, $K = 2$			
lines cut	ϕ^{AC}	ϕ^{SDP}	ϕ^{SOC}	lines cut	ϕ^{AC}	ϕ^{SDP}	ϕ^{SOC}
[29, 79]	0.036	0.003	0	[37, 60]	0.218	0.032	0
[38, 60]	0.267	0.054	0	[38, 61]	0.103	0.007	0
[38, 79]	0.12	0.031	0	[39, 61]	0.171	0.066	0
[39, 65]	0.128	0.069	0	[60, 65]	0.275	0.036	0
[60, 66]	0.597	0.155	0	[61, 65]	0.109	0	0
[61, 66]	0.194	0.014	0				
IEEE Case 57, $K = 3$				IEEE Case 57, $K = 3$			
lines cut	ϕ^{AC}	ϕ^{SDP}	ϕ^{SOC}	lines cut	ϕ^{AC}	ϕ^{SDP}	ϕ^{SOC}
[8, 15, 18]	0.350	0.350	0	[8, 23, 63]	0.185	0.085	0
[8, 37, 79]	0.102	0.032	0	[8, 58, 79]	0.129	0.052	0
[9, 38, 79]	0.125	0.028	0	[10, 38, 79]	0.138	0.041	0
[10, 58, 79]	0.121	0.044	0	[10, 59, 66]	0.131	0.098	0
[10, 72, 79]	0.123	0.048	0	[10, 78, 79]	0.108	0.010	0
[13, 15, 18]	0.256	0.044	0	[14, 25, 28]	0.124	0.082	0
[15, 18, 25]	0.443	0.408	0	[15, 18, 39]	0.155	0.092	0
[15, 60, 66]	0.602	0.209	0	[17, 78, 79]	0.116	0.007	0
[22, 58, 79]	0.154	0.048	0	[22, 72, 79]	0.156	0.052	0
[22, 78, 79]	0.119	0.016	0	[38, 55, 79]	0.201	0.075	0

In Table 2, we present noteworthy examples of solutions that are inaccurately undervalued in attack value due to the use of lower level defender problem convex relaxations. Observations from Table 2 are as follows.

1. For IEEE Case 30, $K = 3$ entries, we see how the computed attack value ϕ^{SOC} can inaccurately take value zero while the corresponding ϕ^{AC} and ϕ^{SDP} values are nonzero and substantial.
2. For IEEE Case 57, $K = 1$, several single-line cut attacks are noted for which

their computed $\bar{\phi}^{AC}$ value is positive and substantial, but at least one of their ϕ^{SDP} and ϕ^{SOC} values is inaccurately valued at zero. Of course, it may be that $\bar{\phi}^{AC} \geq \phi^{AC}$ for any input (i.e., merely an optimal value in a local sense).

(a) But then, the inclusion of these lines persists in substantial attacks with larger budgets $K = 2, 3$ with respect to both $\bar{\phi}^{AC}$ and ϕ^{SDP} value, even when the ϕ^{SDP} values for the corresponding single-line $K = 1$ attacks were zero. This observation is evident, for example, with the attacks $[29, 79]$, $[60, 65]$, and $[60, 66]$.

(b) Shortly, in the discussion of the second set of experiments, we even see the cutting of lines 60 and 65 appearing in the optimal attack and in many of the candidate optimal attacks for the attack budget $K = 4$ with respect to Φ^{AC}, Φ^{SDP}, and Φ^{SOC}.

3. Some of the most notable attacks that are mis-evaluated due to the use of ϕ^{SOC} appear in the IEEE Case 57 attacks $[8, 15, 18]$, $[8, 23, 63]$,$[13, 15, 18]$, $[15, 60, 66]$, and $[38, 55, 79]$. It is noteworthy that the optimal attack realizing the bilevel optimal values even for Φ^{SDP} and Φ^{SOC} with $K = 5$ is $[8, 15, 16, 17, 18]$; this observation suggests that evaluating low budget (here $K = 3$) attacks accurately can provide a better sense of which attacks can be critical with larger budgets (here $K = 5$).
4. Many attacks involving lines $l = 8, 9, 13, 15, 29, 60, 65, 66$, and 79 register substantial nonzero values of ϕ^{AC} and/or ϕ^{SDP} with small budgets $K = 1, 2, 3$, but zero value with respect to ϕ^{SOC}. While these particular lines may become part of substantial attacks even with respect to ϕ^{SOC} for larger budgets, the criticality of these lines can be detected in smaller budget attacks when the lower level defender problem value is obtained either without relaxing the AC power flow equations (i.e., evaluating ϕ^{AC}), or at least using tighter convex relaxations (e.g., evaluating ϕ^{SDP}). This observation can be critical in future research on branch-and-bound branching rules.

The two main conclusions we draw from this first set of experiments are as follows. First, we note a substantial number of attacks that are of significant ϕ^{AC} value, but that are incorrectly valued at zero with respect to ϕ^{SOC} and sometimes even with respect to ϕ^{SDP}. These attacks correspond to power system vulnerabilities that would be missed with any solution approach relying on such relaxations of the PSVA lower level defender subproblem. Second, we note that the computed values of ϕ^{AC} that are nonzero for small K—even if we have not verified global optimality—but which have ϕ^{SOC} and/or ϕ^{SDP} value zero, seem to predict the significant role these line cuts can have for larger budgets with respect to all of ϕ^{AC}, ϕ^{SDP}, and ϕ^{SOC}.

Next, we compare the application of Algorithm 3.1 to problem (2.9) with $\mathcal{W} = \mathcal{W}^{AC}$, referred to as ACBnB, with a baseline mixed-integer SOC relaxation-based approach [51] applied to problem (2.9) with $\mathcal{W} = \mathcal{W}^{SOC}$, referred to as MISOC.

4.2. Numerical experiments for comparing ACBnB with MISOC. We apply ACBnB and MISOC to all of the test instances mentioned at the beginning of this section. For each test instance, we apply ACBnB and MISOC with budgets $K \in \{1, \ldots, 16\}$.

For solving the ACBnB upper bound generating instances of the node subproblems (3.2), we use the open source interior point solver Ipopt [49] with HSL linear solver ma57 [23]. For solving the ACBnB lower bounding instances of the PSVA min level defender subproblem (2.3) with $\mathcal{W} = \mathcal{W}^{SDP}$, we use the commercial Mosek [35] solver with academic license. These same SDP lower bounding node subproblems are formulated with chordal decomposition [33, 30, 55, 56] automatically through the PowerModels [14] interface on which our algorithm implementation is built.

For the MISOC experiments, we also apply the Mosek as a mixed-integer SOC solver to the Lagrangian dual single level reformulation of problem (2.9) with $\mathcal{W} = \mathcal{W}^{SOC}$ in line with [51].

Each of these computational tests are carried out on a single core of a single node of the Argonne National Laboratory Bebop cluster, each consisting of an Intel® Xeon® CPU E5-2695 v4 @ 2.10 GHz processor. We use the GNU `Parallel` utility [47] in submitting jobs in parallel using one core per job. Appropriate parameters for enforcing the use of one thread are set explicitly in the instantiation of the Mosek solver object. The methods were run with the time limit of 24 wall-clock hours (86,400 seconds).

Parameter settings for the use of Ipopt and Mosek are nearly default. For Ipopt, the only nondefault parameter settings are for specifying the use of the ma57 linear solver and for setting the maximum number of iterations to 50,000. For Mosek, the only nondefault parameter setting is for specifying the use of one thread.

When Ipopt is applied to nonconvex subproblems, the initial point is generated internally through the PowerModels.jl interface. Resolves with random initial bus voltage solutions are only applied in two cases where global optimality is obviously *not* realized, (1) solver status indicating numerical error (happened, but rare) or (2) due to the computed optimal value exceeding a known upper bound. In generating the random voltage settings, we always set for each bus $i \in N$ initial value $v_i^R = 1$ and randomly selected $v_i^I \in \{-0.5, 0, 0.5\}$ with equal probability, and then we use the initial bus voltage value obtained after rescaling so that $\|[v_i^R, v_i^I]\|_2 = 1$.

For the baseline experiments using the approach in [51] (MISOC) and for the experiments using Algorithm 3.1 (ACBnB), we report the following in Table 3–Table 5:

1. number of nodes (Nodes) processed at termination;
2. average time (in seconds) spent solving each node (Secs/N);
3. total running time for solving the instance, in seconds (Secs); experiments for which the maximum of 86,400 seconds was reached are marked with 'T' in this entry;
4. either an inteval $[LB, UB]$ in which the optimal value is known to be located, or in the case that the interval reduces to a singleton, a bracketed optimal value $[OPT]$.

For the Table 3 experiments, we make the following observations and conclusions:

1. As we shall see in both Table 3 and Table 5, the advantage in terms of less computational time of using MISOC over ACBnB is always present. In terms of number of node evaluations, MISOC usually requires fewer nodes to reach termination.
2. For IEEE Case 30, the optimal attacks and values found using the MISOC and ACBnB approach do not differ, and the consequence of relaxating the PSVA min level AC power flow equations is not significant for this instance.
3. However, for IEEE Case 57 with $K = 1, \ldots, 5$, the consequences of this relaxation are more apparent:
   (a) The optimal value for MISOC underestimates the optimal value interval for ACBnB. This is in line with the observations from the first set of experiments tabulated in Table 2.
   (b) The optimal solutions with respect to Φ^{SDP} computed in the ACBnB experiments match with the optimal solutions with respect to Φ^{SOC} computed in the MISOC experiments.

Table 3
Comparing MISOC and ACBnB for the smaller test instances (IEEE 30-, 57-bus systems with $K = 1, \ldots, 16$)

		MISOC				ACBnB			
Case	K	Nodes	Secs/N	Secs	[LB,UB]	Nodes	Secs/N	Secs	[LB,UB]
30	1	33	0.06	2	[0.153]	57	1.4	80	[0.153]
	2	21	0.05	1	[0.600]	39	1.97	77	[0.600]
	3	149	0.03	5	[0.658]	275	0.4	110	[0.658]
	4	285	0.04	10	[0.937]	821	0.23	186	[0.937]
	5	859	0.03	28	[0.995]	3191	0.17	543	[0.995]
	6	1195	0.03	38	[1.224]	4581	0.17	768	[1.253]
	7	977	0.03	31	[1.510]	4443	0.17	764	[1.510]
	8	347	0.03	12	[1.939]	1429	0.18	264	[1.939]
	9	171	0.04	7	[2.158]	783	0.25	192	[2.158]
	10	367	0.04	13	[2.216]	2399	0.19	444	[2.216]
	11	765	0.03	23	[2.316]	2927	0.19	564	[2.316]
	12	455	0.03	13	[2.374]	3945	0.2	785	[2.374]
	13	965	0.03	30	[2.407]	5461	0.18	974	[2.407]
	14	653	0.03	19	[2.528]	1467	0.2	296	[2.528]
	15	1027	0.03	31	[2.561]	845	0.24	200	[2.561]
	16	563	0.03	19	[2.572]	755	0.35	262	[2.572]
57	1	47	0.06	3	[0.343]	71	1.3	92	[0.368, 0.398]
	2	223	0.05	12	[0.760]	411	0.53	217	[0.777, 0.812]
	3	967	0.05	47	[1.018]	2073	0.45	934	[1.024, 1.233]
	4	1203	0.06	69	[1.579]	2923	0.46	1330	[1.600, 1.821]
	5	1341	0.05	71	[2.419]	2493	0.45	1122	[2.420, 2.462]
	6	331	0.07	23	[3.601]	493	0.54	264	[3.601]
	7	203	0.07	15	[4.092]	637	0.49	309	[4.092]
	8	503	0.07	34	[4.218]	1823	0.42	768	[4.218]
	9	905	0.06	58	[4.613]	1977	0.43	857	[4.613]
	10	1559	0.06	100	[4.739]	7127	0.41	2903	[4.739]
	11	1091	0.07	71	[5.052]	5681	0.43	2470	[5.052]
	12	2137	0.06	137	[5.177]	9677	0.38	3723	[5.177]
	13	653	0.06	41	[5.436]	4919	0.43	2092	[5.436, 5.437]
	14	665	0.06	41	[5.579]	5683	0.4	2274	[5.579]
	15	513	0.06	30	[5.816]	2327	0.4	942	[5.816, 5.817]
	16	365	0.05	19	[5.959]	2145	0.41	876	[5.959]

(c) However, the ACBnB experiments also provide a list of candidate optimal solutions with respect to Φ^{AC} that are distinct from those with respect to either Φ^{SDP} or Φ^{SOC}, many of which, contingent on verifying global optimality, can have substantially larger value than the reported Φ^{SDP} and Φ^{SOC} values. Furthermore, the line cuts that recur in many of these candidate solutions for $K = 3$ even constitute the optimal $K = 4$ attack with respect to Φ^{SDP} and Φ^{SOC}. See Table 4.

(d) For the $K = 5$ rows of Table 4, it is noteworthy that the optimal solution with respect to Φ^{SDP} and Φ^{SOC} is substantially different from the $K = 1, \ldots, 4$ optimal solutions. There is also a substantial jump in corresponding optimal value from $K = 4$ to $K = 5$. While noteworthy, this apparent absence of "nesting" observed in the comparison of the $K = 4$ and $K = 5$ optimal solution is not unusual in itself. What is even more noteworthy is an observation from the first set of experiments, where $\phi^{AC}(x^{[8,15,18]}) = \phi^{SDP}(x^{[8,15,18]}) = 0.35$ but $\phi^{SOC}(x^{[8,15,18]}) = 0$, and $[8, 15, 18]$ nests within the optimal $K = 5$ attack. From this, we see that avoiding relaxation or at least using tighter SDP relaxation of the PSVA lower level problem can provide substantial hints of power system vulnerabilities that are missed with the use of the weaker SOC relaxation.

4. It is noteworthy that with the larger attack budgets $K = 6, \ldots, 16$, the consequences of relaxing the AC power flow equations for IEEE Case 57 diminish, where the attack values and attack solutions are the same between MISOC and ACBnB.

TABLE 4
Case 57 comparison of MISOC and ACBnB solutions.

	MISOC		ACBnB		
K	opt attack	Φ^{SOC}	opt attack	Φ^{SDP}	Φ^{AC}
3	$[33, 41, 80]$	1.018	$[33, 41, 80]$	1.024	$[1.024, 1.033]$
			$[60, 66, 72]$	0.797	$[0.797, 1.075]$
			$[41, 60, 66]$	0.782	$[0.782, 1.233]$
			$[41, 60, 80]$	0.843	$[0.843, 1.075]$
			$[60, 65, 66]$	0.797	$[0.797, 1.147]$
			$[58, 60, 66]$	0.792	$[0.792, 1.072]$
4	$[60, 65, 66, 72]$	1.579	$[60, 65, 66, 72]$	1.600	$[1.600, 1.752]$
			$[57, 60, 65, 66]$	1.464	$[1.464, 1.611]$
			$[58, 60, 65, 66]$	1.595	$[1.595, 1.748]$
			$[41, 60, 65, 66]$	1.470	$[1.470, 1.821]$
			$[41, 57, 60, 66]$	1.391	$[1.391, 1.626]$
			$[41, 58, 60, 66]$	1.521	$[1.521, 1.766]$
			$[41, 60, 66, 80]$	1.174	$[1.174, 1.638]$
			$[58, 59, 65, 66]$	1.573	$[1.573, 1.632]$
			$[59, 65, 66, 72]$	1.579	$[1.579, 1.643]$
			$[41, 60, 66, 72]$	1.527	$[1.527, 1.769]$
5	$[8, 15, 16, 17, 18]$	2.419	$[8, 15, 16, 17, 18]$	2.420	$[2.420]$
			$[41, 58, 60, 65, 66]$		$[2.326, 2.459]$
			$[41, 60, 65, 66, 72]$		$[2.332, 2.462]$

We make the following observations for the IEEE Case 118 and Case 300 experiments.

Case 118 For the ACBnB experiments that terminated before the time limit ($K = 1, \ldots, 7$), the optimal values and solutions were the same as computed for the MISOC experiments. This suggests that the relaxation of the PSVA min level defender problem AC power flow equations may not be significant for Case 118. Furthmore, the ACBnB incumbent SDP values for the $K = 8, 9, 10$ experiments were the same as the optimal values for the corresponding MISOC experiments.

Case 300 For the budgets $K = 2, 6, 7$, we have not only different optimal values, but also different optimal solutions. Furthermore, for $K = 3$, we have a candidate optimal solution with respect to Φ^{AC} that is different. See Table 6.

5. Conclusions. We make contributions toward the problem of power system vulnerability analysis (PSVA) for identifying the most substantial contingencies of non-relaxed AC power flow networks. The problem is modeled as a bilevel maximin problem, where the lower-level problem seeks to optimally respond to a parameterized attack, which is a decision variable in the upper-level attacker's maximization problem seeking to inflict a maximal amount of damage in spite of the defender's optimal recourse. First, we established concavity and continuity of the optimal *value function* for the lower level defender problem as a function of the parameterized attack, which suggests the existence of an effective branch-and-bound approach based on a non-relaxed formulation of the lower level defender problem. In addressing the orig-

TABLE 5
Comparing MISOC and ACBnB for the medium size test instances (IEEE 118-, 300-bus systems with $K = 1, \dots, 16$)

Case	K	MISOC Nodes	MISOC Secs/N	MISOC Secs	MISOC [LB,UB]	ACBnB Nodes	ACBnB Secs/N	ACBnB Secs	ACBnB [LB,UB]
118	1	5	0.8	4	[0.840]	15	5.4	81	[0.840]
	2	19	0.37	7	[1.448]	81	1.56	126	[1.448]
	3	7	0.71	5	[2.288]	81	1.53	124	[2.288]
	4	171	0.19	33	[2.568]	819	0.79	644	[2.568]
	5	83	0.3	25	[3.018]	2403	0.74	1777	[3.018]
	6	857	0.17	142	[3.298]	13691	0.71	9694	[3.298]
	7	869	0.17	144	[3.697]	40227	0.73	29523	[3.697]
	8	2335	0.21	479	[3.977]	142197	0.61	T	[3.977, 4.216]
	9	2354	0.2	460	[4.427]	151459	0.57	T	[4.427, 4.962]
	10	1399	0.21	294	[4.859]	158367	0.55	T	[4.859, 5.660]
	11	1921	0.21	396	[5.156]	160397	0.54	T	[4.019, 6.343]
	12	3081	0.22	683	[5.439]	153061	0.56	T	[4.316, 6.967]
	13	3189	0.23	721	[5.783]	153123	0.56	T	[4.316, 7.548]
	14	2437	0.19	466	[6.268]	158481	0.55	T	[4.103, 8.102]
	15	3831	0.22	850	[6.565]	158827	0.54	T	[4.019, 8.696]
	16	2385	0.22	528	[6.848]	154891	0.56	T	[2.792, 9.259]
300	1	25	0.68	17	[8.047]	41	7.68	315	[8.047]
	2	203	0.44	89	[12.571]	303	5.25	1590	[12.913, 13.145]
	3	731	0.43	314	[17.532]	1563	5.02	7848	[17.587, 17.668]
	4	1495	0.51	758	[23.772]	3929	4.38	17194	[23.790, 23.797]
	5	2455	0.55	1339	[30.455]	8173	3.86	31508	[30.454, 30.455]
	6	7327	0.53	3857	[35.185]	24871	3.47	T	[35.321, 36.291]
	7	16639	0.55	9109	[39.940]	29145	2.96	T	[40.052, 46.484]
	8	6001	0.55	3294	[46.180]	31303	2.76	T	[44.717, 55.653]
	9	14577	0.55	7946	[50.911]	30759	2.81	T	[49.390, 64.588]
	10	36493	0.55	19899	[55.572]	30531	2.83	T	[49.390, 73.566]
	11	76845	0.54	41505	[60.303]	30573	2.83	T	[49.390, 82.133]
	12	132837	0.54	72240	[64.758]	30603	2.82	T	[52.751, 90.28]
	13	135191	0.54	72857	[70.407]	30749	2.81	T	[49.500, 98.271]
	14	154903	0.56	T	[73.835, 82.422]	30303	2.85	T	[46.783, 106.353]
	15	161884	0.53	T	[79.800, 88.091]	30675	2.82	T	[39.990, 114.100]
	16	153309	0.56	T	[83.962, 95.400]	31005	2.79	T	[42.010, 121.634]

TABLE 6
Case 300 comparison of MISOC and ACBnB solutions.

K	MISOC opt attack	MISOC Φ^{SOC}	ACBnB opt attack	ACBnB Φ^{SDP}	ACBnB Φ^{AC}
2	[208, 316]	12.571	[181, 208]	12.913	[12.913, 13.415]
3	[177, 181, 208]	17.532	[177, 181, 208]	17.587	[17.587, 17.592]
			[181, 316, 208]	17.437	[17.437, 17.668]
4	[177, 181, 182, 208]	23.772	[177, 181, 182, 208]	23.79	[23.790, 23.797]
5	[208, 268, 305, 308, 309]	30.455	[208, 268, 305, 308, 309]	30.454	[30.454, 30.455]
6	[208, 266, 268, 305, 309, 317]	35.185	[181, 208, 268, 305, 308, 309]	35.321	[35.321, 35.553]
7	[177, 181, 208, 268, 305, 308, 309]	39.94	[181, 208, 266, 268, 305, 309, 317]	40.052	[40.052, 46.484]

inal nonconvex lower level defender problem, we cannot obtain an equivalent single level reformulation based on the KKT conditions or Lagrangian dual reformulation of the lower level problem. Otherwise, if we could obtain such an equivalent single level reformulation, then we could apply standard branch-and-bound ideas based on the upper bounds (in maximization context) due to the relaxation of integrality constraints. We find an alternative upper bounding problem by solving the problem obtained by switching max and min. Such a problem is reformulated in a tractable form while preserving the nonlinear nonconvex AC power flow equations of the lower level defender problem. We show conditions under which the maxmin problem has equal value to the minmax upper bounding relaxation that informs a well-defined branch-and-bound algorithm which we implement and apply in comparison with a previous approach based on the convex relaxation of the lower level defender problem. For lower bounding, we solve instances of the semidefinite program (SDP) relaxed lower level defender problem. Such lower and upper bounds help to either verify global optimality of the AC maxmin problem, or otherwise provide a well-vetted list of candidate global optimal solutions. We also demonstrate in standard test cases power system vulnerabilities that are identified due to the use of the non-relaxed lower level defender problem that are not identified with the use of convex relaxations of the lower lever defender problem.

Our paper opens a number of interesting future research directions. First, we have partially addressed the validation of the PSVA lower level defender problem global optimality using lower bounds to solving the semidefinite programming (SDP) relaxation of the lower level problem. But we have not fully addressed this issue of verifying global optimality, which is an important and active area of research [31, 34, 26, 12, 11] that needs to be incorporated into our contribution.

Second, the various areas for improvement in our contributed approach are motivated by our initial experience applying the algorithm to the larger Pegase 1354 and 2869 instances, which we did not tabulate in Section 4. In particular, for these two instances, the computational burden at each node (as would be given in the Secs/N column) due to solving the upper and lower bounding subproblems became significant, and so not many branch-and-bound tree nodes were processed withing the allotted 24 hours. Improvements can thus be realized by reducing the required number of nodes to process through improved branching rules. We gave a simple rule based on dual solutions for constraints specific to the upper bounding problem. (Rules based on integrality violation were not applicable in our approach.) Analogs to strong and pseudo-branching may reduce the number of nodes to process. Further improvements can of course be obtained through improvement in solution technology, and especially technology that improves the utilization of sparse network structures for efficient parallelization. Furthermore, future application of tree-wide parallelization in branch-and-bound that efficiently addresses the issues of load-balancing [52] may also yield computational improvement.

Lastly, other features and contexts of PSVA need to be incorporated into our contribution, such as the use of additional techniques for verifying global optimality of nonconvex node subproblems, use of heuristic and metaheuristic approaches [10, 3, 29] for nonconvex node subproblems, the assumption of probabilistic line failure [46] adaptation for unit commitment applications [45, 13], allowing for the defender to use line de-activation as a defensive recourse [53], use of trilevel defender-attacker-defender [1, 18, 27] frameworks.

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