

# Dual Sufficient Characterizations of Transversality Properties

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**Abstract** This paper continues the study of ‘good arrangements’ of collections of sets near a point in their intersection. Our aim is to develop a general scheme for quantitative analysis of several *transversality* properties within the same framework. We consider a general nonlinear setting and establish dual (subdifferential and normal cone) sufficient characterizations of transversality properties of collections of sets in Banach/Asplund spaces. Besides quantitative estimates for the rates/moduli of the corresponding properties, we establish here also estimates for the other parameters involved in the definitions, particularly the size of the neighbourhood where a property holds. Interpretations of the main general nonlinear characterizations for the case of Hölder transversality are provided. Some characterizations are new even in the linear setting. As an application, we provide dual sufficient conditions for nonlinear extensions of the new transversality properties of a set-valued mapping to a set in the range space due to Ioffe.

**Keywords** Transversality · Subtransversality · Semitransversality · Regularity · Subregularity · Semiregularity · Sum rule · Chain rule

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## 1 Introduction

In this paper, we continue our study of ‘good arrangements’ of collections of sets in normed vector spaces near a point in their intersection, known as *transversality (regularity)* properties and playing an important role in optimization and variational analysis, e.g., as constraint qualifications in optimality conditions, and qualification conditions in subdifferential, normal cone and coderivative calculus, and convergence analysis of computational algorithms [1–5, 7, 11, 16, 20, 23–25, 27–37, 39–41, 43, 44, 50, 51]. Following Ioffe [20], such arrangements are now commonly referred to as *transversality* properties. Here we refer to transversality broadly as a group of ‘good arrangement’ properties, which includes *semitransversality*, *subtransversality*, *transversality* (a specific property) and some others. The term *regularity* was extensively used for the same purpose in the earlier publications by the second author, and is still preferred by many other authors. For a survey of applications of transversality properties in convergence analysis of projection algorithms we refer the readers to [31].

The three mentioned above transversality properties of collections of sets are direct counterparts of the (metric) *semiregularity*, *subregularity* and *regularity* properties of set-valued mappings, respectively; see [25, 34]. For set-valued mappings, up until recently, the ubiquitous (metric) regularity property has dominated the scene as

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the key property ensuring ‘good’ behaviour of parameters of (optimization and other) problems under consideration. Many primal and dual necessary and sufficient characterizations of this property have been developed; see [15, 20, 21, 38]. It was realized relatively recently that for many applications the weaker subregularity property was still sufficient for the job, and the property itself attracted significant attention of researchers; see [15, 20]. Unlike its two well-established siblings, the semiregularity property has only started attracting attention of researchers thanks to its importance, e.g., in the convergence analysis of inexact Newton-type schemes for generalized equations; see [8, 25]. Being also weaker than (full) regularity, it is in general incomparable with subregularity.

For models involving collections of sets, the historical sequence has been different. Starting with the work of Bauschke and Borwein [3] on projection algorithms, the scene has been strongly dominated by the subtransversality property (also known under many other names; cf. [30, 31]), although the property itself had been used much earlier [14, 19]. The stronger transversality property (known under other names too) came into play later when it was realized that in the nonconvex setting subtransversality is not sufficient to ensure (local) linear convergence of alternating projections; cf. [36]. In general, the needs of nonconvex optimization have caused an expansion of the range of transversality properties under consideration. Apart from the three properties studied in the current paper, we want to mention the important *intrinsic transversality* being an intermediate property between subtransversality and transversality; cf. [16, 27]. Significant efforts have been invested into studying various transversality properties and establishing their primal and dual necessary and/or sufficient characterizations in various settings (convex and nonconvex, finite and infinite dimensional, finite and infinite collections of sets).

Up until recently, mostly ‘linear’ transversality properties have been studied, although it has been observed that such properties often fail in very simple situations, for instance, when it comes to convergence analysis of computational algorithms. The first attempts to consider nonlinear extensions of the transversality properties have been made recently in [11, 33, 40, 43]. Even in the linear setting, some dual sufficient conditions have been established only for subtransversality and transversality properties (cf. [23–25, 30–32, 34]).

Our aim in this paper, which continues [11], is to develop a general scheme for quantitative analysis of all three transversality properties within the same framework. We consider a general nonlinear setting. Primal space (geometric, metric and slope) quantitative characterizations of the properties in normed vector spaces have been established in [6, 11, 12]. Here we utilize the slope estimates from [11] to establish a full set of general dual (subdifferential and normal cone) quantitative sufficient characterizations of the nonlinear transversality properties of collections of sets in Banach/Asplund spaces. Dual space necessary characterizations of these properties are going to appear in [13].

Unlike our earlier publications [33, 34] (dedicated mostly to the subtransversality), besides quantitative estimates for the rates/moduli of the corresponding properties, we establish here also estimates for the other parameters involved in the definitions, particularly the size of the neighbourhood where a property holds, which is important from the computational point of view.

Interpretations of the main general nonlinear characterizations for the cases of Hölder transversality properties are provided. Some characterizations are new even in the linear setting.

When formulating dual sufficient characterizations, the underlying space is assumed Banach or Asplund, and the sets are assumed closed. In the setting of a general Banach space, dual characterizations are formulated in terms of Clarke normals and subdifferentials. When the space is Asplund, Fréchet normals and subdifferentials are used in the statements. The characterizations can be easily reformulated in terms of more general abstract subdifferentials possessing some natural properties (in particular, appropriate sum rules) in a reference space (*trustworthy* subdifferentials [18]).

The rest of the paper is organized as follows. Section 2 introduces notation and presents some basic facts from variational analysis and generalized differentiation used in the formulations and proofs of the results. Among other things, we provide new chain rules for Fréchet and Clarke subdifferentials which seem to improve the existing assertions of this type. In Section 3, we briefly discuss the three types of non-

linear transversality properties of finite collections of sets in normed vector spaces, and recall their slope characterizations from [11]. Sections 4–6 are dedicated to dual characterizations of the nonlinear semitransversality, subtransversality and transversality, respectively, in terms of Clarke (in general Banach spaces) or Fréchet (in Asplund spaces) subdifferentials and normals. In Section 7, we consider the most important realizations of the general nonlinear transversality properties corresponding to the Hölder setting. As an application, we provide in Section 8 dual sufficient characterizations for nonlinear extensions of the new *transversality properties of a set-valued mapping to a set in the range space* due to Ioffe [20].

## 2 Notation and Preliminaries

Our basic notation is standard, see, e.g., [15, 38, 47]. Throughout the paper,  $X$  and  $Y$  are normed spaces. The topological dual of the space  $X$  is denoted by  $X^*$ , while  $\langle \cdot, \cdot \rangle$  denotes the bilinear form defining the pairing between the two spaces. The open unit balls in  $X$  and  $X^*$  are denoted by  $\mathbb{B}$  and  $\mathbb{B}^*$ , respectively, and  $B_\delta(x)$  stands for the open ball with center  $x$  and radius  $\delta > 0$ . If not explicitly stated otherwise, products of normed spaces are assumed equipped with the maximum norm  $\|(x, y)\| := \max\{\|x\|, \|y\|\}$ ,  $(x, y) \in X \times Y$ . Symbols  $\mathbb{R}$ ,  $\mathbb{R}_+$  and  $\mathbb{N}$  denote the real line (with the usual norm), the set of all nonnegative real number and the set of all positive integers, respectively.

For a set  $\Omega \subset X$ , its boundary is denoted by  $\text{bd}\Omega$ . The *indicator function*  $i_\Omega$  of  $\Omega$  is defined as follows:  $i_\Omega(x) = 0$  if  $x \in \Omega$  and  $i_\Omega(x) = +\infty$  if  $x \notin \Omega$ . The distance from a point  $x \in X$  to  $\Omega$  is defined by  $d(x, \Omega) := \inf_{u \in \Omega} \|u - x\|$ , and we use the convention  $d(x, \emptyset) = +\infty$ .

For an extended-real-valued function  $f : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$  on a normed space  $X$ , its domain and epigraph are defined, respectively, by  $\text{dom } f := \{x \in X \mid f(x) < +\infty\}$  and  $\text{epi } f := \{(x, \alpha) \in X \times \mathbb{R} \mid f(x) \leq \alpha\}$ . The inverse of  $f$  (if it exists) is denoted by  $f^{-1}$ . Note that  $f$  is allowed to take the value  $-\infty$ . This convention is needed only to accommodate for the general chain rule in Proposition 2.1.

A set-valued mapping  $F : X \rightrightarrows Y$  between two sets  $X$  and  $Y$  is a mapping, which assigns to every  $x \in X$  a subset (possibly empty)  $F(x)$  of  $Y$ . We use the notations  $\text{gph } F := \{(x, y) \in X \times Y \mid y \in F(x)\}$  and  $\text{dom } F := \{x \in X \mid F(x) \neq \emptyset\}$  for the graph and the domain of  $F$ , respectively, and  $F^{-1} : Y \rightrightarrows X$  for the inverse of  $F$ . This inverse (which always exists with possibly empty values at some  $y$ ) is defined by  $F^{-1}(y) := \{x \in X \mid y \in F(x)\}$ ,  $y \in Y$ . Obviously  $\text{dom } F^{-1} = F(X)$ .

Dual characterizations of transversality properties require dual tools: normal cones and subdifferentials. Given a subset  $\Omega$  of a normed space  $X$  and a point  $\bar{x} \in \Omega$ , the sets (cf. [10, 22])

$$N_\Omega^F(\bar{x}) := \left\{ x^* \in X^* \mid \limsup_{\Omega \ni x \rightarrow \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}, \quad (1)$$

$$N_\Omega^C(\bar{x}) := \{x^* \in X^* \mid \langle x^*, z \rangle \leq 0 \text{ for all } z \in T_\Omega^C(\bar{x})\} \quad (2)$$

are the *Fréchet* and *Clarke normal cones* to  $\Omega$  at  $\bar{x}$ . In the last definition,  $T_\Omega^C(\bar{x})$  stands for the *Clarke tangent cone* [10] to  $\Omega$  at  $\bar{x}$ . The sets (1) and (2) are nonempty closed convex cones satisfying  $N_\Omega^F(\bar{x}) \subset N_\Omega^C(\bar{x})$ . If  $\Omega$  is a convex set, they reduce to the normal cone in the sense of convex analysis:

$$N_\Omega(\bar{x}) := \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq 0 \text{ for all } x \in \Omega\}.$$

Given a function  $f : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$  and a point  $\bar{x} \in X$  with  $|f(\bar{x})| < +\infty$ , the *Fréchet* and *Clarke subdifferentials* of  $f$  at  $\bar{x}$  are defined as (cf. [10, 22])

$$\partial^F f(\bar{x}) := \left\{ x^* \in X^* \mid \liminf_{x \rightarrow \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}, \quad (3)$$

$$\partial^C f(\bar{x}) := \{x^* \in X^* \mid \langle x^*, z \rangle \leq f^\circ(\bar{x}, z) \text{ for all } z \in X\}, \quad (4)$$

where  $f^\circ(\bar{x}, z)$  is the *Clarke–Rockafellar directional derivative* [46] of  $f$  at  $\bar{x}$  in the direction  $z \in X$ :

$$f^\circ(\bar{x}; z) := \lim_{\varepsilon \downarrow 0} \limsup_{\substack{(x, \alpha) \rightarrow (\bar{x}, f(\bar{x})) \\ f(x) \leq \alpha, t \downarrow 0}} \inf_{\|z' - z\| < \varepsilon} \frac{f(x + tz') - \alpha}{t}.$$

The last definition admits simplifications if  $f$  is lower semicontinuous at  $\bar{x}$  and especially if it is Lipschitz continuous near  $\bar{x}$ ; cf. [46]. The sets (3) and (4) are closed and convex, and satisfy  $\partial^F f(\bar{x}) \subset \partial^C f(\bar{x})$ . If  $f$  is convex, they reduce to the subdifferential in the sense of convex analysis (cf., e.g., [10, 22]):

$$\partial f(\bar{x}) := \{x^* \in X^* \mid f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle \geq 0 \text{ for all } x \in X\}.$$

By convention, we set  $N_\Omega^F(\bar{x}) = N_\Omega^C(\bar{x}) := \emptyset$  if  $\bar{x} \notin \Omega$  and  $\partial^F f(\bar{x}) = \partial^C f(\bar{x}) := \emptyset$  if  $|f(\bar{x})| = +\infty$ . It is easy to check that  $N_\Omega^F(\bar{x}) = \partial^F i_\Omega(\bar{x})$  and  $N_\Omega^C(\bar{x}) = \partial^C i_\Omega(\bar{x})$ ; cf., e.g., [10, 22].

We often use the generic notations  $N$  and  $\partial$  for both Fréchet and Clarke objects, specifying wherever necessary that either  $N := N^F$  and  $\partial := \partial^F$ , or  $N := N^C$  and  $\partial := \partial^C$ .

The proofs of the main results in this paper rely on two kinds of subdifferential sum rules. Below we provide these rules for completeness.

**Lemma 2.1 (Subdifferential sum rules)** *Suppose  $X$  is a normed space,  $f_1, f_2 : X \rightarrow \mathbb{R} \cup \{+\infty\}$ , and  $\bar{x} \in \text{dom } f_1 \cap \text{dom } f_2$ .*

(i) **Clarke–Rockafellar sum rule.** *Suppose  $f_1$  is Lipschitz continuous and  $f_2$  is lower semicontinuous in a neighbourhood of  $\bar{x}$ . Then*

$$\partial^C(f_1 + f_2)(\bar{x}) \subset \partial^C f_1(\bar{x}) + \partial^C f_2(\bar{x}).$$

(ii) **Fuzzy sum rule.** *Suppose  $X$  is Asplund,  $f_1$  is Lipschitz continuous and  $f_2$  is lower semicontinuous in a neighbourhood of  $\bar{x}$ . Then, for any  $\varepsilon > 0$ , there exist  $x_1, x_2 \in X$  with  $\|x_i - \bar{x}\| < \varepsilon$ ,  $|f_i(x_i) - f_i(\bar{x})| < \varepsilon$  ( $i = 1, 2$ ), such that*

$$\partial^F(f_1 + f_2)(\bar{x}) \subset \partial^F f_1(x_1) + \partial^F f_2(x_2) + \varepsilon \mathbb{B}^*.$$

Both sum rules in the above lemma are valid in general only as inclusions. The first rule is formulated in terms of Clarke subdifferentials, and is an example of *exact* sum rule. It was established in Rockafellar [46, Theorem 2]. The second rule is known as the *fuzzy* or *approximate* sum rule (Fabian [17]) for Fréchet subdifferentials in Asplund spaces; cf., e.g., [22, Rule 2.2] and [38, Theorem 2.33]. Note that, unlike the sum rule in part (i) of the lemma, the subdifferentials in the right-hand side of the inclusion in part (ii) are computed not at the reference point, but at some points nearby. This explains the name.

Recall that a Banach space is *Asplund* if every continuous convex function on an open convex set is Fréchet differentiable on a dense subset [45], or equivalently, if the dual of each its separable subspace is separable. We refer the reader to [38, 45] for discussions about and characterizations of Asplund spaces. All reflexive, particularly, all finite dimensional Banach spaces are Asplund.

The following two facts are immediate consequences of the definitions of the Fréchet and Clarke subdifferentials and normal cones (cf., e.g., [10, 22, 38]).

**Lemma 2.2** *Suppose  $X$  is a normed space, and  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ . If  $\bar{x} \in \text{dom } f$  is a point of local minimum of  $f$ , then  $0 \in \partial^F f(\bar{x})$ .*

**Lemma 2.3** *Let  $\Omega_1, \Omega_2$  be subsets of a normed space  $X$ , and  $\omega_i \in \Omega_i$  ( $i = 1, 2$ ). Then  $N_{\Omega_1 \times \Omega_2}(\omega_1, \omega_2) = N_{\Omega_1}(\omega_1) \times N_{\Omega_2}(\omega_2)$ , where in both parts of the equality  $N$  stands for either the Fréchet ( $N := N^F$ ) or the Clarke ( $N := N^C$ ) normal cone.*

We are going to use a representation of the subdifferential of a special convex function on  $X^{n+1}$  given in the next lemma; cf. [30, Lemma 3].

**Lemma 2.4** *Let  $X$  be a normed space and*

$$\psi(u_1, \dots, u_n, u) := \max_{1 \leq i \leq n} \|u_i - a_i - u\|, \quad u_1, \dots, u_n, u \in X, \quad (5)$$

where  $a_i \in X$  ( $i = 1, \dots, n$ ). Let  $x_1, \dots, x_n, x \in X$  and  $\max_{1 \leq i \leq n} \|x_i - a_i - x\| > 0$ . Then

$$\begin{aligned} \partial \psi(x_1, \dots, x_n, x) = \left\{ (x_1^*, \dots, x_n^*, x^*) \in (X^*)^{n+1} \mid x^* + \sum_{i=1}^n x_i^* = 0, \right. \\ \left. \sum_{i=1}^n \|x_i^*\| = 1, \sum_{i=1}^n \langle x_i^*, x_i - a_i - x \rangle = \max_{1 \leq i \leq n} \|x_i - a_i - x\| \right\}. \quad (6) \end{aligned}$$

*Proof* The convex function  $\psi$  given by (5) is a composition of the continuous linear mapping  $A : X^{n+1} \rightarrow X^n$ :  $(x_1, \dots, x_n, x) \mapsto (x_1 - a_1 - x, \dots, x_n - a_n - x)$  and the maximum norm on  $X^n$ :  $g(x_1, \dots, x_n) := \max_{1 \leq i \leq n} \|x_i\|$ . The adjoint mapping  $A^*$  is in the form  $(x_1^*, \dots, x_n^*) \mapsto (x_1^*, \dots, x_n^*, -\sum_{i=1}^n x_i^*)$ , while (cf., e.g., [49])

$$\begin{aligned} \partial g(x_1, \dots, x_n) = \left\{ (x_1^*, \dots, x_n^*) \in X^* \mid \sum_{i=1}^n \|x_i^*\| = 1, \right. \\ \left. \langle (x_1^*, \dots, x_n^*), (x_1, \dots, x_n) \rangle = \max_{1 \leq i \leq n} \|x_i\| \right\}. \end{aligned}$$

The conclusion is a consequence of the convex chain rule (cf., e.g., [49]).  $\square$

- Remark 2.1* (i) It is easy to notice that in the representation (6), for any  $i = 1, \dots, n$ , either  $\langle x_i^*, x_i - a_i - x \rangle = \max_{1 \leq j \leq n} \|x_j - a_j - x\|$  or  $x_i^* = 0$ .  
(ii) The maximum norm on  $X^n$  used in (5) is a composition of the given norm on  $X$  and the maximum norm on  $\mathbb{R}^n$ . The corresponding dual norm produces the sum of the norms in (6). Any other norm on  $\mathbb{R}^n$  can replace the maximum in (5) as long as the corresponding dual norm is used to replace the sum in (6).

Next we formulate chain rules for Fréchet and Clarke subdifferentials, which are going to be used in the sequel. Such rules are extensively used when proving dual characterizations of nonlinear transversality, regularity and error bound properties. The next statement seems to present the chain rules under the weakest assumptions, compared to the existing assertions of this type, cf. [9, 10, 22, 42, 44, 48]. We provide an elementary direct proof based only on the definitions of the respective subdifferentials.

In the statement below,  $X$  is a general normed space,  $\psi : X \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $\varphi : \mathbb{R} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ . The composition function  $\varphi \circ \psi$  is defined in the usual way:

$$(\varphi \circ \psi)(x) := \begin{cases} \varphi(\psi(x)) & \text{if } x \in \text{dom } \psi, \\ +\infty & \text{if } x \notin \text{dom } \psi. \end{cases} \quad (7)$$

The outer function  $\varphi$  is assumed differentiable (or strictly differentiable) at the reference point, while the inner function  $\psi$  is arbitrary and does not have to be even (semi-)continuous. Note that  $\varphi \circ \psi$  can take the value  $-\infty$ .

Recall that  $\varphi$  is *strictly differentiable* at  $\bar{t} \in \mathbb{R}$  if it is finite near  $\bar{t}$  and

$$\varphi'(\bar{t}) = \lim_{t', t'' \rightarrow \bar{t}, t'' \neq t'} \frac{\varphi(t'') - \varphi(t')}{t'' - t'},$$

which automatically holds if  $\varphi$  is continuously differentiable at  $\bar{t}$ .

**Proposition 2.1** *Let  $X$  be a normed space,  $\psi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $\varphi : \mathbb{R} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ , and  $\bar{x} \in \text{dom } \psi$ . Suppose that  $\varphi$  is nondecreasing on  $\mathbb{R}$ , and finite and differentiable at  $\varphi(\bar{x})$  with  $\varphi'(\varphi(\bar{x})) > 0$ . Then*

$$\partial(\varphi \circ \psi)(\bar{x}) = \varphi'(\varphi(\bar{x})) \partial \psi(\bar{x}), \quad (8)$$

where  $\partial$  stands for the Fréchet subdifferential ( $\partial := \partial^F$ ). If  $\varphi$  is strictly differentiable at  $\varphi(\bar{x})$ , then

$$(\varphi \circ \psi)^\circ(\bar{x}; z) = \varphi'(\varphi(\bar{x})) \psi^\circ(\bar{x}; z) \quad \text{for all } z \in X, \quad (9)$$

and equality (8) holds true with  $\partial$  standing for the Clarke subdifferential ( $\partial := \partial^C$ ).

*Proof* In view of the assumptions on  $\varphi$ , we have  $\varphi(t) > \varphi(\psi(\bar{x}))$  for all  $t > \psi(\bar{x})$  and  $\varphi(t) < \varphi(\psi(\bar{x}))$  for all  $t < \psi(\bar{x})$ . Set  $\gamma := \liminf_{u \rightarrow 0} \varphi(\bar{x} + u)$ . If  $\gamma \neq \psi(\bar{x})$ , then

$$\begin{aligned} \liminf_{u \rightarrow 0} \frac{\varphi(\bar{x} + u) - \varphi(\bar{x})}{\|u\|} &= \lim_{u \rightarrow 0} \frac{\gamma - \varphi(\bar{x})}{\|u\|}, \\ \liminf_{u \rightarrow 0} \frac{\varphi(\psi(\bar{x} + u)) - \varphi(\psi(\bar{x}))}{\|u\|} &= \lim_{u \rightarrow 0} \frac{\varphi(\gamma) - \varphi(\psi(\bar{x}))}{\|u\|}. \end{aligned}$$

If  $\gamma < \psi(\bar{x})$ , then both limits above equal  $-\infty$  and  $\partial^F(\varphi \circ \psi)(\bar{x}) = \partial^F \varphi(\psi(\bar{x})) = \emptyset$ , while if  $\gamma > \psi(\bar{x})$ , then both limits equal  $+\infty$  and  $\partial^F(\varphi \circ \psi)(\bar{x}) = \partial^F \varphi(\psi(\bar{x})) = X^*$ .

Let  $\gamma = \psi(\bar{x})$  and  $x^* \in X^*$ . Since  $\varphi'(\psi(\bar{x})) > 0$ , we have

$$\begin{aligned} &\liminf_{u \rightarrow 0} \frac{\varphi(\psi(\bar{x} + u)) - \varphi(\psi(\bar{x})) - \varphi'(\psi(\bar{x}))\langle x^*, u \rangle}{\|u\|} \\ &= \liminf_{\substack{u \rightarrow 0 \\ \psi(\bar{x} + u) \rightarrow \psi(\bar{x})}} \frac{\varphi(\psi(\bar{x} + u)) - \varphi(\psi(\bar{x})) - \varphi'(\psi(\bar{x}))\langle x^*, u \rangle}{\|u\|} \\ &= \varphi'(\psi(\bar{x})) \liminf_{u \rightarrow 0} \frac{\psi(\bar{x} + u) - \psi(\bar{x}) - \langle x^*, u \rangle}{\|u\|}. \end{aligned}$$

It follows that  $x^* \in \partial^F \psi(\bar{x})$  if and only if  $\varphi'(\psi(\bar{x}))x^* \in \partial^F(\varphi \circ \psi)(\bar{x})$ . Hence, by the definition (3), equality (8) holds true with  $\partial$  standing for the Fréchet subdifferential ( $\partial := \partial^F$ ).

Now suppose  $\varphi$  is strictly differentiable at  $\psi(\bar{x})$ . Then  $\varphi$  is strictly increasing and invertible in a neighbourhood of  $\psi(\bar{x})$ . If  $(x_k, \alpha_k) \rightarrow (\bar{x}, \psi(\bar{x}))$  and  $\psi(x_k) \leq \alpha_k$  ( $k = 1, 2, \dots$ ), then  $\beta_k := \varphi(\alpha_k) \rightarrow \varphi(\psi(\bar{x}))$  and  $\varphi(\psi(x_k)) \leq \beta_k$  ( $k = 1, 2, \dots$ ). Conversely, if  $(x_k, \beta_k) \rightarrow (\bar{x}, \varphi(\psi(\bar{x})))$  and  $\varphi(\psi(x_k)) \leq \beta_k$  ( $k = 1, 2, \dots$ ), then  $\alpha_k := \varphi^{-1}(\beta_k) \rightarrow \psi(\bar{x})$  and  $\psi(x_k) \leq \alpha_k$  for all sufficiently large  $k$ .

Let  $z \in X$ . If for some  $\varepsilon > 0$  and sequences  $(x_k, \alpha_k) \rightarrow (\bar{x}, \psi(\bar{x}))$  and  $t_k \downarrow 0$ , it holds  $\psi(x_k) \leq \alpha_k$  ( $k = 1, 2, \dots$ ) and

$$\lim_{k \rightarrow +\infty} \inf_{\|z' - z\| < \varepsilon} (\psi(x_k + t_k z') - \alpha_k) \neq 0,$$

then  $|\varphi(\alpha_k)| < +\infty$  for all sufficiently large  $k$ ,

$$\liminf_{k \rightarrow +\infty, \|z' - z\| < \varepsilon} (\varphi(\psi(x_k + t_k z')) - \varphi(\alpha_k)) \neq 0,$$

and the above limits are either both positive or both negative. Set

$$\gamma := \lim_{\varepsilon \downarrow 0} \limsup_{\substack{(x, \alpha) \rightarrow (\bar{x}, \psi(\bar{x})) \\ \psi(x) \leq \alpha, t \downarrow 0}} \inf_{\|z' - z\| < \varepsilon} (\psi(x + tz') - \alpha).$$

If  $\gamma < 0$ , then, in view of the above observation,  $(\varphi \circ \psi)^\circ(\bar{x}; z) = \psi^\circ(\bar{x}; z) = -\infty$ . Similarly, if  $\gamma > 0$ , then  $(\varphi \circ \psi)^\circ(\bar{x}; z) = \psi^\circ(\bar{x}; z) = +\infty$ . If  $\gamma = 0$ , then

$$\begin{aligned} (\varphi \circ \psi)^\circ(\bar{x}; z) &= \lim_{\varepsilon \downarrow 0} \limsup_{\substack{(x, \alpha) \rightarrow (\bar{x}, \psi(\bar{x})) \\ \psi(x) \leq \alpha, t \downarrow 0}} \inf_{\|z' - z\| < \varepsilon} \frac{\varphi(\psi(x + tz')) - \varphi(\alpha)}{t} \\ &= \varphi'(\psi(\bar{x})) \lim_{\varepsilon \downarrow 0} \limsup_{\substack{(x, \alpha) \rightarrow (\bar{x}, \psi(\bar{x})) \\ \psi(x) \leq \alpha, t \downarrow 0}} \inf_{\|z' - z\| < \varepsilon} \frac{\psi(x + tz') - \alpha}{t} = \varphi'(\psi(\bar{x})) \psi^\circ(\bar{x}; z). \end{aligned}$$

This proves (9). In view of the definition (4), equality (8) with  $\partial$  standing for the Clarke subdifferential ( $\partial := \partial^C$ ) is a consequence of (9).  $\square$

*Remark 2.2* (i) The chain rules in Proposition 2.1 are local results. Instead of assuming that  $\varphi$  is defined on the whole real line with possibly infinite values, one can assume that  $\varphi$  is defined and finite on a closed interval  $[\alpha, \beta]$  around the point  $\psi(\bar{x})$ :  $\alpha < \psi(\bar{x}) < \beta$ . The proof above remains valid if the definition (7) of the composition function is slightly modified. The ‘if’ condition in the first line should be changed to  $x \in \text{dom } \psi$  and  $\psi(x) \in \text{dom } \varphi$ , and another two lines should be added:  $(\varphi \circ \psi)(x) := \varphi(\alpha)$  if  $\psi(x) < \alpha$ , and  $(\varphi \circ \psi)(x) := \varphi(\beta)$  if  $\beta < \psi(x) < +\infty$ . This change does not affect the conclusion of the proposition.

- (ii) If, additionally,  $\psi$  is assumed lower semicontinuous at  $\bar{x}$ , then the proof of the proposition can be shortened as the cases  $\gamma < \psi(\bar{x})$  in the first part of the proof and  $\gamma < 0$  in the second part cannot happen. In the lower semicontinuous setting, it is sufficient to assume that  $\varphi$  is nondecreasing only on  $[\psi(\bar{x}), +\infty[$  and there is no need to allow  $\varphi$  to take the value  $-\infty$ .

The next statement taken from [7, Lemma 2.4] presents several elementary relations between groups of vectors in a normed space, which are frequently used when deducing dual characterizations of extremality, stationarity and transversality properties of collections of sets, although often hidden in numerous proofs.

**Lemma 2.5** *Let  $K_1, \dots, K_n$  be cones in a normed space,  $\varepsilon > 0$ ,  $\rho > 0$ , and  $\mu > 0$ . Suppose that vectors  $z_1, \dots, z_n$  satisfy*

$$\rho \left\| \sum_{i=1}^n z_i \right\| + \mu \sum_{i=1}^n d(z_i, K_i) < \varepsilon, \quad \sum_{i=1}^n \|z_i\| = 1. \quad (10)$$

- (i) *If  $\varepsilon + \rho \leq \mu$ , then there exist vectors  $\hat{z}_i$  ( $i = 1, \dots, n$ ) such that*

$$\hat{z}_i \in K_i \ (i = 1, \dots, n), \quad \sum_{i=1}^n \|\hat{z}_i\| = 1, \quad \left\| \sum_{i=1}^n \hat{z}_i \right\| < \frac{\varepsilon}{\rho}. \quad (11)$$

- (ii) *If  $\varepsilon + \mu \leq \rho$ , then there exist vectors  $\hat{z}_i$  ( $i = 1, \dots, n$ ) such that*

$$\sum_{i=1}^n \hat{z}_i = 0, \quad \sum_{i=1}^n \|\hat{z}_i\| = 1, \quad \sum_{i=1}^n d(\hat{z}_i, K_i) < \frac{\varepsilon}{\mu}.$$

- (iii) *Moreover, if the underlying space is dual to a normed space, and*

$$\sum_{i=1}^n \langle z_i, x_i \rangle \geq \tau \max_{1 \leq i \leq n} \|x_i\| \quad (12)$$

*for some vectors  $x_i$  ( $i = 1, \dots, n$ ), not all zero, and a number  $\tau \in ]0, 1]$ , then the vectors  $\hat{z}_i$  ( $i = 1, \dots, n$ ) in parts (i) or (ii) satisfy*

$$\sum_{i=1}^n \langle \hat{z}_i, x_i \rangle > \hat{\tau} \max_{1 \leq i \leq n} \|x_i\|, \quad (13)$$

*where  $\hat{\tau} := \frac{\tau\mu - \varepsilon}{\mu + \varepsilon}$  under the assumptions in part (i), and  $\hat{\tau} := \frac{\tau\rho - \varepsilon}{\rho + \varepsilon}$  under the assumptions in part (ii).*

### 3 Definitions and Basic Facts

The next definition from [11] introduces nonlinear transversality properties of collections of sets. It extends the definitions of the corresponding Hölder transversality properties in [33, Definition 1]. The nonlinearity in the definitions of the properties is determined by a continuous strictly increasing function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $\varphi(0) = 0$  and  $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$ . The family of all such functions is denoted by  $\mathcal{C}$ . We denote by  $\mathcal{C}^1$  the subfamily of functions from  $\mathcal{C}$  which are continuously differentiable on  $]0, +\infty[$  with  $\varphi'(t) > 0$  for all  $t > 0$ . Obviously, if  $\varphi \in \mathcal{C}$  ( $\varphi \in \mathcal{C}^1$ ), then  $\varphi^{-1} \in \mathcal{C}$  ( $\varphi^{-1} \in \mathcal{C}^1$ ). Observe that, for any  $\alpha > 0$  and  $q > 0$ , the function  $t \mapsto \alpha t^q$  on  $\mathbb{R}_+$  belongs to  $\mathcal{C}^1$ . Note that in all assertions in the current paper involving Fréchet subdifferentials and normal cones it is sufficient to assume functions from  $\mathcal{C}^1$  to be (not necessarily continuously) differentiable on  $]0, +\infty[$ . Continuous differentiability is only needed for assertions involving Clarke subdifferentials.

*Remark 3.1* For the purposes of this paper, it suffices to assume that functions  $\varphi \in \mathcal{C}$  are defined and invertible near 0.

From now on,  $\Omega_1, \dots, \Omega_n$  are subsets of a normed space  $X$  and  $\bar{x} \in \bigcap_{i=1}^n \Omega_i$ , and we assume that  $\varphi \in \mathcal{C}$ .

**Definition 3.1** The collection  $\{\Omega_1, \dots, \Omega_n\}$  is

- (i)  $\varphi$ -semitransversal at  $\bar{x}$  if there exists a  $\delta > 0$  such that  $\bigcap_{i=1}^n (\Omega_i - x_i) \cap B_\rho(\bar{x}) \neq \emptyset$  for all  $\rho \in ]0, \delta[$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \rho$ ;
- (ii)  $\varphi$ -subtransversal at  $\bar{x}$  if there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that  $\bigcap_{i=1}^n \Omega_i \cap B_\rho(x) \neq \emptyset$  for all  $\rho \in ]0, \delta_1[$  and  $x \in B_{\delta_2}(\bar{x})$  with  $\varphi(\max_{1 \leq i \leq n} d(x, \Omega_i)) < \rho$ ;
- (iii)  $\varphi$ -transversal at  $\bar{x}$  if there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that  $\bigcap_{i=1}^n (\Omega_i - \omega_i - x_i) \cap (\rho \mathbb{B}) \neq \emptyset$  for all  $\rho \in ]0, \delta_1[$ ,  $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  and  $x_i \in X$  ( $i = 1, \dots, n$ ) with  $\varphi(\max_{1 \leq i \leq n} \|x_i\|) < \rho$ .

The most important realization of the three properties in Definition 3.1 corresponds to the Hölder setting, i.e.  $\varphi$  being a power function, given for all  $t \geq 0$  by  $\varphi(t) := \alpha^{-1}t^q$  with  $\alpha > 0$  and  $q > 0$ . In this case, Definition 3.1 sharpens [33, Definition 1]. In the Hölder setting, we refer to the three properties above as  $\alpha$ -(semi-, sub-)transversality of order  $q$ . With  $q = 1$  (the linear case), the properties were discussed, e.g., in [24, 25, 34].

Each of the properties in Definition 3.1 is determined by a function  $\varphi \in \mathcal{C}$ , and either a number  $\delta > 0$  in item (i) or two numbers  $\delta_1 > 0$  and  $\delta_2 > 0$  in items (ii) and (iii). The function plays the role of a kind of rate or modulus of the respective property, while the role of the  $\delta$ 's is more technical: they control the size of the interval for the values of  $\rho$  and, in the case of  $\varphi$ -subtransversality and  $\varphi$ -transversality in parts (ii) and (iii), the size of the neighbourhoods of  $\bar{x}$  involved in the respective definitions. Of course, if a property is satisfied with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , it is satisfied also with the single  $\delta := \min\{\delta_1, \delta_2\}$  in place of both  $\delta_1$  and  $\delta_2$ . We consider in the current paper two different parameters to emphasise their different roles in the definitions and the corresponding characterizations. Moreover, we are going to provide quantitative estimates for the values of these parameters.

Note that the definitions of the nonlinear transversality properties and their primal space characterizations discussed in this section do not require  $\varphi$  to be differentiable.

The next proposition provides two simple facts about the properties in Definition 3.1; cf. [11].

- Proposition 3.1** (i) *If  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then it is  $\varphi$ -semitransversal at  $\bar{x}$  with  $\delta_1$ , and  $\varphi$ -subtransversal at  $\bar{x}$  with any  $\delta'_1 \in ]0, \delta_1[$  and  $\delta'_2 > 0$  such that  $\varphi^{-1}(\delta'_1) + \delta'_2 \leq \delta_2$ .*
- (ii) *Suppose  $\bigcap_{i=1}^n \Omega_i$  is closed, and  $\bar{x} \in \text{bd } \bigcap_{i=1}^n \Omega_i$ . If  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal (particularly, if it is  $\varphi$ -transversal) at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , then there exists a  $\bar{t} \in ]0, \min\{\delta_2, \varphi^{-1}(\delta_1)\}[$  such that  $\varphi(t) \geq t$  for all  $t \in ]0, \bar{t}[$ . As a consequence,  $\liminf_{t \downarrow 0} \varphi(t)/t \geq 1$ .*

*Remark 3.2* In the Hölder setting, i.e. when  $\varphi(t) = \alpha^{-1}t^q$  with  $\alpha > 0$  and  $q > 0$ , the conditions on  $\varphi$  in Proposition 3.1(ii) can only be satisfied if either  $q < 1$ , or  $q = 1$  and  $\alpha \leq 1$ . This reflects the well known fact that the Hölder subtransversality and transversality properties are only meaningful when  $q \leq 1$  and, moreover, the linear case ( $q = 1$ ) is only meaningful when  $\alpha \leq 1$ ; cf. [30, p. 705], [27, p. 118]. Note that the Hölder semitransversality can hold also with  $q > 1$ ; see an example in [11].

The next three propositions, established in [11] as consequences of the Ekeland variational principle, provide sufficient ‘slope’ characterizations of the three nonlinear transversality properties in Definition 3.1. They employ the following norm on  $X^{n+1}$  depending on a parameter  $\gamma > 0$ :

$$\|(x_1, \dots, x_n, x)\|_\gamma := \max \left\{ \|x\|, \gamma \max_{1 \leq i \leq n} \|x_i\| \right\}, \quad x_1, \dots, x_n, x \in X. \quad (14)$$

**Proposition 3.2** *Let  $X$  be Banach, and  $\Omega_1, \dots, \Omega_n$  be closed. The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$  if, for some  $\gamma > 0$  and any  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying*

$$0 < \max_{1 \leq i \leq n} \|x_i\| < \varphi^{-1}(\delta), \quad (15)$$

*there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  such that*

$$\limsup_{\substack{\Omega_i \\ u_i \rightarrow x_i \ (i=1, \dots, n), \ u \rightarrow x \\ (u_1, \dots, u_n, u) \neq (\omega_1, \dots, \omega_n, x)}} \frac{\varphi \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) - \varphi \left( \max_{1 \leq i \leq n} \|u_i - x_i - u\| \right)}{\|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma} \geq 1 \quad (16)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying

$$\|x - \bar{x}\| < \lambda, \quad \max_{1 \leq i \leq n} \|\omega_i - \bar{x}\| < \frac{\lambda}{\gamma}, \quad (17)$$

$$0 < \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \leq \max_{1 \leq i \leq n} \|x_i\|. \quad (18)$$

**Proposition 3.3** *Let  $X$  be Banach, and  $\Omega_1, \dots, \Omega_n$  be closed. The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $x' \in X$  satisfying*

$$\|x' - \bar{x}\| < \delta_2, \quad 0 < \max_{1 \leq i \leq n} d(x', \Omega_i) < \varphi^{-1}(\delta_1), \quad (19)$$

there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  such that

$$\limsup_{\substack{u_i \rightarrow \omega_i \ (i=1, \dots, n), \ u \rightarrow x \\ (u_1, \dots, u_n, u) \neq (\omega_1, \dots, \omega_n, x)}} \frac{\varphi\left(\max_{1 \leq i \leq n} \|\omega_i - x\|\right) - \varphi\left(\max_{1 \leq i \leq n} \|u_i - u\|\right)}{\|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma} \geq 1 \quad (20)$$

for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying

$$\|x - x'\| < \lambda, \quad \max_{1 \leq i \leq n} \|\omega_i - \omega'_i\| < \frac{\lambda}{\gamma}, \quad (21)$$

$$0 < \max_{1 \leq i \leq n} \|\omega_i - x\| \leq \max_{1 \leq i \leq n} \|\omega'_i - x'\| < \varphi^{-1}(\lambda). \quad (22)$$

**Proposition 3.4** *Let  $X$  be Banach, and  $\Omega_1, \dots, \Omega_n$  be closed. The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\xi \in ]0, \varphi^{-1}(\delta_1)[$ , there exists a  $\lambda \in ]\varphi(\xi), \delta_1[$  such that inequality (16) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying*

$$\|x - \bar{x}\| < \lambda, \quad \max_{1 \leq i \leq n} \|\omega_i - \omega'_i\| < \frac{\lambda}{\gamma}, \quad (23)$$

$$0 < \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \leq \max_{1 \leq i \leq n} \|\omega'_i - x_i - \bar{x}\| = \xi. \quad (24)$$

**Remark 3.3** (i) The expressions in the left-hand sides of the inequalities (16) and (20) are the  $\gamma$ -slopes (cf. [26, p. 61]) computed at the respective points of the extended real-valued function

$$\widehat{f} := f + i_{\Omega_1 \times \dots \times \Omega_n}, \quad (25)$$

with  $f : X^{n+1} \rightarrow \mathbb{R}_+$  being a continuous function defined for all  $u_1, \dots, u_n, u \in X$  by one of the expressions:

$$f(u_1, \dots, u_n, u) := \varphi\left(\max_{1 \leq i \leq n} \|u_i - u\|\right), \quad (26)$$

$$f(u_1, \dots, u_n, u) := \varphi\left(\max_{1 \leq i \leq n} \|u_i - x_i - u\|\right), \quad (27)$$

where  $x_1, \dots, x_n$  in (27) are given vectors in  $X$ . Note that (26) is a particular case of (27) corresponding to setting  $x_i := 0$  ( $i = 1, \dots, n$ ).

(ii) The sufficient conditions for  $\varphi$ -semitransversality and  $\varphi$ -transversality in Propositions 3.2 and 3.4, respectively, use the same inequality (16) involving slopes. Nevertheless, the sufficient condition in Proposition 3.4 is stronger than the corresponding one in Proposition 3.2 as it requires condition (16) to be satisfied on a larger set of points. This is natural as  $\varphi$ -transversality is a stronger property than  $\varphi$ -semitransversality.

#### 4 Semitransversality

In this and the subsequent sections, the nonlinear transversality properties of collections of sets are characterized in terms of Clarke and Fréchet subdifferentials and normals. If not explicitly stated otherwise, the space  $X$  is assumed to be Banach, and the sets  $\Omega_1, \dots, \Omega_n$  are assumed closed.

The dual norm on  $(X^*)^{n+1}$  corresponding to (14) has the following form:

$$\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma = \|x^*\| + \frac{1}{\gamma} \sum_{i=1}^n \|x_i^*\|, \quad x_1^*, \dots, x_n^*, x^* \in X^*. \quad (28)$$

We will denote by  $d_\gamma$  the distance in  $(X^*)^{n+1}$  determined by (28).

In this section, we use the function  $\widehat{f}$  given by (25) with  $f : X^{n+1} \rightarrow \mathbb{R}_+$  defined by (27). The next statement provides dual characterizations of  $\varphi$ -semitransversality in terms of subdifferentials of  $\widehat{f}$ .

**Proposition 4.1** (i) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$  if, for some  $\gamma > 0$  and any  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying (15), there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  such that*

$$d_\gamma(0, \partial \widehat{f}(\omega_1, \dots, \omega_n, x)) \geq 1 \quad (29)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (17) and (18), where  $\partial$  in (29) stands for the Clarke subdifferential ( $\partial := \partial^C$ ).

(ii) *If  $X$  is Asplund, then the above assertion is valid with  $\partial$  in (29) standing for the Fréchet subdifferential ( $\partial := \partial^F$ ), and condition (18) replaced by*

$$0 < \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| < \varphi^{-1}(\lambda). \quad (30)$$

*Proof* (i) Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is not  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$ , and let  $\gamma > 0$  be given. By Proposition 3.2, there exist points  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying (15) such that, for any  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$ , there exist  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (17) and (18), and a number  $\tau \in ]0, 1[$  such that

$$\begin{aligned} \varphi \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) - \varphi \left( \max_{1 \leq i \leq n} \|u_i - x_i - u\| \right) \\ \leq \tau \|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma \end{aligned} \quad (31)$$

for all  $u_i \in \Omega_i$  near  $\omega_i$  ( $i = 1, \dots, n$ ) and all  $u$  near  $x$ . In other words,  $(\omega_1, \dots, \omega_n, x)$  is a local minimizer of the function

$$(u_1, \dots, u_n, u) \mapsto \widehat{f}(u_1, \dots, u_n, u) + \tau \|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma. \quad (32)$$

By Lemma 2.2, its Fréchet and, as a consequence, Clarke subdifferential at this point contains 0. Observe that (32) is the sum of the function  $\widehat{f}$  and the Lipschitz continuous convex function  $(u_1, \dots, u_n, u) \mapsto \tau \|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma$ , and at any point all subgradients  $(x_1^*, \dots, x_n^*, x^*)$  of the latter function satisfy  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma \leq \tau$ . By the Clarke–Rockafellar sum rule (Lemma 2.1(i)), there exists a subgradient  $(x_1^*, \dots, x_n^*, x^*) \in \partial^C \widehat{f}(\omega_1, \dots, \omega_n, x)$  such that  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma \leq \tau < 1$ . The last inequality contradicts (29).

(ii) If  $X$  is Asplund, then one can employ the fuzzy sum rule (Lemma 2.1(ii)): for any  $\varepsilon > 0$ , there exist points  $x' \in B_\varepsilon(x)$ ,  $\omega'_i \in \Omega_i \cap B_\varepsilon(\omega_i)$  ( $i = 1, \dots, n$ ), and a subgradient  $(x_1^*, \dots, x_n^*, x^*) \in \partial^F \widehat{f}(\omega'_1, \dots, \omega'_n, x')$  such that  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma < \tau + \varepsilon$ . The number  $\varepsilon$  can be chosen small enough so that  $\|x' - \bar{x}\| < \lambda$ ,  $\max_{1 \leq i \leq n} \|\omega'_i - \bar{x}\| < \lambda/\gamma$ ,  $0 < \max_{1 \leq i \leq n} \|\omega'_i - x_i - x'\| < \varphi^{-1}(\lambda)$ , and  $\tau + \varepsilon < 1$ . The last inequality again yields  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma < 1$ , which contradicts (29).  $\square$

The key condition (29) in Proposition 4.1 involves subdifferentials of the function  $\widehat{f}$  given by (25) with  $f : X^{n+1} \rightarrow \mathbb{R}_+$  defined by (27). Subgradients of  $\widehat{f}$  have  $n+1$  component vectors  $x_1^*, \dots, x_n^*, x^*$ . As it can be seen from the representation (28) of the dual norm, the contribution of the vectors  $x_1^*, \dots, x_n^*$  on one hand and  $x^*$  on the other hand to condition (29) is different. The next corollary exposes this difference.

- Corollary 4.1** (i) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$  if, for some  $\gamma > 0$  and any  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying (15), there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  such that  $\|x^*\| \geq 1$  for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (17) and (18), and all  $(x_1^*, \dots, x_n^*, x^*) \in \partial \widehat{f}(\omega_1, \dots, \omega_n, x)$  with  $\sum_{i=1}^n \|x_i^*\| < \gamma$ , where  $\partial$  stands for the Clarke subdifferential ( $\partial := \partial^C$ ).*
- (ii) *If  $X$  is Asplund, then the above assertion is valid with  $\partial$  standing for the Fréchet subdifferential ( $\partial := \partial^F$ ), and condition (18) replaced by (30).*

*Proof* Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is not  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$ . Let  $\gamma > 0$ . By Proposition 4.1, there exist  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying (15) such that, for any  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$ , there exist  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (17) and (18) ((30) if  $X$  is Asplund), and a subgradient  $(x_1^*, \dots, x_n^*, x^*) \in \partial^C \widehat{f}(\omega_1, \dots, \omega_n, x)$  ( $(x_1^*, \dots, x_n^*, x^*) \in \partial^F \widehat{f}(\omega_1, \dots, \omega_n, x)$  if  $X$  is Asplund) such that  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma < 1$ . By the definition (28) of the dual norm, this implies  $\sum_{i=1}^n \|x_i^*\| < \gamma$  and  $\|x^*\| < 1$ . The latter inequality contradicts the assumption.  $\square$

The next ‘ $\delta$ -free’ statement is a direct consequence of Corollary 4.1.

- Corollary 4.2** (i) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  if*

$$\lim_{\gamma \downarrow 0} \liminf_{t \downarrow 0} \lim_{\lambda \downarrow \varphi(t)} \inf_{\substack{0 < \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \leq t \\ (x_1^*, \dots, x_n^*, x^*) \in \partial \widehat{f}(\omega_1, \dots, \omega_n, x), \sum_{i=1}^n \|x_i^*\| < \gamma}} \|x^*\| > 1, \quad (33)$$

or, in particular, if

$$\liminf_{\substack{x \rightarrow \bar{x}, \omega_i \xrightarrow{\Omega_i} \bar{x}, x_i \rightarrow 0, x_i^* \rightarrow 0 \ (i=1, \dots, n) \\ \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| > 0, (x_1^*, \dots, x_n^*, x^*) \in \partial \widehat{f}(\omega_1, \dots, \omega_n, x)}} \|x^*\| > 1, \quad (34)$$

where  $\partial$  stands for the Clarke subdifferential ( $\partial := \partial^C$ ).

- (ii) *If  $X$  is Asplund, then the above assertion is valid with  $\partial$  standing for the Fréchet subdifferential ( $\partial := \partial^F$ ).*

**Remark 4.1** Condition (34) is obviously stronger than (33). Moreover, it is in fact sufficient for the stronger  $\varphi$ -transversality property of  $\Omega_1, \dots, \Omega_n$ ; cf. Remark 6.1.

The proof of Proposition 4.1 utilizes the sum rules in Lemma 2.1 to obtain representations of subgradients of the sum function (32) in terms of subgradients of the summand functions. Note that one of the summands – the function  $\widehat{f}$  – is itself a sum of functions; see (25). Next we apply these sum rules again to obtain characterizations of the nonlinear semitransversality in terms of normals to the given individual sets. This time the difference between the exact sum rule in Lemma 2.1(i) and the approximate sum rule in Lemma 2.1(ii) becomes explicit in the conclusions of the next theorem: compare the exact condition (38) and the corresponding approximate condition (39).

**Theorem 4.1** *Let  $\varphi \in \mathcal{C}^1$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$  if, for some  $\mu > 0$  and any  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying (15), one of the following conditions is satisfied:*

- (i) *there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  such that*

$$\varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) \left( \left\| \sum_{i=1}^n x_i^* \right\| + \mu \sum_{i=1}^n d(x_i^*, N_{\Omega_i}(\omega_i)) \right) \geq 1 \quad (35)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (18) and

$$\|x - \bar{x}\| < \lambda, \quad \max_{1 \leq i \leq n} \|\omega_i - \bar{x}\| < \mu \lambda, \quad (36)$$

and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying

$$\sum_{i=1}^n \|x_i^*\| = 1, \quad (37)$$

$$\sum_{i=1}^n \langle x_i^*, x + x_i - \omega_i \rangle = \max_{1 \leq i \leq n} \|x + x_i - \omega_i\|, \quad (38)$$

where  $N$  in (35) stands for the Clarke normal cone ( $N := N^C$ );

- (ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  and a  $\tau \in ]0, 1[$  such that inequality (35) holds with  $N$  standing for the Fréchet normal cone ( $N := N^F$ ) for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (36) and (30), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying condition (37) and

$$\sum_{i=1}^n \langle x_i^*, x + x_i - \omega_i \rangle > \tau \max_{1 \leq i \leq n} \|x + x_i - \omega_i\|. \quad (39)$$

*Proof* Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is not  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$ , and let  $\mu > 0$  be given. Set  $\gamma := \mu^{-1}$ . By Proposition 4.1, there exist points  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying (15) such that, for any  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$ , there exist  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (18) ((30) if  $X$  is Asplund) and (36), and a subgradient  $(\hat{x}_1^*, \dots, \hat{x}_n^*, \hat{x}^*) \in \partial \hat{f}(\omega_1, \dots, \omega_n, x)$  such that

$$\|(\hat{x}_1^*, \dots, \hat{x}_n^*, \hat{x}^*)\|_\gamma < 1, \quad (40)$$

where  $\partial$  stands for either the Clarke subdifferential (if  $X$  is a general Banach space) or the Fréchet subdifferential (if  $X$  is Asplund). Recall from (25) that  $\hat{f}$  is a sum of two functions: the function  $f$  given by (27) and the indicator function of the set  $\Omega_1 \times \dots \times \Omega_n$ . Since  $\max_{1 \leq i \leq n} \|\omega_i - x_i - x\| > 0$ ,  $f$  is locally Lipschitz continuous near  $(\omega_1, \dots, \omega_n, x)$ .

- (i)  $X$  is a general Banach space, and  $(\hat{x}_1^*, \dots, \hat{x}_n^*, \hat{x}^*)$  in (40) is a Clarke subgradient. By the Clarke–Rockafellar sum rule (Lemma 2.1(i)), there exist vectors  $(\tilde{x}_1^*, \dots, \tilde{x}_n^*, \tilde{x}^*) \in \partial^C f(\omega_1, \dots, \omega_n, x)$  and  $(u_1^*, \dots, u_n^*) \in N_{\Omega_1 \times \dots \times \Omega_n}^C(\omega_1, \dots, \omega_n)$  such that

$$(\hat{x}_1^*, \dots, \hat{x}_n^*, \hat{x}^*) = (\tilde{x}_1^*, \dots, \tilde{x}_n^*, \tilde{x}^*) + (u_1^*, \dots, u_n^*, 0). \quad (41)$$

By Lemma 2.3, Proposition 2.1 and Remark 2.2,

$$u_i^* \in N_{\Omega_i}^C(\omega_i) \quad (i = 1, \dots, n), \quad (42)$$

$$(\tilde{x}_1^*, \dots, \tilde{x}_n^*, \tilde{x}^*) = \varphi'(\psi(\omega_1, \dots, \omega_n, x))(-x_1^*, \dots, -x_n^*, x^*), \quad (43)$$

and  $(-x_1^*, \dots, -x_n^*, x^*) \in \partial \psi(\omega_1, \dots, \omega_n, x)$ , where

$$\psi(u_1, \dots, u_n, u) := \max_{1 \leq i \leq n} \|u_i - x_i - u\|, \quad u_1, \dots, u_n, u \in X. \quad (44)$$

By Lemma 2.4, conditions (37) and (38) are satisfied, and

$$x^* = \sum_{i=1}^n x_i^*. \quad (45)$$

Combining (40), (41), (42), (43), (45) and (28), we obtain

$$\varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) \left( \left\| \sum_{i=1}^n x_i^* \right\| + \mu \sum_{i=1}^n d(x_i^*, N_{\Omega_i}^C(\omega_i)) \right) < 1.$$

This contradicts (35).

- (ii) Let  $X$  be Asplund and a number  $\tau \in ]0, 1[$  be given. Instead of the Clarke–Rockafellar sum rule, one can employ the fuzzy sum rule (Lemma 2.1(ii)): for any  $\varepsilon > 0$ , there exist points  $x' \in B_\varepsilon(x)$ ,  $y_i \in B_\varepsilon(\omega_i)$ ,  $\omega'_i \in \Omega_i \cap B_\varepsilon(\omega_i)$  ( $i = 1, \dots, n$ ), and vectors  $(\tilde{x}_1^*, \dots, \tilde{x}_n^*, \tilde{x}^*) \in \partial^F f(y_1, \dots, y_n, x')$  and  $(u_1^*, \dots, u_n^*) \in N_{\Omega_1 \times \dots \times \Omega_n}^F(\omega'_1, \dots, \omega'_n)$  such that

$$\|(\tilde{x}_1^*, \dots, \tilde{x}_n^*, \tilde{x}^*) + (u_1^*, \dots, u_n^*, 0) - (\hat{x}_1^*, \dots, \hat{x}_n^*, \hat{x}^*)\|_\gamma < \varepsilon. \quad (46)$$

Denote

$$\beta := \left| \varphi' \left( \max_{1 \leq i \leq n} \|\omega'_i - x_i - x'\| \right) - \varphi' \left( \max_{1 \leq i \leq n} \|y_i - x_i - x'\| \right) \right|, \quad (47)$$

and observe that  $\beta \rightarrow 0$  as  $\varepsilon \downarrow 0$ . Recall that  $0 < \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \leq \max_{1 \leq i \leq n} \|x_i\| < \varphi^{-1}(\lambda)$ . The number  $\varepsilon$  can be chosen small enough so that

$$\begin{aligned} \|x' - \bar{x}\| < \lambda, \quad \max_{1 \leq i \leq n} \|\omega'_i - \bar{x}\| < \mu\lambda, \quad 0 < \max_{1 \leq i \leq n} \|\omega'_i - x_i - x'\| < \varphi^{-1}(\lambda), \\ \|(\hat{x}_1^*, \dots, \hat{x}_n^*, \hat{x}^*)\|_\gamma + (1 + \mu)\beta + \varepsilon < 1, \end{aligned} \quad (48)$$

$$\max_{1 \leq i \leq n} \|y_i - \omega'_i\| < \frac{1 - \tau}{2} \max_{1 \leq i \leq n} \|x' - x_i - \omega'_i\|. \quad (49)$$

By Lemma 2.3, Proposition 2.1, Remark 2.2 and Lemma 2.4,

$$u_i^* \in N_{\Omega_i}^F(\omega'_i) \quad (i = 1, \dots, n), \quad (50)$$

$$(\tilde{x}_1^*, \dots, \tilde{x}_n^*, \tilde{x}^*) = \varphi' \left( \max_{1 \leq i \leq n} \|y_i - x_i - x'\| \right) (-x_1^*, \dots, -x_n^*, x^*), \quad (51)$$

where vectors  $x_1^*, \dots, x_n^*, x^* \in X^*$  satisfy (37), (45) and

$$\sum_{i=1}^n \langle x_i^*, x' + x_i - y_i \rangle = \max_{1 \leq i \leq n} \|x' + x_i - y_i\|. \quad (52)$$

It follows from (37), (49) and (52) that

$$\begin{aligned} \sum_{i=1}^n \langle x_i^*, x' + x_i - \omega'_i \rangle &\geq \sum_{i=1}^n \langle x_i^*, x' + x_i - y_i \rangle - \max_{1 \leq i \leq n} \|y_i - \omega'_i\| \\ &= \max_{1 \leq i \leq n} \|x' + x_i - y_i\| - \max_{1 \leq i \leq n} \|y_i - \omega'_i\| \\ &\geq \max_{1 \leq i \leq n} \|x' + x_i - \omega'_i\| - 2 \max_{1 \leq i \leq n} \|y_i - \omega'_i\| \\ &> \tau \max_{1 \leq i \leq n} \|x' + x_i - \omega'_i\|. \end{aligned}$$

Combining (45), (46), (50), (51) and (28), we obtain

$$\begin{aligned} \varphi' \left( \max_{1 \leq i \leq n} \|y_i - x_i - x'\| \right) &\left( \left\| \sum_{i=1}^n x_i^* \right\| + \mu \sum_{i=1}^n d(x_i^*, N_{\Omega_i}^F(\omega'_i)) \right) \\ &\stackrel{(45),(51)}{=} \|\tilde{x}^*\| + \mu \sum_{i=1}^n d(-\tilde{x}_i^*, N_{\Omega_i}^F(\omega'_i)) \stackrel{(50)}{\leq} \|\tilde{x}^*\| + \mu \sum_{i=1}^n \|\tilde{x}_i^* + u_i^*\| \\ &\stackrel{(28)}{=} \|(\tilde{x}_1^*, \dots, \tilde{x}_n^*, \tilde{x}^*) + (u_1^*, \dots, u_n^*, 0)\|_\gamma \stackrel{(46)}{<} \|(\hat{x}_1^*, \dots, \hat{x}_n^*, \hat{x}^*)\|_\gamma + \varepsilon. \end{aligned}$$

Hence, thanks to (47), (48) and (52),

$$\begin{aligned} \varphi' \left( \max_{1 \leq i \leq n} \|\omega'_i - x_i - x'\| \right) &\left( \left\| \sum_{i=1}^n x_i^* \right\| + \mu \sum_{i=1}^n d(x_i^*, N_{\Omega_i}^F(\omega'_i)) \right) \\ &< \|(\hat{x}_1^*, \dots, \hat{x}_n^*, \hat{x}^*)\|_\gamma + (1 + \mu)\beta + \varepsilon < 1. \end{aligned}$$

This contradicts (35).  $\square$

The key dual transversality condition (35) in Theorem 4.1 combines two conditions on the dual vectors  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ): either their sum must be sufficiently far from 0, or the vectors themselves must be sufficiently far from the corresponding normal cones. From the point of view of applications, it can be convenient to have these conditions separated. The next corollary shows that it can be easily done. It collects four separate sufficient conditions for  $\varphi$ -semitransversality.

**Corollary 4.3** *Let  $\varphi \in \mathcal{C}^1$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$  if, for some  $\gamma > 0$  and any  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying (15), one of the following conditions is satisfied:*

(i) *there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  such that*

$$\varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) \left\| \sum_{i=1}^n x_i^* \right\| \geq 1 \quad (53)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (17) and (18), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (38) and

$$\varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) \sum_{i=1}^n d(x_i^*, N_{\Omega_i}(\omega_i)) < \gamma, \quad (54)$$

where  $N$  stands for the Clarke normal cone ( $N := N^C$ );

- (ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  and a  $\tau \in ]0, 1[$  such that inequality (53) holds for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (17) and (30), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (39) and (54), where  $N$  stands for the Fréchet normal cone ( $N := N^F$ );
- (iii) there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  such that

$$\varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) \sum_{i=1}^n d(x_i^*, N_{\Omega_i}(\omega_i)) \geq \gamma \quad (55)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (17) and (18), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (38) and

$$\varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) \left\| \sum_{i=1}^n x_i^* \right\| < 1, \quad (56)$$

where  $N$  in (55) stands for the Clarke normal cone ( $N := N^C$ );

- (iv)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  and a  $\tau \in ]0, 1[$  such that inequality (55) holds for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (17) and (30), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (39) and (56), where  $N$  in (55) stands for the Fréchet normal cone ( $N := N^F$ ).

*Proof* It is sufficient to notice that the assumptions in parts (i)–(iv) of the above corollary imply those in the respective parts of Theorem 4.1 with  $\mu := \gamma^{-1}$ . Indeed, the corollary replaces condition (35) by a stronger condition: either (53) in parts (i) and (ii) or (55) in parts (iii) and (iv). These conditions only need to be satisfied by  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (54) in the case of (53) or (56) in the case of (55). If any of the conditions (54) and (56) is violated, then condition (35) is automatically satisfied.  $\square$

Corollary 4.3 ‘separating’ the two dual transversality conditions hidden in the key combined condition (35) in Theorem 4.1 still has a drawback. It does not allow for the two ‘exact’ cases:  $x_i^* \in N_{\Omega_i}(\omega_i)$  ( $i = 1, \dots, n$ ) (with appropriate normal cones) or  $\sum_{i=1}^n x_i^* = 0$ , very important for the transversality (as well as extremality and stationarity) theory; cf. conditions (54) and (56). Employing Lemma 2.5, we can accommodate for these important cases.

**Corollary 4.4** *Let  $\varphi \in \mathcal{C}^1$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$  if, for some  $\mu > 0$  and any  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying (15), one of the following conditions is satisfied:*

- (i) there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  such that

$$\left[ \varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) \right]^{-1} + 1 \leq \mu, \quad (57)$$

and condition (53) holds true for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (18) and (36), and all  $x_i^* \in N_{\Omega_i}^C(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37) and (39) with

$$\tau := \frac{\mu \varphi'(\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|) - 1}{\mu \varphi'(\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|) + 1}, \quad (58)$$

- (ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  and a  $\hat{\tau} \in ]0, 1[$  such that condition (53) hold true and

$$\left[ \varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) \right]^{-1} + 1 \leq \hat{\tau} \mu \quad (59)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (30) and (36), and all  $x_i^* \in N_{\Omega_i}^F(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37) and (39) with

$$\tau := \frac{\hat{\tau} \mu \varphi'(\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|) - 1}{\mu \varphi'(\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|) + 1}; \quad (60)$$

(iii) there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  such that

$$\left[ \varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) \right]^{-1} + \mu \leq 1, \quad (61)$$

and condition (55) holds true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Clarke normal cone ( $N := N^C$ ) for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (18) and (36), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$ , and conditions (37) and (39) with

$$\tau := \frac{\varphi'(\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|) - 1}{\varphi'(\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|) + 1}; \quad (62)$$

(iv)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  and a  $\hat{\tau} \in ]0, 1[$  such that condition (55) hold true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Fréchet normal cone ( $N := N^F$ ), and

$$\left[ \varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right) \right]^{-1} + \mu \leq \hat{\tau} \quad (63)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (30) and (36), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$ , and conditions (37) and (39) with

$$\tau := \frac{\hat{\tau} \varphi'(\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|) - 1}{\varphi'(\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|) + 1}. \quad (64)$$

*Proof* (i) and (ii). Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is not  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$ , and let a  $\mu > 0$  be given. Set  $\gamma := \mu^{-1}$ . By Theorem 4.1, there exist points  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying (15) such that both conditions (i) and (ii) in the theorem are not satisfied. Hence, for any  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  (as well as  $\tau \in ]0, 1[$  in case (ii)), there exist  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (18) (or (30) in case (ii)) and (36), and  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37) and (38) (or (39) in case (ii)) such that inequality (35) is violated with  $N$  standing for the Clarke (Fréchet in case (ii)) normal cone. Thus, conditions (10) hold true with  $\varepsilon := [\varphi'(\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|)]^{-1}$ ,  $\rho := 1$ ,  $z_i := x_i^*$  and  $K_i$  standing for  $N_{\Omega_i}^C(\omega_i)$  (or  $N_{\Omega_i}^F(\omega_i)$  in case (ii)) ( $i = 1, \dots, n$ ). Besides, if inequality (57) is satisfied, we have  $\varepsilon + \rho \leq \mu$  and, by Lemma 2.5(i), there exist vectors  $\hat{x}_i^* := \hat{z}_i \in X^*$  ( $i = 1, \dots, n$ ) such that conditions (11) hold true, i.e.  $\hat{x}_i^* \in N_{\Omega_i}^C(\omega_i)$  (or  $\hat{x}_i^* \in N_{\Omega_i}^F(\omega_i)$  in case (ii)) ( $i = 1, \dots, n$ ),  $\sum_{i=1}^n \|\hat{x}_i^*\| = 1$ , while condition (53), with  $\hat{x}_i^*$  in place of  $x_i^*$  ( $i = 1, \dots, n$ ), is violated.

Moreover, setting  $\tau := 1$  in case (i), we have inequality (12) satisfied in both cases and, by Lemma 2.5(iii), inequality (13) holds with  $\hat{\tau} := \frac{\tau\mu - \varepsilon}{\mu + \varepsilon}$  in place of  $\tau$ . Observe that the above definition of  $\hat{\tau}$  is exactly the definition of  $\tau$  in (58) in case (i) or in (60) in case (ii). Since condition (53) is violated, both conditions (i) and (ii) are not satisfied.

(iii) and (iv). The proof proceeds as above, replacing the application of part (i) of Lemma 2.5 with that of its part (ii).  $\square$

*Remark 4.2* Since  $\tau$  defined by any of the formulas (58), (60), (62) and (64) is not in general a constant, but depends on  $x$ ,  $\omega_i$  and  $x_i$  ( $i = 1, \dots, n$ ), checking condition (39) with such a  $\tau$  does not seem practical. Such a check becomes meaningful in the linear setting, i.e. when  $\varphi'$ , and consequently, also  $\tau$  is constant; cf. Remark 7.1.

Dropping condition (39) in all parts of Corollary 4.4, we can formulate its simplified and easier to use version.

**Corollary 4.5** *Let  $\varphi \in \mathcal{C}^1$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -semitransversal at  $\bar{x}$  with some  $\delta > 0$  if, for some  $\mu > 0$  and any  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying (15), one of the following conditions is satisfied:*

(i) *there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  such that conditions (57) and (53) hold true for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (18) and (36), and all  $x_i^* \in N_{\Omega_i}^C(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37);*

- (ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  and a  $\hat{\tau} \in ]0, 1[$  such that conditions (59) and (53) hold true for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (30) and (36), and all  $x_i^* \in N_{\Omega_i}^F(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37);
- (iii) there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  such that conditions (61) and (55) hold true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Clarke normal cone ( $N := N^C$ ) for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (18) and (36), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$  and condition (37);
- (iv)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  and a  $\hat{\tau} \in ]0, 1[$  such that conditions (63) and (55) hold true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Fréchet normal cone ( $N := N^F$ ), for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (30) and (36), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$  and condition (37).

## 5 Subtransversality

The dual characterizations in this section follow the pattern of those in Section 4 with appropriate adjustments in the proofs. If not explicitly stated otherwise, the space  $X$  is assumed to be Banach, and the sets  $\Omega_1, \dots, \Omega_n$  are assumed closed. We use the function  $\hat{f}$  given by (25) with  $f : X^{n+1} \rightarrow \mathbb{R}_+$  defined by (26). The next statement provides dual characterizations of  $\varphi$ -subtransversality in terms of subdifferentials of  $\hat{f}$ .

- Proposition 5.1** (i) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $x' \in X$  satisfying (19), there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  such that condition (29) is satisfied for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (21) and (22), where  $\partial$  in (29) stands for the Clarke subdifferential ( $\partial := \partial^C$ ).*
- (ii) *If  $X$  is Asplund, then the above assertion is valid with  $\partial$  in (29) standing for the Fréchet subdifferential ( $\partial := \partial^F$ ), and condition (22) replaced by*

$$0 < \max_{1 \leq i \leq n} \|\omega_i - x\| < \varphi^{-1}(\lambda). \quad (65)$$

*Proof* (i) Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is not  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , and let  $\gamma > 0$  be given. By Proposition 3.3, there exists a point  $x' \in X$  satisfying (19) such that, for any  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$ , there exist points  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (21) and (22), and a number  $\tau \in ]0, 1[$  such that

$$\varphi \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right) - \varphi \left( \max_{1 \leq i \leq n} \|u_i - u\| \right) \leq \tau \|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma$$

for all  $u_i \in \Omega_i$  near  $\omega_i$  ( $i = 1, \dots, n$ ) and all  $u$  near  $x$ . In other words,  $(\omega_1, \dots, \omega_n, x)$  is a local minimizer of the function (32). By Lemma 2.2, its Fréchet and, as a consequence, Clarke subdifferential at this point contains 0. Observe that (32) is the sum of the function  $\hat{f}$  and the Lipschitz continuous convex function  $(u_1, \dots, u_n, u) \mapsto \tau \|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma$ . At any point, all subgradients  $(x_1^*, \dots, x_n^*, x^*)$  of the latter function satisfy  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma \leq \tau$ . By the Clarke–Rockafellar sum rule (Lemma 2.1(i)), there exists a subgradient  $(x_1^*, \dots, x_n^*, x^*) \in \partial^C \hat{f}(\omega_1, \dots, \omega_n, x)$  such that  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma \leq \tau$ . The last inequality contradicts (29).

- (ii) If  $X$  is Asplund, then one can employ the fuzzy sum rule (Lemma 2.1(ii)): for any  $\varepsilon > 0$ , there exist points  $y \in B_\varepsilon(x)$ ,  $y_i \in \Omega_i \cap B_\varepsilon(\omega_i)$  ( $i = 1, \dots, n$ ), and a subgradient  $(x_1^*, \dots, x_n^*, x^*) \in \partial^F \hat{f}(y_1, \dots, y_n, y)$  such that  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma < \tau + \varepsilon$ . The number  $\varepsilon$  can be chosen small enough so that  $\|y - x'\| < \lambda$ ,  $\max_{1 \leq i \leq n} \|y_i - \omega'_i\| < \lambda/\gamma$ ,  $0 < \max_{1 \leq i \leq n} \|y_i - y\| < \varphi^{-1}(\lambda)$ , and  $\tau + \varepsilon < 1$ . The last inequality again yields  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma < 1$ , which contradicts (29).  $\square$

Similar to the case of nonlinear semitransversality, the difference in the contribution of components of subgradients of  $\hat{f}$  to the key dual condition (29) in Proposition 5.1 can be exposed.

- Corollary 5.1** (i) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $x' \in X$  satisfying (19), there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  such that  $\|x^*\| \geq 1$  for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (21) and (22), and all  $(x_1^*, \dots, x_n^*, x^*) \in \partial \widehat{f}(\omega_1, \dots, \omega_n, x)$  with  $\sum_{i=1}^n \|x_i^*\| < \gamma$ , where  $\partial$  stands for the Clarke subdifferential ( $\partial := \partial^C$ ).*
- (ii) *If  $X$  is Asplund, then the above assertion is valid with  $\partial$  standing for the Fréchet subdifferential ( $\partial := \partial^F$ ), and condition (22) replaced by (65).*

*Proof* Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is not  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , and let  $\gamma > 0$  be given. By Proposition 5.1, there exists a point  $x' \in X$  satisfying (19) such that, for any  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$ , there exist points  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (21) and (22), and a subgradient  $(x_1^*, \dots, x_n^*, x^*) \in \partial^C \widehat{f}(\omega_1, \dots, \omega_n, x)$  ( $(x_1^*, \dots, x_n^*, x^*) \in \partial^F \widehat{f}(\omega_1, \dots, \omega_n, x)$  if  $X$  is Asplund) such that  $\|(x_1^*, \dots, x_n^*, x^*)\| < \gamma$ . By the representation (28) of the dual norm, this implies  $\sum_{i=1}^n \|x_i^*\| < \gamma$  and  $\|x^*\| < 1$ . The latter inequality contradicts the assumption.  $\square$

*Remark 5.1* In the Hölder setting, i.e. when  $\varphi(t) = \alpha^{-1}t^q$  with  $\alpha > 0$  and  $q > 0$ , Corollary 5.1 improves [33, Proposition 7].

The next ‘ $\delta$ -free’ statement is a direct consequence of Corollary 5.1.

- Corollary 5.2** (i) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  if*

$$\lim_{\gamma \downarrow 0} \liminf_{\substack{x' \rightarrow \bar{x}, t \downarrow 0 \\ t = \max_{1 \leq i \leq n} d(x', \Omega_i) > 0}} \lim_{\lambda \downarrow \varphi(t)} \inf_{\substack{\|x-x'\| < \lambda, \|\omega_i - \omega'_i\| < \lambda/\gamma, \omega_i, \omega'_i \in \Omega_i (i=1, \dots, n) \\ 0 < \max_{1 \leq i \leq n} \|\omega_i - x\| \leq \max_{1 \leq i \leq n} \|\omega'_i - x'\| < \varphi^{-1}(\lambda) \\ (x_1^*, \dots, x_n^*, x^*) \in \partial \widehat{f}(\omega_1, \dots, \omega_n, x), \sum_{i=1}^n \|x_i^*\| < \gamma}} \|x^*\| > 1, \quad (66)$$

or, in particular, if

$$\liminf_{\substack{x \rightarrow \bar{x}, \omega_i \rightarrow \bar{x}, x_i^* \rightarrow 0 (i=1, \dots, n) \\ \max_{1 \leq i \leq n} \|\omega_i - x\| > 0, (x_1^*, \dots, x_n^*, x^*) \in \partial \widehat{f}(\omega_1, \dots, \omega_n, x)}} \|x^*\| > 1, \quad (67)$$

where  $\partial$  stands for the Clarke subdifferential ( $\partial := \partial^C$ ).

- (ii) *If  $X$  is Asplund, then the above assertion is valid with  $\partial$  being the Fréchet subdifferential ( $\partial := \partial^F$ ).*

*Remark 5.2* Condition (67) is obviously stronger than (66).

Next, similar to the case of nonlinear semitransversality, we apply the sum rules again to the function  $\widehat{f}$  to obtain characterizations of nonlinear subtransversality in terms of normals to the given individual sets.

**Theorem 5.1** *Let  $\varphi \in \mathcal{C}^1$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\mu > 0$  and any  $x' \in X$  satisfying (19), one of the following conditions is satisfied:*

- (i) *there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  such that*

$$\varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right) \left( \left\| \sum_{i=1}^n x_i^* \right\| + \mu \sum_{i=1}^n d(x_i^*, N_{\Omega_i}(\omega_i)) \right) \geq 1 \quad (68)$$

for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (22) and

$$\|x - x'\| < \lambda, \quad \max_{1 \leq i \leq n} \|\omega_i - \omega'_i\| < \mu \lambda, \quad (69)$$

and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37) and

$$\sum_{i=1}^n \langle x_i^*, x - \omega_i \rangle = \max_{1 \leq i \leq n} \|x - \omega_i\|, \quad (70)$$

where  $N$  in (68) stands for the Clarke normal cone ( $N := N^C$ );

- (ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  and a  $\tau \in ]0, 1[$  such that inequality (68) holds with  $N$  standing for the Fréchet normal cone ( $N := N^F$ ) for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (65) and (69), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37) and

$$\sum_{i=1}^n \langle x_i^*, x - \omega_i \rangle > \tau \max_{1 \leq i \leq n} \|x - \omega_i\|. \quad (71)$$

*Proof* Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is not  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , and let  $\mu > 0$  be given. By Proposition 5.1, there exists a point  $x' \in X$  satisfying (19) such that, for any  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$ , there exist points  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (21) and (22) ((65) if  $X$  is Asplund), and a subgradient  $(\hat{x}_1^*, \dots, \hat{x}_n^*, \hat{x}^*) \in \partial \hat{f}(\omega_1, \dots, \omega_n, x)$  such that condition (40) holds, where  $\gamma := \mu^{-1}$  and  $\partial$  stands for either the Clarke subdifferential (if  $X$  is a general Banach space) or the Fréchet subdifferential (if  $X$  is Asplund). Recall from (25) that  $\hat{f}$  is a sum of two functions: the function  $f$  given by (26) and the indicator function of the set  $\Omega_1 \times \dots \times \Omega_n$ . Since  $\max_{1 \leq i \leq n} \|\omega_i - x\| > 0$ ,  $f$  is locally Lipschitz continuous near  $(\omega_1, \dots, \omega_n, x)$ .

- (i)  $X$  is a general Banach space, and  $(\hat{x}_1^*, \dots, \hat{x}_n^*, \hat{x}^*)$  in (40) is a Clarke subgradient. By the Clarke–Rockafellar sum rule (Lemma 2.1(i)), there exist vectors  $(\tilde{x}_1^*, \dots, \tilde{x}_n^*, \tilde{x}^*) \in \partial^C f(\omega_1, \dots, \omega_n, x)$  and  $(u_1^*, \dots, u_n^*) \in N_{\Omega_1 \times \dots \times \Omega_n}^C(\omega_1, \dots, \omega_n)$  such that equality (41) holds, where  $f$  is given by (26). By Lemma 2.3, Proposition 2.1 and Remark 2.2, these vectors satisfy (42), (43) and  $(-x_1^*, \dots, -x_n^*, x^*) \in \partial \psi(\omega_1, \dots, \omega_n, x)$ , where  $\psi(u_1, \dots, u_n, u) := \max_{1 \leq i \leq n} \|u_i - u\|$  ( $u_1, \dots, u_n, u \in X$ ). By Lemma 2.4, conditions (37), (45) and (70) are satisfied. Combining (40), (41), (42), (43), (45) and (28), we obtain

$$\varphi \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right) \left( \left\| \sum_{i=1}^n x_i^* \right\| + \mu \sum_{i=1}^n d(x_i^*, N_{\Omega_i}^C(\omega_i)) \right) < 1.$$

This contradicts (68).

- (ii) Let  $X$  be Asplund and a number  $\tau \in ]0, 1[$  be given. Instead of the Clarke–Rockafellar sum rule, one can employ the fuzzy sum rule (Lemma 2.1(ii)): for any  $\varepsilon > 0$ , there exist points  $y \in B_\varepsilon(x)$ ,  $y_i \in B_\varepsilon(\omega_i)$ ,  $\omega_i'' \in \Omega_i \cap B_\varepsilon(\omega_i)$  ( $i = 1, \dots, n$ ), and vectors  $(\tilde{x}_1^*, \dots, \tilde{x}_n^*, \tilde{x}^*) \in \partial^F f(y_1, \dots, y_n, y)$  and  $(u_1^*, \dots, u_n^*) \in N_{\Omega_1 \times \dots \times \Omega_n}^F(\omega_1'', \dots, \omega_n'')$  satisfying (46). Denote

$$\beta := \left| \varphi' \left( \max_{1 \leq i \leq n} \|\omega_i'' - y\| \right) - \varphi' \left( \max_{1 \leq i \leq n} \|y_i - y\| \right) \right|, \quad (72)$$

and observe that  $\beta \rightarrow 0$  as  $\varepsilon \downarrow 0$ . The number  $\varepsilon$  can be chosen small enough so that

$$\|y - x'\| < \lambda, \quad \max_{1 \leq i \leq n} \|y_i' - \omega_i''\| < \frac{\lambda}{\gamma}, \quad 0 < \max_{1 \leq i \leq n} \|y_i' - y\| < \varphi^{-1}(\lambda),$$

condition (48) is satisfied, and

$$\max_{1 \leq i \leq n} \|y_i - \omega_i''\| < \frac{1 - \tau}{2} \max_{1 \leq i \leq n} \|y - \omega_i''\|. \quad (73)$$

By Lemma 2.3, Proposition 2.1, Remark 2.2 and Lemma 2.4, we have (50) and

$$(\tilde{x}_1^*, \dots, \tilde{x}_n^*, \tilde{x}^*) = \varphi' \left( \max_{1 \leq i \leq n} \|y_i - y\| \right) (-x_1^*, \dots, -x_n^*, x^*), \quad (74)$$

where vectors  $x_1^*, \dots, x_n^*, x^* \in X^*$  satisfy (37), (45) and

$$\sum_{i=1}^n \langle x_i^*, y - y_i \rangle = \max_{1 \leq i \leq n} \|y - y_i\|. \quad (75)$$

It follows from (73) and (75) that

$$\begin{aligned} \sum_{i=1}^n \langle x_i^*, y - \omega_i'' \rangle &\geq \sum_{i=1}^n \langle x_i^*, y - y_i \rangle - \max_{1 \leq i \leq n} \|y_i - \omega_i''\| \\ &= \max_{1 \leq i \leq n} \|y - y_i\| - \max_{1 \leq i \leq n} \|y_i - \omega_i''\| \\ &\geq \max_{1 \leq i \leq n} \|y - \omega_i''\| - 2 \max_{1 \leq i \leq n} \|y_i - \omega_i''\| > \tau \max_{1 \leq i \leq n} \|y - \omega_i''\|. \end{aligned}$$

Combining (45), (46), (50), (74) and (28), we obtain

$$\begin{aligned} \varphi' \left( \max_{1 \leq i \leq n} \|y_i - y\| \right) & \left( \left\| \sum_{i=1}^n x_i^* \right\| + \mu \sum_{i=1}^n d(x_i^*, N_{\Omega_i}^F(\omega_i'')) \right) \\ & \stackrel{(45),(74)}{=} \|\bar{x}^*\| + \mu \sum_{i=1}^n d(-\bar{x}_i^*, N_{\Omega_i}^F(\omega_i'')) \stackrel{(50)}{\leq} \|\bar{x}^*\| + \mu \sum_{i=1}^n \|\hat{x}_i^* + u_i^*\| \\ & \stackrel{(28)}{=} \|(\hat{x}_1^*, \dots, \hat{x}_n^*, \bar{x}^*) + (u_1^*, \dots, u_n^*, 0)\|_\gamma \stackrel{(46)}{<} \|(\hat{x}_1^*, \dots, \hat{x}_n^*, \bar{x}^*)\|_\gamma + \varepsilon. \end{aligned}$$

Hence, thanks to (72), (48) and (75),

$$\begin{aligned} \varphi' \left( \max_{1 \leq i \leq n} \|\omega_i'' - y\| \right) & \left( \left\| \sum_{i=1}^n x_i^* \right\| + \mu \sum_{i=1}^n d(x_i^*, N_{\Omega_i}^F(\omega_i'')) \right) \\ & < \|(\hat{x}_1^*, \dots, \hat{x}_n^*, \bar{x}^*)\|_\gamma + (1 + \mu)\beta + \varepsilon < 1. \end{aligned}$$

This contradicts (68).  $\square$

The next corollary ‘separates’ the two conditions on the dual vectors  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) combined in the key dual transversality condition (68) in Theorem 5.1.

**Corollary 5.3** *Let  $\varphi \in \mathcal{C}^1$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $x' \in X$  satisfying (19), one of the following conditions is satisfied:*

- (i) *there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  such that*

$$\varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right) \left\| \sum_{i=1}^n x_i^* \right\| \geq 1 \quad (76)$$

*for all  $x \in X$  and  $\omega_i, \omega_i' \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (21) and (22), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (70) and*

$$\varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right) \sum_{i=1}^n d(x_i^*, N_{\Omega_i}(\omega_i)) < \gamma, \quad (77)$$

*where  $N$  stands for the Clarke normal cone ( $N := N^C$ );*

- (ii)  *$X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  and a  $\tau \in ]0, 1[$  such that inequality (76) holds for all  $x \in X$  and  $\omega_i, \omega_i' \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (21) and (65), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (71) and (77), where  $N$  stands for the Fréchet normal cone ( $N := N^F$ );*
- (iii) *there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  such that*

$$\varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right) \sum_{i=1}^n d(x_i^*, N_{\Omega_i}(\omega_i)) \geq \gamma \quad (78)$$

*for all  $x \in X$  and  $\omega_i, \omega_i' \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (21) and (22), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (70) and*

$$\varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right) \left\| \sum_{i=1}^n x_i^* \right\| < 1, \quad (79)$$

*where  $N$  in (78) stands for the Clarke normal cone ( $N := N^C$ );*

- (iv)  *$X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  and a  $\tau \in ]0, 1[$  such that inequality (78) holds for all  $x \in X$  and  $\omega_i, \omega_i' \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (21) and (65), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (71) and (79), where  $N$  in (78) stands for the Fréchet normal cone ( $N := N^F$ ).*

*Proof* It is sufficient to notice that the assumptions in parts (i)–(iv) imply those in the respective parts of Theorem 5.1.  $\square$

The next corollary complements Corollary 5.3 and accommodates for the two ‘exact’ cases  $x_i^* \in N_{\Omega_i}(\omega_i)$  ( $i = 1, \dots, n$ ) and  $\sum_{i=1}^n x_i^* = 0$ . It is a consequence of Theorem 5.1 and Lemma 2.5; cf. the proof of Corollary 4.4.

**Corollary 5.4** Let  $\varphi \in \mathcal{C}^1$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\mu > 0$  and any  $x' \in X$  satisfying (19), one of the following conditions is satisfied:

(i) there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  such that

$$\left[ \varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right) \right]^{-1} + 1 \leq \mu, \quad (80)$$

and condition (76) holds true for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (22) and (69), and all  $x_i^* \in N_{\Omega_i}^C(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37) and (71) with

$$\tau := \frac{\mu \varphi'(\max_{1 \leq i \leq n} \|\omega_i - x\|) - 1}{\mu \varphi'(\max_{1 \leq i \leq n} \|\omega_i - x\|) + 1}; \quad (81)$$

(ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  and a  $\hat{\tau} \in ]0, 1[$  such that condition (76) hold true and

$$\left[ \varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right) \right]^{-1} + 1 \leq \hat{\tau} \mu \quad (82)$$

for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (65) and (69), and all  $x_i^* \in N_{\Omega_i}^F(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37) and (71) with

$$\tau := \frac{\hat{\tau} \mu \varphi'(\max_{1 \leq i \leq n} \|\omega_i - x\|) - 1}{\mu \varphi'(\max_{1 \leq i \leq n} \|\omega_i - x\|) + 1}; \quad (83)$$

(iii) there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  such that

$$\left[ \varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right) \right]^{-1} + \mu \leq 1, \quad (84)$$

and condition (78) holds true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Clarke normal cone ( $N := N^C$ ) for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (22) and (69), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$ , and conditions (37) and (71) with

$$\tau := \frac{\varphi'(\max_{1 \leq i \leq n} \|\omega_i - x\|) - 1}{\varphi'(\max_{1 \leq i \leq n} \|\omega_i - x\|) + 1}; \quad (85)$$

(iv)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta[$  and a  $\hat{\tau} \in ]0, 1[$  such that condition (78) hold true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Fréchet normal cone ( $N := N^F$ ), and

$$\left[ \varphi' \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right) \right]^{-1} + \mu \leq \hat{\tau} \quad (86)$$

for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (65) and (69), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$ , and conditions (37) and (71) with

$$\tau := \frac{\hat{\tau} \varphi'(\max_{1 \leq i \leq n} \|\omega_i - x\|) - 1}{\varphi'(\max_{1 \leq i \leq n} \|\omega_i - x\|) + 1}. \quad (87)$$

*Remark 5.3* Since  $\tau$  defined by any of the formulas (81), (83), (85) and (87) is not in general a constant, but depends on  $x$  and  $\omega_i$  ( $i = 1, \dots, n$ ), checking condition (71) with such a  $\tau$  does not seem practical. Such a check becomes meaningful in the linear setting, i.e. when  $\varphi'$ , and consequently, also  $\tau$  is constant; cf. Remark 7.1.

Dropping condition (71) in all parts of Corollary 5.4, we can formulate its simplified and easier to use version.

**Corollary 5.5** Let  $\varphi \in \mathcal{C}^1$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -subtransversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\mu > 0$  and any  $x' \in X$  satisfying (19), one of the following conditions is satisfied:

- (i) *there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  such that conditions (76) and (80) hold true for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (22) and (69), and all  $x_i^* \in N_{\Omega_i}^C(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37);*
- (ii)  *$X$  is Asplund, and there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  such that conditions (76) and (82) hold true for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (65) and (69), and all  $x_i^* \in N_{\Omega_i}^F(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37);*
- (iii) *there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} d(x', \Omega_i)), \delta_1[$  such that conditions (78) and (84) hold true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Clarke normal cone ( $N := N^C$ ) for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (22) and (69), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$  and condition (37);*
- (iv)  *$X$  is Asplund, and there exists a  $\lambda \in ]\varphi(\max_{1 \leq i \leq n} \|x_i\|), \delta_1[$  such that conditions (86) and (78) hold true, the latter with  $\gamma := \mu^{-1}$  and  $N$  standing for the Fréchet normal cone ( $N := N^F$ ), for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (65) and (69), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$  and condition (37).*

## 6 Transversality

The dual characterizations in this section follow the pattern of those in Sections 4 and 5 with appropriate adjustments in the proofs. If not explicitly stated otherwise, the space  $X$  is assumed to be Banach, and the sets  $\Omega_1, \dots, \Omega_n$  are assumed closed. We use the function  $\hat{f}$  given by (25) with  $f : X^{n+1} \rightarrow \mathbb{R}_+$  defined by (27). The next statement provides dual characterizations of  $\varphi$ -transversality in terms of subdifferentials of  $\hat{f}$ .

- Proposition 6.1** (i) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\xi \in ]0, \varphi^{-1}(\delta_1)[$ , there exists a  $\lambda \in ]\varphi(\xi), \delta_1[$  such that condition (29) is satisfied for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (23) and (24), where  $\partial$  in (29) stands for the Clarke subdifferential ( $\partial := \partial^C$ ).*
- (ii) *If  $X$  is Asplund, then the above assertion is valid with  $\partial$  in (29) standing for the Fréchet subdifferential ( $\partial := \partial^F$ ), and condition (24) replaced by (30).*

*Proof* (i) Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is not  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , and let  $\gamma > 0$  be given. By Proposition 3.4, there exist points  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and a number  $\xi \in ]0, \varphi^{-1}(\delta_1)[$  such that, for any  $\lambda \in ]\varphi(\xi), \delta_1[$ , there exist points  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ), and a number  $\tau \in ]0, 1[$  such that conditions (23) and (24) are satisfied, and inequality (31) holds for all  $u_i \in \Omega_i$  near  $\omega_i$  ( $i = 1, \dots, n$ ) and all  $u$  near  $x$ . In other words,  $(\omega_1, \dots, \omega_n, x)$  is a local minimizer of the function (32). By Lemma 2.2, its Fréchet and, as a consequence, Clarke subdifferential at this point contains 0. Observe that (32) is the sum of the function  $\hat{f}$  and the Lipschitz continuous convex function  $(u_1, \dots, u_n, u) \mapsto \tau \|(u_1, \dots, u_n, u) - (\omega_1, \dots, \omega_n, x)\|_\gamma$ , and at any point all subgradients  $(x_1^*, \dots, x_n^*, x^*)$  of the latter function satisfy  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma \leq \tau$ . By the Clarke–Rockafellar sum rule (Lemma 2.1(i)), there exists a subgradient  $(x_1^*, \dots, x_n^*, x^*) \in \partial^C \hat{f}(\omega_1, \dots, \omega_n, x)$  such that  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma \leq \tau < 1$ . The last inequality contradicts (29).

- (ii) If  $X$  is Asplund, then one can employ the fuzzy sum rule (Lemma 2.1(ii)): for any  $\varepsilon > 0$ , there exist points  $x' \in B_\varepsilon(x)$  and  $y_i \in \Omega_i \cap B_\varepsilon(\omega_i)$  ( $i = 1, \dots, n$ ), and a subgradient  $(x_1^*, \dots, x_n^*, x^*) \in \partial^F \hat{f}(y_1, \dots, y_n, x')$  such that  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma < \tau + \varepsilon$ . The number  $\varepsilon$  can be chosen small enough so that  $\|x' - \bar{x}\| < \lambda$ ,  $\max_{1 \leq i \leq n} \|y_i - \omega'_i\| < \lambda/\gamma$ ,  $0 < \max_{1 \leq i \leq n} \|y_i - x_i - x'\| < \varphi^{-1}(\lambda)$ , and  $\tau + \varepsilon < 1$ . The last inequality again yields  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma < 1$ , which contradicts (29).  $\square$

Similar to the cases of the nonlinear semitransversality and subtransversality, the difference in the contribution of components of subgradients of  $\hat{f}$  to the key dual condition (29) in Proposition 6.1 can be exposed.

- Corollary 6.1** (i) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\xi \in ]0, \varphi^{-1}(\delta_1)[$ , there exists a  $\lambda \in ]\varphi(\xi), \delta_1[$  such that  $\|x^*\| \geq 1$  for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (23) and (24), and all  $(x_1^*, \dots, x_n^*, x^*) \in \partial \hat{f}(\omega_1, \dots, \omega_n, x)$  with  $\sum_{i=1}^n \|x_i^*\| < \gamma$ , where  $\partial$  stands for the Clarke subdifferential ( $\partial := \partial^C$ ).*

- (ii) If  $X$  is Asplund, then the above assertion is valid with  $\partial$  standing for the Fréchet subdifferential ( $\partial := \partial^F$ ), and condition (24) replaced by (30).

*Proof* Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is not  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , and let  $\gamma > 0$  be given. By Proposition 6.1, there exist points  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and a number  $\xi \in ]0, \varphi^{-1}(\delta_1)[$  such that, for any  $\lambda \in ]\varphi(\xi), \delta_1[$ , there exist  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (23) and (24) ((30) if  $X$  is Asplund), and a subgradient  $(x_1^*, \dots, x_n^*, x^*) \in \partial^C \widehat{f}(\omega_1, \dots, \omega_n, x)$  ( $(x_1^*, \dots, x_n^*, x^*) \in \partial^F \widehat{f}(\omega_1, \dots, \omega_n, x)$  if  $X$  is Asplund) such that  $\|(x_1^*, \dots, x_n^*, x^*)\|_\gamma < 1$ . By the representation (28) of the dual norm, this implies  $\sum_{i=1}^n \|x_i^*\| < \gamma$  and  $\|x^*\| < 1$ . The latter inequality contradicts the assumption.  $\square$

The next ‘ $\delta$ -free’ statement is a direct consequence of Corollary 6.1.

**Corollary 6.2** (i) *The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  if*

$$\lim_{\gamma \downarrow 0} \liminf_{\omega'_i \rightarrow \bar{x} (i=1, \dots, n)} \lim_{\lambda \downarrow \varphi(\xi)} \inf_{\substack{0 < \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \leq \xi \\ (x_1^*, \dots, x_n^*, x^*) \in \partial \widehat{f}(\omega_1, \dots, \omega_n, x), \sum_{i=1}^n \|x_i^*\| < \gamma}} \|x^*\| > 1, \quad (88)$$

or, in particular, if condition (34) is satisfied, where  $\partial$  in both conditions stands for the Clarke subdifferential ( $\partial := \partial^C$ ).

- (ii) If  $X$  is Asplund, then the above assertion is valid with  $\partial$  standing for the Fréchet subdifferential ( $\partial := \partial^F$ ).

*Remark 6.1* Condition (34) is obviously stronger than (88).

Next, similar to the cases of the nonlinear semitransversality and subtransversality, we apply the sum rules again to the function  $\widehat{f}$  to obtain characterizations of the nonlinear transversality in terms of normals to the given individual sets.

**Theorem 6.1** *Let  $\varphi \in \mathcal{C}^1$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\mu > 0$  and any  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\xi \in ]0, \varphi^{-1}(\delta_1)[$ , one of the following conditions is satisfied:*

- (i) *there exists a  $\lambda \in ]\varphi(\xi), \delta_1[$  such that inequality (35) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (24) and*

$$\|x - \bar{x}\| < \lambda, \quad \max_{1 \leq i \leq n} \|\omega_i - \omega'_i\| < \mu \lambda, \quad (89)$$

and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37) and (38), where  $N$  stands for the Clarke normal cone ( $N := N^C$ );

- (ii)  *$X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\xi), \delta_1[$  and a  $\tau \in ]0, 1[$  such that inequality (35) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (30) and (89), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37) and (39), where  $N$  stands for the Fréchet normal cone ( $N := N^F$ ).*

*Proof* Suppose  $\{\Omega_1, \dots, \Omega_n\}$  is not  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$ , and let  $\mu > 0$  be given. Set  $\gamma := \mu^{-1}$ . By Proposition 6.1, there exist points  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and a  $\xi \in ]0, \varphi^{-1}(\delta_1)[$  such that, for any  $\lambda \in ]\varphi(\xi), \delta_1[$ , there exist points  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying conditions (24) ((30) if  $X$  is Asplund) and (89), and a subgradient  $(\hat{x}_1^*, \dots, \hat{x}_n^*, \hat{x}^*) \in \partial \widehat{f}(\omega_1, \dots, \omega_n, x)$  such that condition (40) holds, where  $\partial$  stands for either the Clarke subdifferential (if  $X$  is a general Banach space) or the Fréchet subdifferential (if  $X$  is Asplund). The rest of the proof follows that of Theorem 4.1.  $\square$

*Remark 6.2* The sufficient conditions in Theorems 4.1 and 5.1 correspond to setting, respectively,  $\omega'_i := \bar{x}$  ( $i = 1, \dots, n$ ) and  $x_1 = \dots = x_n$  in the sufficient conditions in Theorem 6.1.

The next corollary ‘separates’ the two conditions on the dual vectors  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) combined in the key dual transversality condition (35) in Theorem 6.1.

**Corollary 6.3** *Let  $\varphi \in \mathcal{C}^1$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ -transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\xi \in ]0, \varphi^{-1}(\delta_1)[$ , one of the following conditions is satisfied:*

- (i) there exists a  $\lambda \in ]\varphi(\xi), \delta_1[$  such that inequality (53) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (23) and (24), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (38) and (54), where  $N$  stands for the Clarke normal cone ( $N := N^C$ );
- (ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\xi), \delta_1[$  and a  $\tau \in ]0, 1[$  such that inequality (53) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (23) and (30), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (39) and (54), where  $N$  stands for the Fréchet normal cone ( $N := N^F$ );
- (iii) there exists a  $\lambda \in ]\varphi(\xi), \delta_1[$  such that inequality (55) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (23) and (24), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (38) and (56), where  $N$  stands for the Clarke normal cone ( $N := N^C$ );
- (iv)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\xi), \delta_1[$  and a  $\tau \in ]0, 1[$  such that inequality (55) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (23) and (30), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (39) and (56), where  $N$  stands for the Fréchet normal cone ( $N := N^F$ ).

*Proof* It is sufficient to notice that the assumptions in parts (i)–(iv) imply those in the respective parts of Theorem 6.1; cf. the proof of Corollary 4.3.  $\square$

The next corollary complements Corollary 6.3 and accommodates for the two ‘exact’ cases  $x_i^* \in N_{\Omega_i}(\omega_i)$  ( $i = 1, \dots, n$ ) and  $\sum_{i=1}^n x_i^* = 0$ . They are consequences of Theorem 6.1 and Lemma 2.5; cf. the proof of Corollary 4.4.

**Corollary 6.4** *Let  $\varphi \in \mathcal{C}^1$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\varphi$ –transversal at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\mu > 0$  and any  $\omega_i' \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\xi \in ]0, \varphi^{-1}(\delta_1)[$ , one of the following conditions is satisfied:*

- (i) there exists a  $\lambda \in ]\varphi(\xi), \delta_1[$  such that conditions (57) and (53) hold true for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (24) and (89), and all  $x_i^* \in N_{\Omega_i}^C(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37) and (39) with  $\tau$  defined by (58);
- (ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\xi), \delta_1[$  and a  $\hat{\tau} \in ]0, 1[$  such that conditions (59) and (53) hold true for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (30) and (89), and all  $x_i^* \in N_{\Omega_i}^F(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37) and (39) with  $\tau$  defined by (60);
- (iii) there exists a  $\lambda \in ]\varphi(\xi), \delta_1[$  such that conditions (61) and (55) hold true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Clarke normal cone ( $N := N^C$ ) for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (24) and (89), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$ , and conditions (37) and (39) with  $\tau$  defined by (62);
- (iv)  $X$  is Asplund, and there exist a  $\lambda \in ]\varphi(\xi), \delta_1[$  and a  $\hat{\tau} \in ]0, 1[$  such that conditions (63) and (55) hold true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Fréchet normal cone ( $N := N^F$ ) for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (30) and (89), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$ , and conditions (37) and (39) with  $\tau$  defined by (64).

*Remark 6.3* In all parts of Corollary 6.4, checking condition (39) (and the mentioning of  $\hat{\tau} \in ]0, 1[$  in parts (ii) and (iv)) can be dropped.

## 7 Hölder Transversality Properties

In this section, we consider the most important realizations of the three nonlinear transversality properties corresponding to the Hölder setting, i.e.  $\varphi$  being a power function, given for all  $t \geq 0$  by  $\varphi(t) := \alpha^{-1}t^q$  with  $\alpha > 0$  and  $q > 0$  ( $q \in ]0, 1[$  in the case of Hölder subtransversality and Hölder transversality). If not explicitly stated otherwise, the space  $X$  is assumed to be Banach, and the sets  $\Omega_1, \dots, \Omega_n$  are assumed closed.

The next three statements are direct consequences of Theorem 4.1, and Corollaries 4.3 and 4.4, respectively.

**Corollary 7.1** *Let  $\alpha > 0$  and  $q > 0$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\alpha$ –semitransversal of order  $q$  at  $\bar{x}$  with some  $\delta > 0$  if, for some  $\mu > 0$  and any  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying*

$$0 < \max_{1 \leq i \leq n} \|x_i\| < (\alpha\delta)^{\frac{1}{q}}, \quad (90)$$

*one of the following conditions is satisfied:*

(i) there exists a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} \|x_i\|)^q, \delta[$  such that

$$q \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right)^{q-1} \left( \left\| \sum_{i=1}^n x_i^* \right\| + \mu \sum_{i=1}^n d(x_i^*, N_{\Omega_i}(\omega_i)) \right) \geq \alpha \quad (91)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (18) and (36), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37) and (38), where  $N$  in (91) stands for the Clarke normal cone ( $N := N^C$ );

(ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} \|x_i\|)^q, \delta[$  and a  $\tau \in ]0, 1[$  such that inequality (91) holds with  $N$  standing for the Fréchet normal cone ( $N := N^F$ ) for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (36) and

$$0 < \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| < (\alpha\lambda)^{\frac{1}{q}}, \quad (92)$$

and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37) and (39).

**Corollary 7.2** Let  $\alpha > 0$  and  $q > 0$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\alpha$ -semitransversal at  $\bar{x}$  of order  $q$  with some  $\delta > 0$  if, for some  $\gamma > 0$  and any  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying (90), one of the following conditions is satisfied:

(i) there exists a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} \|x_i\|)^q, \delta[$  such that

$$q \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right)^{q-1} \left\| \sum_{i=1}^n x_i^* \right\| \geq \alpha \quad (93)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (17) and (18), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (38) and

$$q \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right)^{q-1} \sum_{i=1}^n d(x_i^*, N_{\Omega_i}(\omega_i)) < \alpha\gamma, \quad (94)$$

where  $N$  stands for the Clarke normal cone ( $N := N^C$ );

(ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} \|x_i\|)^q, \delta[$  and a  $\tau \in ]0, 1[$  such that inequality (93) holds for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (17) and (92), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (39) and (94), where  $N$  stands for the Fréchet normal cone ( $N := N^F$ );

(iii) there exists a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} \|x_i\|)^q, \delta[$  such that

$$q \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right)^{q-1} \sum_{i=1}^n d(x_i^*, N_{\Omega_i}(\omega_i)) \geq \alpha\gamma \quad (95)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (17) and (18), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (38) and

$$q \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right)^{q-1} \left\| \sum_{i=1}^n x_i^* \right\| < \alpha, \quad (96)$$

where  $N$  in (95) stands for the Clarke normal cone ( $N := N^C$ );

(iv)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} \|x_i\|)^q, \delta[$  and a  $\tau \in ]0, 1[$  such that inequality (95) holds for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (17) and (92), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (39) and (96), where  $N$  in (95) stands for the Fréchet normal cone ( $N := N^F$ ).

**Corollary 7.3** Let  $\alpha > 0$  and  $q > 0$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\alpha$ -semitransversal of order  $q$  at  $\bar{x}$  with some  $\delta > 0$  if, for some  $\mu > 0$  and any  $x_i \in X$  ( $i = 1, \dots, n$ ) satisfying (90), one of the following conditions is satisfied:

(i) there exists a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} \|x_i\|)^q, \delta[$  such that

$$\alpha q^{-1} \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right)^{1-q} + 1 \leq \mu, \quad (97)$$

and condition (93) holds true for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (18) and (36), and all  $x_i^* \in N_{\Omega_i}^C(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying conditions (37) and (39) with

$$\tau := \frac{\mu q (\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|)^{q-1} - \alpha}{\mu q (\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|)^{q-1} + \alpha}; \quad (98)$$

- (ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} \|x_i\|)^q, \delta[$  and a  $\hat{\tau} \in ]0, 1[$  such that condition (93) holds true and

$$\alpha q^{-1} \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right)^{1-q} + 1 \leq \hat{\tau} \mu \quad (99)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (36) and (92), and all  $x_i^* \in N_{\Omega_i}^F(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying conditions (37) and (39) with

$$\tau := \frac{\hat{\tau} \mu q (\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|)^{q-1} - \alpha}{\mu q (\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|)^{q-1} + \alpha}; \quad (100)$$

- (iii) there exists a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} \|x_i\|)^q, \delta[$  such that

$$\alpha q^{-1} \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right)^{1-q} + \mu \leq 1, \quad (101)$$

and condition (95) holds true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Clarke normal cone ( $N := N^C$ ) for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (18) and (36), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$ , and conditions (37) and (39) with

$$\tau := \frac{q (\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|)^{q-1} - \alpha}{q (\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|)^{q-1} + \alpha}; \quad (102)$$

- (iv)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} \|x_i\|)^q, \delta[$  and a  $\hat{\tau} \in ]0, 1[$  such that condition (95) holds true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Fréchet normal cone ( $N := N^F$ ), and

$$\alpha q^{-1} \left( \max_{1 \leq i \leq n} \|\omega_i - x_i - x\| \right)^{1-q} + \mu \leq \hat{\tau} \quad (103)$$

for all  $x \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (36) and (92), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$ , and conditions (37) and (39) with

$$\tau := \frac{\hat{\tau} q (\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|)^{q-1} - \alpha}{q (\max_{1 \leq i \leq n} \|\omega_i - x_i - x\|)^{q-1} + \alpha}. \quad (104)$$

*Remark 7.1* Since  $\tau$  defined by any of the formulas (98), (100), (102) and (104) is not in general a constant, but depends on  $x$ ,  $\omega_i$  and  $x_i$  ( $i = 1, \dots, n$ ), checking condition (39) with such a  $\tau$  does not seem practical unless  $q = 1$ . In the latter case, formulas (98), (100), (102) and (104) reduce, respectively, to

$$\tau := \frac{\mu - \alpha}{\mu + \alpha}, \quad \tau := \frac{\hat{\tau} \mu - \alpha}{\mu + \alpha}, \quad \tau := \frac{1 - \alpha}{1 + \alpha} \quad \text{and} \quad \tau := \frac{\hat{\tau} - \alpha}{1 + \alpha}. \quad (105)$$

In the general nonlinear case, condition (39) can be dropped, leading to more concise but stronger sufficient conditions requiring a larger number of candidates to be checked.

The next three statements are direct consequences of Theorem 5.1, and Corollaries 5.3 and 5.4, respectively.

**Corollary 7.4** *Let  $\alpha > 0$  and  $q \in ]0, 1[$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\alpha$ -subtransversal of order  $q$  at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\mu > 0$  and any  $x' \in X$  satisfying*

$$\|x' - \bar{x}\| < \delta_2, \quad 0 < \max_{1 \leq i \leq n} d(x', \Omega_i) < (\alpha \delta_1)^{\frac{1}{q}}, \quad (106)$$

one of the following conditions is satisfied:

(i) there exists a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} d(x', \Omega_i))^q, \delta_1[$  such that

$$q \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right)^{q-1} \left( \left\| \sum_{i=1}^n x_i^* \right\| + \mu \sum_{i=1}^n d(x_i^*, N_{\Omega_i}(\omega_i)) \right) \geq \alpha \quad (107)$$

for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (69) and

$$0 < \max_{1 \leq i \leq n} \|\omega_i - x\| \leq \max_{1 \leq i \leq n} \|\omega'_i - x'\| < (\alpha\lambda)^{\frac{1}{q}}, \quad (108)$$

and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37) and (70), where  $N$  in (107) stands for the Clarke normal cone ( $N := N^C$ );

(ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} d(x', \Omega_i))^q, \delta_1[$  and a  $\tau \in ]0, 1[$  such that inequality (107) holds with  $N$  standing for the Fréchet normal cone ( $N := N^F$ ) for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (69) and

$$0 < \max_{1 \leq i \leq n} \|\omega_i - x\| < (\alpha\lambda)^{\frac{1}{q}}, \quad (109)$$

and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37) and (71).

**Corollary 7.5** Let  $\alpha > 0$  and  $q \in ]0, 1]$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\alpha$ -subtransversal of order  $q$  at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $x' \in X$  satisfying (106), one of the following conditions is satisfied:

(i) there exists a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} d(x', \Omega_i))^q, \delta_1[$  such that

$$q \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right)^{q-1} \left\| \sum_{i=1}^n x_i^* \right\| \geq \alpha \quad (110)$$

for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (21) and (108), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (70) and

$$q \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right)^{q-1} \sum_{i=1}^n d(x_i^*, N_{\Omega_i}(\omega_i)) < \alpha\gamma, \quad (111)$$

where  $N$  stands for the Clarke normal cone ( $N := N^C$ );

(ii) there exists a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} d(x', \Omega_i))^q, \delta_1[$  such that

$$q \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right)^{q-1} \sum_{i=1}^n d(x_i^*, N_{\Omega_i}(\omega_i)) \geq \alpha\gamma \quad (112)$$

for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (21) and (108), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (70) and

$$q \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right)^{q-1} \left\| \sum_{i=1}^n x_i^* \right\| < \alpha, \quad (113)$$

where  $N$  in (112) stands for the Clarke normal cone ( $N := N^C$ );

(iii)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} d(x', \Omega_i))^q, \delta_1[$  and a  $\tau \in ]0, 1[$  such that inequality (110) holds for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (21) and (109), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (71) and (111), where  $N$  stands for the Fréchet normal cone ( $N := N^F$ );

(iv)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} d(x', \Omega_i))^q, \delta_1[$  and a  $\tau \in ]0, 1[$  such that inequality (112) holds for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (21) and (109), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (71) and (113), where  $N$  in (112) stands for the Fréchet normal cone ( $N := N^F$ ).

**Remark 7.2** The sufficient condition in part (i) of Corollary 7.5 improves [33, Theorem 2].

**Corollary 7.6** Let  $\alpha > 0$  and  $q \in ]0, 1]$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\alpha$ -subtransversal of order  $q$  at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\mu > 0$  and any  $x' \in X$  satisfying (106), one of the following conditions is satisfied:

(i) there exists a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} d(x', \Omega_i))^q, \delta_1[$  such that

$$\alpha q^{-1} \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right)^{1-q} + 1 \leq \mu, \quad (114)$$

and condition (110) holds true for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (69) and (108), and all  $x_i^* \in N_{\Omega_i}^C(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37) and (71) with

$$\tau := \frac{\mu q (\max_{1 \leq i \leq n} \|\omega_i - x\|)^q - \alpha}{\mu q (\max_{1 \leq i \leq n} \|\omega_i - x\|)^{q-1} + \alpha}; \quad (115)$$

(ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} d(x', \Omega_i))^q, \delta_1[$  and a  $\hat{\tau} \in ]0, 1[$  such that condition (110) holds true and

$$\alpha q^{-1} \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right)^{1-q} + 1 \leq \hat{\tau} \mu \quad (116)$$

for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (69) and (109), and all  $x_i^* \in N_{\Omega_i}^F(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37) and (71) with

$$\tau := \frac{\hat{\tau} \mu q (\max_{1 \leq i \leq n} \|\omega_i - x\|)^q - \alpha}{\mu q (\max_{1 \leq i \leq n} \|\omega_i - x\|)^{q-1} + \alpha}; \quad (117)$$

(iii) there exists a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} d(x', \Omega_i))^q, \delta_1[$  such that

$$\alpha q^{-1} \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right)^{1-q} + \mu \leq 1, \quad (118)$$

and condition (112) holds true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Clarke normal cone ( $N := N^C$ ) for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (69) and (108), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$ , and conditions (37) and (71) with

$$\tau := \frac{q (\max_{1 \leq i \leq n} \|\omega_i - x\|)^{q-1} - \alpha}{q (\max_{1 \leq i \leq n} \|\omega_i - x\|)^{q-1} + \alpha}; \quad (119)$$

(iv)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}(\max_{1 \leq i \leq n} \|x_i\|)^q, \delta[$  and a  $\hat{\tau} \in ]0, 1[$  such that condition (112) holds true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Fréchet normal cone ( $N := N^F$ ), and

$$\alpha q^{-1} \left( \max_{1 \leq i \leq n} \|\omega_i - x\| \right)^{1-q} + \mu \leq \hat{\tau} \quad (120)$$

for all  $x \in X$  and  $\omega_i, \omega'_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (69) and (109), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$ , and conditions (37) and (71) with

$$\tau := \frac{\hat{\tau} q (\max_{1 \leq i \leq n} \|\omega_i - x\|)^{q-1} - \alpha}{q (\max_{1 \leq i \leq n} \|\omega_i - x\|)^{q-1} + \alpha}. \quad (121)$$

*Remark 7.3* Since  $\tau$  defined by any of the formulas (115), (117), (119) and (121) is not in general a constant, but depends on  $x$  and  $\omega_i$  ( $i = 1, \dots, n$ ), checking condition (71) with such a  $\tau$  does not seem practical and can be dropped (at the expense of weakening the assertions). Such a check becomes meaningful in the linear setting, i.e. when  $q = 1$ . In this case, (115), (117), (119) and (121) reduce to the corresponding expressions in (105).

The next three statements are direct consequences of Theorem 6.1, and Corollaries 6.3 and 6.4, respectively.

**Corollary 7.7** *Let  $\alpha > 0$  and  $q \in ]0, 1[$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\alpha$ -transversal of order  $q$  at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\mu > 0$  and any  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\xi \in ]0, (\alpha \delta_1)^{\frac{1}{q}}[$ , one of the following conditions is satisfied:*

- (i) there exists a  $\lambda \in ]\alpha^{-1}\xi^q, \delta_1[$  such that inequality (91) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (24) and (89), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37) and (38), where  $N$  stands for the Clarke normal cone ( $N := N^C$ );
- (ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}\xi^q, \delta_1[$  and a  $\tau \in ]0, 1[$  such that inequality (91) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (89) and (92), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37) and (39), where  $N$  stands for the Fréchet normal cone ( $N := N^F$ ).

**Corollary 7.8** Let  $\alpha > 0$  and  $q \in ]0, 1[$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\alpha$ -transversal of order  $q$  at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\gamma > 0$  and any  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  and  $\xi \in ]0, (\alpha\delta_1)^{\frac{1}{q}}[$ , one of the following conditions is satisfied:

- (i) there exists a  $\lambda \in ]\alpha^{-1}\xi^q, \delta_1[$  such that inequality (93) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (23) and (24), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (38) and (94), where  $N$  stands for the Clarke normal cone ( $N := N^C$ );
- (ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}\xi^q, \delta_1[$  and a  $\tau \in ]0, 1[$  such that inequality (93) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (23) and (92), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (39) and (94), where  $N$  stands for the Fréchet normal cone ( $N := N^F$ );
- (iii) there exists a  $\lambda \in ]\alpha^{-1}\xi^q, \delta_1[$  such that inequality (95) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (23) and (24), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (38) and (96), where  $N$  stands for the Clarke normal cone ( $N := N^C$ );
- (iv)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}\xi^q, \delta_1[$  and a  $\tau \in ]0, 1[$  such that inequality (95) holds for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (23) and (92), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying (37), (39) and (96), where  $N$  stands for the Fréchet normal cone ( $N := N^F$ ).

**Corollary 7.9** Let  $\alpha > 0$  and  $q \in ]0, 1[$ . The collection  $\{\Omega_1, \dots, \Omega_n\}$  is  $\alpha$ -transversal of order  $q$  at  $\bar{x}$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\mu > 0$  and any  $\omega'_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$  ( $i = 1, \dots, n$ ) and  $\xi \in ]0, (\alpha\delta_1)^{\frac{1}{q}}[$ , one of the following conditions is satisfied:

- (i) there exists a  $\lambda \in ]\alpha^{-1}\xi^q, \delta_1[$  such that conditions (93) and (97) hold true for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (24) and (89), and all  $x_i^* \in N_{\Omega_i}^C(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37) and (39) with  $\tau$  defined by (98);
- (ii)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}\xi^q, \delta_1[$  and a  $\hat{\tau} \in ]0, 1[$  such that conditions (93) and (99) hold true for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (89) and (92), and all  $x_i^* \in N_{\Omega_i}^F(\omega_i)$  ( $i = 1, \dots, n$ ) satisfying (37) and (39) with  $\tau$  defined by (100);
- (iii) there exists a  $\lambda \in ]\alpha^{-1}\xi^q, \delta_1[$  such that conditions (95) and (101) hold true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Clarke normal cone ( $N := N^C$ ) for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying (24) and (89), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$ , and conditions (37) and (39) with  $\tau$  defined by (102);
- (iv)  $X$  is Asplund, and there exist a  $\lambda \in ]\alpha^{-1}\xi^q, \delta_1[$  and a  $\hat{\tau} \in ]0, 1[$  such that conditions (95) and (103) hold true with  $\gamma := \mu^{-1}$  and  $N$  standing for the Fréchet normal cone ( $N := N^F$ ) for all  $x, x_i \in X$  and  $\omega_i \in \Omega_i$  ( $i = 1, \dots, n$ ) satisfying conditions (89) and (92), and all  $x_i^* \in X^*$  ( $i = 1, \dots, n$ ) satisfying  $\sum_{i=1}^n x_i^* = 0$ , and conditions (37) and (39) with  $\tau$  defined by (104).

## 8 Transversality of a Set-Valued Mapping to a Set

In this section, we provide applications of the dual characterizations of nonlinear transversality properties of collections of sets established in Sections 4–7 to nonlinear extensions of the new transversality properties of a set-valued mapping to a set in the range space due to Ioffe [20].

In this section,  $X$  and  $Y$  are normed spaces,  $F : X \rightrightarrows Y$ ,  $S \subset Y$ ,  $(\bar{x}, \bar{y}) \in \text{gph } F$ ,  $\bar{y} \in S$ , and  $\varphi \in \mathcal{C}$ .

**Definition 8.1** The mapping  $F$  is

- (i)  $\varphi$ -semitransversal to  $S$  at  $(\bar{x}, \bar{y})$  if  $\{\text{gph } F, X \times S\}$  is  $\varphi$ -semitransversal at  $(\bar{x}, \bar{y})$ , i.e. there exists a  $\delta > 0$  such that

$$(\text{gph } F - (u_1, v_1)) \cap (X \times (S - v_2)) \cap B_\rho(\bar{x}, \bar{y}) \neq \emptyset \quad (122)$$

for all  $\rho \in ]0, \delta[$ ,  $u_1 \in X$ ,  $v_1, v_2 \in Y$  with  $\varphi(\max\{\|u_1\|, \|v_1\|, \|v_2\|\}) < \rho$ ;

- (ii)  $\varphi$ -subtransversal to  $S$  at  $(\bar{x}, \bar{y})$  if  $\{\text{gph}F, X \times S\}$  is  $\varphi$ -subtransversal at  $(\bar{x}, \bar{y})$ , i.e. there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that

$$\text{gph}F \cap (X \times S) \cap B_\rho(x, y) \neq \emptyset \quad (123)$$

for all  $\rho \in ]0, \delta_1[$  and  $(x, y) \in B_{\delta_2}(\bar{x}, \bar{y})$  with  $\varphi(\max\{d((x, y), \text{gph}F), d(y, S)\}) < \rho$ ;

- (iii)  $\varphi$ -transversal to  $S$  at  $(\bar{x}, \bar{y})$  if  $\{\text{gph}F, X \times S\}$  is  $\varphi$ -transversal at  $(\bar{x}, \bar{y})$ , i.e. there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that

$$(\text{gph}F - (x_1, y_1) - (u_1, v_1)) \cap (X \times (S - y_2 - v_2)) \cap (\rho\mathbb{B}) \neq \emptyset \quad (124)$$

for all  $\rho \in ]0, \delta_1[$ ,  $(x_1, y_1) \in \text{gph}F \cap B_{\delta_2}(\bar{x}, \bar{y})$ ,  $y_2 \in S \cap B_{\delta_2}(\bar{y})$ ,  $u_1 \in X$ ,  $v_1, v_2 \in Y$  with  $\varphi(\max\{\|u_1\|, \|v_1\|, \|v_2\|\}) < \rho$ .

In the rest of the section,  $X$  and  $Y$  are Banach spaces,  $\text{gph}F$  and  $S$  are closed, and  $\varphi \in \mathcal{C}^1$ .

The dual characterizations of the nonlinear transversality properties in the next three statements are direct consequences of Theorems 4.1, 5.1 and 6.1, respectively. For metric characterizations of these properties we refer the readers to [11].

**Theorem 8.1** *The mapping  $F$  is  $\varphi$ -semitransversal to  $S$  at  $(\bar{x}, \bar{y})$  with some  $\delta > 0$  if, for some  $\mu > 0$  and any  $u_1 \in X$ ,  $v_1, v_2 \in Y$  with  $0 < \max\{\|u_1\|, \|v_1\|, \|v_2\|\} < \varphi^{-1}(\delta)$ , one of the following conditions is satisfied:*

- (i) *there exists a  $\lambda \in ]\varphi(\max\{\|u_1\|, \|v_1\|, \|v_2\|\}), \delta[$  such that*

$$\begin{aligned} & \varphi'(\max\{\|x_1 - u_1 - x\|, \|y_1 - v_1 - y\|, \|y_2 - v_2 - y\|\}) (\|x_1^*\| + \|y_1^* + y_2^*\| \\ & \quad + \mu (d((x_1^*, y_1^*), N_{\text{gph}F}(x_1, y_1)) + d(y_2^*, N_S(y_2)))) \geq 1 \end{aligned} \quad (125)$$

for all  $(x, y) \in X \times Y$ ,  $(x_1, y_1) \in \text{gph}F$  and  $y_2 \in S$  satisfying

$$\begin{aligned} & \max\{\|x - \bar{x}\|, \|y - \bar{y}\|\} < \lambda, \quad \max\{\|x_1 - \bar{x}\|, \|y_1 - \bar{y}\|, \|y_2 - \bar{y}\|\} < \mu\lambda, \quad (126) \\ & 0 < \max\{\|x_1 - u_1 - x\|, \|y_1 - v_1 - y\|, \|y_2 - v_2 - y\|\} \\ & \leq \max\{\|u_1\|, \|v_1\|, \|v_2\|\}, \end{aligned} \quad (127)$$

and all  $x_1^* \in X^*$ ,  $y_1^*, y_2^* \in Y^*$  satisfying

$$\|x_1^*\| + \|y_1^*\| + \|y_2^*\| = 1, \quad (128)$$

$$\begin{aligned} & \langle x_1^*, x + u_1 - x_1 \rangle + \langle y_1^*, y + v_1 - y_1 \rangle + \langle y_2^*, y + v_2 - y_2 \rangle \\ & = \max\{\|x_1 - u_1 - x\|, \|y_1 - v_1 - y\|, \|y_2 - v_2 - y\|\}, \end{aligned} \quad (129)$$

where  $N$  in (125) stands for the Clarke normal cone ( $N := N^C$ );

- (ii)  *$X$  and  $Y$  are Asplund spaces, and there exist a  $\lambda \in ]\varphi(\max\{\|u_1\|, \|v_1\|, \|v_2\|\})$  and a  $\tau \in ]0, 1[$  such that inequality (125) holds with  $N$  standing for the Fréchet normal cone ( $N := N^F$ ) for all  $(x, y) \in X \times Y$ ,  $(x_1, y_1) \in \text{gph}F$  and  $y_2 \in S$  satisfying (126) and*

$$0 < \max\{\|x_1 - u_1 - x\|, \|y_1 - v_1 - y\|, \|y_2 - v_2 - y\|\} < \varphi^{-1}(\lambda), \quad (130)$$

and all  $x_1^* \in X^*$ ,  $y_1^*, y_2^* \in Y^*$  satisfying (128) and

$$\begin{aligned} & \langle x_1^*, x + u_1 - x_1 \rangle + \langle y_1^*, y + v_1 - y_1 \rangle + \langle y_2^*, y + v_2 - y_2 \rangle \\ & > \tau \max\{\|x_1 - u_1 - x\|, \|y_1 - v_1 - y\|, \|y_2 - v_2 - y\|\}. \end{aligned} \quad (131)$$

**Theorem 8.2** *The mapping  $F$  is  $\varphi$ -subtransversal to  $S$  at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\mu > 0$  and any  $(x', y') \in B_{\delta_2}(\bar{x}, \bar{y})$  with  $0 < \max\{d((x', y'), \text{gph}F), d(y', S)\} < \varphi^{-1}(\delta_1)$ , one of the following conditions is satisfied:*

(i) there exists a  $\lambda \in ]\varphi(\max\{d((x', y'), \text{gph}F), d(y', S)\}), \delta_1[$  such that

$$\begin{aligned} & \varphi(\max\{\|x_1 - x\|, \|y_1 - y\|, \|y_2 - y\|\}) (\|x_1^*\| + \|y_1^* + y_2^*\| \\ & \quad + \mu (d((x_1^*, y_1^*), N_{\text{gph}F}(x_1, y_1)) + d(y_2^*, N_S(y_2)))) \geq 1 \end{aligned} \quad (132)$$

for all  $(x, y) \in X \times Y$  and  $(x_1, y_1), (x_1', y_1') \in \text{gph}F, y_2, y_2' \in S$  satisfying

$$\begin{aligned} \max\{\|x - x'\|, \|y - y'\|\} &< \lambda, \max\{\|x_1 - x_1'\|, \|y_1 - y_1'\|, \|y_2 - y_2'\|\} < \mu\lambda, \quad (133) \\ 0 &< \max\{\|x_1 - x\|, \|y_1 - y\|, \|y_2 - y\|\} \\ &\leq \max\{\|x_1' - x'\|, \|y_1' - y'\|, \|y_2' - y'\|\} < \varphi^{-1}(\lambda), \end{aligned}$$

and all  $x_1^* \in X^*, y_1^*, y_2^* \in Y^*$  satisfying (128) and

$$\langle x_1^*, x - x_1 \rangle + \langle y_1^*, y - y_1 \rangle + \langle y_2^*, y - y_2 \rangle = \max\{\|x - x_1\|, \|y - y_1\|, \|y - y_2\|\}, \quad (134)$$

where  $N$  in (132) stands for the Clarke normal cone ( $N := N^C$ );

(ii)  $X$  and  $Y$  are Asplund spaces, and there exist numbers  $\lambda \in ]\varphi(\max\{d((x', y'), \text{gph}F), d(y', S)\}), \delta_1[$  and  $\tau \in ]0, 1[$  such that inequality (132) holds with  $N$  standing for the Fréchet normal cone ( $N := N^F$ ) for all  $(x, y) \in X \times Y$  and  $(x_1, y_1), (x_1', y_1') \in \text{gph}F, y_2, y_2' \in S$  satisfying (133) and  $0 < \max\{\|x_1 - x\|, \|y_1 - y\|, \|y_2 - y\|\} < \varphi^{-1}(\lambda)$ , and all  $x_1^* \in X^*, y_1^*, y_2^* \in Y^*$  satisfying (128) and

$$\begin{aligned} & \langle x_1^*, x - x_1 \rangle + \langle y_1^*, y - y_1 \rangle + \langle y_2^*, y - y_2 \rangle \\ & \quad > \tau \max\{\|x - x_1\|, \|y - y_1\|, \|y - y_2\|\}. \end{aligned} \quad (135)$$

**Theorem 8.3** The mapping  $F$  is  $\varphi$ -transversal to  $S$  at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\mu > 0$  and any  $(x_1', y_1') \in \text{gph}F \cap B_{\delta_2}(\bar{x}, \bar{y}), y_2' \in S \cap B_{\delta_2}(\bar{y})$  and  $\xi \in ]0, \varphi^{-1}(\delta_1)[$ , one of the following conditions is satisfied:

(i) there exists a  $\lambda \in ]\varphi(\xi), \delta_1[$  such that inequality (125) holds with  $N$  standing for the Clarke normal cone ( $N := N^C$ ) for all  $(x, y) \in X \times Y, (x_1, y_1) \in \text{gph}F, y_2 \in S$  and  $u_1 \in X, v_1, v_2 \in Y$  satisfying

$$\max\{\|x - \bar{x}\|, \|y - \bar{y}\|\} < \lambda, \max\{\|x_1 - x_1'\|, \|y_1 - y_1'\|, \|y_2 - y_2'\|\} < \mu\lambda, \quad (136)$$

$$\begin{aligned} 0 &< \max\{\|x_1 - u_1 - x\|, \|y_1 - v_1 - y\|, \|y_2 - v_2 - y\|\} \\ &\leq \max\{\|x_1' - u_1 - \bar{x}\|, \|y_1' - v_1 - \bar{y}\|, \|y_2' - v_2 - \bar{y}\|\} = \xi, \end{aligned} \quad (137)$$

and all  $x_1^* \in X^*, y_1^*, y_2^* \in Y^*$  satisfying (128) and (129);

(ii)  $X$  and  $Y$  are Asplund spaces, and there exist numbers  $\lambda \in ]\varphi(\xi), \delta_1[$  and  $\tau \in ]0, 1[$  such that inequality (125) holds with  $N$  standing for the Fréchet normal cone ( $N := N^F$ ) for all  $(x, y) \in X \times Y, (x_1, y_1) \in \text{gph}F, y_2 \in S$  and  $u_1 \in X, v_1, v_2 \in Y$  satisfying (130) and (136), and all  $x_1^* \in X^*, y_1^*, y_2^* \in Y^*$  satisfying (128) and (131).

In the Hölder setting, Definition 8.1 takes the following form.

**Definition 8.2** Let  $\alpha > 0$ . The mapping  $F$  is

- (i)  $\alpha$ -semitransversal of order  $q > 0$  to  $S$  at  $(\bar{x}, \bar{y})$  if there exists a  $\delta > 0$  such that condition (122) is satisfied for all  $\rho \in ]0, \delta[$ ,  $u_1 \in X, v_1, v_2 \in Y$  with  $(\max\{\|u_1\|, \|v_1\|, \|v_2\|\})^q < \alpha\rho$ ;
- (ii)  $\alpha$ -subtransversal of order  $q \in ]0, 1[$  to  $S$  at  $(\bar{x}, \bar{y})$  if there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that condition (123) is satisfied for all  $\rho \in ]0, \delta_1[$  and  $(x, y) \in B_{\delta_2}(\bar{x}, \bar{y})$  with  $(\max\{d((x, y), \text{gph}F), d(y, S)\})^q < \alpha\rho$ ;
- (iii)  $\alpha$ -transversal of order  $q \in ]0, 1[$  to  $S$  at  $(\bar{x}, \bar{y})$  if there exist  $\delta_1 > 0$  and  $\delta_2 > 0$  such that condition (124) is satisfied for all  $\rho \in ]0, \delta_1[$ ,  $(x_1, y_1) \in \text{gph}F \cap B_{\delta_2}(\bar{x}, \bar{y}), y_2 \in S \cap B_{\delta_2}(\bar{y}), u_1 \in X, v_1, v_2 \in Y$  with  $(\max\{\|u_1\|, \|v_1\|, \|v_2\|\})^q < \alpha\rho$ .

In the linear case, i.e. when  $\varphi(t) = \alpha t$  for some  $\alpha > 0$  and all  $t \geq 0$ , the properties in parts (ii) and (iii) of the above definition reduce, respectively, to the ones in [20, Definitions 7.11 and 7.8]. The property in part (i) is new.

Dual characterizations of the Hölder transversality properties follow immediately.

**Corollary 8.1** *Let  $\alpha > 0$  and  $q > 0$ . The mapping  $F$  is  $\alpha$ -semitransversal of order  $q$  to  $S$  at  $(\bar{x}, \bar{y})$  with some  $\delta > 0$  if, for some  $\mu > 0$  and any  $u_1 \in X$ ,  $v_1, v_2 \in Y$  with  $0 < \max\{\|u_1\|, \|v_1\|, \|v_2\|\} < (\alpha\delta)^{\frac{1}{q}}$ , one of the following conditions is satisfied:*

(i) *there exists a  $\lambda \in ]\alpha^{-1}(\max\{\|u_1\|, \|v_1\|, \|v_2\|\})^q, \delta[$  such that*

$$q(\max\{\|x_1 - u_1 - x\|, \|y_1 - v_1 - y\|, \|y_2 - v_2 - y\|\})^{q-1} (\|x_1^*\| + \|y_1^* + y_2^*\| + \mu(d((x_1^*, y_1^*), N_{\text{gph}F}(x_1, y_1)) + d(y_2^*, N_S(y_2)))) \geq \alpha \quad (138)$$

*for all  $(x, y) \in X \times Y$ ,  $(x_1, y_1) \in \text{gph}F$  and  $y_2 \in S$  satisfying (126) and (127), and all  $x_1^* \in X^*$ ,  $y_1^*, y_2^* \in Y^*$  satisfying (128) and (129), where  $N$  in (138) stands for the Clarke normal cone ( $N := N^C$ );*

(ii)  *$X$  and  $Y$  are Asplund spaces, and there exist a  $\lambda \in ]\alpha^{-1}(\max\{\|u_1\|, \|v_1\|, \|v_2\|\})^q$  and a  $\tau \in ]0, 1[$  such that inequality (138) holds with  $N$  standing for the Fréchet normal cone ( $N := N^F$ ) for all  $(x, y) \in X \times Y$ ,  $(x_1, y_1) \in \text{gph}F$  and  $y_2 \in S$  satisfying (126) and*

$$0 < \max\{\|x_1 - u_1 - x\|, \|y_1 - v_1 - y\|, \|y_2 - v_2 - y\|\} < (\alpha\lambda)^{\frac{1}{q}}, \quad (139)$$

*and all  $x_1^* \in X^*$ ,  $y_1^*, y_2^* \in Y^*$  satisfying (128) and (131).*

**Corollary 8.2** *Let  $\alpha > 0$  and  $q \in ]0, 1[$ . The mapping  $F$  is  $\alpha$ -subtransversal of order  $q$  to  $S$  at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\mu > 0$  and any  $(x', y') \in B_{\delta_2}(\bar{x}, \bar{y})$  with  $0 < \max\{d((x', y'), \text{gph}F), d(y', S)\} < (\alpha\delta_1)^{\frac{1}{q}}$ , one of the following conditions is satisfied:*

(i) *there exists a  $\lambda \in ]\alpha^{-1}(\max\{d((x', y'), \text{gph}F), d(y', S)\})^q, \delta_1[$  such that*

$$q(\max\{\|x_1 - x\|, \|y_1 - y\|, \|y_2 - y\|\})^{q-1} (\|x_1^*\| + \|y_1^* + y_2^*\| + \mu(d((x_1^*, y_1^*), N_{\text{gph}F}(x_1, y_1)) + d(y_2^*, N_S(y_2)))) \geq \alpha \quad (140)$$

*for all  $(x, y) \in X \times Y$  and  $(x_1, y_1), (x'_1, y'_1) \in \text{gph}F$ ,  $y_2, y'_2 \in S$  satisfying (133) and*

$$0 < \max\{\|x_1 - x\|, \|y_1 - y\|, \|y_2 - y\|\} \leq \max\{\|x'_1 - x'\|, \|y'_1 - y'\|, \|y'_2 - y'\|\} < (\alpha\lambda)^{\frac{1}{q}},$$

*and all  $x_1^* \in X^*$ ,  $y_1^*, y_2^* \in Y^*$  satisfying (128) and (129), where  $N$  in (140) stands for the Clarke normal cone ( $N := N^C$ );*

(ii)  *$X$  and  $Y$  are Asplund spaces, and there exist numbers  $\lambda \in ]\alpha^{-1}(\max\{d((x', y'), \text{gph}F), d(y', S)\})^q, \delta_1[$  and  $\tau \in ]0, 1[$  such that inequality (140) holds with  $N$  standing for the Fréchet normal cone ( $N := N^F$ ) for all  $(x, y) \in X \times Y$  and  $(x_1, y_1), (x'_1, y'_1) \in \text{gph}F$ ,  $y_2, y'_2 \in S$  satisfying (133) and  $0 < \max\{\|x_1 - x\|, \|y_1 - y\|, \|y_2 - y\|\} < (\alpha\lambda)^{\frac{1}{q}}$ , and all  $x_1^* \in X^*$ ,  $y_1^*, y_2^* \in Y^*$  satisfying (128) and (135).*

**Corollary 8.3** *Let  $\alpha > 0$  and  $q \in ]0, 1[$ . The mapping  $F$  is  $\alpha$ -transversal of order  $q$  to  $S$  at  $(\bar{x}, \bar{y})$  with some  $\delta_1 > 0$  and  $\delta_2 > 0$  if, for some  $\mu > 0$  and any  $(x'_1, y'_1) \in \text{gph}F \cap B_{\delta_2}(\bar{x}, \bar{y})$ ,  $y'_2 \in S \cap B_{\delta_2}(\bar{y})$  and  $\xi \in ]0, (\alpha\delta_1)^{\frac{1}{q}}[$ , one of the following conditions is satisfied:*

(i) *there exists a  $\lambda \in ]\alpha^{-1}(\xi)^q, \delta_1[$  such that inequality (138) holds with  $N$  standing for the Clarke normal cone ( $N := N^C$ ) for all  $(x, y) \in X \times Y$ ,  $(x_1, y_1) \in \text{gph}F$ ,  $y_2 \in S$  and  $u_1 \in X$ ,  $v_1, v_2 \in Y$  satisfying (136) and (137), and all  $x_1^* \in X^*$ ,  $y_1^*, y_2^* \in Y^*$  satisfying (128) and (129);*

(ii)  *$X$  and  $Y$  are Asplund, and there exist a  $\lambda \in ]\alpha^{-1}(\xi)^q, \delta_1[$  and a  $\tau \in ]0, 1[$  such that inequality (138) holds with  $N$  standing for the Fréchet normal cone ( $N := N^F$ ) for all  $(x, y) \in X \times Y$ ,  $(x_1, y_1) \in \text{gph}F$ ,  $y_2 \in S$  and  $u_1 \in X$ ,  $v_1, v_2 \in Y$  satisfying (136) and (139), and all  $x_1^* \in X^*$ ,  $y_1^*, y_2^* \in Y^*$  satisfying (128) and (131).*

**Remark 8.1** The combined dual transversality conditions (125), (132), (138) and (140) involving normal cones to  $\text{gph}F$  and  $S$  in the statements above can be ‘decomposed’ into components; cf. Corollaries 4.3, 4.4, 5.3, 5.4, 6.3, 6.4, 7.5 and 7.6.

## 9 Conclusions

The nonlinear (Hölder) transversality properties of collections of sets have been studied in a systematic way with the main emphasis on quantitative dual sufficient conditions. General sufficient conditions are established in Theorems 4.1, 5.1 and 6.1 in terms of Clarke normals in general Banach spaces and Fréchet normals in Asplund spaces.

The key dual transversality conditions (35) and (68) combine two common types of conditions on the dual vectors: either their sum must be sufficiently far from 0, or the vectors themselves must be sufficiently far from the corresponding normal cones. In the subsequent series of corollaries, these conditions are separated, thus, producing characterizations of nonlinear transversality properties in a more conventional form. The Hölder setting has been given a special attention.

As an application, dual sufficient characterizations for nonlinear extensions of the new transversality properties of a set-valued mapping to a set in the range space due to Ioffe [20] have been provided.

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