Navigating Concave Regions in Continuous Facility Location Problems

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ABSTRACT

In prior literature regarding facility location problems, there has been little explicit acknowledgement of problems arising from concave (non-convex) regions. This issue extends to computational geometry as a whole, as there is a distinct deficiency in existing center finding techniques amidst work on forbidden regions. In this paper, we present a novel method for finding representative regional center-points, referred to as "Concave Interior Centers", to approximate inter-regional distances for solving optimal facility location problems. The validity of the proposal as a means for solving these placements is discussed on three "special cases", and on a humanitarian focused real-world application. We compare the performance of the Concave Interior Center to the results from using Geometric Centers. We discuss the implications of using externally located representative centers in facility location problems. The context of the application involves maximizing the potential services available to homeless persons in Fairfield County, Connecticut under a given budget through the construction of limited capacity night-by-night shelters and optionally included food pantries. Our computational results show the efficiency of the proposed techniques in solving a critical societal problem.

Keywords: Centers, Concave Regions, Distance measurement, Facility location problems, Geometric distance, Continuous facility location, Homeless shelters.

1 Introduction

Finding regional centers and inter-regional distances is a critical component in solving many optimization problems, specifically facility location problems. The point most often considered representative for a region has been its *Geometric* Center:

$$\left(\lim_{n\to\infty}\frac{\sum_{k=1}^{n} x_k}{n}, \lim_{n\to\infty}\frac{\sum_{k=1}^{n} y_k}{n}\right)$$

The glaring issue for using this point is that it may fall exterior to the region it intends to define. While there has been much work in dealing with forbidden regions, most has only considered convex regions, and very little has focused on finding points to represent regions when running models. This remark is expounded upon in greater detail in the following section, and we do note that the arising issues have been previously noted, though rarely with context[1,2]. The existing workarounds proposed have typically been for very specific case studies, and otherwise require problem relaxation[3–7]. We define a concave region as any region containing two points between which a connecting straight line would fall at least partially exterior to the region itself. This is to say, in other words, a non-convex region. Let us imagine a concave region whose Geometric Center falls externally. A simple example of this occurrence is seen in Figure 1.

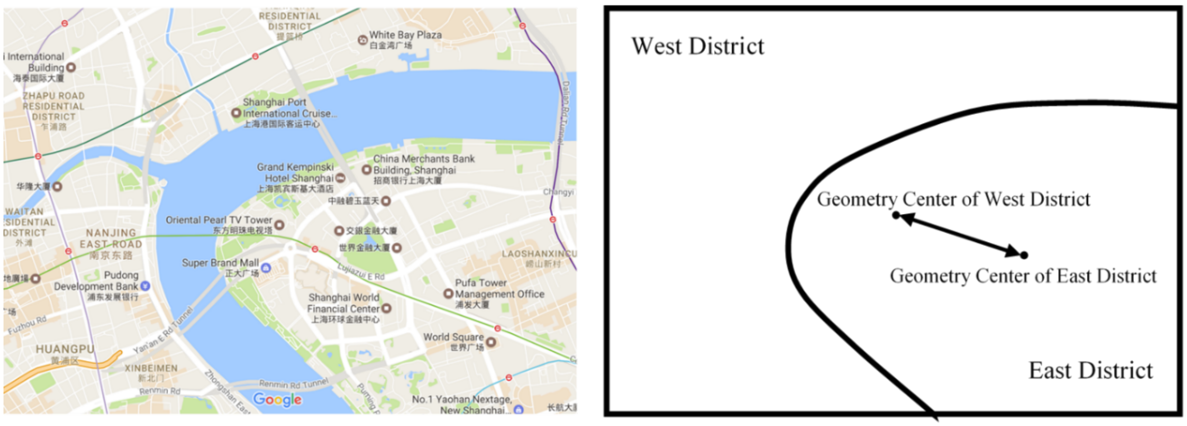

Figure 1. Map and outline of the primary districts of Shanghai (adopted from Google Maps).

The above figure shows Shanghai city map with its outline represented as two concave regions. Clearly, the Geometric Center of the West District falls within the East District. A real-world application later on in this paper includes comments on some specific potential consequences of such an inaccuracy. While Geometric Centers are an adequate representative point in most instances, the presence of severely concave regions can make them an ultimately inaccurate measure, and therefore unsuitable for use. Later in the paper we will explore this statement further. In response to this, we will discuss a different point which stays internal regardless of region shape. This work focuses on the defense of the validity of the proposed point (which we refer to as the *Concave Interior Center*), a recommended associated inter-regional distance measurement, and their applications to continuous facility location problems.

In the vast majority of real-life applications, it will be very difficult to describe all observed regions as non-concave. Though an attempt can be made to cut a map into convex polygons, any application dealing with infeasible sub-regions may run the risk of including concave regions. In fact, any instance of a sub-region existing not against the border of its relevant region will create concavity. The noted issue of using an externally located center is therefore one of under-discussed ubiquity. The potential severity of such errors should not be understated. Any application with sufficient sensitivity is at risk of being greatly affected.

The negative results of decisions made based on externally located regional centers fall into two main camps. Either an assumed solution is actually infeasible, or an assumed solution is feasible but inaccurate. The former may come in the form of a recommendation to construct a warehouse in the middle of a lake, the lake being where the Geometric Center of a certain distribution region happened to fall. This would eventually be very obvious, and would force a manual location choice which would need to be universally agreed upon, thus setting back the construction timeline. The latter is potentially more dangerous, as it may become apparent later on and even after completion. For example, we are considering a project which proposes to build a school located most conveniently to serve the majority of a population, and to be outside of an interior "high crime" area located towards the center of the overall educational district. Though it may not be apparent during the construction process, the long-term safety of children and school faculty may be put at risk because of the unrealized fact that the Geometric Center of the district happened to fall inside that intended exclusion zone. These will be specifically investigated again later with regards to the application in section 5.

The remainder of this paper is separated into five primary sections. The coming section is a literature review and discussion of existing center finding techniques, distance measurements, prior studies into dealing with forbidden regions, and relevant applications, primarily in solving facility location placement problems. Next, we describe our method for center identification and exemplify its use in problem solving. Following that is an analysis of three specific regional occurrences which show notable properties of our proposed Concave Interior Center. Afterwards we study the performance of our center in a real-world continuous facility location application which focuses on maximizing services to a homeless population, a currently critical humanitarian issue. Finally we conclude the paper, summarizing our observations and results and noting the intended impact of the work and its potential for further exploration. In this paper we will show our Concave Interior Center to be not only valid, but explicitly necessary in solving facility location problems with concave regions in cases where other methods would underperform or fail.

2 Literature Review

Our research intends to contribute to the study of center finding methods in the presence of forbidden regions. The findings and results of such pursuits are most often utilized in distance measuring algorithms and within the scope of facility location problems, but can be found in other applications such as stability analysis for electrical systems as seen in the work of Liao et al. (2017) and Zhou et al. (2019)[8,9]. Referring to the problem of representing the regions, point specifying and sub-area designating methods have both been used. Murray (2003) discussed the uncertainty and error in realistic facility locations compared to initially suggested ones based on land suitability, access, and costs[1]. Murray & Kelly (2002) developed an approach for evaluating representational appropriateness of the geographic space[2]. These papers opened discussion on the problem on which our work focuses. Katz & Cooper (1979) brought about one of the earliest mentions of forbidden regions in location analysis[3]. Their focus looked at the relatively simple case of placement in the case of a single circular forbidden region. Much later, we see similar work on the case of two elliptical regions by Prakash et al. (2017)[4]. The same team expanded to the inclusion of general polygonal and elliptical forbidden regions in Prakas et al. (2018)[5]. The sparsity of work in this scope is clear in the time it took for more complex study to arise. Notable results on the computational complexity of multifacility problems in this context have only recently been studied, as seen in Maier & Hamacher (2019)[10].

Wei et al. (2006) tried to solve the p-center problem in continuous space. They looked at the case that a region is non-convex and possibly containing holes, thus causing an infeasible center. They proposed generating a constrained minimum covering circle and using its center as the new location for calculating distance to other centers[6]. Dinler et al. (2015) also used this point to represent regions and mentioned techniques for studying to minimum and maximum region-facility distances, as well as expected distances. They used the maximum distance to represent the region to calculate the distance of a demand region to a facility location point and all the demand regions are closed and convex, which is a rectangle[11]. Love (1972) considered the weighted sum of density and frequency of demand in an area to represent its center, and exclusively represented regions and their infeasible sub areas as rectangles as an attempt avoid non-convexity. Instead of using points to represent the region, Wei & Murray (2015) discussed using collections of polygons to represent demand regions. Unlike a single point, which only assumes no coverage or all coverage, polygonal representation takes into account percentage coverings when deciding the facility locations[12]. Similarly, Murray & Tong (2007) depicted the region by square polygons to avoid non-convexity. They used the intersect points of covering boundary of the polygon to be the potential centers for the facility in the covering problem[13].

P-norm distance, or "l_p" distance, is widely used in the facility location problem to calculate distance between two points. The following equation for "l_p distance" was proposed by Francis (1967)[14].

$$l_p(q,r) = \left[\sum_{i=1}^{n} |q_i - r_i|^p\right]^{\frac{1}{p}} = ||q-r||_p = ||\overrightarrow{rq}||_p \tag{1}$$

In equation 1, $q = (q_1, \ldots, q_n)$ and $r = (r_1, \ldots, r_n)$ are two points in n-dimensional space (i.e. $q, r \in \mathbb{R}^n$).

Love & Morris (1972) compared seven different distance functions including l_p functions, weighted l_p functions, and weighted Euclidean functions on two-road distance samples to determine their accuracies[15]. They considered two criteria with respect to the accuracy of estimation:

- Sum of absolute deviations (the difference between actual road miles and estimated road miles).
- Sum of squares (more sensitive and require more accuracy on shorter distance).

Love & Morris (1979) further compared different parameters for the "l_p distance" model to make the distance calculation more accurate[16]. They used statistical tests to compare the results on seven samples and concluded that "the more parameters used in the model, the more accurate and complexity will get". Love Dowling (1985) used an "$l_{k,p}$ distance" model for floor layout problems to find the appropriate k, p values compared to the "l_1 distance" (or rectangular distance) model[17]. In the literature, authors stated that the "l_1 distance" model was suitable for obtaining optimal facility locations while "$l_{k,p}$ distance" model should be used when total cost function is considered by adjusting k values rather than p values. None of the several mentioned generalizations of this algorithm, unfortunately, can successfully determine the exact location of the potential point in each region.

Batta et al. (1989) solved the location problem with convex forbidden regions using the Manhattan metric ($p = 1$)[18]. Amiri-Aref et al. (2011) used a P-norm distance model to find the location for a new facility which has maximum distance to other existing facilities with respect to the same weights and line barriers[19]. They emphasized the rectilinear distance metric ($p = 1$) in the formulation and stated that further research can be extended on different distance functions. Amiri-Aref et al. (2013) also presented the same distance metric ($p = 1$) to solve the Weber location problem with respect to a probabilistic polyhedral barrier[20]. They transferred the barrier zone from a central point of a barrier with extreme points to a probabilistic rectangular-shaped barrier. Later, Amiri-Aref et al. (2015) utilized the "l_1 distance" model to deal with relocation problems with a probabilistic barrier[21]. Three approaches for solving the restricted location-relocation problem with finite time horizon have been well-established in prior literature. Dearing Segars (2002) presented that for solving single facility location problems with barriers using rectilinear distance model ($p = 1$), the result remains the same for original barriers and modified barriers[22]. Aneja & Parlar (1994) discussed two algorithms (Convex FR and Location with BTT) that can solve Weber Location problems with forbidden regions for $1 < p \leq 2$ when using the l_p distance model to calculate the distance between demand points and facility location points[23]. Bischoff & Klamrot (2007) illustrated a solution method for locating a new facility while given a set of existing facilities in the presence of convex polyhedral forbidden areas[24]. This method first decomposes the problem, then uses a genetic algorithm to find appropriate intermediate points. In existing literature, only Euclidean distance ($p = 2$) has been considered, and it has been shown that this algorithm is very efficient. Altınel et al. (2009) considered the heuristic approach for the multi-facility location problem with probabilistic customer locations[25]. To calculate the expected distance between probabilistic demand points and facility locations, they used Euclidean distance, squared Euclidean distance, rectilinear distance, and weighted $l_{1,2}$-norm distance models.

By way of network theory, we can find another way of calculating distances for location problem. Blanquero et al. (2016) represented the competitive location problem as a network with customers as the nodes[26]. They put the facilities on the edges and calculated the distances as the lengths of the shortest path from each customer to the nearest facility. Peeters (1998) discussed a new algorithm to solve the location problem by calculating distance as the sum of the lengths of arcs between two vertices and the length between a vertex and an arbitrary point on that arc[27]. The author used nine different models to test the new algorithms, and showed much better performances than the old algorithm that calculates the distance between each node first. One of the earlier uses of graphs in the context of the p-median problem was in the work of Teitz & Bart (1968)[28]. Though their method of single vertex substitutions was relatively inefficient, it was of significant importance for expansions on network based solutions for these problems. Eilon & Galvão (1978) looked at double vertex substitutions in the uncapacitated case of location analysis[29]. Jacobsen (1983) later used several heuristics and explored the context of the capacitated problem[30].

The intent of this paper is to further contribute to the study of continuous consideration of problems on concave polyhedral maps, issues surrounding concave regions have been somewhat overlooked. Satyanarayana et al. (2004) explored using an iterative line search method to find near-optimal placements, which was extendable to non-convex settings[31]. Another iterative approach for near-optimal solutions in this context was in the big triangle small triangle method proposed in Drezner & Suzuki (2004), which itself was an extension of the generalized big square small square method of Plastria (1992)[32,33].

The work that has gone into solving optimal placements in concave settings has been restricted to iterative approximating methods such as those described above, and advancements have been relatively sparse. To the best of our knowledge, there is no current method which focuses on extracting points which most accurately represent concave regions. Our view is that if a point is used to describe a region, but gives unrealistic or unusable results in a model's output, that point is an inaccurate. We assert the nature of the application to be novel. Facility location research is typically tested on cost-minimizing full covering models, most commonly with uncapacitated facilities. The application proposed in this paper concerns capacitated facilities in the setting of an objective function to maximize reward under a specified budget. The goal of this paper is to help fill these existing gaps, to expand the scope of applications on which facility location algorithms may be tested, and to draw attention to the needs of existing homeless populations.

3 Concave Interior Center

We now mathematically define our proposed Concave interior center and distance measuring metrics. The distance between two points in two dimensional space (x_0, y_0) and (x_1, y_1) can be defined as $\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}$. The point which minimizes the potential distance between itself and anywhere else in the region can be viewed functionally as a center. In light of this, we can use the aggregation of distances between some (x^*, y^*) and the rest of the points the surrounding region R as a metric of centrality.

Suppose we have a region S_i. We intend to use a point $(x_{S_i}, y_{S_i}) \in S_i$, such that the value of

$$\iint_{(x,y)\in S_i} \sqrt{(x_{S_i}-x)^2+(y_{S_i}-y)^2}dxdy \tag{2}$$

will be minimized. The distance between any two regions S_1 and S_2 with Concave Interior Centers (x_{S_1}, y_{S_1}) and (x_{S_2}, y_{S_2}) is approximated as

$$d(S_1,S_2) = \sqrt{(x_{S_1}-x_{S_2})^2+(y_{S_1}-y_{S_2})^2} \tag{3}$$

We see a discrete example of this in Figure 2 below. The black squares below all represent the same given region. We look at the performances of three of its points. This is done by finding the sum of the distances from the tested point (marked in red) to all other points within the region. Unchecked edge points will yield identical results to what those shown in the middle and rightmost boxes respectively. Since the Concave Interior Center is defined as the point minimizing aggregate distance, it is chosen as the one highlighted in the leftmost box.

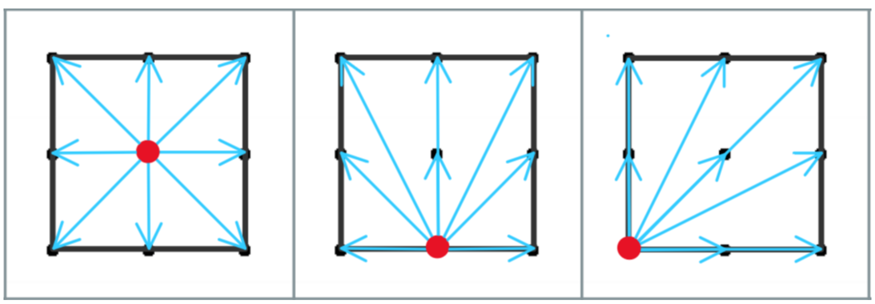

Figure 2. Example points. From left to right they have aggregate distances of 9.7, 12.3, and 14.7 units.

In most applications, the point (x_{S_i}, y_{S_i}) will be found in a discrete context. The density of points to chose from in region S_i, then, is fully up to the discretion of the operator. When utilized in this paper, the number of available points is set as a bijection with the pixels in the displayed maps. For more precise results, the map could have just as well been cut into an arbitrarily finer mesh. *We assert that the center point (x_{S_i}, y_{S_i}) we defined above well represents region i.*

Our general argument that the point well represents the region is that by construction (x_{S_i}, y_{S_i}) is the point which minimizes aggregate travel distance over the regional space.

4 Special Case Analysis

In this section, we present three special cases to showcase notable qualities of our proposed point. In each, we present a particular set of regions, and compare the resulting Concave Interior Centers and Geometric

Centers. We find immediate and concrete differences in the performances. A more detailed discussion of how these differences reflect our center's importance can be found in following section.

Case 1

A frequent occurrence faced in location placements is the existence of annular or annular-like concave regions. We exemplify this in the following images. In Figure 3, we see that when using Geometric Center, Regions I and II will act as having zero distance from each other, when clearly this is not the case. From Figure 4, we can avoid this inaccuracy by using our Concave Interior Centers as representative points. We can see from the generated maps that the new point gives a much more accurate distance inter-regional distance.

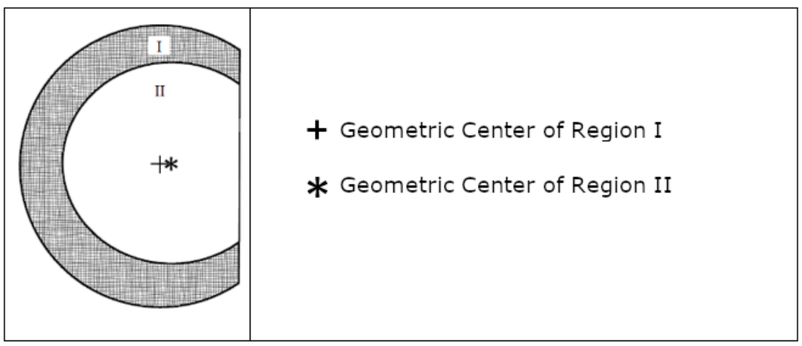

Figure 3. Geometric Centers for case 1.

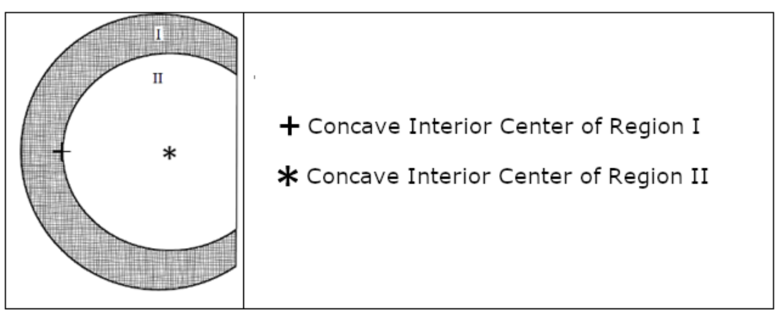

Figure 4. Concave Interior Centers for case 1.

Case 2

We can see from Figure 5 and Figure 6 that this is an interesting case where the Geometric Center 1, Geometric Center 2, Concave Interior Center 1, and Concave Interior Center 2 all fall identically. This is a very specific case which shows the one weakness of our algorithm: that the algorithm is not able to identify the boundary of a region as potentially invalid, and therefore two separate regions may be seen as having a distance of zero. A practical solution to this potential error is to make the boundaries of all regions itself a region with any nonzero width, as any regions resulting in this situation will exist in extremely close proximity. Though the resulting distance may be inaccurate, it will likely function in a truthful manner. It is important to recognize that boundary positioned Concave Interior Centers will rarely cause a critical error, as was seen in the annular-like case.

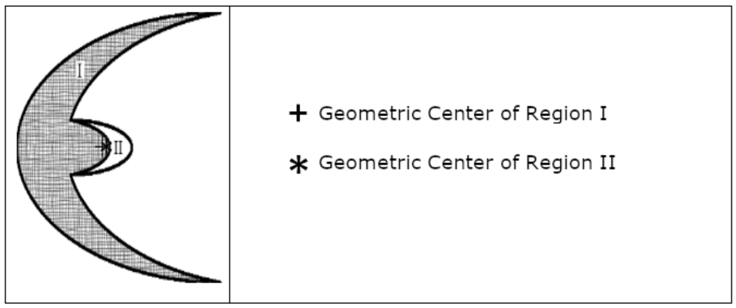

Figure 5. Geometric Centers for case 2.

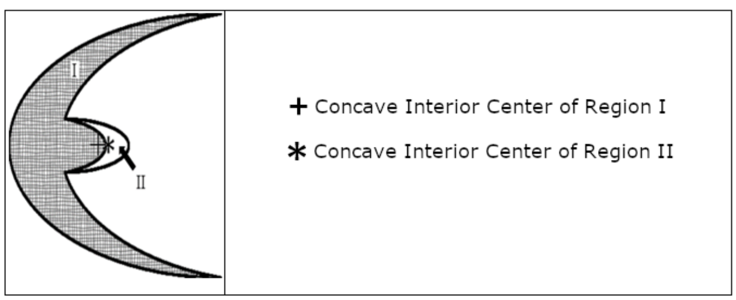

Figure 6. Concave Interior Centers for case 2.

Case 3

In this case, we have 3 regions. We present this case in the context that region 2 has a demand needing to be served by either region 1 or 3. From Figure 7, we can see that the distance between the Geometric Centers generated for regions 1 and 2 is smaller than that for regions 2 and 3, which implies that region 1 would be chosen as the servicer. It is shown in Figure 8 that under Concave Interior Centers the distance between regions 2 and 3 is seen as smaller than the distance between regions 1 and 2. The demand at region 2 would then be served by region 1. The latter can be visually confirmed as the better interpretation, as the shortest distance between one point in each of regions 2 and 3 is smaller than that between regions 2 and 1 aside from the Concave Interior Center of region 1. The only case where this is not true is when considering the region 2 point to be on a short excerpt of the theoretically single point wide horizontal line extending from where regions 1 and 2 actually touch (which is the marked Concave Interior Center of region 1) up until approximately one-third of the way into region 2.

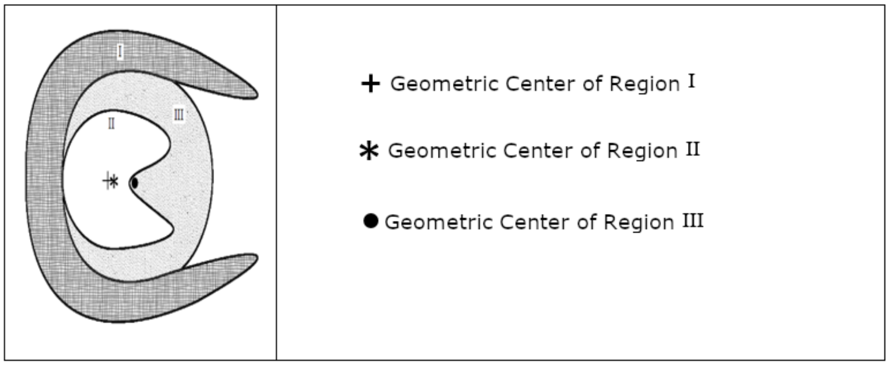

Figure 7. Geometric Centers for case 3.

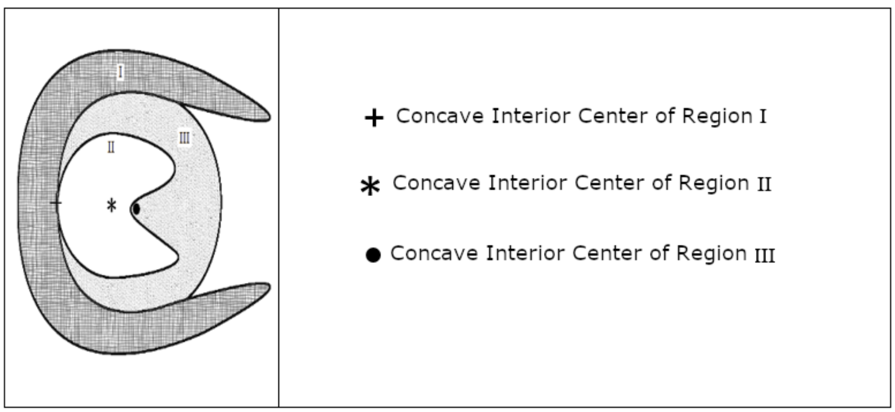

Figure 8. Concave Interior Centers for case 3.

5 Real World Model

In this section, we describe an important societal location problem to show the importance of the proposed center. The model we have developed to solve this problem provides an opportunity to show the explicit advantages of using the Concave Interior Center.

5.1 Problem Description

The goal of this model is to find optimal placement of fixed capacity night-by-night homeless shelters and decide whether or not to allocate funding for running food pantries at one or more of them in order to best serve the needing populations of a given set of regions under offered funding.

Any shelters constructed will have a universally set capacity. Funds received will be considered as a one-time lump sum. The cost of shelter construction in a region will be approximated based on that of comparable local buildings. Pantry cost is determined as the price of one daily meal for each homeless resident of its constructed region, funded for the span of ten years. Regional borders for the map will be drawn in response to the following phenomena: existing town boundaries, areas unavailable for construction due to natural features, and areas not zoned for construction of such shelters.

The region observed is Fairfield County, Connecticut. This county well shows the importance of accurate regional representation for a number of reasons. The difference in approximated costs, described below, between even adjacent towns is seen notably between Stamford ($339,000) and Darien ($1,595,000). The map features many lakes, mountainous areas, wetlands, and protected forests in which construction is impossible. Zoning regulations, additionally, vary greatly. Greenwich ordinances, for example, flat out ban such facilities, while Danbury is largely open for construction. Ridgefield, however, does not currently have defined legislation for this form of shelter, as is often the case. When information is not available it will be assumed that exclusively residential, full industrial, agricultural, historic, and aesthetic zoning districts will be marked as infeasible, as these areas are the most tightly restricted in terms of building type.

Set up for this application required researching four topics. Specifically, we needed to set up a map to depict areas valid and not valid for construction, find homeless populations for each region, and approximate build cost for each. The valid districts are set in the map as tan, invalid build districts are marked red, and ignored areas of the map are marked grey (which will be treated the same as invalid districts, but with no demand). Specific reasons for ineligibility are not noted for two reasons: there is significant overlap between aforementioned causes (many lakes already lie in exclusive residential zones), and the only practical difference between causes is whether or not there could still be a homeless population in the area. Homeless populations are approximated based on data from Point-in-Time counts by the Connecticut Coalition to End Homelessness, and split into regions within towns based on their relative population densities[34]. Facility cost was found for each town by increasing or decreasing the approximated construction cost for Danbury based on the relative percent increase or decrease housing costs (land and construction). Danbury's cost was an approximation based on that of its own homeless shelter, the Dorothy Day Hospitality House. Data for pantry costs is approximated by statistics from Feeding America and the Regional Food Bank of Northeastern New York[35–37]. When information is not available or has not yet been legislated on for zoning validity, as is the case with Ridgefield, it will be assumed that any area marked as exclusive residential, full industrial, agricultural, historic, and aesthetic type zones will be marked as infeasible, as these areas are the most tightly restricted in terms of building type. For our initial application, shelters will universally have a 45 bed capacity, and the budget for the overall project will be $2,000,000. Figure 9 below is the section of southwestern Connecticut we are referring to. A table containing relevant attributes of each region is shown in tables 3,4, and 5 following the discussion in section 5.3

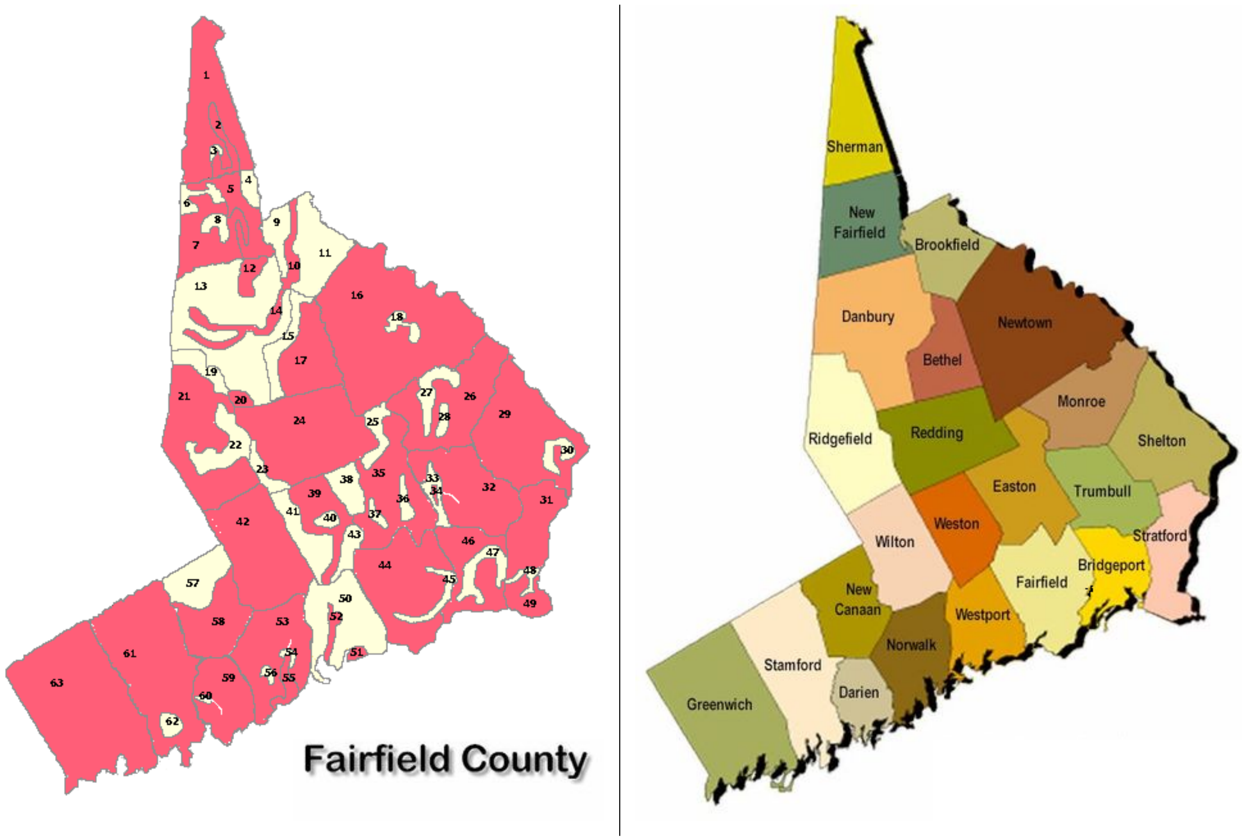

Figure 9. Left: Fairfield County split into valid (tan) and invalid (red) regions. Right: A map of the county split on town borders.

5.2 Model

We now define the model we will use to solve our decision problem. Tables 1 and 2 list the associated variables and parameters along with their definitions in the context of the problem. The objective function and constraints are presented in equations (4) through (16).

Table 1. Definitions for model parameters.

parameters	definition
c_i	cost of constructing a shelter at center of region i
D_i	Demand (homeless population) of region i
f_i	$f_i = 1$ if construction of a shelter at region center i is feasible, else $f_i = 0$
R	Set of indices of Region centers for r regions $\{1, 2, ..., r\}$
L	Set capacity of all shelters
v	Weighting of value added by inclusion of a pantry
p	Cost of running a pantry
B	Overall budget for construction of shelters
M	Large cost parameter

Table 2. Definitions for model variables.

variables	definition
e_i	$e_i = 1$ if a shelter is to be constructed at region center i, else $e_i = 0$
F_i	$F_i = 1$ if a pantry is to be operated through the shelter at region i, else $F_i = 0$
$s_{m,n}$	$s_{m,n} = 1$ if demand of region m is satisfied by facility at region center n, else $s_{m,n} = 0$
$d_{m,n}$	distance between center point of regions n and m
h_i	$h_i = 1$ if $D_i \geq L$, else $h_i = 0$

$$\max \sum_{i=1}^{r}\sum_{n=1}^{r}[Ls_{i,n}h_i + D_i s_{i,n}(1-h_i)] + v\sum_{i=1}^{r}[LF_i h_i + D_i F_i(1-h_i)] \tag{4}$$

$$\begin{aligned}
s_{m,n} &\leq e_n && \forall m,n \in R && (5)\\
F_i &\leq e_i && \forall i \in R && (6)\\
e_i &\leq f_i && \forall i \in R && (7)\\
s_{m,n} &\leq D_m && \forall m,n \in R && (8)\\
d_{m,n}s_{m,n} &\leq d_{m,i} + M(1-e_i) && \forall m,n,i \in R && (9)\\
\sum_{m=1}^{r}\sum_{n=1}^{r} s_{m,n} &= \sum_{i=1}^{r} e_i && && (10)\\
\sum_{i=1}^{r}(c_i e_i + 2100 D_i F_i) &\leq B && && (11)\\
\sum_{n=1}^{r} s_{m,n} &\leq 1 && \forall m \in R && (12)\\
\sum_{m=1}^{r} s_{m,n} &\leq 1 && \forall n \in R && (13)\\
\sum_{m=1}^{r} s_{m,n} &\geq e_n && \forall n \in R && (14)\\
d_{m,n}s_{m,n} &\leq \frac{1}{r^2}\sum_{i=1}^{r}\sum_{j=1}^{r} d_{i,j} && \forall m,n \in R && (15)\\
\frac{d_{m,n}e_m + d_{m,n}e_n}{2} + M(1-e_m) + M(1-e_m) &\geq \frac{1}{2r^2}\sum_{i=1}^{r}\sum_{j=1}^{r} d_{i,j} && \forall m,n \in R && (16)
\end{aligned}$$

The goal of the objective function (4) is to maximize the homeless population served, automatically setting the population served by any shelter to equal capacity L if the region being served by that shelter has a demand greater or equal to said than said capacity. Feeding a person through the pantry is considered a fraction (v) of the value of sheltering the person. Constraint (5) ensures that only existing shelters can serve a region. Constraint (6) requires that pantries can only be created at shelter sites. Constraint (7) stops shelters from being constructed in feasible regions. Because it is unrealistic that a person would travel an unnecessary additional distance, constraint (9) states that a region can only be served by its nearest shelter. If an optimal solution is found with sufficient room left within the budget, the model may request the construction of a non servicing shelter. Constraints (10) and (14) prevents this. If, in addition, there is a low cost feasible construction zone near on or near a region with no demand, the model may request construction of a shelter which does not contribute any objective improvement. This is inhibited by constraint (8). Per constraint (11), the sum of the construction and food pantry costs must not exceed the available budget. Constraint (12) limits a region's population to be served by at most one shelter. Otherwise, any constructed shelter would be automatically assumed to have demand exceeding capacity. For the same reason, constraint (13) requires that any shelter serves at most one region. Constraints (15) and (16) bound the distance between a shelter and the region it services as less than a fraction of the aggregate of all inter-regional distances, and distance between any two shelters as greater than that same fraction.

5.3 Computational Result Analysis

The proposed algorithm for center calculation is implemented with MATLAB R 2019a on a computer with Intel(R) Xeon(R) E3-1231 v3 CPU running at 3.4 GHz with 16 GB memory and a 64-bit operating system. We get the Geometric Center for a region in practice by running the function $(\frac{\sum_{k=1}^{n} x_k}{n}, \frac{\sum_{k=1}^{n} y_k}{n})$ for the set of n points in a given region, with the ith point represented as (x_i, y_i). The Concave Interior Center of a region is found as the point (x_{S_i}, y_{S_i}) which minimizes the output of the function $\sum_b^n \sum_a^m \sqrt{(x_{S_i} - x_a)^2 + (y_{S_i} - y_b)^2}$ over the finite set $(x_a, y_b) \in S_i$. The model is run on the regioned map in Figure 9 scaled to 638 x 474 pixels, with pixels catalogued as matrix form (i.e. origin at top left as (1,1) and address as (row, column)). The model is implemented with Julia Language 0.6 on a Macbook pro with 3.5 GHz Intel Core I-7 processor and 16 GB memory with Julia Language 0.6 and solved with Gurobi 7.5.2 solver.

Tables 3,4, and 5 show each region's homeless population, cost of construction, construction feasibility, Geometric Center, Concave Interior Center, and respective town. For use in the model constraints, construction cost in infeasible regions is set as 0. Tables 6 and 7 show the results of running the model. The Parameters column show the parameter sets v, L, and B for each run. The Geometric column shows the objective value, the regions at which shelters should be constructed n, the region which each constructed shelter should serve m, and whether or not a food pantry should be run at the shelter, assuming the model uses the Geometric Centers from tables 3,4, and 5. The Concave column shows the same information when using the Concave Interior Centers.

A picture by picture inspection shows that all discrepancies between region centers occur in cases where the Geometric Center falls external to that region. The listed centers in tables 3,4, and 5 for regions 1, 2, 5, 6, 16, 22, 30, 45, 46, 47, and 48 are highlighted to note their differences. It is immediately apparent that under all parameter sets, regions with populations which met or exceeded shelter assigned shelter capacity were favored. This is most evident in regions 13, 46, 47, and 61 as being most commonly chosen for parameter sets when $L = 100$. When shelter capacities are reduced to 45, it is notable that highest priority is placed on low cost construction regions. This can be attributed to having a larger pool of capacity fulfilling choices. Under this same capacity with $B = 4,000,000$ and $v = 0.5$ using Geometric Centers, it is notable that fewer than half of the regions serviced have less demand than the shelter capacity, and that 6 of the shelters include food pantries. When changing to $v = 0.125$, we see that more than half of the serviced regions have demand greater than capacity, and that only 4 of the 7 shelters have pantries. It is evident that this occurs because less priority is placed on the benefit which results from the objective addition of the chosen construction regions' population being served through associated food pantries. Had the same solution set been for $v = 0.125$ as was chosen for $v = 0.5$, the objective value would have fallen to 274.375, as opposed from the achieved 282.5. This difference is illustrated in Figures 10 and 11.

The differences when using our concave interior center when compared to the Geometric Center are of critical importance. In each run with differing objective values, we see that one or more of the regions chosen under the Geometric Center have construction locations which are different from their concave interior centers and which, as previously noted, are actually infeasible. It is notable that this results in a higher objective value which, in practice, would be impossible to obtain. We can look at the implications of this in practice through our model. Under the parameters $v = 0.5, L = 45, B = 4,000,000$, construction for regions 45, 47, and 48 would be infeasible. Removing these would drop the resulting objective from 329.5 to only 174.5. We note that the objective will not always be improved. Looking to Special Case 1 we can imagine a third, much smaller concentric region creating a third innermost region. If we wanted to maximize distance between regions while choosing only two regions, Geometric Center would imply any pair would give identical results. concave interior centers, however, would imply that the two regions which are furthest apart radii, the inner most and outermost ones, would give the maximum distance.

It is easy to imagine a scenario in which the executor of this project is chosen based on projected performance. In such an event, a less considerate party which assumes efficacy of the Geometric Center may be granted the contract. The results would be disastrous, as construction may be started before realizing the impending error. Following that, the board in charge of the approved grant would need to make one of two choices. They could either nullify the contract and give it to the more considerate firm which uses Concave Interior Centers, now with a much lower budget, or allow the initial company to only commit to the construction of their feasible shelter choices, resulting in significantly reduced objective value. Our model explicitly shows the importance of our proposed point in avoiding the critical error that can arise from a naïve choice of estimator.

Table 3. Data and calculated centers for regions 1-21.

Region	Di	Ci	Fi	Geometric	Concave	Town
1	20	0	0	**(85,167)**	**(85,166)**	Sherman
2	0	0	0	**(113,178)**	**(112,179)**	Sherman
3	15	480000	1	(115,171)	(115,171)	Sherman
4	0	384000	1	(144,198)	(144,198)	New Fairfield
5	0	0	0	**(164,192)**	**(163,192)**	New Fairfield
6	6	384000	1	**(151,156)**	**(150,156)**	New Fairfield
7	42	0	0	(183,166)	(183,166)	New Fairfield
8	23	384000	1	(173,172)	(173,172)	New Fairfield
9	32	394000	1	(179,220)	(179,220)	Brookfield
10	27	0	0	(196,231)	(196,231)	Brookfield
11	25	394000	1	(192,251)	(192,251)	Brookfield
12	0	0	0	(212,198)	(212,198)	Danbury
13	123	320000	1	(247,184)	(247,184)	Danbury
14	0	0	0	(255,183)	(255,183)	Danbury
15	20	357000	1	(267,223)	(267,223)	Bethel
16	5	0	0	**(253,307)**	**(252,307)**	Newtown
17	40	0	0	(277,245)	(277,245)	Bethel
18	2	428000	1	(250,317)	(250,317)	Newtown
19	10	686000	1	(283,160)	(283,160)	Ridgefield
20	0	0	0	(309,191)	(309,191)	Danbury
21	10	0	0	(339,159)	(339,159)	Ridgefield

Table 4. Data and calculated centers for regions 22-42.

Region	Di	Ci	Fi	Geometric	Concave	Town
22	20	686000	1	**(346,178)**	**(346,179)**	Ridgefield
23	9	627000	1	(367,209)	(367,209)	Redding
24	11	0	0	(336,243)	(336,243)	Redding
25	0	707000	1	(334,297)	(334,297)	Easton
26	4	0	0	(311,357)	(311,357)	Monroe
27	2	418000	1	(302,340)	(302,340)	Monroe
28	6	418000	1	(326,349)	(326,349)	Monroe
29	32	0	0	(337,411)	(337,411)	Shelton
30	24	364000	1	**(351,438)**	**(351,435)**	Shelton
31	80	0	0	(409,419)	(409,419)	Stratford
32	50	0	0	(380,370)	(380,370)	Trumbull
33	35	424000	1	(383,343)	(383,343)	Trumbull
34	7	0	0	(383,342)	(383,342)	Trumbull
35	8	0	0	(382,309)	(382,309)	Easton
36	5	707000	1	(390,319)	(390,319)	Easton
37	0	707000	1	(400,297)	(400,297)	Easton
38	2	707000	1	(378,274)	(378,274)	Easton
39	0	0	0	(396,253)	-396,253	Weston
40	0	873000	1	(403,259)	(403,259)	Weston
41	0	873000	1	(418,242)	(418,242)	Weston
42	11	0	0	(427,206)	(427,206)	Wilton

Table 5. Data and calculated centers for regions 43-63.

Region	Di	Ci	Fi	Geometric	Concave	Town
43	0	873000	1	(426,275)	(426,275)	Weston
44	19	0	0	(454,319)	(454,319)	Fairfield
45	21	617000	1	**(465,345)**	**(466,346)**	Fairfield
46	162	0	0	**(441,372)**	**(443,374)**	Bridgeport
47	150	192000	1	**(444,378)**	**(443,375)**	Bridgeport
48	42	278400	1	**(449,414)**	**(448,414)**	Stratford
49	48	0	0	(465,415)	(465,415)	Stratford
50	7	1288000	1	(482,271)	(482,271)	Westport
51	0	0	0	(508,281)	(508,281)	Westport
52	0	0	0	(488,263)	(488,263)	Westport
53	94	0	0	(505,217)	(505,217)	Norwalk
54	45	466000	1	(509,229)	(509,229)	Norwalk
55	0	0	0	(534,227)	(534,227)	Norwalk
56	65	466000	1	(523,212)	(523,212)	Norwalk
57	2	1508000	1	(450,158)	(450,158)	New Canaan
58	3	0	0	(483,171)	(483,171)	New Canaan
59	6	0	0	(542,178)	(542,178)	Darien
60	0	1595000	1	(543,163)	(543,163)	Darien
61	180	0	0	(514,122)	(514,122)	Stamford
62	40	339000	1	(563,136)	(563,136)	Stamford
63	35	0	0	(549,67)	(549,67)	Greenwich

Table 6. Changes in chosen serviced regions m and respective servicing shelter construction regions n under Geometric and Concave Interior Centers for differing shelter capacities L and Budgets B with $v = 0.5$.

Parameters			Geometric Results			Concave Results		
v	L	**B**	**Objective**	$s_{m,n}$	**Pantry**	**Objective**	$s_{m,n}$	**Pantry**
0.5	100	2,000,000	428.5	13,6	yes	428.5	13,6	yes
				32,18	yes		32,18	yes
				46,23	yes		46,23	yes
				53,62	yes		53,62	yes
0.5	100	4,000,000	547	10,3	yes	547	10,3	yes
				13,22	yes		13,22	yes
				46,38	yes		46,38	yes
				47,28	yes		47,28	yes
				53,40	no		53,40	no
				61,23	yes		61,23	yes
0.5	45	2,000,000	232.5	13,6	yes	223.5	12,3	yes
				17,4	no		17,27	yes
				29,47	no		49,33	yes
				30,48	yes		61,54	yes
				61,54	yes			
0.5	45	4,000,000	329.5	7,19	yes	300.5	13,6	yes
				8,15	yes		15,4	no
				29,47	yes		17,28	yes
				30,48	yes		22,41	no
				53,37	no		46,43	no
				54,45	yes		49,30	yes
				61,23	yes		61,54	yes

Table 7. Changes in chosen serviced regions m and respective servicing shelter construction regions n under Geometric and Concave Interior Centers for differing shelter capacities L and Budgets B with $v = 0.125$.

Parameters			Geometric Results			Concave Results		
v	L	**B**	**Objective**	$s_{m,n}$	**Pantry**	**Objective**	$s_{m,n}$	**Pantry**
0.125	100	2,000,000	407.125	13,6	yes	407.125	13,6	yes
				32,18	yes		32,18	yes
				47,23	yes		46,23	yes
				53,62	yes		53,62	yes
0.125	100	4,000,000	530.875	9,27	yes	530.875	9,27	yes
				13,22	yes		13,22	yes
				46,38	yes		46,38	yes
				47,28	yes		47,28	yes
				53,40	no		53,40	yes
				61,23	yes		61,23	yes
0.125	45	2,000,000	273.5	13,6	yes	273.5	13,6	yes
				31,28	yes		31,28	yes
				49,37	no		49,37	no
				53,62	yes		53,62	yes
0.125	45	4,000,000	282.5	9,18	no	273.25	13,6	yes
				13,22	yes		17,27	yes
				29,47	yes		31,28	yes
				30,48	yes		46,43	no
				53,37	no		49,50	yes
				54,45	no		61,54	yes
				61,23	yes			

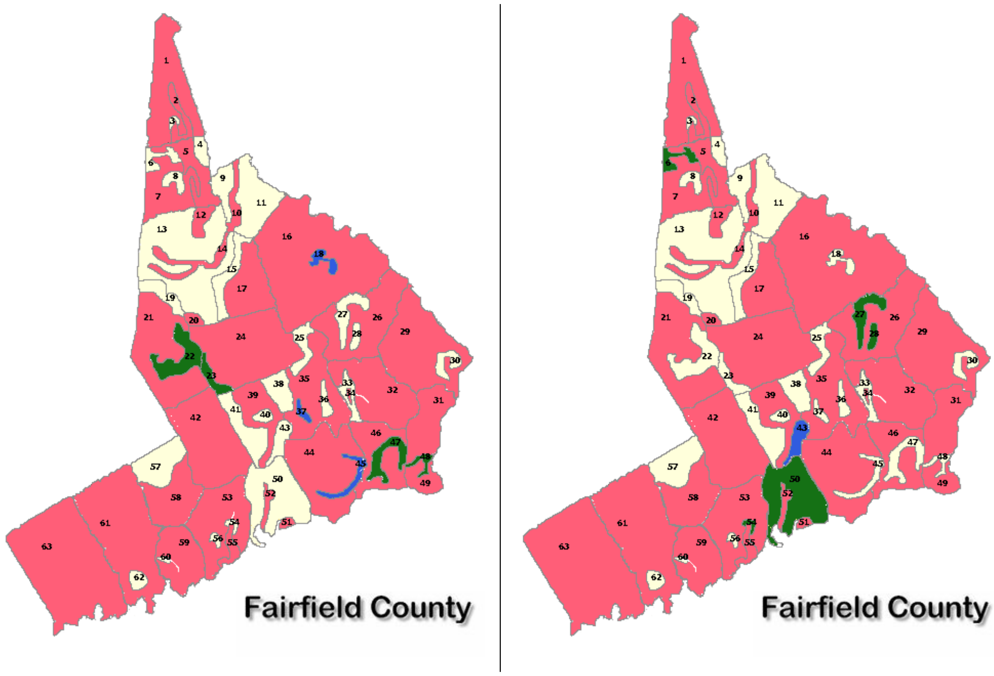

Figure 10. Results from $v = 0.125; L = 45; B = 4,000,000$ and Geometric Centers (left) and Concave Interior Center (right). Shelters with food pantries are marked green, Shelters without are marked blue.

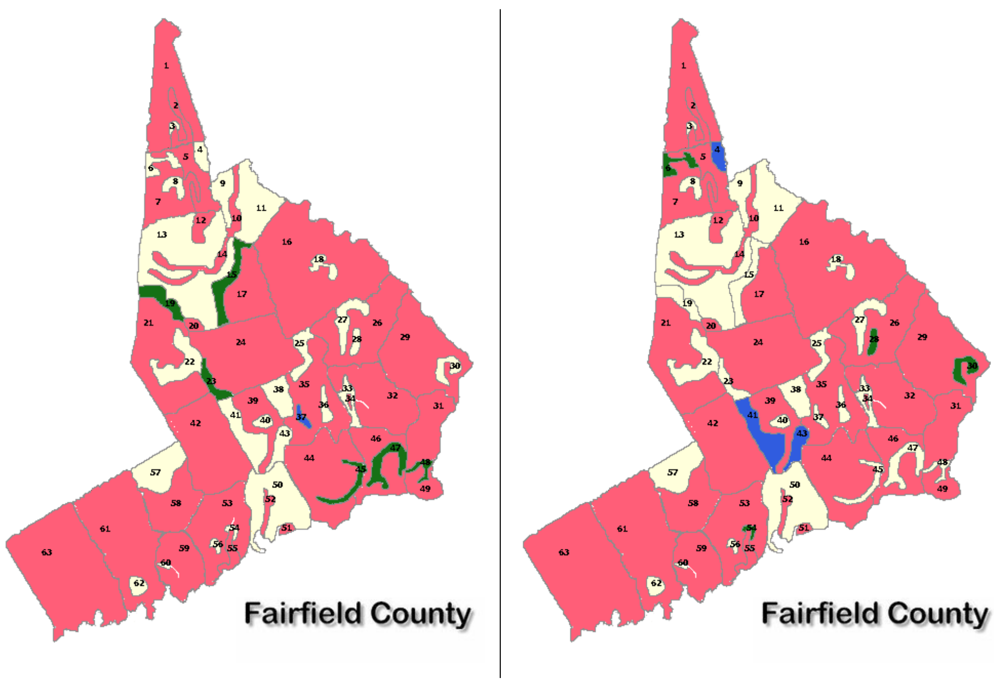

Figure 11. Results from $v = 0.5; L = 45; B = 4,000,000$ and Geometric Centers (left) and Concave Interior Center (right). Shelters with food pantries are marked green, Shelters without are marked blue.

6 Conclusion

Center finding methods are of critical importance in location analysis and related humanitarian and scientific applications. Despite the amount of research existing on dealing with forbidden regions in such problems, there has been a distinct lack of focus on the issues arising from nonconvex contexts. In this work, we presented a novel method for finding representative regional center points, which we referred to as Concave Interior Centers, by solving a minimization function over the chosen mesh of possible locations. We created a real-world model to test the validity of our generated points against that of the set of Geometric Centers. Three special cases were analyzed to show notable characteristics of our proposed center. These included a commonly occurring instance of severe inaccuracy of the Geometric Center, the possible inaccuracy of border points being chosen as Concave Interior Centers, and a less severe spacial inaccuracy which still causes a significantly different conclusion for an optimized solution. We showed this in an application which had the goal of maximizing the amount of social services offered to homeless people sheltered in the context of limited capacity shelters and a limiting shelter construction budget.

In practice, we would hope to have more guaranteed data for our model. The point-in-time counts used for our model were not given with respect to each region described on our map, but were rather grouped by one or more towns. The splitting of populations had to be done by estimation based on nature and approximate population density of each region and its respective town. Furthermore, the approximation of shelter costs could be greatly impacted by potentially preexisting sites which could be renovated for use. Regardless, our model results offered explicit proof that, in practice, our centers avoided the unrealistic and flawed solutions which were offered by more naïve methods. Further study could be made into the computational complexity for finding the Concave Interior Center. Going forward it may also be worthwhile to look into its mathematical relationship to other methods, specifically to see what circumstances would cause identical points to be generated by, say, the centers of regionally inscribing or circumscribing circles.

The center finding method proposed in this paper has been shown to be sound, effective, and advantageous in use. We put forth that it is not only appropriate for use in any application which hinges on finding regionally representative points. Moreover we argue that it is the most appropriate existing method for finding such points in any context which desires to minimize travel distance. This extends to any number of applications from supply chains to disaster relief efforts.

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