

# RELATIONS BETWEEN ABS-NORMAL NLPs AND MPCCs

## PART 2: WEAK CONSTRAINT QUALIFICATIONS

L.C. HEGERHORST-SCHULTCHEN, C. KIRCHES, AND M.C. STEINBACH

ABSTRACT. This work continues an ongoing effort to compare non-smooth optimization problems in abs-normal form to Mathematical Programs with Complementarity Constraints (MPCCs). We study general Nonlinear Programs with equality and inequality constraints in abs-normal form, so-called Abs-Normal NLPs, and their relation to equivalent MPCC reformulations. We introduce the concepts of Abadie's and Guignard's kink qualification and prove relations to MPCC-ACQ and MPCC-GCQ for the counterpart MPCC formulations. Due to non-uniqueness of a specific slack reformulation suggested in [10], the relations are non-trivial. It turns out that constraint qualifications of Abadie type are preserved. We also prove the weaker result that equivalence of Guignard's (and Abadie's) constraint qualifications for all branch problems hold, while the question of GCQ preservation remains open. Finally, we introduce M-stationarity and B-stationarity concepts for abs-normal NLPs and prove first order optimality conditions corresponding to MPCC counterpart formulations.

### 1. INTRODUCTION

Non-smooth nonlinear optimization problems of the form

$$\min_x f(x) \quad \text{s.t.} \quad g(x) = 0, \quad h(x) \geq 0, \quad (\text{NLP})$$

where  $D^x \subseteq \mathbb{R}^n$  is open, the objective  $f \in C^d(D^x, \mathbb{R})$  is a smooth function and the equality and inequality constraints  $g \in C_{\text{abs}}^d(D^x, \mathbb{R}^{m_1})$  and  $h \in C_{\text{abs}}^d(D^x, \mathbb{R}^{m_2})$  are level-1 non-smooth functions that can be written in *abs-normal form* [3] have been considered by the authors in [10]. In this problem class, the non-smoothness is caused by finitely many occurrences of the absolute value function, the branches of which we represent by signature matrices  $\Sigma = \text{diag}(\sigma)$  with  $\sigma \in \{-1, 0, 1\}^s$ . We find functions  $c_{\mathcal{E}} \in C^d(D^{x,|z|}, \mathbb{R}^{m_1})$ ,  $c_{\mathcal{I}} \in C^d(D^{x,|z|}, \mathbb{R}^{m_2})$  and  $c_{\mathcal{Z}} \in C^d(D^{x,|z|}, \mathbb{R}^s)$  with  $D^{x,|z|} = D^x \times D^{|z|}$ ,  $D^{|z|} \subseteq \mathbb{R}^s$  open and *symmetric* (i.e.,  $z \in D^{|z|}$  implies  $\Sigma z \in D^{|z|}$  for every signature matrix  $\Sigma$ ) such that

$$\begin{aligned} g(x) &= c_{\mathcal{E}}(x, |z|), \\ h(x) &= c_{\mathcal{I}}(x, |z|), \\ z &= c_{\mathcal{Z}}(x, |z|) \quad \text{with } \partial_2 c_{\mathcal{Z}}(x, |z|) \text{ strictly lower triangular.} \end{aligned} \quad (\text{ANF})$$

Here we use a single joint *switching constraint*  $c_{\mathcal{Z}}$  for both  $g$  and  $h$ , and reuse *switching variables*  $z_i$  if the same argument repeats as an absolute value argument in  $g$  or  $h$ . Due to the strictly lower triangular form of  $\partial_2 c_{\mathcal{Z}}(x, |z|)$ , component  $z_j$  of  $z$  can be computed from  $x$  and the components  $z_i$ ,  $i < j$ . Hence, the variable  $z$  is implicitly defined by  $z = c_{\mathcal{Z}}(x, |z|)$ , and to denote this dependence explicitly, we write  $z(x)$  in the following. Whenever we address questions of solvability of this system, we make use of the reformulation  $|z_i| = \text{sign}(z_i)z_i$ .

**Definition 1** (Signature of  $z$ ). Let  $x \in D^x$ . We define the *signature*  $\sigma(x)$  and the associated *signature matrix*  $\Sigma(x)$  as

$$\sigma(x) := \text{sign}(z(x)) \in \{-1, 0, 1\}^s \quad \text{and} \quad \Sigma(x) := \text{diag}(\sigma(x)).$$

A signature vector  $\sigma(x) \in \{-1, 1\}^s$  is called *definite*, otherwise *indefinite*.

---

1991 *Mathematics Subject Classification.* 90C30, 90C33, 90C46.

*Key words and phrases.* Non-smooth NLPs, abs-normal form, MPCCs, Abadie and Guignard type constraint qualifications, optimality conditions.

For signatures  $\sigma, \hat{\sigma} \in \{-1, 0, 1\}^s$ , it is convenient to use the partial order

$$\hat{\sigma} \succeq \sigma \iff \hat{\sigma}_i \sigma_i \geq \sigma_i^2 \quad \text{for } i = 1, \dots, s,$$

i.e.,  $\hat{\sigma}_i$  is arbitrary if  $\sigma_i = 0$  and  $\hat{\sigma}_i = \sigma_i$  otherwise. Thus, we may write  $|z(x)| = \Sigma z(x)$  for every  $\sigma \succeq \sigma(x)$ . Further, we may consider the system  $z = c_{\mathcal{Z}}(x, \Sigma z)$  **for fixed signature**  $\Sigma = \Sigma(\hat{x})$  around a point of interest  $\hat{x}$ . By the implicit function theorem, the system has a locally unique solution  $z(x)$  for fixed signature  $\Sigma$ , and the associated Jacobian at  $\hat{x}$  reads

$$\partial_x z(\hat{x}) = [I - \partial_2 c_{\mathcal{Z}}(\hat{x}, |z(\hat{x})|)\Sigma]^{-1} \partial_1 c_{\mathcal{Z}}(\hat{x}, |z(\hat{x})|) \in \mathbb{R}^{s \times n}.$$

**Definition 2** (Active Switching Set). We call the switching variable  $z_i$  *active* if  $z_i(x) = 0$ . The *active switching set*  $\alpha$  consists of all indices of active switching variables,

$$\alpha(x) := \{1 \leq i \leq s: z_i(x) = 0\}.$$

The numbers of active and inactive switching variables are  $|\alpha(x)|$  and  $|\sigma(x)| := s - |\alpha(x)|$ .

**1.1. Literature.** Griewank and Walther have developed a class of unconstrained abs-normal problems in [3, 4]. These problems offer particularly attractive theoretical features when generalizing KKT theory and stationarity concepts to non-smooth problems. Under certain regularity conditions, they are computationally tractable by active-set type algorithms with guaranteed convergence based on piecewise linearizations and using algorithmic differentiation techniques [5, 6].

Another important class of non-smooth optimization problems are Mathematical Programs with Complementarity (or Equilibrium) Constraints (MPCCs, MPECs); an overview can be found in the book [12]. Since standard theory for smooth optimization problems cannot be applied, new constraint qualifications and corresponding optimality conditions were introduced. By now, there is a large body of literature on MPCCs, and we refer to [15] for an overview of the basic concepts and theory. In this paper, constraint qualifications for MPCCs in the sense of Abadie and Guignard and corresponding stationarity concepts (in particular M-stationarity and MPCC-linearized B-stationarity) are considered. Details can be found in [14], [12] and [1].

In [9] we have shown that unconstrained abs-normal problems constitute a subclass of MPCCs. In addition, we have studied regularity concepts of linear independence and of Mangasarian-Fromovitz type. As a direct generalization of unconstrained abs-normal problems we have considered NLPs with abs-normal constraints, which turned out to be equivalent to the class of MPCCs. In [10] we have extended optimality conditions of unconstrained abs-normal problems to general abs-normal NLPs under the linear independence kink qualification using a reformulation of inequalities with absolute value slacks. We have compared these optimality conditions to concepts of MPCCs in [8]. We have also shown that the above slack reformulation preserves kink qualifications of linear independence type but not of Mangasarian-Fromovitz type. More details and additional information about these results as well as about the results in this paper can be found in [7].

*Contributions.* In the present article we extend our detailed comparative study of general abs-normal NLPs and MPECs, considering constraint qualifications of Abadie and Guignard type both for the standard formulation and for the reformulation with absolute value slacks. In particular, we show that constraint qualifications of Abadie type are equivalent for abs-normal NLPs and MPCCs and that they are preserved under the slack reformulation. For constraint qualifications of Guignard type we cannot prove equivalence but only certain implications. However, when considering branch problems of abs-normal NLPs and MPCCs, we again obtain equivalence of constraint qualifications of Abadie and Guignard type, even under the slack reformulation. Finally we introduce Mordukhovich and Bouligand stationarity concepts for abs-normal NLPs and prove first order optimality conditions using the corresponding concepts for MPCCs.

*Structure.* The remainder of this article is structured as follows. In Section 2 we state the general abs-normal NLP and its reformulation with absolute value slacks that permits to dispose of inequalities. We also present the branch structure of both formulations and introduce appropriate definitions of the tangent cone and the linearized cone. Using these tools, we introduce kink qualifications in the sense of Abadie and Guignard. In terms of these two kink qualifications, we then compare the regularity of the equality-constrained form of an abs-normal NLP to the

inequality-constrained one. In Section 3 we introduce counterpart MPCCs for the two formulations of abs-normal NLPs and discuss the associated MPCC-constraint qualifications, namely MPCC-ACQ and MPCC-GCQ. In Section 4 we investigate the interrelation of the regularity concepts for abs-normal NLPs and MPECs and find the situation to be more intricate than under LICQ and MFCQ discussed in [10]. Finally, in Section 5 we introduce abs-normal variants of M-stationarity and B-stationarity as first order necessary optimality conditions for abs-normal NLPs and prove equivalence relations for the respective MPCC stationarity conditions. We conclude with Section 6.

## 2. INEQUALITY AND EQUALITY CONSTRAINED FORMULATIONS

In this section we consider two different treatments of inequality constraints for non-smooth NLPs in abs-normal form.

**2.1. General Abs-Normal NLPs.** Substituting the representation (ANF) of constraints in abs-normal form into the general problem (NLP), we obtain a general abs-normal NLP. Here, we use the variables  $(t, z^t)$  instead of  $(x, z)$  and analogously  $\sigma^t(t)$  and  $\alpha^t(t)$  instead of  $\sigma(x)$  and  $\alpha(x)$ .

**Definition 3** (Abs-Normal NLP). Let  $D^t$  be an open subset of  $\mathbb{R}^{n_t}$ . A non-smooth NLP is called an *abs-normal* NLP if functions  $f \in C^d(D^t, \mathbb{R})$ ,  $c_{\mathcal{E}} \in C^d(D^t, |z^t|, \mathbb{R}^{m_1})$ ,  $c_{\mathcal{I}} \in C^d(D^t, |z^t|, \mathbb{R}^{m_2})$ , and  $c_{\mathcal{Z}} \in C^d(D^t, |z^t|, \mathbb{R}^{s_t})$  with  $d \geq 1$  exist such that it reads

$$\min_{t, z^t} f(t) \quad \text{s.t.} \quad c_{\mathcal{E}}(t, |z^t|) = 0, \quad c_{\mathcal{I}}(t, |z^t|) \geq 0, \quad c_{\mathcal{Z}}(t, |z^t|) - z^t = 0, \quad (\text{I-NLP})$$

where  $D^t$  is open and symmetric and  $\partial_2 c_{\mathcal{Z}}(x, |z^t|)$  is strictly lower triangular. The feasible set of (I-NLP) is  $\mathcal{F}_{\text{abs}} := \{(t, z^t) : c_{\mathcal{E}}(t, |z^t|) = 0, c_{\mathcal{I}}(t, |z^t|) \geq 0, c_{\mathcal{Z}}(t, |z^t|) = z^t\}$ .

**Definition 4** (Active Inequality Set). Let  $(t, z^t(t)) \in \mathcal{F}_{\text{abs}}$ . We call the inequality constraint  $i \in \mathcal{I}$  *active* if  $c_i(t, |z^t(t)|) = 0$ . The *active set*  $\mathcal{A}(t)$  consists of all indices of active inequality constraints,  $\mathcal{A}(t) = \{i \in \mathcal{I} : c_i(t, |z^t(t)|) = 0\}$ . We set  $c_{\mathcal{A}} := [c_i]_{i \in \mathcal{A}(t)}$  and denote the number of active inequality constraints by  $|\mathcal{A}(t)|$ .

With the goal of considering kink qualifications in the spirit of Abadie and Guignard, we define the tangent cone and the abs-normal-linearized cone.

**Definition 5** (Tangent Cone and Abs-Normal-Linearized Cone for (I-NLP)). Consider a feasible point  $(t, z^t)$  of (I-NLP). The *tangent cone* to  $\mathcal{F}_{\text{abs}}$  at  $(t, z^t)$  is

$$\mathcal{T}_{\text{abs}}(t, z^t) := \left\{ (\delta t, \delta z^t) \mid \begin{array}{l} \exists \tau_k \searrow 0, \mathcal{F}_{\text{abs}} \ni (t_k, z_k^t) \rightarrow (t, z^t): \\ \tau_k^{-1}(t_k - t, z_k^t - z^t) \rightarrow (\delta t, \delta z^t) \end{array} \right\}.$$

With  $\delta \zeta_i := |\delta z_i^t|$  if  $i \in \alpha^t(t)$  and  $\delta \zeta_i := \sigma_i^t(t) \delta z_i^t$  if  $i \notin \alpha^t(t)$ , the *abs-normal-linearized cone* is

$$\mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t) := \left\{ (\delta t, \delta z^t) \mid \begin{array}{l} \partial_1 c_{\mathcal{E}}(t, |z^t|) \delta t + \partial_2 c_{\mathcal{E}}(t, |z^t|) \delta \zeta = 0, \\ \partial_1 c_{\mathcal{A}}(t, |z^t|) \delta t + \partial_2 c_{\mathcal{A}}(t, |z^t|) \delta \zeta \geq 0, \\ \partial_1 c_{\mathcal{Z}}(t, |z^t|) \delta t + \partial_2 c_{\mathcal{Z}}(t, |z^t|) \delta \zeta = \delta z^t \end{array} \right\}.$$

To prove that the tangent cone is a subset of the abs-normal-linearized cone, we follow an idea from [1], where an analogous result for MPCCs was obtained. First, we need the definition of the smooth branch NLPs for (I-NLP) with their standard tangent cones and linearized cones.

**Definition 6** (Branch NLPs for (I-NLP)). Consider a feasible point  $(\hat{t}, \hat{z}^t)$  of (I-NLP). Choose  $\sigma^t \in \{-1, 1\}^{s_t}$  with  $\sigma^t \succeq \sigma^t(\hat{t})$  and set  $\Sigma^t = \text{diag}(\sigma^t)$ . The branch problem NLP( $\Sigma^t$ ) is defined as

$$\min_{t, z^t} f(t) \quad \text{s.t.} \quad c_{\mathcal{E}}(t, \Sigma^t z^t) = 0, \quad c_{\mathcal{I}}(t, \Sigma^t z^t) \geq 0, \\ c_{\mathcal{Z}}(t, \Sigma^t z^t) - z^t = 0, \quad \Sigma^t z^t \geq 0. \quad (\text{NLP}(\Sigma^t))$$

The feasible set of (NLP( $\Sigma^t$ )), which always contains  $(\hat{t}, \hat{z}^t)$ , is denoted by  $\mathcal{F}_{\Sigma^t}$ .

**Definition 7** (Tangent Cone and Linearized Cone for (NLP( $\Sigma^t$ ))). Given (NLP( $\Sigma^t$ )), consider a feasible point  $(t, z^t)$ . The *tangent cone* to  $\mathcal{F}_{\Sigma^t}$  at  $(t, z^t)$  is

$$\mathcal{T}_{\Sigma^t}(t, z^t) := \left\{ (\delta t, \delta z^t) \mid \begin{array}{l} \exists \tau_k \searrow 0, \mathcal{F}_{\Sigma^t} \ni (t_k, z_k^t) \rightarrow (t, z^t): \\ \tau_k^{-1}(t_k - t, z_k^t - z^t) \rightarrow (\delta t, \delta z^t) \end{array} \right\}.$$

The *linearized cone* is

$$\mathcal{T}_{\Sigma^t}^{\text{lin}}(t, z^t) := \left\{ (\delta t, \delta z^t) \left| \begin{array}{l} \partial_1 c_{\mathcal{E}}(t, \Sigma^t z^t) \delta t + \partial_2 c_{\mathcal{E}}(t, \Sigma^t z^t) \Sigma^t \delta z^t = 0, \\ \partial_1 c_{\mathcal{A}}(t, \Sigma^t z^t) \delta t + \partial_2 c_{\mathcal{A}}(t, \Sigma^t z^t) \Sigma^t \delta z^t \geq 0, \\ \partial_1 c_{\mathcal{Z}}(t, \Sigma^t z^t) \delta t + \partial_2 c_{\mathcal{Z}}(t, \Sigma^t z^t) \Sigma^t \delta z^t = \delta z^t, \\ \sigma_i^t \delta z_i^t \geq 0, \quad i \in \alpha^t(t) \end{array} \right. \right\}.$$

*Remark 8.* Observe that  $|z^t| = \Sigma^t z^t$  in Definitions 6 and 7, and for every  $\Sigma^t$  we have  $\mathcal{F}_{\Sigma^t} \subseteq \mathcal{F}_{\text{abs}}$ ,  $\mathcal{T}_{\Sigma^t}(t, z^t) \subseteq \mathcal{T}_{\text{abs}}(t, z^t)$ , and  $\mathcal{T}_{\Sigma^t}^{\text{lin}}(t, z^t) \subseteq \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t)$ .

**Lemma 9.** *Consider a feasible point  $(\hat{t}, \hat{z}^t)$  of (I-NLP) with associated branch problems  $(\text{NLP}(\Sigma^t))$ . Then, the following decompositions of the tangent cone and of the abs-normal-linearized cone of (I-NLP) hold:*

$$\mathcal{T}_{\text{abs}}(\hat{t}, \hat{z}^t) = \bigcup_{\Sigma^t} \mathcal{T}_{\Sigma^t}(\hat{t}, \hat{z}^t) \quad \text{and} \quad \mathcal{T}_{\text{abs}}^{\text{lin}}(\hat{t}, \hat{z}^t) = \bigcup_{\Sigma^t} \mathcal{T}_{\Sigma^t}^{\text{lin}}(\hat{t}, \hat{z}^t).$$

*Proof.* We first consider the tangent cones and show that a neighborhood  $\mathcal{N}$  of  $(\hat{t}, \hat{z}^t)$  exists such that

$$\mathcal{F}_{\text{abs}} \cap \mathcal{N} = \bigcup_{\Sigma^t} (\mathcal{F}_{\Sigma^t} \cap \mathcal{N}).$$

The inclusion  $\supseteq$  holds for every neighborhood  $\mathcal{N}$  since  $\mathcal{F}_{\Sigma^t} \subseteq \mathcal{F}_{\text{abs}}$  for all  $\Sigma^t$ . To show the inclusion  $\subseteq$  we consider an index  $i \notin \alpha^t(\hat{t})$ . Then, by continuity,  $\epsilon_i > 0$  exists with  $\sigma_i^t(t) = \sigma_i^t(\hat{t}) \in \{-1, +1\}$  for all  $t \in B_{\epsilon_i}(\hat{t})$ . Now set  $\epsilon := \min_{i \notin \alpha^t(\hat{t})} \epsilon_i$ ,  $\mathcal{N} := B_{\epsilon} \times \mathbb{R}^{n^t}$ , and consider  $(t, z^t) \in \mathcal{N} \cap \mathcal{F}_{\text{abs}}$ . With the choice  $\sigma_i^t = \sigma_i^t(t)$  for  $i \notin \alpha^t(t)$  and  $\sigma_i^t = 1$  for  $i \in \alpha^t(t)$  we find  $\Sigma^t = \text{diag}(\sigma^t)$  such that  $(t, z^t) \in \mathcal{N} \cap \mathcal{F}_{\Sigma^t}$  since  $\alpha^t(t) \subseteq \alpha^t(\hat{t})$ . Thus,

$$\mathcal{F}_{\text{abs}} \cap \mathcal{N} = \bigcup_{\Sigma^t} (\mathcal{F}_{\Sigma^t} \cap \mathcal{N}).$$

Now, let  $\mathcal{T}(\hat{t}, \hat{z}^t; \mathcal{F})$  generically denote the tangent cone to  $\mathcal{F}$  at  $(\hat{t}, \hat{z}^t)$ . Then,

$$\begin{aligned} \mathcal{T}_{\text{abs}}(\hat{t}, \hat{z}^t) &= \mathcal{T}(\hat{t}, \hat{z}^t; \mathcal{F}_{\text{abs}}) = \mathcal{T}(\hat{t}, \hat{z}^t; \mathcal{F}_{\text{abs}} \cap \mathcal{N}) = \mathcal{T}(\hat{t}, \hat{z}^t; \bigcup_{\Sigma^t} (\mathcal{F}_{\Sigma^t} \cap \mathcal{N})) \\ &= \bigcup_{\Sigma^t} \mathcal{T}(\hat{t}, \hat{z}^t; \mathcal{F}_{\Sigma^t} \cap \mathcal{N}) = \bigcup_{\Sigma^t} \mathcal{T}(\hat{t}, \hat{z}^t; \mathcal{F}_{\Sigma^t}) = \bigcup_{\Sigma^t} \mathcal{T}_{\Sigma^t}(\hat{t}, \hat{z}^t). \end{aligned}$$

Here the fourth equality holds since the number of branch problems is finite. The decomposition of  $\mathcal{T}_{\text{abs}}^{\text{lin}}$  follows directly by comparing definitions of  $\mathcal{T}_{\text{abs}}^{\text{lin}}$  and  $\mathcal{T}_{\Sigma^t}^{\text{lin}}$ .  $\square$

**Lemma 10.** *Let  $(t, z^t)$  be feasible for (I-NLP). Then,*

$$\mathcal{T}_{\text{abs}}(t, z^t) \subseteq \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t) \quad \text{and} \quad \mathcal{T}_{\text{abs}}(t, z^t)^* \supseteq \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t)^*.$$

*Proof.* The branch NLPs are smooth, hence the inclusion  $\mathcal{T}_{\Sigma^t}(t, z^t) \subseteq \mathcal{T}_{\Sigma^t}^{\text{lin}}(t, z^t)$  holds by standard NLP theory. Then, the first inclusion follows directly from Lemma 9 and the second inclusion follows by dualization of the cones.  $\square$

In general, the reverse inclusions do not hold. This leads to the following definitions.

**Definition 11** (Abadie's and Guignard's Kink Qualifications for (I-NLP)). Consider a feasible point  $(t, z^t(t))$  of (I-NLP). We say that *Abadie's Kink Qualification (AKQ)* holds at  $t$  if  $\mathcal{T}_{\text{abs}}(t, z^t(t)) = \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t(t))$ , and that *Guignard's Kink Qualification (GKQ)* holds at  $t$  if  $\mathcal{T}_{\text{abs}}(t, z^t(t))^* = \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t(t))^*$ .

The decomposition of cones in Lemma 9 and its dualization immediately lead to the next theorem.

**Theorem 12** (ACQ/GCQ for all  $(\text{NLP}(\Sigma^t))$  implies AKQ/GKQ for (I-NLP)). *Consider a feasible point  $(t, z^t(t))$  of (I-NLP) with associated branch problems  $(\text{NLP}(\Sigma^t))$ . Then, AKQ respectively GKQ holds for (I-NLP) at  $t$  if ACQ respectively GCQ holds for all  $(\text{NLP}(\Sigma^t))$  at  $(t, z^t(t))$ .*

**2.2. Abs-Normal NLPs with Inequality Slacks.** Here, we use absolute values of slack variables to get rid of the inequality constraints. This idea is due to Griewank. It has been introduced in [10] and has been further investigated in [8]. With slack variables  $w \in \mathbb{R}^{m_2}$ , we reformulate (NLP) as follows:

$$\min_{t,w} f(t) \quad \text{s.t.} \quad g(t) = 0, \quad h(t) - |w| = 0.$$

Then, we express  $g$  and  $h$  in abs-normal form as in (ANF) and introduce additional switching variables  $z^w$  to handle  $|w|$ . We obtain a class of purely equality-constrained abs-normal NLPs.

**Definition 13** (Abs-Normal NLP with Inequality Slacks). An abs-normal NLP posed in the following form is called an *abs-normal NLP with inequality slacks*:

$$\begin{aligned} \min_{t,w,z^t,z^w} f(t) \quad \text{s.t.} \quad & c_{\mathcal{E}}(t, |z^t|) = 0, \quad c_{\mathcal{I}}(t, |z^t|) - |z^w| = 0, \\ & c_{\mathcal{Z}}(t, |z^t|) = z^t, \quad w = z^w, \end{aligned} \quad (\text{E-NLP})$$

where  $D^{|z^t|}$  is open and symmetric and  $\partial_2 c_{\mathcal{Z}}(t, |z^t|)$  is strictly lower triangular. The feasible set of (E-NLP) is denoted by  $\mathcal{F}_{\text{e-abs}}$  and is a lifting of  $\mathcal{F}_{\text{abs}}$ .

*Remark 14.* Introducing  $|w|$  converts inequalities to pure equalities without a nonnegativity condition for the slack variables  $w$ . In [10] we have used this formulation to simplify the presentation of first and second order conditions for the general abs-normal NLP under the linear independence kink qualification (LIKQ). Later we will see that constraint qualifications of Abadie type are preserved under reformulation. Nevertheless, this representation causes some problems. In [8] we have shown that, in contrast to LIKQ, constraint qualifications of Mangasarian-Fromovitz type are not preserved. Moreover, we cannot prove compatibility of constraint qualifications of Guignard type. Also, note that the equation  $w - z^w = 0$  (and hence  $w$ ) cannot be eliminated as this would destroy the abs-normal form. Finally, the signs of nonzero components  $w_i$  can be chosen arbitrarily and thus the slack  $w$  is not uniquely determined. This needs to be taken into account when formulating kink qualifications (KQ) for (E-NLP).

We are now interested in deriving Abadie's and Guignard's KQ for (E-NLP). To this end, we observe that the formulation (E-NLP) can be seen as a special case of (I-NLP): Let  $x = (t, w)$ ,  $z = (z^t, z^w)$ ,  $\bar{f}(x) = f(t)$ ,  $\bar{c}_{\mathcal{E}}(x, |z|) = (c_{\mathcal{E}}(t, |z^t|), c_{\mathcal{I}}(t, |z^t|) - |z^w|)$ , and  $\bar{c}_{\mathcal{Z}}(x, |z|) = (c_{\mathcal{Z}}(t, |z^t|), w)$ . Then, we can rewrite (E-NLP) as

$$\min_{x,z} f(x) \quad \text{s.t.} \quad \bar{c}_{\mathcal{E}}(x, |z|) = 0, \quad \bar{c}_{\mathcal{Z}}(x, |z|) - z = 0.$$

Hence, the following material is readily obtained by specializing the definitions and results in the previous section.

With  $\delta = (\delta t, \delta w, \delta z^t, \delta z^w)$ , Definition 5 and  $w = z^w$  give the tangent cone to  $\mathcal{F}_{\text{e-abs}}$  at  $(t, w, z^t, z^w)$  as

$$\mathcal{T}_{\text{e-abs}}(t, w, z^t, z^w) = \left\{ \delta \left| \begin{array}{l} \exists \tau_k \searrow 0, \mathcal{F}_{\text{e-abs}} \ni (t_k, w_k, z_k^t, z_k^w) \rightarrow (t, w, z^t, z^w): \\ \tau_k^{-1}(t_k - t, w_k - w, z_k^t - z^t) \rightarrow (\delta t, \delta w, \delta z^t), \delta z^w = \delta w \end{array} \right. \right\},$$

and the abs-normal-linearized cone reads

$$\mathcal{T}_{\text{e-abs}}^{\text{lin}}(t, w, z^t, z^w) = \left\{ \delta \left| \begin{array}{l} \partial_1 c_{\mathcal{I}}(t, |z^t|) \delta t + \partial_2 c_{\mathcal{I}}(t, |z^t|) \delta \zeta = \delta \omega, \\ (\delta t, \delta z^t) \in \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t), \delta z^w = \delta w \end{array} \right. \right\},$$

where  $\alpha = (\alpha^t, \alpha^w)$  and

$$\delta \zeta_i = \begin{cases} \sigma_i^t(t) \delta z_i^t, & i \notin \alpha^t(t) \\ |\delta z_i^t|, & i \in \alpha^t(t) \end{cases}, \quad \delta \omega_i = \begin{cases} \sigma_i^w(w) \delta z_i^w, & i \notin \alpha^w(w) \\ |\delta z_i^w|, & i \in \alpha^w(w) \end{cases}.$$

In Definition 6, consider a feasible point  $(\hat{t}, \hat{w}, \hat{z}^t, \hat{z}^w)$  of (E-NLP). Choose  $\sigma^t \in \{-1, 1\}^{s_t}$  with  $\sigma^t \succeq \sigma^t(\hat{t})$  and  $\sigma^w \in \{-1, 1\}^{m_2}$  with  $\sigma^w \succeq \sigma^w(\hat{w})$ . Set  $\Sigma^t = \text{diag}(\sigma^t)$  and  $\Sigma^w = \text{diag}(\sigma^w)$ . Then, the branch problem  $\text{NLP}(\Sigma^{t,w})$  for  $\Sigma^{t,w} := \text{diag}(\Sigma^t, \Sigma^w)$  reads

$$\begin{aligned} \min_{t,w,z^t,z^w} f(t) \quad \text{s.t.} \quad & c_{\mathcal{E}}(t, \Sigma^t z^t) = 0, \quad c_{\mathcal{I}}(t, \Sigma^t z^t) - \Sigma^w z^w = 0, \\ & c_{\mathcal{Z}}(t, \Sigma^t z^t) - z^t = 0, \quad w - z^w = 0, \\ & \Sigma^t z^t \geq 0, \quad \Sigma^w z^w \geq 0. \end{aligned} \quad (\text{NLP}(\Sigma^{t,w}))$$

The feasible set of  $(\text{NLP}(\Sigma^{t,w}))$ , which always contains  $(\hat{t}, \hat{w}, \hat{z}^t, \hat{z}^w)$ , is denoted by  $\mathcal{F}_{\Sigma^{t,w}}$  and is a lifting of  $\mathcal{F}_{\Sigma^t}$ . By Definition 7, the tangent cone to  $\mathcal{F}_{\Sigma^{t,w}}$  at  $(t, w, z^t, z^w)$  reads

$$\mathcal{T}_{\Sigma^{t,w}}(t, w, z^t, z^w) = \left\{ \delta \mid \begin{array}{l} \exists \tau_k \searrow 0, \mathcal{F}_{\Sigma^{t,w}} \ni (t_k, w_k, z_k^t, z_k^w) \rightarrow (t, w, z^t, z^w): \\ \tau_k^{-1}(t_k - t, w_k - w, z_k^t - z^t) \rightarrow (\delta t, \delta w, \delta z^t), \delta z^w = \delta w \end{array} \right\}$$

with  $\delta = (\delta t, \delta w, \delta z^t, \delta z^w)$ . The linearized cone reads

$$\mathcal{T}_{\Sigma^{t,w}}^{\text{lin}}(t, w, z^t, z^w) = \left\{ \delta \mid \begin{array}{l} \partial_1 c_{\mathcal{I}} \delta t + \partial_2 c_{\mathcal{I}} \Sigma^t \delta z^t - \Sigma^w \delta z^w = 0, \delta z^w = \delta w, \\ (\delta t, \delta z^t) \in \mathcal{T}_{\Sigma^t}^{\text{lin}}(t, z^t), \sigma_i^w \delta z_i^w \geq 0, i \in \alpha^w(w) \end{array} \right\}.$$

Here, all partial derivatives are evaluated at  $(t, \Sigma^t z^t)$ .

Moreover, we obtain the following decompositions by applying Lemma 9 to  $(\text{E-NLP})$  at  $y = (t, w, z^t, z^w)$  with associated branch problems  $(\text{NLP}(\Sigma^{t,w}))$ :

$$\mathcal{T}_{\text{e-abs}}(y) = \bigcup_{\Sigma^{t,w}} \mathcal{T}_{\Sigma^{t,w}}(y) \quad \text{and} \quad \mathcal{T}_{\text{e-abs}}^{\text{lin}}(y) = \bigcup_{\Sigma^{t,w}} \mathcal{T}_{\Sigma^{t,w}}^{\text{lin}}(y).$$

As before, the tangent cone is a subset of the linearized cone and the reverse inclusion holds for the dual cones:

$$\mathcal{T}_{\text{e-abs}}(y) \subseteq \mathcal{T}_{\text{e-abs}}^{\text{lin}}(y) \quad \text{and} \quad \mathcal{T}_{\text{e-abs}}(y)^* \supseteq \mathcal{T}_{\text{e-abs}}^{\text{lin}}(y)^*.$$

This follows directly by applying Lemma 10 to  $(\text{E-NLP})$ . Again, equality does not hold in general, and we consider Abadie's Kink Qualification (AKQ) and Guignard's Kink Qualification (GKQ) for  $(\text{E-NLP})$ .

Given a feasible point  $y = (t, w, z^t(t), z^w(w))$  of  $(\text{E-NLP})$ , Definition 11 gives AKQ and GKQ at  $(t, w)$ , respectively, as

$$\mathcal{T}_{\text{e-abs}}(y) = \mathcal{T}_{\text{e-abs}}^{\text{lin}}(y) \quad \text{and} \quad \mathcal{T}_{\text{e-abs}}(y)^* = \mathcal{T}_{\text{e-abs}}^{\text{lin}}(y)^*.$$

*Remark 15.* The possible slack values  $w \in W(t) := \{w : |w| = c_{\mathcal{I}}(t, |z^t(t)|)\}$  just differ by the signs of components  $w_i$  for  $i \in \mathcal{A}(t)$ . Thus, neither AKQ nor GKQ depends on the particular choice of  $w$ , and both conditions are well-defined for  $(\text{E-NLP})$ .

Now Theorem 12 takes the following form.

**Theorem 16** (ACQ/GCQ for all  $(\text{NLP}(\Sigma^{t,w}))$  implies AKQ/GKQ for  $(\text{E-NLP})$ ). *Consider a feasible point  $y = (t, w, z^t(t), z^w(w))$  of  $(\text{E-NLP})$  with associated branch problems  $(\text{NLP}(\Sigma^{t,w}))$ . Then, AKQ respectively GKQ for  $(\text{E-NLP})$  holds at  $(t, w)$  if ACQ respectively GCQ holds for all  $(\text{NLP}(\Sigma^{t,w}))$  at  $y$ .*

**2.3. Relations of Kink Qualifications for Abs-Normal NLPs.** In this paragraph we discuss the relations of kink qualifications for the two different formulations introduced above. Here, equality of the cones and of the dual cones just needs to be considered for one element of the set  $W(t) = \{w : |w| = c_{\mathcal{I}}(t, |z^t(t)|)\}$ . Then, it holds directly for all other elements by Remark 15.

**Theorem 17.** *AKQ for  $(\text{I-NLP})$  holds at  $(t, z^t(t)) \in \mathcal{F}_{\text{abs}}$  if and only if AKQ for  $(\text{E-NLP})$  holds at  $(t, w, z^t(t), z^w(w)) \in \mathcal{F}_{\text{e-abs}}$  for any (and hence all)  $w \in W(t)$ .*

*Proof.* As  $\mathcal{T}_{\text{abs}}(t, z^t) \subseteq \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t)$  and  $\mathcal{T}_{\text{e-abs}}(t, z^t) \subseteq \mathcal{T}_{\text{e-abs}}^{\text{lin}}(t, z^t)$  always hold, we just need to prove

$$\mathcal{T}_{\text{abs}}(t, z^t) \supseteq \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t) \iff \mathcal{T}_{\text{e-abs}}(t, w, z^t, z^w) \supseteq \mathcal{T}_{\text{e-abs}}^{\text{lin}}(t, w, z^t, z^w).$$

We start with the implication “ $\Rightarrow$ ”. Let  $\delta = (\delta t, \delta w, \delta z^t, \delta z^w) \in \mathcal{T}_{\text{e-abs}}^{\text{lin}}(t, w, z^t, z^w)$ . Then, we have  $\tilde{\delta} = (\delta t, \delta z^t) \in \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t) = \mathcal{T}_{\text{abs}}(t, z^t)$ . Hence, there exist sequences  $(t_k, z_k^t) \in \mathcal{F}_{\text{abs}}$  and  $\tau_k \searrow 0$  with  $(t_k, z_k^t) \rightarrow (t, z^t)$  and  $\tau_k^{-1}(t_k - t, z_k^t - z^t) \rightarrow (\delta t, \delta z^t)$ . Now, define

$$\Sigma^w = \text{diag}(\sigma) \quad \text{with} \quad \sigma_i = \begin{cases} \sigma^w(w_i), & i \notin \alpha^w(w), \\ \text{sign}(\delta z_i^w), & i \in \alpha^w(w), \end{cases}$$

and set  $z_k^w := w_k := \Sigma^w c_{\mathcal{I}}(t_k, |z_k^t|)$ . Then, we have  $z^w = w = \Sigma^w c_{\mathcal{I}}(t, |z^t|)$  and obtain

$$\begin{aligned} z_k^w - z^w &= \Sigma^w [c_{\mathcal{I}}(t_k, |z_k^t|) - c_{\mathcal{I}}(t, |z^t|)] \\ &= \Sigma^w [\partial_1 c_{\mathcal{I}}(t, |z^t|)(t_k - t) + \partial_2 c_{\mathcal{I}}(t, |z^t|)(|z_k^t| - |z^t|) + o(\|(t_k - t, |z_k^t| - |z^t|)\|)]. \end{aligned}$$

Further, for  $k$  large enough we have  $|z_k^t| - |z^t| = \Sigma_k^t z_k^t - \Sigma^t z^t$  using  $\Sigma_k^t = \text{diag}(\sigma_k^t)$  with  $\sigma_k^t = \sigma(t_k)$  and  $\Sigma^t = \text{diag}(\sigma^t)$  with  $\sigma^t = \sigma(t)$ . Then, we obtain for  $z_i^t \neq 0$

$$\tau_k^{-1}(|(z_k^t)_i| - |z_i^t|) = \tau_k^{-1} \sigma_i^t((z_k^t)_i - z_i^t) \rightarrow \sigma_i^t \delta z_i^t.$$

For  $z_i^t = 0$  we have  $\tau_k^{-1}(z_k^t)_i \rightarrow \delta z_i^t$  and hence

$$\tau_k^{-1}(|(z_k^t)_i| - |z_i^t|) = \tau_k^{-1} |(z_k^t)_i| \rightarrow |\delta z_i^t|.$$

Thus,  $\tau_k^{-1}(|(z_k^t)| - |z^t|) \rightarrow \delta \zeta$  holds, and in total

$$\tau_k^{-1}(z_k^w - z^w) \rightarrow \Sigma^w [\partial_1 c_{\mathcal{I}}(t, |z^t|) \delta t + \partial_2 c_{\mathcal{I}}(t, |z^t|) \delta \zeta] = \Sigma^w \delta \zeta = \delta z^w.$$

Additionally, we obtain  $\tau_k^{-1}(w_k - w) \rightarrow \delta w$  and finally  $d \in \mathcal{T}_{\text{e-abs}}(t, w, z^t, z^w)$ . To prove the implication “ $\Leftarrow$ ”, consider  $\delta = (\delta t, \delta z^t) \in \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t)$ . We define

$$\Sigma^w = \text{diag}(\sigma) \quad \text{with} \quad \sigma_i = \begin{cases} \pm 1, & i \in \mathcal{A}(t), \\ \text{sign}([\partial_1 c_{\mathcal{I}}(t, |z^t|) \delta t + \partial_2 c_{\mathcal{I}}(t, |z^t|) \delta \zeta]_i), & i \notin \mathcal{A}(t), \end{cases}$$

and set  $\delta w = \delta z^w = \Sigma^w [\partial_1 c_{\mathcal{I}}(t, |z^t|) \delta t + \partial_2 c_{\mathcal{I}}(t, |z^t|) \delta \zeta]$ . Then we have  $\tilde{\delta} = (\delta t, \delta w, \delta z^t, \delta z^w) \in \mathcal{T}_{\text{e-abs}}^{\text{lin}}(t, w, z^t, z^w)$  for  $w = z^w = \Sigma^w c_{\mathcal{I}}(t, |z^t|)$ . By assumption,  $\tilde{\delta} \in \mathcal{T}_{\text{e-abs}}(t, w, z^t, z^w)$  holds, and this directly implies  $\delta = (\delta t, \delta z^t) \in \mathcal{T}_{\text{abs}}(t, z^t)$ .  $\square$

**Theorem 18.** *GKQ for (I-NLP) holds at the point  $(t, z^t(t)) \in \mathcal{F}_{\text{abs}}$  if GKQ for (E-NLP) holds at  $(t, w, z^t(t), z^w(w)) \in \mathcal{F}_{\text{e-abs}}$  for any (and hence all)  $w \in W(t)$ .*

*Proof.* The inclusion  $\mathcal{T}_{\text{abs}}(t, z^t)^* \supseteq \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t)^*$  is always satisfied. Thus, we just have to show

$$\mathcal{T}_{\text{abs}}(t, z^t)^* \subseteq \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t)^*.$$

Let  $\omega = (\omega t, \omega z^t) \in \mathcal{T}_{\text{abs}}(t, z^t)^*$ , i.e.  $\omega^T \delta \geq 0$  for all  $\delta = (\delta t, \delta z^t) \in \mathcal{T}_{\text{abs}}(t, z^t)$ . Then, set  $\tilde{\omega} = (\omega t, 0, \omega z^t, 0)$  and obtain  $\tilde{\omega}^T \tilde{\delta} = \omega^T \delta \geq 0$  for all  $\tilde{\delta} \in \mathcal{T}_{\text{e-abs}}(t, w, z^t, z^w)$  where  $w \in W(t)$  is arbitrary. By assumption, then  $\tilde{\omega}^T \tilde{\delta} \geq 0$  for all  $\tilde{\delta} \in \mathcal{T}_{\text{e-abs}}^{\text{lin}}(t, w, z^t, z^w)$  holds. This implies  $\omega^T \delta = \tilde{\omega}^T \tilde{\delta} \geq 0$  for all  $\delta \in \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t)$ .  $\square$

The converse is unlikely to hold, but we are, at the same time, not aware of a counterexample. Next, we consider the branch problems and relations of ACQ and GCQ for all branch problems. Here, we can exploit sign information to show equivalence of GCQ for the branch problems of (I-NLP) and (E-NLP).

**Theorem 19.** *ACQ for (NLP( $\Sigma^t$ )) holds at  $(t, z^t(t)) \in \mathcal{F}_{\Sigma^t}$  if and only if ACQ for (NLP( $\Sigma^t, w$ )) holds at  $(t, w, z^t(t), z^t(w)) \in \mathcal{F}_{\Sigma^t, w}$  for any (and hence all)  $w \in W(t)$ .*

*Proof.* The proof proceeds as in Theorem 17.  $\square$

**Theorem 20.** *GCQ for (NLP( $\Sigma^t$ )) holds at  $(t, z^t(t)) \in \mathcal{F}_{\Sigma^t}$  if and only if GCQ for (NLP( $\Sigma^t, w$ )) holds at  $(t, w, z^t(t), z^t(w)) \in \mathcal{F}_{\Sigma^t, w}$  for any (and hence all)  $w \in W(t)$ .*

*Proof.* The inclusions  $\mathcal{T}_{\Sigma^t}(t, z^t)^* \supseteq \mathcal{T}_{\Sigma^t}^{\text{lin}}(t, z^t)^*$  and  $\mathcal{T}_{\Sigma^t, w}(t, z^t)^* \supseteq \mathcal{T}_{\Sigma^t, w}^{\text{lin}}(t, z^t)^*$  are always satisfied. Thus, we just need to prove

$$\mathcal{T}_{\Sigma^t}(t, z^t)^* \subseteq \mathcal{T}_{\Sigma^t}^{\text{lin}}(t, z^t)^* \iff \mathcal{T}_{\Sigma^t, w}(t, w, z^t, z^w)^* \subseteq \mathcal{T}_{\Sigma^t, w}^{\text{lin}}(t, w, z^t, z^w)^*.$$

We start with the implication “ $\Rightarrow$ ”. Let  $\omega = (\omega t, \omega w, \omega z^t, \omega z^w) \in \mathcal{T}_{\Sigma^t, w}(t, w, z^t, z^w)^*$ , i.e.  $\omega^T \delta \geq 0$  for all  $\delta = (\delta t, \delta w, \delta z^t, \delta z^w) \in \mathcal{T}_{\Sigma^t, w}(t, w, z^t, z^w)$ . Set

$$\tilde{\omega} = (\tilde{\omega} t, \tilde{\omega} z^t) = (\omega t, \omega z^t) + (\omega w + \omega z^w) \Sigma^w (\partial_1 c_{\mathcal{I}}(t, \Sigma^t z^t), \partial_2 c_{\mathcal{I}}(t, \Sigma^t z^t) \Sigma^t).$$

Then, we have  $\tilde{\omega}^T \tilde{\delta} = \omega^T \delta \geq 0$  for all  $\delta = (\delta t, \delta z^t) \in \mathcal{T}_{\Sigma^t}(t, z^t)$  and thus  $\tilde{\omega} \in \mathcal{T}_{\Sigma^t}^{\text{lin}}(t, z^t)$ . Then,  $\omega^T \delta \geq 0$  for all  $\delta = (\delta t, \delta w, \delta z^t, \delta z^w) \in \mathcal{T}_{\Sigma^t, w}^{\text{lin}}(t, w, z^t, z^w)$  as  $\omega^T \delta = \tilde{\omega}^T \tilde{\delta}$  holds. The reverse implication may be proved as shown in Theorem 18.  $\square$

### 3. COUNTERPART MPCCS

In this section we restate the MPCC counterpart problems for the two formulations (I-NLP) and (E-NLP) and we present the relations between them.

**3.1. Counterpart MPCC for the General Abs-Normal NLP.** To reformulate **(I-NLP)** as an MPCC, we split  $z^t$  into its nonnegative part and the modulus of its nonpositive part,  $u^t := [z^t]^+ := \max(z^t, 0)$  and  $v^t := [z^t]^- := \max(-z^t, 0)$ . Then, we add complementarity of these two variables to replace  $|z^t|$  by  $u^t + v^t$  and  $z^t$  itself by  $u^t - v^t$ .

**Definition 21** (Counterpart MPCC of **(I-NLP)**). The *counterpart MPCC* of the non-smooth NLP **(I-NLP)** reads

$$\begin{aligned} \min_{t, u^t, v^t} f(t) \quad \text{s.t.} \quad & c_{\mathcal{E}}(t, u^t + v^t) = 0, \quad c_{\mathcal{I}}(t, u^t + v^t) \geq 0, \\ & c_{\mathcal{Z}}(t, u^t + v^t) - (u^t - v^t) = 0, \\ & 0 \leq u^t \perp v^t \geq 0, \end{aligned} \tag{I-MPCC}$$

where  $u^t, v^t \in \mathbb{R}^{s_t}$ . The feasible set of **(I-MPCC)** is denoted by  $\mathcal{F}_{\text{mpcc}}$ .

Given an abs-normal NLP **(I-NLP)** and its counterpart MPCC **(I-MPCC)**, the mapping  $\phi: \mathcal{F}_{\text{mpcc}} \rightarrow \mathcal{F}_{\text{abs}}$  defined by

$$\phi(t, u^t, v^t) = (t, u^t - v^t) \quad \text{and} \quad \phi^{-1}(t, z^t) = (t, [z^t]^+, [z^t]^-)$$

is a homeomorphism. This result was obtained in [8, Lemma 31].

Corresponding to the active switching set in the previous section, we introduce index sets for MPCCs.

**Definition 22** (Index Sets). We denote by  $\mathcal{U}_0^t := \{i \in \{1, \dots, s_t\}: u_i^t = 0\}$  the set of indices of active inequalities  $u_i^t \geq 0$ , and by  $\mathcal{U}_+^t := \{i \in \{1, \dots, s_t\}: u_i^t > 0\}$  the set of indices of inactive inequalities  $u_i^t \geq 0$ . Analogous definitions hold of  $\mathcal{V}_0^t$  and  $\mathcal{V}_+^t$ . By  $\mathcal{D}^t := \mathcal{U}_0^t \cap \mathcal{V}_0^t$  we denote the set of indices of non-strict (degenerate) complementarity pairs. Thus we have the partitioning  $\{1, \dots, s_t\} = \mathcal{U}_+^t \cup \mathcal{V}_+^t \cup \mathcal{D}^t$ .

In order to define MPCC-CQs in the spirit of Abadie and Guignard, we introduce the tangent cone, the complementarity cone, and the MPCC-linearized cone.

**Definition 23** (Tangent Cone and MPCC-Linearized Cone for **(I-MPCC)**, see [1]). Consider a feasible point  $(t, u^t, v^t)$  of **(I-MPCC)** with associated index sets  $\mathcal{U}_+^t$ ,  $\mathcal{V}_+^t$  and  $\mathcal{D}^t$ . The *tangent cone* to  $\mathcal{F}_{\text{mpcc}}$  at  $(t, u^t, v^t)$  is

$$\mathcal{T}_{\text{mpcc}}(t, u^t, v^t) := \left\{ (\delta t, \delta u^t, \delta v^t) \mid \begin{array}{l} \exists \tau_k \searrow 0, \mathcal{F}_{\text{mpcc}} \ni (t_k, u_k^t, v_k^t) \rightarrow (t, u^t, v^t): \\ \tau_k^{-1}(t_k - t, u_k^t - u^t, v_k^t - v^t) \rightarrow (\delta t, \delta u^t, \delta v^t) \end{array} \right\}.$$

The *MPCC-linearized cone* at  $(t, u^t, v^t)$  is

$$\mathcal{T}_{\text{mpcc}}^{\text{lin}}(t, u^t, v^t) := \left\{ \begin{pmatrix} \delta t \\ \delta u^t \\ \delta v^t \end{pmatrix} \mid \begin{array}{l} \partial_1 c_{\mathcal{E}} \delta t + \partial_2 c_{\mathcal{E}}(\delta u^t + \delta v^t) = 0, \\ \partial_1 c_{\mathcal{A}} \delta t + \partial_2 c_{\mathcal{A}}(\delta u^t + \delta v^t) \geq 0, \\ \partial_1 c_{\mathcal{Z}} \delta t + \partial_2 c_{\mathcal{Z}}(\delta u^t + \delta v^t) = \delta u^t - \delta v^t, \\ (\delta u^t, \delta v^t) \in \mathcal{T}_{\perp}(u^t, v^t) \end{array} \right\}$$

with *complementarity cone*

$$\mathcal{T}_{\perp}(u^t, v^t) := \left\{ (\delta u^t, \delta v^t) \mid \begin{array}{l} \delta u_i^t = 0, \quad i \in \mathcal{V}_+^t, \quad \delta v_i^t = 0, \quad i \in \mathcal{U}_+^t, \\ 0 \leq \delta u_i^t \perp \delta v_i^t \geq 0, \quad i \in \mathcal{D}^t \end{array} \right\}.$$

Here, all partial derivatives are evaluated at  $(t, u^t + v^t)$ .

Note that the MPCC-linearized cone was originally stated in [11] and [14], but was not further investigated there. Moreover, we modified the definition in [1] by introducing the complementarity cone which is studied in the next lemma.

**Lemma 24.** *The complementarity cone  $\mathcal{T}_{\perp}(\hat{u}^t, \hat{v}^t)$  is the tangent cone and also the linearized cone to the complementarity set  $\{(u^t, v^t): 0 \leq u^t \perp v^t \geq 0\}$  at  $(\hat{u}^t, \hat{v}^t)$ .*

*Proof.* Given a tangent vector  $(\delta u^t, \delta v^t) = \lim \tau_k^{-1}(u_k^t - \hat{u}^t, v_k^t - \hat{v}^t)$  where  $0 \leq u_k^t \perp v_k^t \geq 0$  and  $\tau_k \searrow 0$ , we have for  $k$  large enough:

$$\begin{aligned} u_{ki}^t > 0, \quad v_{ki}^t &= 0, & i \in \mathcal{U}_+^t \quad (\hat{u}_i^t > 0, \hat{v}_i^t &= 0), \\ u_{ki}^t &= 0, \quad v_{ki}^t > 0, & i \in \mathcal{V}_+^t \quad (\hat{u}_i^t &= 0, \hat{v}_i^t > 0), \\ 0 \leq u_{ki}^t \perp v_{ki}^t &\geq 0, & i \in \mathcal{D}^t \quad (\hat{u}_i^t &= 0, \hat{v}_i^t = 0). \end{aligned}$$

This implies  $(\delta u^t, \delta v^t) \in \mathcal{T}_\perp(\hat{u}^t, \hat{v}^t)$ . Conversely, every  $(\delta u^t, \delta v^t) \in \mathcal{T}_\perp(\hat{u}^t, \hat{v}^t)$  is a tangent vector generated by the sequence  $(u_k^t, v_k^t) = (\hat{u}^t, \hat{v}^t) + \tau_k(\delta u^t, \delta v^t)$  with  $\tau_k = 1/k$ ,  $k \in \mathbb{N}_{>0}$ . The linearized cone clearly coincides with the tangent cone.  $\square$

**Lemma 25.** *Given (I-NLP) with counterpart MPCC (I-MPCC), consider  $(t, z^t) \in \mathcal{F}_{\text{abs}}$  with  $\sigma^t = \sigma^t(t)$  and  $(t, u^t, v^t) = \phi^{-1}(t, z^t) \in \mathcal{F}_{\text{mpcc}}$  with associated index sets  $\mathcal{U}_+^t$ ,  $\mathcal{V}_+^t$  and  $\mathcal{D}^t$ . Define  $\psi: \mathcal{T}_{\text{mpcc}}(t, u^t, v^t) \rightarrow \mathcal{T}_{\text{abs}}(t, z^t)$  and  $\psi: \mathcal{T}_{\text{mpcc}}^{\text{lin}}(t, u^t, v^t) \rightarrow \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t)$  as*

$$\psi(\delta t, \delta u^t, \delta v^t) = (\delta t, \delta u^t - \delta v^t) \quad \text{and} \quad \psi^{-1}(\delta t, \delta z^t) = (\delta t, \langle \delta z^t \rangle^+, \langle \delta z^t \rangle^-).$$

Here,  $\langle \delta z^t \rangle^+, \langle \delta z^t \rangle^-$  map  $\delta z^t$  into the complementarity cone via

$$\langle \delta z_i^t \rangle^+ = \begin{cases} +\delta z_i^t, & i \in \mathcal{U}_+^t \quad (\sigma_i^t > 0) \\ 0, & i \in \mathcal{V}_+^t \quad (\sigma_i^t < 0) \\ [\delta z_i^t]^+, & i \in \mathcal{D}^t \quad (\sigma_i^t = 0) \end{cases}, \quad \langle \delta z_i^t \rangle^- = \begin{cases} 0, & i \in \mathcal{U}_+^t \quad (\sigma_i^t > 0) \\ -\delta z_i^t, & i \in \mathcal{V}_+^t \quad (\sigma_i^t < 0) \\ [\delta z_i^t]^-, & i \in \mathcal{D}^t \quad (\sigma_i^t = 0) \end{cases}.$$

Then, both functions  $\psi$  are homeomorphisms.

*Proof.* First, consider  $\psi: \mathcal{T}_{\text{mpcc}}(t, u^t, v^t) \rightarrow \mathcal{T}_{\text{abs}}(t, z^t)$ : Given a tangent vector  $(\delta t, \delta u^t, \delta v^t) = \lim \tau_k^{-1}(t_k - t, u_k^t - u^t, v_k^t - v^t) \in \mathcal{T}_{\text{mpcc}}(t, u^t, v^t)$ , set  $(t_k, z_k^t) = \phi(t_k, u_k^t, v_k^t) = (t_k, u_k^t - v_k^t) \in \mathcal{F}_{\text{abs}}$  to obtain

$$\lim \frac{z_k^t - z^t}{\tau_k} = \lim \frac{(u_k^t - u^t) - (v_k^t - v^t)}{\tau_k} = \delta u^t - \delta v^t \implies (\delta t, \delta u^t - \delta v^t) \in \mathcal{T}_{\text{abs}}(t, z^t).$$

Conversely, given a vector  $(\delta t, \delta z^t) = \lim \tau_k^{-1}(t_k - t, z_k^t - z^t) \in \mathcal{T}_{\text{abs}}(t, z^t)$ , define  $(t_k, u_k^t, v_k^t) = \phi^{-1}(t_k, z_k^t) = (t_k, [z_k^t]^+, [z_k^t]^-) \in \mathcal{F}_{\text{mpcc}}$ . Then,  $\tau_k^{-1}((u_k - u) - (v_k - v)) \rightarrow \langle \delta z^t \rangle^+ - \langle \delta z^t \rangle^-$  holds. Thus, it remains to show  $\tau_k^{-1}(u_k - u, v_k - v) \rightarrow (\langle \delta z^t \rangle^+, \langle \delta z^t \rangle^-)$  which is done componentwise:

- $i \in \mathcal{U}_+^t$ :  $v_i^t = 0$  holds by feasibility and  $\langle \delta z^t \rangle^- = 0$  by definition. Thus,  $(u_k^t)_i > 0$  holds for  $k$  large enough and by complementarity  $(v_k^t)_i = 0$  holds. Then,  $\tau_k^{-1}((u_k^t)_i - u_i^t) \rightarrow \langle \delta z^t \rangle_i^+$  follows.
- $i \in \mathcal{V}_+^t$ :  $\tau_k^{-1}((v_k^t)_i - v_i^t) \rightarrow \langle \delta z^t \rangle_i^-$  follows as in the previous case.
- $i \in \mathcal{D}^t$  and  $\langle \delta z^t \rangle_i^+ > 0$ :  $\langle \delta z^t \rangle_i^- = 0$  holds by complementarity and so  $\tau_k^{-1}((u_k^t)_i - u_i^t) \rightarrow \langle \delta z^t \rangle_i^+$ . Then,  $\tau_k^{-1}(u_i^t)_k \rightarrow \langle \delta z^t \rangle_i^+$  and  $\tau_k^{-1}(v_i^t)_k \rightarrow 0$  because of sign constraints.
- $i \in \mathcal{D}^t$  and  $\langle \delta z^t \rangle_i^- > 0$ :  $\tau_k^{-1}(u_i^t)_k \rightarrow 0$  and  $\tau_k^{-1}(v_i^t)_k \rightarrow \langle \delta z^t \rangle_i^-$  follow as in the previous case.
- $i \in \mathcal{D}^t$  and  $\langle \delta z^t \rangle_i^+ = \langle \delta z^t \rangle_i^- = 0$ : Then,  $\tau_k^{-1}((u_k^t)_i - (v_k^t)_i) \rightarrow 0$  holds. Because of sign constraints and complementarity, this can only hold if  $\tau_k^{-1}(u_i^t)_k \rightarrow 0$ ,  $\tau_k^{-1}(v_i^t)_k \rightarrow 0$ .

Altogether, this implies

$$\lim \frac{(t_k - t, u_k^t - u^t, v_k^t - v^t)}{\tau_k} = (\delta t, \langle \delta z^t \rangle^+, \langle \delta z^t \rangle^-) \in \mathcal{T}_{\text{mpcc}}(t, u^t, v^t).$$

By construction,  $\psi$  and  $\psi^{-1}$  are both continuous and inverse to each other.

Second, consider  $\psi: \mathcal{T}_{\text{mpcc}}^{\text{lin}}(t, u^t, v^t) \rightarrow \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t)$ : Given  $(\delta t, \delta u^t, \delta v^t) \in \mathcal{T}_{\text{mpcc}}^{\text{lin}}(t, u^t, v^t)$ , the vectors  $\delta z^t = \delta u^t - \delta v^t$  and  $\delta \zeta = \delta u^t + \delta v^t$  satisfy

$$\delta z_i^t = \begin{cases} \delta u_i^t, & i \in \mathcal{U}_+^t \\ -\delta v_i^t, & i \in \mathcal{V}_+^t \\ \delta u_i^t - \delta v_i^t, & i \in \mathcal{D}^t \end{cases}, \quad \delta \zeta_i = \begin{cases} \delta u_i^t = \sigma_i \delta z_i^t, & i \in \mathcal{U}_+^t \\ \delta v_i^t = \sigma_i^t \delta z_i^t, & i \in \mathcal{V}_+^t \\ \delta u_i^t + \delta v_i^t = |\delta z_i^t|, & i \in \mathcal{D}^t \end{cases}.$$

Thus,  $(\delta t, \delta z^t) = \psi(\delta t, \delta u^t, \delta v^t) \in \mathcal{T}_{\text{abs}}^{\text{lin}}(t, z^t)$ .

Conversely, the same case distinction yields  $(\delta t, \delta u^t, \delta v^t) = \psi^{-1}(\delta t, \delta z^t) \in \mathcal{T}_{\text{mpcc}}^{\text{lin}}(\hat{t}, \hat{u}^t, \hat{v}^t)$  for every  $(\delta t, \delta z^t) \in \mathcal{T}_{\text{abs}}^{\text{lin}}(\hat{t}, \hat{z}^t)$ . Again,  $\psi$  and  $\psi^{-1}$  are both continuous and inverse to each other by construction.  $\square$

**Definition 26** (Branch NLPs for **(I-MPCC)**, see [11]). Consider a feasible point  $(\hat{t}, \hat{u}^t, \hat{v}^t)$  of **(I-MPCC)** with associated index sets  $\mathcal{U}_+^t$ ,  $\mathcal{V}_+^t$ , and  $\mathcal{D}^t$  and choose  $\mathcal{P}^t \subseteq \mathcal{D}^t$  with complement  $\bar{\mathcal{P}}^t = \mathcal{D}^t \setminus \mathcal{P}^t$ . The branch problem NLP( $\mathcal{P}^t$ ) is defined as

$$\begin{aligned} \min_{t, u^t, v^t} f(t) \quad \text{s.t.} \quad & c_{\mathcal{E}}(t, u^t + v^t) = 0, \\ & c_{\mathcal{I}}(t, u^t + v^t) \geq 0, \\ & c_{\mathcal{Z}}(t, u^t + v^t) - (u^t - v^t) = 0, \\ & 0 = u_i^t, \quad 0 \leq v_i^t, \quad i \in \mathcal{V}_+^t \cup \mathcal{P}^t, \\ & 0 \leq u_i^t, \quad 0 = v_i^t, \quad i \in \mathcal{U}_+^t \cup \bar{\mathcal{P}}^t. \end{aligned} \tag{NLP( $\mathcal{P}^t$ )}$$

The feasible set of **(NLP( $\mathcal{P}^t$ ))**, which always contains  $(\hat{t}, \hat{u}^t, \hat{v}^t)$ , is denoted by  $\mathcal{F}_{\mathcal{P}^t}$ .

Clearly, the homeomorphism  $\phi$  can be restricted to the branch problems **(NLP( $\Sigma^t$ ))** and **(NLP( $\mathcal{P}^t$ ))** where  $\mathcal{P}^t = \{i \in \alpha^t(\hat{t}) : \sigma_i^t = -1\}$ . Thus, the mapping  $\phi_{\mathcal{P}^t} : \mathcal{F}_{\mathcal{P}^t} \rightarrow \mathcal{F}_{\Sigma^t}$  defined by

$$\phi_{\mathcal{P}^t} := \phi|_{\mathcal{P}^t} \quad \text{and} \quad \phi_{\mathcal{P}^t}^{-1} := \phi^{-1}|_{\Sigma^t}$$

is a homeomorphism. The tangent cone to  $\mathcal{F}_{\mathcal{P}^t}$  at  $(t, u^t, v^t)$  is

$$\mathcal{T}_{\mathcal{P}^t}(t, u^t, v^t) := \left\{ (\delta t, \delta u^t, \delta v^t) \mid \begin{array}{l} \exists \tau_k \searrow 0, \mathcal{F}_{\mathcal{P}^t} \ni (t_k, u_k^t, v_k^t) \rightarrow (t, u^t, v^t): \\ \tau_k^{-1}(t_k - t, u_k^t - u^t, v_k^t - v^t) \rightarrow (\delta t, \delta u^t, \delta v^t) \end{array} \right\}.$$

The linearized cone is

$$\mathcal{T}_{\mathcal{P}^t}^{\text{lin}}(t, u^t, v^t) := \left\{ \begin{array}{l} \begin{pmatrix} \delta t \\ \delta u^t \\ \delta v^t \end{pmatrix} \mid \begin{array}{l} \partial_1 c_{\mathcal{E}} \delta t + \partial_2 c_{\mathcal{E}}(\delta u^t + \delta v^t) = 0, \\ \partial_1 c_{\mathcal{A}} \delta t + \partial_2 c_{\mathcal{A}}(\delta u^t + \delta v^t) \geq 0, \\ \partial_1 c_{\mathcal{Z}} \delta t + \partial_2 c_{\mathcal{Z}}(\delta u^t + \delta v^t) = \delta u^t - \delta v^t, \\ 0 = \delta u_i^t \text{ for } i \in \mathcal{V}_+^t \cup \mathcal{P}, \quad 0 = \delta v_i^t \text{ for } i \in \mathcal{U}_+^t \cup \bar{\mathcal{P}}, \\ 0 \leq \delta u_i^t \text{ for } i \in \bar{\mathcal{P}}, \quad 0 \leq \delta v_i^t \text{ for } i \in \mathcal{P} \end{array} \end{array} \right\}.$$

Here, all partial derivatives are evaluated at  $(t, u^t + v^t)$ .

**Lemma 27.** Given **(NLP( $\Sigma^t$ ))** and **(NLP( $\mathcal{P}^t$ ))** with  $\mathcal{P}^t = \{i \in \alpha^t(\hat{t}) : \sigma_i^t = -1\}$ . Consider  $(t, z^t) \in \mathcal{F}_{\Sigma^t}$  and  $(t, u^t, v^t) = \phi_{\mathcal{P}^t}^{-1}(t, z^t)$ . Define  $\psi_{\mathcal{P}^t} := \psi|_{\mathcal{F}_{\mathcal{P}^t}}$ ,  $\psi_{\mathcal{P}^t}^{-1} := \psi^{-1}|_{\Sigma^t}$  and  $\psi_{\mathcal{P}^t} := \psi|_{\mathcal{T}_{\mathcal{P}^t}^{\text{lin}}}$ ,  $\psi_{\mathcal{P}^t}^{-1} := \psi^{-1}|_{\mathcal{T}_{\Sigma^t}^{\text{lin}}}$ . Then,

$$\psi_{\mathcal{P}^t} : \mathcal{T}_{\mathcal{P}^t}(t, u^t, v^t) \rightarrow \mathcal{T}_{\Sigma^t}(t, \hat{z}^t) \quad \text{and} \quad \psi_{\mathcal{P}^t} : \mathcal{T}_{\mathcal{P}^t}^{\text{lin}}(t, \hat{u}^t, \hat{v}^t) \rightarrow \mathcal{T}_{\Sigma^t}^{\text{lin}}(t, \hat{z}^t)$$

are homeomorphisms.

*Proof.* By construction and since  $\alpha^t(\hat{t}) = \mathcal{D}^t$ , the following equalities of sets hold:

$$\begin{aligned} \mathcal{P}^t &= \{i \in \alpha^t(\hat{t}) : \sigma_i^t = -1\}, & \mathcal{V}_+^t &= \{i \notin \alpha^t(\hat{t}) : \sigma_i^t = -1\}, \\ \bar{\mathcal{P}}^t &= \{i \in \alpha^t(\hat{t}) : \sigma_i^t = +1\}, & \mathcal{U}_+^t &= \{i \notin \alpha^t(\hat{t}) : \sigma_i^t = +1\}. \end{aligned}$$

Thus, the claim follows directly from Lemma 25.  $\square$

Consider a feasible point  $(t, u^t, v^t)$  of **(I-MPCC)** with associated branch problems **(NLP( $\mathcal{P}^t$ ))**. Then, the following decompositions of the tangent cone and of the abs-normal-linearized cone of **(I-MPCC)** hold (for a proof see [1]):

$$\mathcal{T}_{\text{mpcc}}(t, u^t, v^t) = \bigcup_{\mathcal{P}^t} \mathcal{T}_{\mathcal{P}^t}(t, u^t, v^t) \quad \text{and} \quad \mathcal{T}_{\text{mpcc}}^{\text{lin}}(t, u^t, v^t) = \bigcup_{\mathcal{P}^t} \mathcal{T}_{\mathcal{P}^t}^{\text{lin}}(t, u^t, v^t). \tag{1}$$

The following inclusions are also proved in [1]:

$$\mathcal{T}_{\text{mpcc}}(t, u^t, v^t) \subseteq \mathcal{T}_{\text{mpcc}}^{\text{lin}}(t, u^t, v^t) \quad \text{and} \quad \mathcal{T}_{\text{mpcc}}(t, u^t, v^t)^* \supseteq \mathcal{T}_{\text{mpcc}}^{\text{lin}}(t, u^t, v^t)^*.$$

In general, the converses do not hold. This motivates the definition of MPCC-ACQ and MPCC-GCQ.

**Definition 28** (Abadie's and Guignard's Constraint Qualifications for **(I-MPCC)**, see [1]). Consider a feasible point  $(t, u^t, v^t)$  of **(I-MPCC)**. We say that *Abadie's Constraint Qualification for MPCC (MPCC-ACQ)* holds at  $(t, u^t, v^t)$  if  $\mathcal{T}_{\text{mpcc}}(t, u^t, v^t) = \mathcal{T}_{\text{mpcc}}^{\text{lin}}(t, u^t, v^t)$ , and that *Guignard's Constraint Qualification for MPCC (MPCC-GCQ)* holds at  $(t, u^t, v^t)$  if  $\mathcal{T}_{\text{mpcc}}(t, u^t, v^t)^* = \mathcal{T}_{\text{mpcc}}^{\text{lin}}(t, u^t, v^t)^*$ .

The decomposition (1) and its dualization imply that both MPCC-CQs hold if the corresponding CQ holds for all branch problems.

**Theorem 29** (ACQ/GCQ for all **(NLP( $\mathcal{P}^t$ ))** implies MPCC-ACQ/MPCC-GCQ for **(I-MPCC)**). Consider a feasible point  $(t, u^t, v^t)$  of **(I-MPCC)**. Then, MPCC-ACQ respectively MPCC-GCQ holds at  $(t, u^t, v^t)$  if ACQ respectively GCQ holds for all **(NLP( $\mathcal{P}^t$ ))** at  $(t, u^t, v^t)$ .

**3.2. Counterpart MPCC for the Abs-Normal NLP with Inequality Slacks.** By Definition 21, the *counterpart MPCC* of the non-smooth NLP **(E-NLP)** reads:

$$\begin{aligned} \min_{t, w, u^t, v^t, u^w, v^w} f(t) \quad \text{s.t.} \quad & c_{\mathcal{E}}(t, u^t + v^t) = 0, \\ & c_{\mathcal{I}}(t, u^t + v^t) - (u^w + v^w) = 0, \\ & c_{\mathcal{Z}}(t, u^t + v^t) - (u^t - v^t) = 0, \\ & w - (u^w - v^w) = 0, \\ & 0 \leq u^t \perp v^t \geq 0, \quad 0 \leq u^w \perp v^w \geq 0, \end{aligned} \quad (\text{E-MPCC})$$

where  $u^t, v^t \in \mathbb{R}^{s^t}$  and  $u^w, v^w \in \mathbb{R}^{m^2}$ . The feasible set is denoted by  $\mathcal{F}_{\text{e-mpcc}}$  and is a lifting of  $\mathcal{F}_{\text{mpcc}}$ .

Clearly, the homeomorphism between  $\mathcal{F}_{\text{mpcc}}$  and  $\mathcal{F}_{\text{abs}}$  extends to  $\mathcal{F}_{\text{e-mpcc}}$  and  $\mathcal{F}_{\text{e-abs}}$ . It is given by

$$\begin{aligned} \bar{\phi}(t, w, u^t, v^t, u^w, v^w) &= (t, w, u^t - v^t, u^w - v^w), \\ \bar{\phi}^{-1}(t, w, z^t, z^w) &= (t, w, [z^t]^+, [z^t]^-, [z^w]^+, [z^w]^-). \end{aligned}$$

Just like in the abs-normal case, problem **(E-MPCC)** is a special case of **(I-MPCC)**. Hence, we obtain the following material by specializing the definitions and results for **(I-MPCC)**.

By Definition 23, the tangent cone to  $\mathcal{F}_{\text{e-mpcc}}$  at  $y$  reads

$$\mathcal{T}_{\text{e-mpcc}}(y) = \left\{ \delta \left| \begin{array}{l} \exists \tau_k \searrow 0, \mathcal{F}_{\text{e-mpcc}} \ni y_k = (t_k, w_k, u_k^t, v_k^t, u_k^w, v_k^w) \rightarrow y: \\ \tau_k^{-1}(y_k - y) \rightarrow \delta = (\delta t, \delta w, \delta u^t, \delta v^t, \delta u^w, \delta v^w) \end{array} \right. \right\}.$$

The MPCC-linearized cone reads

$$\mathcal{T}_{\text{e-mpcc}}^{\text{lin}}(\hat{y}) = \left\{ \delta \left| \begin{array}{l} \partial_1 c_{\mathcal{I}} \delta t + \partial_2 c_{\mathcal{I}} (\delta u^t + \delta v^t) = \delta u^w + \delta v^w, \quad \delta w = \delta u^w - \delta v^w, \\ (\delta t, \delta u^t, \delta v^t) \in \mathcal{T}_{\text{mpcc}}^{\text{lin}}(t, u^t, v^t), \quad (\delta u^w, \delta v^w) \in \mathcal{T}_{\perp}(\hat{u}^w, \hat{v}^w) \end{array} \right. \right\}.$$

Here, all partial derivatives are evaluated at  $(t, u^t + v^t)$ . The associated homeomorphisms of Lemma 25,

$$\begin{aligned} \bar{\psi}: \mathcal{T}_{\text{e-mpcc}}(t, w, u^t, v^t, u^w, v^w) &\rightarrow \mathcal{T}_{\text{e-abs}}(t, w, z^t, z^w), \\ \bar{\psi}: \mathcal{T}_{\text{e-mpcc}}^{\text{lin}}(t, w, u^t, v^t, u^w, v^w) &\rightarrow \mathcal{T}_{\text{e-abs}}^{\text{lin}}(t, w, z^t, z^w), \end{aligned}$$

now take the form

$$\begin{aligned} \bar{\psi}(\delta t, \delta w, \delta u^t, \delta v^t, \delta u^w, \delta v^w) &= (\delta t, \delta w, \delta u^t - \delta v^t, \delta u^w - \delta v^w), \\ \bar{\psi}^{-1}(\delta t, \delta w, \delta z^t, \delta z^w) &= (\delta t, \delta w, \langle \delta z^t \rangle^+, \langle \delta z^t \rangle^-, \langle \delta z^w \rangle^+, \langle \delta z^w \rangle^-). \end{aligned}$$

Given  $\hat{y} = (\hat{t}, \hat{w}, \hat{u}^t, \hat{v}^t, \hat{u}^w, \hat{v}^w)$ , a feasible point of **(E-MPCC)** with associated index sets  $\mathcal{U}_+^t, \mathcal{V}_+^t, \mathcal{D}^t, \mathcal{U}_+^w, \mathcal{V}_+^w$ , and  $\mathcal{D}^w$ , choose  $\mathcal{P}^t \subseteq \mathcal{D}^t$  as well as  $\mathcal{P}^w \subseteq \mathcal{D}^w$  and set  $\mathcal{P}^{t,w} = \mathcal{P}^t \cup \mathcal{P}^w$ . The branch

problem  $\text{NLP}(\mathcal{P}^{t,w})$  of Definition 26 then reads

$$\begin{aligned} \min_{t,w,u^t,v^t,u^w,v^w} f(t) \quad \text{s.t.} \quad & c_{\mathcal{I}}(t, u^t + v^t) = 0, \quad c_{\mathcal{I}}(t, u^t + v^t) - (u^w + v^w) = 0, \\ & c_{\mathcal{Z}}(t, u^t + v^t) - (u^t - v^t) = 0, \quad w - (u^w - v^w) = 0, \\ & 0 = u_i^t, \quad 0 \leq v_i^t, \quad i \in \mathcal{V}_+^t \cup \mathcal{P}^t, \\ & 0 \leq u_i^t, \quad 0 = v_i^t, \quad i \in \mathcal{U}_+^t \cup \bar{\mathcal{P}}^t, \\ & 0 = u_i^w, \quad 0 \leq v_i^w, \quad i \in \mathcal{V}_+^w \cup \mathcal{P}^w, \\ & 0 \leq u_i^w, \quad 0 = v_i^w, \quad i \in \mathcal{U}_+^w \cup \bar{\mathcal{P}}^w. \end{aligned} \quad (\text{NLP}(\mathcal{P}^{t,w}))$$

The feasible set of  $(\text{NLP}(\mathcal{P}^{t,w}))$ , which always contains  $\hat{y}$ , is denoted by  $\mathcal{F}_{\mathcal{P}^{t,w}}$  and is a lifting of  $\mathcal{F}_{\mathcal{P}^t}$ .

Again, the homeomorphism between feasible sets can be restricted to the branch problems  $(\text{NLP}(\Sigma^{t,w}))$  and  $(\text{NLP}(\mathcal{P}^{t,w}))$  where  $\mathcal{P}^t = \{i \in \alpha^t(\hat{t}): \sigma_i^t = -1\}$  and  $\mathcal{P}^w = \{i \in \alpha^w(\hat{w}): \sigma_i^w = -1\}$ . Thus, the mapping  $\bar{\phi}_{\mathcal{P}^{t,w}}: \mathcal{F}_{\mathcal{P}^{t,w}} \rightarrow \mathcal{F}_{\Sigma^{t,w}}$  given as

$$\bar{\phi}_{\mathcal{P}^{t,w}} := \bar{\phi}|_{\mathcal{P}^{t,w}} \quad \text{and} \quad \bar{\phi}_{\mathcal{P}^{t,w}}^{-1} := \bar{\phi}^{-1}|_{\Sigma^{t,w}}$$

is a homeomorphism.

The tangent cone to  $\mathcal{F}_{\mathcal{P}^{t,w}}$  at  $y$  reads

$$\mathcal{T}_{\mathcal{P}^{t,w}}(y) = \left\{ \delta \mid \begin{array}{l} \exists \tau_k \searrow 0, \mathcal{F}_{\mathcal{P}^{t,w}} \ni (t_k, w_k, u_k^t, v_k^t, u_k^w, v_k^w) \rightarrow (t, w, u^t, v^t, u^w, v^w): \\ \tau_k^{-1}(t_k - t, w_k - w, u_k^t - u^t, v_k^t - v^t, u_k^w - u^w, v_k^w - v^w) \rightarrow \delta \end{array} \right\}$$

where  $\delta = (\delta t, \delta w, \delta u^t, \delta v^t, \delta u^w, \delta v^w)$ . The linearized cone reads

$$\mathcal{T}_{\mathcal{P}^{t,w}}^{\text{lin}}(y) = \left\{ \delta \mid \begin{array}{l} (\delta t, \delta u^t, \delta v^t) \in \mathcal{T}_{\mathcal{P}^t}^{\text{lin}}, \\ \partial_1 c_{\mathcal{I}} \delta t + \partial_2 c_{\mathcal{I}} (\delta u^t + \delta v^t) = \delta u^w + \delta v^w, \quad \delta w = \delta u^w - \delta v^w, \\ 0 = \delta u_i^w \text{ for } i \in \mathcal{V}_+^w \cup \mathcal{P}^w, \quad 0 = \delta v_i^w \text{ for } i \in \mathcal{U}_+^w \cup \bar{\mathcal{P}}^w, \\ 0 \leq \delta u_i^w \text{ for } i \in \bar{\mathcal{P}}^w, \quad 0 \leq \delta v_i^w \text{ for } i \in \mathcal{P}^w \end{array} \right\}.$$

Here, all partial derivatives are evaluated at  $(t, u^t + v^t)$ . The associated cone homeomorphisms of Lemma 27 are now obtained as follows. Given  $(\text{NLP}(\Sigma^{t,w}))$  and  $(\text{NLP}(\mathcal{P}^{t,w}))$  with  $\mathcal{P}^t = \{i \in \alpha^t(\hat{t}): \sigma_i^t = -1\}$  and  $\mathcal{P}^w = \{i \in \alpha^w(\hat{w}): \sigma_i^w = -1\}$ , consider  $(t, w, z^t, z^w) \in \mathcal{F}_{\text{e-abs}}$  and  $(t, w, u^t, v^t, u^w, v^w) = \bar{\phi}^{-1}(t, w, z^t, z^w)$ . Define  $\bar{\psi}_{\mathcal{P}^{t,w}} := \bar{\psi}|_{\mathcal{T}_{\mathcal{P}^{t,w}}}$ ,  $\bar{\psi}_{\mathcal{P}^{t,w}}^{-1} := \bar{\psi}^{-1}|_{\mathcal{T}_{\Sigma^{t,w}}}$  and  $\bar{\psi}_{\mathcal{P}^{t,w}} := \bar{\psi}|_{\mathcal{T}_{\mathcal{P}^{t,w}}^{\text{lin}}}$ ,  $\bar{\psi}_{\mathcal{P}^{t,w}}^{-1} := \bar{\psi}^{-1}|_{\mathcal{T}_{\Sigma^{t,w}}^{\text{lin}}}$ . Then, we have

$$\begin{aligned} \bar{\psi}_{\mathcal{P}^{t,w}}: \mathcal{T}_{\mathcal{P}^{t,w}}(t, w, u^t, v^t, u^w, v^w) &\rightarrow \mathcal{T}_{\Sigma^{t,w}}(t, w, z^t, z^w), \\ \bar{\psi}_{\mathcal{P}^{t,w}}: \mathcal{T}_{\mathcal{P}^{t,w}}^{\text{lin}}(t, w, u^t, v^t, u^w, v^w) &\rightarrow \mathcal{T}_{\Sigma^{t,w}}^{\text{lin}}(t, w, z^t, z^w). \end{aligned}$$

By applying (1) to  $(\text{E-MPCC})$  with associated branch problems  $(\text{NLP}(\mathcal{P}^{t,w}))$ , we obtain the following decomposition of cones at  $y = (t, w, u^t, v^t, u^w, v^w)$ :

$$\mathcal{T}_{\text{e-mpcc}}(y) = \bigcup_{\mathcal{P}^{t,w}} \mathcal{T}_{\mathcal{P}^{t,w}}(y) \quad \text{and} \quad \mathcal{T}_{\text{e-mpcc}}^{\text{lin}}(y) = \bigcup_{\mathcal{P}^{t,w}} \mathcal{T}_{\mathcal{P}^{t,w}}^{\text{lin}}(y).$$

Moreover, the tangent cone is contained in the linearized cone and the converse holds for the dual cones:

$$\mathcal{T}_{\text{e-mpcc}}(y) \subseteq \mathcal{T}_{\text{e-mpcc}}^{\text{lin}}(y) \quad \text{and} \quad \mathcal{T}_{\text{e-mpcc}}(y)^* \supseteq \mathcal{T}_{\text{e-mpcc}}^{\text{lin}}(y)^*.$$

Once again, the converses do not hold in general and we consider Abadie's and Guignard's Constraint Qualifications for  $(\text{E-MPCC})$  at  $y = (t, w, u^t, v^t, u^w, v^w)$ . Recalling Definition 28, MPCC-ACQ and MPCC-GCQ simply read

$$\mathcal{T}_{\text{e-mpcc}}(y) = \mathcal{T}_{\text{e-mpcc}}^{\text{lin}}(y) \quad \text{and} \quad \mathcal{T}_{\text{e-mpcc}}(y)^* = \mathcal{T}_{\text{e-mpcc}}^{\text{lin}}(y)^*.$$

*Remark 30.* Let

$$W(t, u^t, v^t) = \{(w, u^w, v^w): |w| = c_{\mathcal{I}}(t, u^t + v^t), u^w = [w]^+, v^w = [w]^-\}.$$

Due to symmetry, the above equality of cones (respectively dual cones) holds for all elements  $(w, u^w, v^w) \in W(t, u^t, v^t)$  if it holds for any element.

Now Theorem 29 reads as follows.

**Theorem 31** (ACQ/GCQ for all  $(\text{NLP}(\mathcal{P}^{t,w}))$  implies MPCC-ACQ/MPCC-GCQ for  $(\text{E-MPCC})$ ). Consider a feasible point  $y = (t, w, u^t, v^t, u^w, v^w)$  of  $(\text{E-MPCC})$  with associated branch problems  $(\text{NLP}(\mathcal{P}^{t,w}))$ . Then, MPCC-ACQ respectively MPCC-GCQ holds for  $(\text{E-MPCC})$  at  $y$  if ACQ respectively GCQ holds for all  $(\text{NLP}(\mathcal{P}^{t,w}))$  at  $y$ .

**3.3. Relations of MPCC-CQs for Different Formulations.** In this paragraph we prove relations between constraint qualifications for the two different formulations  $(\text{I-MPCC})$  and  $(\text{E-MPCC})$ . Some relations follow from the results in the previous section and in the two following sections.

**Theorem 32.** MPCC-ACQ for  $(\text{I-MPCC})$  holds at  $(t, u^t, v^t) \in \mathcal{F}_{\text{mpcc}}$  if and only if MPCC-ACQ for  $(\text{E-MPCC})$  holds at  $(t, w, u^t, u^w, v^t, v^w) \in \mathcal{F}_{\text{e-mpcc}}$  for any (and hence all)  $(w, u^w, v^w) \in W(t, u^t, v^t)$ .

*Proof.* This follows immediately from Theorem 17, Theorem 36 and Theorem 40.  $\square$

**Theorem 33.** MPCC-GCQ for  $(\text{I-MPCC})$  holds at  $(t, u^t, v^t) \in \mathcal{F}_{\text{mpcc}}$  if MPCC-GCQ for  $(\text{E-MPCC})$  holds at  $(t, w, u^t, v^t, u^w, v^w) \in \mathcal{F}_{\text{e-mpcc}}$  for any (and hence all)  $(w, u^w, v^w) \in W(t, u^t, v^t)$ .

*Proof.* The inclusion  $\mathcal{T}_{\text{mpcc}}(t, u^t, v^t)^* \supseteq \mathcal{T}_{\text{mpcc}}^{\text{lin}}(t, u^t, v^t)^*$  holds always. Thus, it is left to show that

$$\mathcal{T}_{\text{mpcc}}(t, u^t, v^t)^* \subseteq \mathcal{T}_{\text{mpcc}}^{\text{lin}}(t, u^t, v^t)^*.$$

Let  $\omega = (\omega t, \omega u^t, \omega v^t) \in \mathcal{T}_{\text{mpcc}}(t, u^t, v^t)^*$ , i.e.  $\omega^T \delta \geq 0$  for all  $\delta = (\delta t, \delta u^t, \delta v^t) \in \mathcal{T}_{\text{mpcc}}(t, u^t, v^t)$ . Then, let  $\tilde{\omega} = (\omega t, 0, \omega u^t, \omega v^t, 0, 0)$  to obtain  $\tilde{\omega}^T \tilde{\delta} = \omega^T \delta \geq 0$  for all  $\tilde{\delta} \in \mathcal{T}_{\text{e-mpcc}}(t, w, u^t, v^t, u^w, v^w)$  where  $w \in W(t)$  is arbitrary. By assumption, we have  $\tilde{\omega}^T \tilde{\delta} \geq 0$  for all  $\tilde{\delta} \in \mathcal{T}_{\text{e-mpcc}}^{\text{lin}}(t, w, u^t, v^t, u^w, v^w)$  which implies  $\omega^T \delta = \tilde{\omega}^T \tilde{\delta} \geq 0$  for all  $\delta \in \mathcal{T}_{\text{mpcc}}^{\text{lin}}(t, u^t, v^t)$ .  $\square$

The converse of the previous theorem is unlikely to hold, but we do not know how to construct a counterexample. However, equivalence of ACQ or GCQ for corresponding branch problems holds.

**Theorem 34.** ACQ for  $(\text{NLP}(\mathcal{P}^t))$  holds at  $(t, u^t, v^t) \in \mathcal{F}_{\mathcal{P}^t}$  if and only if ACQ for  $(\text{NLP}(\mathcal{P}^{t,w}))$  holds at  $(t, w, u^t, v^t, u^w, v^w) \in \mathcal{F}_{\mathcal{P}^{t,w}}$  for any (and hence all)  $(w, u^w, v^w) \in W(t, u^t, v^t)$ .

*Proof.* This follows immediately from Theorem 19, Theorem 38 and Theorem 42.  $\square$

**Theorem 35.** GCQ for  $(\text{NLP}(\mathcal{P}^t))$  holds at  $(t, u^t, v^t) \in \mathcal{F}_{\mathcal{P}^t}$  if and only if GCQ for  $(\text{NLP}(\mathcal{P}^{t,w}))$  holds at  $(t, w, u^t, v^t, u^w, v^w) \in \mathcal{F}_{\mathcal{P}^{t,w}}$  for any (and hence all)  $(w, u^w, v^w) \in W(t, u^t, v^t)$ .

*Proof.* This follows immediately from Theorem 20, Theorem 39 and Theorem 43.  $\square$

#### 4. KINK QUALIFICATIONS AND MPCC CONSTRAINT QUALIFICATIONS

In this section we show relations between abs-normal NLPs and counterpart MPCCs. Here, we consider both treatments of inequality constraints.

**4.1. Relations of General Abs-Normal NLP and MPCC.** In the following the variables  $x$  and  $z$  instead of  $t$  and  $z^t$  are used. Thus, the abs-normal NLP  $(\text{I-NLP})$  reads:

$$\min_{x,z} f(x) \quad \text{s.t.} \quad c_{\mathcal{E}}(x, |z|) = 0, \quad c_{\mathcal{I}}(x, |z|) \geq 0, \quad c_{\mathcal{Z}}(x, |z|) - z = 0.$$

The counterpart MPCC  $(\text{I-MPCC})$  becomes:

$$\min_{x,u,v} f(x) \quad \text{s.t.} \quad c_{\mathcal{E}}(x, u+v) = 0, \quad c_{\mathcal{I}}(x, u+v) \geq 0, \\ c_{\mathcal{Z}}(x, u+v) - (u-v) = 0, \quad 0 \leq u \perp v \geq 0.$$

Then, the subsequent relations of kink qualifications and MPCC constraint qualifications can be shown.

**Theorem 36** (AKQ for  $(\text{I-NLP}) \iff \text{MPCC-ACQ for } (\text{I-MPCC})$ ). AKQ for  $(\text{I-NLP})$  holds at  $(x, z(x)) \in \mathcal{F}_{\text{abs}}$  if and only if MPCC-ACQ for  $(\text{I-MPCC})$  holds at  $(x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in \mathcal{F}_{\text{mpcc}}$ .

*Proof.* We need to show

$$\mathcal{T}_{\text{abs}}(x, z) = \mathcal{T}_{\text{abs}}^{\text{lin}}(x, z) \iff \mathcal{T}_{\text{mpcc}}(x, u, v) = \mathcal{T}_{\text{mpcc}}^{\text{lin}}(x, u, v).$$

This is obvious from the homeomorphisms  $\psi$  in Lemma 25.  $\square$

**Theorem 37** (MPCC-GCQ for **(I-MPCC)** implies GKQ for **(I-NLP)**). *GKQ for **(I-NLP)** holds at  $(x, z(x)) \in \mathcal{F}_{\text{abs}}$  if MPCC-GCQ for **(I-MPCC)** holds at  $(x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in \mathcal{F}_{\text{mpcc}}$ .*

*Proof.* The inclusion  $\mathcal{T}_{\text{abs}}^{\text{lin}}(x, z)^* \subseteq \mathcal{T}_{\text{abs}}(x, z)^*$  hold always by Lemma 10. Thus, we just have to show

$$\mathcal{T}_{\text{abs}}(x, z)^* \subseteq \mathcal{T}_{\text{abs}}^{\text{lin}}(x, z)^*.$$

Consider  $\omega = (\omega x, \omega z) \in \mathcal{T}_{\text{abs}}(x, z)^*$ , i.e.  $\omega^T \delta \geq 0$  for all  $\delta = (\delta x, \delta z) \in \mathcal{T}_{\text{abs}}(x, z)$ . Set  $\tilde{\omega} = (\omega x, \omega z, -\omega z)$ . For every  $\delta \in \mathcal{T}_{\text{abs}}(x, z)$  we then have

$$\tilde{\omega}^T \psi^{-1}(\delta) = \omega x^T \delta x + \omega z^T \langle \delta z \rangle^+ - \omega z^T \langle \delta z \rangle^- = \omega x^T \delta x + \omega z^T \delta z = \omega^T \delta \geq 0.$$

This means  $\tilde{\omega} \in \mathcal{T}_{\text{mpcc}}(x, u, v)^*$  and hence, by assumption,  $\tilde{\omega} \in \mathcal{T}_{\text{mpcc}}^{\text{lin}}(x, u, v)^*$ . We thus have  $\omega^T \delta = \tilde{\omega}^T \psi^{-1}(\delta) \geq 0$  for every  $\delta \in \mathcal{T}_{\text{abs}}^{\text{lin}}(x, z)$ , which means  $\omega \in \mathcal{T}_{\text{abs}}^{\text{lin}}(x, z)^*$ .  $\square$

The converse is unlikely to hold, although we are not, at this time, aware of a counterexample. Once again, moving to the branch problems allows to exploit additional sign information.

**Theorem 38** (ACQ for **(NLP( $\Sigma^t$ ))**  $\iff$  ACQ for **(NLP( $\mathcal{P}^t$ ))**). *ACQ for **(NLP( $\Sigma^t$ ))** holds at  $(x, z(x)) \in \mathcal{F}_{\Sigma^t}$  if and only if ACQ for the corresponding **(NLP( $\mathcal{P}^t$ ))** holds at  $(x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in \mathcal{F}_{\mathcal{P}^t}$ .*

*Proof.* We need to show

$$\mathcal{T}_{\Sigma^t}(x, z) = \mathcal{T}_{\Sigma^t}^{\text{lin}}(x, z) \iff \mathcal{T}_{\mathcal{P}^t}(x, u, v) = \mathcal{T}_{\mathcal{P}^t}^{\text{lin}}(x, u, v).$$

This is obvious from the homeomorphisms  $\psi_{\mathcal{P}}$  in Lemma 27.  $\square$

**Theorem 39** (GCQ for **(NLP( $\Sigma^t$ ))**  $\iff$  GCQ for **(NLP( $\mathcal{P}^t$ ))**). *GCQ for **(NLP( $\Sigma^t$ ))** holds at  $(x, z(x)) \in \mathcal{F}_{\Sigma^t}$  if and only if GCQ for the corresponding **(NLP( $\mathcal{P}^t$ ))** holds at  $(x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in \mathcal{F}_{\mathcal{P}^t}$ .*

*Proof.* The inclusions  $\mathcal{T}_{\mathcal{P}^t}^{\text{lin}}(x, u, v)^* \subseteq \mathcal{T}_{\mathcal{P}^t}(x, u, v)^*$  and  $\mathcal{T}_{\Sigma^t}^{\text{lin}}(x, z)^* \subseteq \mathcal{T}_{\Sigma^t}(x, z)^*$  hold always. Thus, we just have to show

$$\mathcal{T}_{\Sigma^t}(x, z)^* \supseteq \mathcal{T}_{\Sigma^t}^{\text{lin}}(x, z)^* \iff \mathcal{T}_{\mathcal{P}^t}(x, u, v)^* \supseteq \mathcal{T}_{\mathcal{P}^t}^{\text{lin}}(x, u, v)^*.$$

First, we show the implication “ $\supseteq$ ”. Consider  $\omega = (\omega x, \omega u, \omega v) \in \mathcal{T}_{\mathcal{P}^t}(x, u, v)^*$ , i.e.  $\omega^T \delta \geq 0$  for all  $\delta = (\delta x, \delta u, \delta v) \in \mathcal{T}_{\mathcal{P}^t}(x, u, v)$ . Set  $\tilde{\omega} = (\omega x, \omega z)$  with

$$\omega z_i = \begin{cases} +\omega u_i, & i \in \mathcal{U}_+ \cup \mathcal{P}, \\ -\omega v_i, & i \in \mathcal{V}_+ \cup \bar{\mathcal{P}}. \end{cases}$$

This leads to

$$\tilde{\omega}^T \psi_{\mathcal{P}}(\delta) = \omega x^T \delta x + \omega z^T (\delta u - \delta v) = \omega x^T \delta x + \omega u^T \delta u + \omega v^T \delta v = \omega^T \delta \geq 0$$

for every  $\delta \in \mathcal{T}_{\mathcal{P}^t}(x, u, v)$ , i.e.  $\tilde{\omega} \in \mathcal{T}_{\Sigma^t}(x, z)^*$ . Then, the assumption yields  $\tilde{\omega} \in \mathcal{T}_{\Sigma^t}^{\text{lin}}(x, z)^*$ . As we have  $\omega^T \delta = \tilde{\omega}^T \psi_{\mathcal{P}}(\delta) \geq 0$  for every  $\delta \in \mathcal{T}_{\mathcal{P}^t}^{\text{lin}}(x, u, v)$ , we obtain  $\omega \in \mathcal{T}_{\mathcal{P}^t}^{\text{lin}}(x, u, v)^*$ . The reverse implication follows as in Theorem 37.  $\square$

**4.2. Relations of Abs-Normal NLP and MPCC with Inequality Slacks.** Now, the relations for the slack reformulations are stated. These are special cases of the general problem formulations, hence we obtain the following four theorems that correspond to Theorems 36–39.

**Theorem 40** (AKQ for (E-NLP)  $\iff$  MPCC-ACQ for (E-MPCC)). *AKQ for (E-NLP) holds at  $(x, z(x)) \in \mathcal{F}_{\text{e-abs}}$  if and only if MPCC-ACQ for (E-MPCC) holds at  $(x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in \mathcal{F}_{\text{e-mpcc}}$ .*

**Theorem 41** (MPCC-GCQ for (E-MPCC) implies GKQ for (E-NLP)). *GKQ for (E-NLP) holds at  $(x, z(x)) \in \mathcal{F}_{\text{e-abs}}$  if MPCC-GCQ for (E-MPCC) holds at  $(x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in \mathcal{F}_{\text{e-mpcc}}$ .*

The converse is unlikely to hold, but to date we are not aware of a counterexample.

**Theorem 42** (ACQ for (NLP( $\Sigma^{t,w}$ ))  $\iff$  ACQ for (NLP( $\mathcal{P}^{t,w}$ ))). *ACQ for (NLP( $\Sigma^{t,w}$ )) at  $(x, z(x)) \in \mathcal{F}_{\Sigma^{t,w}}$  is equivalent to ACQ for the corresponding (NLP( $\mathcal{P}^{t,w}$ )) at  $(x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in \mathcal{F}_{\mathcal{P}^{t,w}}$ .*

**Theorem 43** (GCQ for (NLP( $\Sigma^{t,w}$ ))  $\iff$  GCQ for (NLP( $\mathcal{P}^{t,w}$ ))). *GCQ for (NLP( $\Sigma^{t,w}$ )) at  $(x, z(x)) \in \mathcal{F}_{\Sigma^{t,w}}$  is equivalent to GCQ for (NLP( $\mathcal{P}^{t,w}$ )) at  $(x, u, v) = (x, [z(x)]^+, [z(x)]^-) \in \mathcal{F}_{\mathcal{P}^{t,w}}$ .*

All the relations discussed in Sections 2–4 are illustrated in Figure 1. In the inner square (containing (I-NLP) and (E-NLP) as well as the counterpart MPCCs (I-MPCC) and (E-MPCC)) there are four single-headed arrows, which indicate that only one direction has been proved and we do not know if the converses hold as well. Therefore we considered the branch problems given on the outer right and left in the figure. Since ACQ respectively GCQ for all branch problems imply the corresponding kink qualification or MPCC-constraint qualification, there are further single-headed arrows pointing to the inner square. Results that follow directly from other equivalences have arrows with the label (implied).

## 5. FIRST ORDER STATIONARITY CONCEPTS

In this section, we introduce definitions of Mordukhovich stationarity and Bouligand stationarity for abs-normal NLPs and compare these definitions to M-stationarity and B-stationarity for MPCCs. We give proofs based on the general formulation.

**5.1. Mordukhovich Stationarity.** In this paragraph we have a closer look at M-stationarity [13], which is a necessary optimality condition for MPCCs under MPCC-ACQ [2].

**Definition 44** (M-Stationarity for (I-MPCC), see [13]). Consider a feasible point  $(x^*, u^*, v^*)$  of (I-MPCC) with associated index sets  $\mathcal{U}_+$ ,  $\mathcal{V}_+$  and  $\mathcal{D}$ . It is an *M-stationary* point if there exist multipliers  $\lambda = (\lambda_{\mathcal{E}}, \lambda_{\mathcal{I}}, \lambda_{\mathcal{Z}})$  and  $\mu = (\mu_u, \mu_v)$  such that the following conditions are satisfied:

$$\partial_{x,u,v} \mathcal{L}_{\perp}(x^*, u^*, v^*, \lambda, \mu) = 0, \quad (2a)$$

$$((\mu_u)_i > 0, (\mu_v)_i > 0) \vee (\mu_u)_i (\mu_v)_i = 0, \quad i \in \mathcal{D} \quad (2b)$$

$$(\mu_u)_i = 0, \quad i \in \mathcal{U}_+, \quad (2c)$$

$$(\mu_v)_i = 0, \quad i \in \mathcal{V}_+, \quad (2d)$$

$$\lambda_{\mathcal{I}} \geq 0, \quad (2e)$$

$$\lambda_{\mathcal{Z}}^T c_{\mathcal{Z}}(x^*, u^*, v^*) = 0. \quad (2f)$$

Herein,  $\mathcal{L}_{\perp}$  is the MPCC-Lagrangian function

$$\begin{aligned} \mathcal{L}_{\perp}(x, u, v, \lambda, \mu) := & f(x) + \lambda_{\mathcal{E}}^T c_{\mathcal{E}}(x, u + v) - \lambda_{\mathcal{I}}^T c_{\mathcal{I}}(x, u + v) \\ & + \lambda_{\mathcal{Z}}^T [c_{\mathcal{Z}}(x, u + v) - (u - v)] - \mu_u^T u^t - \mu_v^T v^t. \end{aligned}$$

Local minimizers of (I-MPCC) are M-stationary points under MPCC-ACQ, as shown in [1, 2].

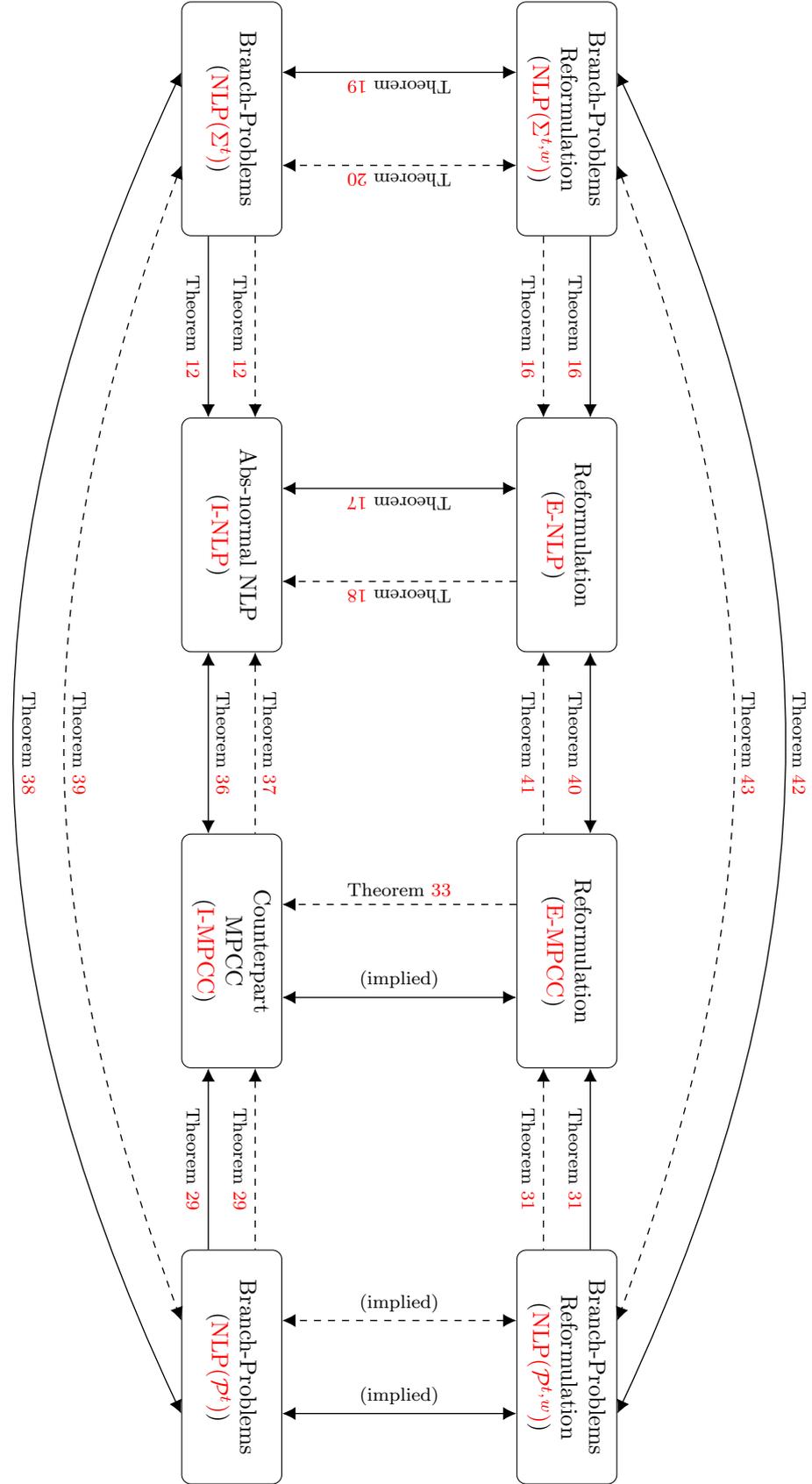


FIGURE 1. Solid arrows: relations between AKQ and MPCC-ACQ; dashed arrows: relations between GKQ and MPCC-GCQ. Note that in Theorem 18, Theorem 33, Theorem 37 and Theorem 41 we have only proved one-sided implications and it is open whether the reverse implications hold.

**Definition 45** (M-Stationarity for (I-NLP)). Consider a feasible point  $(x^*, z^*)$  of (I-NLP). It is an *M-stationary* point if there exist multipliers  $\lambda = (\lambda_{\mathcal{E}}, \lambda_{\mathcal{I}}, \lambda_{\mathcal{Z}})$  such that the following conditions are satisfied:

$$f'(x^*) + \lambda_{\mathcal{E}}^T \partial_1 c_{\mathcal{E}} - \lambda_{\mathcal{I}}^T \partial_1 c_{\mathcal{I}} + \lambda_{\mathcal{Z}}^T \partial_1 c_{\mathcal{Z}} = 0, \quad (3a)$$

$$[\lambda_{\mathcal{E}}^T \partial_2 c_{\mathcal{E}} - \lambda_{\mathcal{I}}^T \partial_2 c_{\mathcal{I}} + \lambda_{\mathcal{Z}}^T \partial_2 c_{\mathcal{Z}}]_i = (\lambda_{\mathcal{Z}})_i \sigma_i^*, \quad i \notin \alpha(x^*), \quad (3b)$$

$$(\mu_i^-)(\mu_i^+) = 0 \vee [\lambda_{\mathcal{E}}^T \partial_2 c_{\mathcal{E}} - \lambda_{\mathcal{I}}^T \partial_2 c_{\mathcal{I}} + \lambda_{\mathcal{Z}}^T \partial_2 c_{\mathcal{Z}}]_i > |(\lambda_{\mathcal{Z}})_i|, \quad i \in \alpha(x^*), \quad (3c)$$

$$\lambda_{\mathcal{I}} \geq 0, \quad (3d)$$

$$\lambda_{\mathcal{I}}^T c_{\mathcal{I}} = 0. \quad (3e)$$

Here we use the notation

$$\mu_i^+ := [\lambda_{\mathcal{E}}^T \partial_2 c_{\mathcal{E}} - \lambda_{\mathcal{I}}^T \partial_2 c_{\mathcal{I}} + \lambda_{\mathcal{Z}}^T [\partial_2 c_{\mathcal{Z}} - I]]_i,$$

$$\mu_i^- := [\lambda_{\mathcal{E}}^T \partial_2 c_{\mathcal{E}} - \lambda_{\mathcal{I}}^T \partial_2 c_{\mathcal{I}} + \lambda_{\mathcal{Z}}^T [\partial_2 c_{\mathcal{Z}} + I]]_i,$$

and the constraints and the partial derivatives are evaluated at  $(x^*, |z^*|)$ .

**Theorem 46** (M-Stationarity for (I-MPCC) is M-Stationarity for (I-NLP)). *A feasible point  $(x^*, z^*)$  of (I-NLP) is M-stationary if and only if  $(x^*, u^*, v^*) = (x^*, [z^*]^+, [z^*]^-)$  of (I-MPCC) is M-stationary.*

*Proof.* For indices that satisfy the first condition in (2b), the equivalence with the second condition in (3c) was shown in [8, Theorem 33]. Thus, we just need to consider the alternative conditions. For (I-MPCC) we have the relations

$$[\lambda_{\mathcal{E}}^T \partial_2 c_{\mathcal{E}} - \lambda_{\mathcal{I}}^T \partial_2 c_{\mathcal{I}} + \lambda_{\mathcal{Z}}^T [\partial_2 c_{\mathcal{Z}} - I]]_i = (\mu_u)_i, \quad i \in \mathcal{D},$$

$$[\lambda_{\mathcal{E}}^T \partial_2 c_{\mathcal{E}} - \lambda_{\mathcal{I}}^T \partial_2 c_{\mathcal{I}} + \lambda_{\mathcal{Z}}^T [\partial_2 c_{\mathcal{Z}} + I]]_i = (\mu_v)_i, \quad i \in \mathcal{D},$$

which was also shown in [8, Theorem 33]. These are exactly the definitions of  $\mu_i^+$  and  $\mu_i^-$  in the definition of M-Stationarity for (I-NLP).  $\square$

Consequently, we may now rephrase the result by [1, 2] in the language of abs-normal forms.

**Theorem 47** (Minimizers and M-Stationarity for (I-NLP)). *Assume that  $(x^*, z^*)$  is a local minimizer of (I-NLP) and that AKQ holds at  $x^*$ . Then,  $(x^*, z^*)$  is M-stationary for (I-NLP).*

*Proof.* First, note that  $(x^*, z^*)$  is a local minimizer of (I-NLP) if and only if  $(x^*, u^*, v^*) = (x^*, [z^*]^+, [z^*]^-)$  is a local minimizer of (I-MPCC). Then, the point  $(x^*, u^*, v^*)$  is a local minimizer of the counterpart MPCC, and MPCC-ACQ holds by Theorem 36. Thus,  $(x^*, u^*, v^*)$  is M-stationary for (I-MPCC) and Theorem 46 implies that  $(x^*, z^*)$  is M-stationary for (I-NLP).  $\square$

**5.2. MPCC-linearized Bouligand Stationarity.** Finally, we introduce MPCC-linearized Bouligand stationarity, which is defined via smooth subproblems.

**Definition 48** (MPCC-linearized B-Stationarity for (I-MPCC), see [14]). Consider a feasible point  $(x^*, u^*, v^*)$  of (I-MPCC) with associated index sets  $\mathcal{U}_+$ ,  $\mathcal{V}_+$  and  $\mathcal{D}$ . It is a *B-stationary* point if it is a stationary point of all branch problems  $(\text{NLP}(\mathcal{P}^t))$  for  $\mathcal{P}^t = \mathcal{P} \subseteq \mathcal{D}$ . Here,  $\bar{\mathcal{P}}$  denotes the complement of  $\mathcal{P}$  in  $\mathcal{D}(x^*)$ .

Note that there exist different names for the previous defined variant of B-stationarity, i.e., it is called B-stationarity in [14]. But we use here the name MPCC-linearized B-stationarity as suggested in [1] to prevent confusion with the definition of B-stationarity in the smooth case.

**Theorem 49.** *If GCQ holds for all  $(\text{NLP}(\mathcal{P}^t))$ , then all local minimizers of (I-MPCC) are MPCC-linearized B-stationary points.*

*Proof.* This follows directly by KKT theory for smooth optimization problems.  $\square$

**Definition 50** (Abs-Normal-Linearized B-Stationarity for (I-NLP)). Consider a feasible point  $(x^*, z^*)$  of (I-NLP). It is an *abs-normal-linearized B-stationary* point if it is a stationary point of the branch problems  $(\text{NLP}(\Sigma^t))$  for  $\Sigma^t = \text{diag}(\sigma)$  with  $\sigma \succeq \sigma(x)$ .

**Theorem 51** (MPCC-linearized B-stationarity for **(I-MPCC)** is abs-normal-linearized B-stationarity for **(I-NLP)**). *A feasible point  $(x^*, z^*)$  of **(I-NLP)** is abs-normal-linearized B-stationary if and only if  $(x^*, u^*, v^*) = (x^*, [z^*]^+, [z^*]^-)$  of **(I-MPCC)** is MPCC-linearized B-stationary.*

*Proof.* This follows directly from Lemma 27.  $\square$

**Theorem 52** (Minimizers and abs-normal-linearized B-Stationarity for **(I-NLP)**). *Assume that  $(x^*, z^*)$  is a local minimizer of **(I-NLP)** and that GCQ holds at  $(x^*, z^*)$  for all **(NLP( $\Sigma^t$ ))**. Then,  $(x^*, z^*)$  is abs-normal-linearized B-stationary for **(I-NLP)**.*

*Proof.* The point  $(x^*, z^*)$  is a local minimizer of **(I-NLP)** if and only if  $(x^*, u^*, v^*) = (x^*, [z^*]^+, [z^*]^-)$  is a local minimizer of **(I-MPCC)**. Moreover, GCQ for all **(NLP( $\Sigma^t$ ))** and GCQ for all **(NLP( $\mathcal{P}^t$ ))** are equivalent by Theorem 39. Thus,  $(x^*, u^*, v^*)$  is a local minimizer of the counterpart MPCC and GCQ holds for all **(NLP( $\mathcal{P}^t$ ))**. Then, it is MPCC-linearized B-stationary by Lemma 49 and finally  $(x^*, z^*)$  is abs-normal-linearized B-stationary by Theorem 51.  $\square$

*Remark 53.* In [6], Griewank and Walther have presented a stationarity concept that holds without any kink qualification for minimizers of the *unconstrained* abs-normal NLP

$$\min_x f(x), \quad f \in C_{\text{abs}}^d(D^x, \mathbb{R}). \quad (4)$$

Indeed, this concept is precisely abs-normal-linearized Bouligand stationarity: it requires the conditions of Definition 50 specialized to (4). Now, the question arises why no regularity assumption is needed. The answer is that the abs-normal form provides a certain degree of built-in regularity: we have shown in [9] that MPCC-ACQ is always satisfied for the counterpart MPCC of (4) (and thus every local minimizer is an M-stationary point). Analogously one can show that ACQ for all branch problems **(NLP( $\mathcal{P}^t$ ))** is always satisfied for (4). Now, ACQ for all branch problems **(NLP( $\mathcal{P}^t$ ))** is equivalent to ACQ for all branch problems **(NLP( $\Sigma^t$ ))** by Theorem 38, which in turn implies GCQ for all branch problems **(NLP( $\Sigma^t$ ))**. Thus, GCQ for all branch problems **(NLP( $\Sigma^t$ ))** is always satisfied for (4) and Theorem 52 holds.

## 6. CONCLUSIONS

We have shown that general abs-normal NLPs are essentially the same problem class as MPCCs. The two problem classes permit the definition of corresponding constraint qualifications, and optimality conditions of first order under weak constraint qualifications. We have also shown that the slack reformulation from [10] preserves constraint qualifications of Abadie type, whereas for Guignard type we could only prove some implications. Here, one subtle drawback is the non-uniqueness of slack variables. Thus, we have introduced branch formulations of general abs-normal NLPs and counterpart MPCCs. Then, constraint qualifications of Abadie and Guignard type are preserved.

## ACKNOWLEDGEMENT

C. Kirches discloses support by the German Federal Ministry of Education and Research through grants n° 05M17MBA-MoPhaPro, 05M18MBA-MORENet, and 01/S17089C-ODINE, and by Deutsche Forschungsgemeinschaft (DFG) through Priority Programme 1962, grant Ki1839/1-2.

## REFERENCES

- [1] M. Flegel, *Constraint qualifications and stationarity concepts for mathematical programs with equilibrium constraints*, Dissertation, Universität Würzburg, 2005.
- [2] M. Flegel and C. Kanzow, *A direct proof for M-stationarity under MPEC-ACQ for mathematical programs with equilibrium constraints*, in *Optimization with Multivalued Mappings*, S. Dempe and V. Kalashnikov, eds., Springer Optimization and Its Applications Vol. 2, Springer, Boston, MA, 2006.
- [3] A. Griewank, *On stable piecewise linearization and generalized algorithmic differentiation*, *Optimization Methods and Software* 28 (2013), pp. 1139–1178. Available at <https://doi.org/10.1080/10556788.2013.796683>.

- [4] A. Griewank and A. Walther, *First- and second-order optimality conditions for piecewise smooth objective functions*, Optimization Methods and Software 31 (2016), pp. 904–930. Available at <https://doi.org/10.1080/10556788.2016.1189549>.
- [5] A. Griewank and A. Walther, *Characterizing and testing subdifferential regularity for piecewise smooth objective functions*, SIAM Journal on Optimization 29 (2019), pp. 1473–1501. Available at <https://doi.org/10.1137/17M115520X>.
- [6] A. Griewank and A. Walther, *Relaxing kink qualifications and proving convergence rates in piecewise smooth optimization*, SIAM Journal on Optimization 29 (2019), pp. 262–289. Available at <https://doi.org/10.1080/10556788.2016.1189549>.
- [7] L.C. Hegerhorst-Schultchen, *Optimality conditions for abs-normal NLPs*, Dissertation, Leibniz Universität Hannover, 2020.
- [8] L.C. Hegerhorst-Schultchen, C. Kirches, and M.C. Steinbach, *Relations Between Abs-Normal NLPs and MPCCs. Part 1: Strong Constraint Qualifications*, submitted (2019). Available at [http://www.optimization-online.org/DB\\_HTML/2019/07/7302.html](http://www.optimization-online.org/DB_HTML/2019/07/7302.html).
- [9] L.C. Hegerhorst-Schultchen, C. Kirches, and M.C. Steinbach, *On the relation between MPECs and optimization problems in abs-normal form*, Opt. Methods Softw. 35 (2020), pp. 565–575. DOI: 10.1080/10556788.2019.1588268 (2019).
- [10] L.C. Hegerhorst-Schultchen and M.C. Steinbach, *On first and second order optimality conditions for abs-normal NLP*, Optimization (2019). Available at <https://doi.org/10.1080/02331934.2019.1626386>.
- [11] J.Pang and M.Fukushima, *Complementarity constraint qualifications and simplified b-stationarity conditions for mathematical programs with equilibrium constraints*, Computational Optimization and Applications 13 (1999), pp. 111–136.
- [12] Z. Luo, J. Pang, and D. Ralph, *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, 1996.
- [13] J.V. Outrata, *Optimality conditions for a class of mathematical programs with equilibrium constraints*, Mathematics of Operations Research 24 (1999), pp. 627–644.
- [14] H. Scheel and S. Scholtes, *Mathematical programs with complementarity constraints: Stationarity, optimality, and sensitivity*, Mathematics of Operations Research 25 (2000), pp. 1–22. Available at <https://doi.org/10.1287/moor.25.1.1.15213>.
- [15] J. Ye, *Necessary and sufficient optimality conditions for mathematical programs with equilibrium constraints*, Journal of Mathematical Analysis and Applications 307 (2005), pp. 350 – 369. Available at <https://doi.org/10.1016/j.jmaa.2004.10.032>.

L.C. HEGERHORST-SCHULTCHEN, LEIBNIZ UNIVERSITÄT HANNOVER, INSTITUT FÜR ANGEWANDTE MATHEMATIK, WELFENGARTEN 1, 30167 HANNOVER, GERMANY.

*Email address:* hegerhorst@ifam.uni-hannover.de

C. KIRCHES, TECHNISCHE UNIVERSITÄT CAROLO-WILHELMINA ZU BRAUNSCHWEIG, INSTITUT FÜR MATHEMATISCHE OPTIMIERUNG, UNIVERSITÄTSPLATZ 2, 38106 BRAUNSCHWEIG, GERMANY.

*Email address:* c.kirches@tu-bs.de

M.C. STEINBACH, LEIBNIZ UNIVERSITÄT HANNOVER, INSTITUT FÜR ANGEWANDTE MATHEMATIK, WELFENGARTEN 1, 30167 HANNOVER, GERMANY.

*Email address:* mcs@ifam.uni-hannover.de