Weak sharpness and finite termination for variational inequalities on Hadamard manifolds

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ABSTRACT
We first introduce the notion of weak sharpness for the solution sets of variational inequality problems (in short, VIP) on Hadamard spaces. We then study the finite convergence property of sequences generated by the inexact proximal point algorithm with different error terms for solving VIP under weak sharpness of the solution set. We also give an upper bound on the number of iterations by which the sequence generated by exact proximal point algorithm converges to a solution of VIP. An example is also given to illustrate our results.

KEYWORDS
Variational inequalities; Proximal point algorithm; Weak sharpness; Finite termination; Hadamard manifolds

1. Introduction

Ferris [1] introduced the notion of weak sharp minima for a convex optimization problem which is a generalization of the notion of sharp minima due to Polyak [2] to include the case of non-singleton sets of solutions. This notion was later extensively investigated by many researcher because it plays an important role in sensitivity analysis, error bounds and (finite) convergence analysis of a large number of optimization algorithms, see, e.g., [3–6] and the references therein. Here, the finite convergence of an algorithm means that the sequence generated by the algorithm terminates after a finite number of iterations. Extending this notion, Patricksson [7] introduced the notion of weak sharp solutions for variational inequality problems. Marcotte and Zhu [8] established the necessary and sufficient condition for a solution set to be weakly sharp in term of its dual gap function and also studied finite convergence of the sequences generated by some algorithms for solving VIP under the weak sharpness of solution sets. Afterwards, weak sharpness of the solution set and its applications to the finite convergence property of various algorithms for solving (generalized) variational inequality problems have been investigated by many authors, see, e.g., [9–16] and references therein.

During the last decade, many important concepts and results in nonlinear analysis and optimization theory have been extended from Euclidean spaces to the setting of manifolds. For example, theory of subdifferential calculus for nonsmooth functions on

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Riemannian manifolds is developed in [17–19]. Various numerical methods for solving variational inequalities, equilibrium problems, optimization problems, etc. on manifolds can be found in, e.g., [20–26] and references therein. The notion of monotonicity in linear spaces were extended to manifold setting and have been studied intensively in [23,27–29] and references therein. We refer the reader to [30,31] for the study of weak sharp minima for constrained optimization problems on Riemannian manifolds and to [26,32] for the study of finite convergence of proximal point algorithms for solving optimization problems and for finding singular points of multivalued vector fields on Hadamard manifolds under weak sharp minima-like conditions. The main advantages of these extensions from Euclidean spaces to Riemannian manifolds are that nonconvex (constrained, nonmonotone, respectively) problems can be transformed to convex (unconstrained, monotone, respectively) problems from Riemannian point of view. Therefore, the extension of the concepts, techniques and results about weak sharpness and finite convergence for variational inequality problems from Euclidean spaces to Riemannian manifolds is natural and interesting.

Variational inequality problems on manifolds were first introduced and established by Németh in [33] for single-valued vector fields on Hadamard manifolds. The results were extended to Riemannian manifolds in [34] by Li et al. and to the set-valued vector fields on Riemannian manifolds in [35] by Li and Yao. Numerical methods for solving variational inequality problems on manifolds can be seen, for instances, in [23,35–38] and references therein. The purpose of this paper is to introduce the notion of weak sharp solution for variational inequality problems on Hadamard manifolds and study the finite convergence of the inexact proximal point algorithms for solving monotone variational inequalities under weak sharpness of the solution set extending some results from linear context to manifold context. More precisely, we extend the notion of weak sharp solution for variational inequality problems from Euclidean setting to Hadamard manifolds and present an abstract result on finite termination for the inexact proximal point algorithm under weak sharpness of the solution set. We then present finite convergence results with different error terms. Finally, we give an upper bound for number of iterations by which the sequence generated by the exact proximal point algorithm converges to a solution of VIP. An example is also presented to illustrate the latter result.

2. Preliminaries

In this section we recall some basic definitions and notations of Riemannian geometry which can be found in, for instances, [39–41].

Let M be a connected m-dimensional manifolds. If M is endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$, with the corresponding norm denoted by $||\cdot||$, then M is a Riemannian manifold. Let $p \in M$. The tangent space of M at p is denoted by T_pM and the tangent bundle of M by $TM = \cup_{x\in M} T_pM$ which is naturally a manifold. We denote by $\mathbb{B}_p$ the closed unit ball of T_pM, i.e.,

$$\mathbb{B}_p := \{v \in T_pM : ||v|| \leq 1\}.$$

For a piecewise smooth curve $\gamma : [a, b] \to M$ joining p to q (i.e., $\gamma(a) = p$ and $\gamma(b) = q$), we define the length of γ by using the metric as

$$L(\gamma) := \int_a^b ||\gamma'(t)||dt.$$

The Riemannian distance $d(p, q)$ is defined by minimizing this length functional over the set of all such curves joining p and q. This distance induces the original topology on M. Given a nonempty set $S \subset M$, the distance from $p \in M$ to S is defined by

$$d(p, S) := \inf\{d(p, q) : q \in S\}.$$

Let ∇ be the Levi-Civita connection associated to $(M, \langle \cdot, \cdot \rangle)$ and γ be a smooth curve in M. A vector filed V is said to be parallel along γ if $\nabla_{\gamma'} V = 0$. If γ' itself is parallel along γ, we say that γ is a geodesic and in this case $||\gamma'||$ is constant. If $||\gamma'|| = 1$, then we say that γ is normalized. A geodesic joining p to q is said to be minimal if its length equals $d(p, q)$ and this geodesic is called a minimizing geodesic. A Riemannian manifold is complete if geodesic is defined for all $-\infty < t < +\infty$. The Hopf-Rinow theorem asserts that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space and bounded closed subsets are compact.

Assume that M is complete. The exponential map $\exp_x : T_pM \to M$ at p is defined by $\exp_p v := \gamma_v(1, p)$ for each $v \in T_pM$, where $\gamma_v(\cdot, p)$ is the geodesic starting from p with velocity v, that is, $\gamma(0) = p$ and $\gamma'(0) = v$. It is easy to see that $\exp_p tv = \gamma_v(t, p)$ for any real number t and $\exp_p \mathbf{0} = \gamma_v(0, p) = p$, where $\mathbf{0}$ is the zero tangent vector. Note that the map $\exp_p$ is differentiable on T_pM for any $p \in M$. Moreover, for any $p, q \in M$, we have $d(p, q) = ||\exp_p^{-1} q||$.

A complete, simply connected Riemannian manifold of non-positive sectional curvature is called a Hadamard manifolds. Throughout this paper, unless otherwise stated, we always assume that M is an m-dimensional Hadamard manifold. Let $p \in M$. It is known that the map $\exp : T_pM \to M$ is a diffeomorphism and for any two point $p, q \in M$, there exists a unique normalized geodesic joining p to q which is , in fact, a minimal geodesic. Furthermore, for Hadamard manifolds, one of the most important properties is the following comparison result which is taken from Proposition 4.5 of [40] and will be useful in the sequel. Recall that a geodesic triangle $\Delta(p_1 p_2 p_3)$ of a Riemannian manifold is a set consisting of three points p_1, p_2 and p_3 and three minimal geodesics joining these points.

Proposition 2.1. *(Comparison result for triangle geodesic). Let $\Delta(p_1 p_2 p_3)$ be a geodesic triangle. Denote, for each $i = 1, 2, 3 (mod\, 3)$, by $\gamma_i : [0, \ell_i] \to \mathbb{M}$ the geodesic joining p_i to p_{i+1}, and set $\ell_i = L(\gamma_i)$ and $\alpha_i = \angle(\gamma_i'(0), -\gamma_{i-1}'(\ell_{i-1}))$. Then*

(i) $\alpha_1 + \alpha_2 + \alpha_3 \leq \pi$;
(ii) $\ell_i^2 + \ell_{i+1}^2 - 2\ell_i \ell_{i+1} \cos \alpha_{i+1} \leq \ell_{i-1}^2$;
(iii) $\ell_{i+1} \cos \alpha_{i+2} + \ell_i \cos \alpha_i \geq \ell_{i+2}$.

In terms of the distance and the exponential map, the inequality (ii) of Proposition 2.1 can be rewritten as follows

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle \leq d^2(p_{i-1}, p_i) \quad (1)$$

since

$$\langle \exp^{-1}_{p_{i+1}} p_i, \exp^{-1}_{p_{i+1}} p_{i+2} \rangle = d(p_i, p_{i+1}) d(p_{i+1}, p_{i+2}) \cos \alpha_{i+1}.$$

A subset $K \subset M$ is said to be convex if for any two point p and q in K, the geodesic joining p to q is contained in K, that is, if $\gamma : [a, b] \to M$ is a geodesic such that $\gamma(a) = p$ and $\gamma(b) = q$, then $\gamma(ta + (1 - t)b) \in K$ for all $t \in [0, 1]$. The projection of a point $x \in M$ onto a subset K of a Hadamard manifold M is defined by

$$P(x, K) := \{p \in K : d(x, p) = d(x, K)\}.$$

Proposition 2.2. *[42] Let K be a closed convex subset of a Hadamard manifold M. Then, for any $x \in M$, $P(x, K)$ is a singleton set. Also, for any $p \in M$, the following assertions are equivalent:*

(i) $y = P(p, K)$;
(ii) $\langle \exp^{-1}_y p, \exp^{-1}_y q \rangle \leq 0$ *for all* $q \in K$.

We now recall a definition of the normal cone and the tangent cone to a closed convex subset of a Hadamard manifold; for more details see [18]. Let K be a closed convex subset of M and let $x \in K$. The normal cone to K at x, denoted by $N_K(x)$, is defined by

$$N_K(x) = \{v \in T_x M : \langle v, \exp^{-1}_x y \rangle \leq 0 \quad \text{for all} \;\; y \in K\}.$$

The tangent cone to K at x, denoted by $T_K(x)$, is defined by

$$T_K(x) = \{v \in T_x M : \langle v, w \rangle \leq 0 \text{ for all } \; w \in N_K(x)\}.$$

That is, $T_K(x) = [N_K(x)]^\circ$. Recalling that, if Z is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and S is subset of Z, then the polar S° of S is defined by $S^\circ = \{v \in Z : \langle v, w \rangle \leq 0 \; \forall w \in S\}$. If C be a closed convex subset of Z and $x \in C$, the normal cone to C at x and the tangent cone to C at x are defined, respectively, by

$$N_C(x) = \{v \in Z : \langle v, y - x \rangle \leq 0 \quad \forall y \in C\},$$

and $T_C(x) = [N_C(x)]^\circ$.

Let $\mathcal{X}(M)$ denote the set of all univalued vector fields $V : M \to TM$ such that $V(x) \in T_x M$ for each $x \in M$.

Definition 2.3. Let M be a Hadamard manifold and X be a convex subset of M. A vector field $V \in \mathcal{X}(M)$ is said to be monotone on X if for any $x, y \in X$, it holds

$$\langle V(x), \exp^{-1}_x y \rangle + \leq \langle V(y), -\exp^{-1}_y x \rangle.$$

Let $V \in \mathcal{X}(M)$ and X be a closed convex subset of M. Németh [33] introduced the variational inequality problem on Hadamard manifolds: find $x \in X$ such that

$$\langle V(x), \exp^{-1}_x y \rangle \geq 0 \quad \text{for all} \;\; y \in X. \tag{2}$$

We denote by X^* the solution set of VIP (2). Throughout this paper, we always assume that X^* is nonempty. For nonemptiness of the solution set of variational inequality problems on manifolds, we refer the reader to, for instance, [33–35]. Note that the variational inequality problem (2) on Hadamard manifolds is an extension of classical variational inequality problems. More precisely, if $M = \mathbb{R}^m$, then (2) reduces to

$$\text{find } \ x \in X: \qquad \langle V(x), y - x\rangle \geq 0, \quad \text{for all } \ y \in X. \tag{3}$$

From [35, Corollary 4.7], we have the following result about the convexity of the solution set of (2).

Proposition 2.4. *If the vector field V is monotone on X, then the solution set X^* of (2) is convex.*

To conclude this section, we recall the following result which will be useful in the sequel.

Lemma 2.5. *[43] Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

3. Weak sharpness and finite convergence results

In this section, we introduce the notion of weak sharpness for the solution set of variational inequality problems on Hadamard manifolds and study the finite convergence of sequences generated by proximal point algorithm.

Definition 3.1. The solution set X^* of VIP (2) is said to be weakly sharp if there is a constant $\alpha > 0$ such that

$$\alpha\mathbb{B}_z \subset V(z) + [T_X(z) \cap N_{X^*}(z)]^\circ, \quad \text{for each } \ z \in X^*. \tag{4}$$

The constant α is called the modulus of the weak sharpness of X^*.

Remark 1. When $M = \mathbb{R}^n$, then (4) reduces to

$$\alpha\mathbb{B} \subset V(z) + [T_X(z) \cap N_{X^*}(z)]^\circ, \quad \text{for each } \ z \in X^*, \tag{5}$$

where $\mathbb{B}$ is the closed unit ball in $\mathbb{R}^n$. In this linear setting, if X^* is weakly in the sense of Marcotte and Zhu [8], i.e.,

$$-V(z) \in \operatorname{int}\left(\bigcap_{x\in X^*} [T_X(x) \cap N_{X^*}(x)]^\circ\right), \quad \forall z \in X^*,$$

then (5) holds.

Proposition 3.2. *If X^* is weakly sharp with modulus $\alpha > 0$, then*

$$\langle V(P(x, X^*)), \exp^{-1}_{P(x,X^*)} x\rangle \geq \alpha d(x, X^*), \quad \textit{for all } \ x \in X. \tag{6}$$

If, in addition, V is monotone on X, then

$$-\langle V(x), \exp_x^{-1} P(x, X^*)\rangle \geq \alpha d(x, X^*), \quad \textit{for all } x \in X. \tag{7}$$

Proof. Let $x \in X$ and set $z = P(x, X^*)$. If $x \in X^*$, i.e., $z = x$, then (6) holds. Assume that $x \notin X^*$. By the convexity of X and X^*, we have

$$\exp_z^{-1} x \in T_X(z) \cap N_{X^*}(z) \quad \text{and} \quad d(x, z) = d(x, X^*) > 0.$$

Set

$$w = \frac{\exp_z^{-1} x}{d(x, z)}.$$

Then, $w \in \mathbb{B}_z$. By (4), one has $\alpha w - V(z) \in [T_X(z) \cap N_{X^*}(z)]^\circ$. Thus,

$$\langle \alpha w - V(z), \exp_z^{-1} x\rangle \leq 0,$$

or, equivalently,

$$\begin{aligned} \langle V(z), \exp_z^{-1} x\rangle &\geq \langle \alpha w, \exp_z^{-1} x\rangle = \left\langle \alpha \frac{\exp_z^{-1} x\rangle}{d(x, z)}, \exp_z^{-1} x \right\rangle \\ &= \alpha \frac{||\exp_z^{-1} x||^2}{d(x, z)} = \alpha d(x, z) = \alpha d(x, X^*). \end{aligned}$$

Hence, (6) holds. If V is monotone, then

$$-\langle V(x), \exp_x^{-1} P(x, X^*)\rangle \geq \langle V(P(x, X^*)), \exp_{P(x, X^*)}^{-1} x\rangle \geq \alpha d(x, X^*),$$

i.e., (7) holds. This ends the proof. □

Remark 2. In the linear case, i.e., when $M = \mathbb{R}^n$, then (6) and (7) reduce respectively to

$$\langle V(P(x, X^*)), x - P(x, X^*)\rangle \geq \alpha d(x, X^*), \quad \forall x \in X, \tag{8}$$

and

$$\langle V(x), x - P(x, X^*)\rangle \geq \alpha d(x, X^*), \quad \forall x \in X. \tag{9}$$

As in [10], if V is continuous, then both (8) and (9) imply (5). So it is natural to rise the following open question: *Does (6) (or, (7)) imply the weak sharpness of the solution set X^*?*

We are now going to study the finite convergence for sequences generated by the proximal point algorithm. Consider the following inexact proximal point algorithm (in short, IPPA): let $x_0 \in X$ and for each $n \geq 0$,

$$\langle \lambda_n V(x_{n+1}) - (e_{n+1} + \exp_{x_{n+1}}^{-1} x_n), \exp_{x_{n+1}}^{-1} y\rangle \geq 0, \quad \forall\, y \in X, \tag{10}$$

where $\{e_{n+1}\}$ is regarded as an error sequence and $\{\lambda_n\} \subset (0, \infty)$ is a stepsize sequence. We note that when V is monotone, then the algorithm (10) is well defined (see, e.g., [24, Remark 3.2]).

We first present a result on the finite convergence of the above IPPA under an abstract condition when the solution set of VIP (2) is weakly sharp.

Proposition 3.3. *Let V be monotone and $\{x_n\}$ be a sequence generated by (10). Assume that*

$$\lim_{n\to\infty} \frac{d(x_n, x_{n+1}) + ||e_{n+1}||}{\lambda_n} = 0. \tag{11}$$

If X^* is weakly sharp, then $x_n \in X^*$ for all n sufficiently large.

Proof. Let z be a solution of the variational inequality problem (2). By (10) we have

$$\langle \lambda_n V(x_{n+1}) - (e_{n+1} + \exp^{-1}_{x_{n+1}} x_n), \exp^{-1}_{x_{n+1}} z \rangle \geq 0,$$

which implies that

$$\begin{aligned} -\langle V(x_{n+1}), \exp^{-1}_{x_{n+1}} z \rangle &\leq -\frac{1}{\lambda_n} \left(\langle e_{n+1} + \exp^{-1}_{x_{n+1}} x_n, \exp^{-1}_{x_{n+1}} z \rangle \right) \\ &\leq \frac{1}{\lambda_n} \left(||e_{n+1}|| + ||\exp^{-1}_{x_{n+1}} x_n|| \right) .||\exp^{-1}_{x_{n+1}} z|| \\ &= \frac{1}{\lambda_n} \left(||e_{n+1}|| + d(x_{n+1}, x_n) \right) .d(x_{n+1}, z) \end{aligned} \tag{12}$$

Assume that the conclusion is not true. Then, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \notin X^*$ for all k. For each k, set $z_{n_k} = P(x_{n_k}, X^*)$. Then, $d(x_{n_k}, z_{n_k}) = d(x_{n_k}, X^*) > 0$. By Proposition 3.2, one has

$$-\langle V(x_{n_k}), \exp^{-1}_{x_{n_k}} z_{n_k} \rangle \geq \alpha d(x_{n_k}, X^*) = \alpha d(x_{n_k}, z_{n_k}).$$

This, together with (12), yields

$$\alpha d(x_{n_k}, z_{n_k}) \leq \frac{1}{\lambda_{n_k-1}} \left(||e_{n_k}|| + d(x_{n_k}, x_{n_k-1}) \right) .d(x_{n_k}, z_{n_k}),$$

and then

$$\alpha \leq \frac{1}{\lambda_{n_k-1}} \left(||e_{n_k}|| + d(x_{n_k}, x_{n_k-1}) \right).$$

Letting $k \to \infty$ and using (11), one obtains that $\alpha \leq 0$. This is a contradiction. Thus, $x_n \in X^*$ for all n sufficiently large. □

Remark 3. We can replace the condition (11) by the following stronger conditions:

$$\liminf_{n\to\infty} \lambda_n > 0, \quad \text{and} \quad \lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} ||e_{n+1}|| = 0.$$

We next consider some special cases of the error terms. The first case is when the sequence $\{||e_n||\}$ is summable (see [14] for an analogous result in Hilbert spaces).

Theorem 3.4. *Let V be monotone and $\{x_n\}$ be a sequence generated by the IPPA (10) with*

$$\liminf_{n\to\infty} \lambda_n > 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} ||e_n|| < \infty \tag{13}$$

If X^* is weakly sharp, then $x_n \in X^*$ for all n sufficiently large.

Proof. Since $\{||e_n||\}$ is summable, $\lim_{n\to\infty} ||e_n|| = 0$. By Proposition 3.3 and the condition (13), it is enough to show that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Let z be a solution of VIP (2). We have

$$\langle V(z), \exp_z^{-1} x_{n+1}\rangle \geq 0.$$

Then, by the monotonicity of V,

$$\langle V(x_{n+1}), -\exp_{x_{n+1}}^{-1} z\rangle \geq \langle V(z), \exp_z^{-1} x_{n+1}\rangle \geq 0.$$

Since $\lambda_n > 0$, it follows from (10) that

$$\langle e_{n+1} + \exp_{x_{n+1}}^{-1} x_n, -\exp_{x_{n+1}}^{-1} z\rangle \geq \langle V(x_{n+1}), -\exp_{x_{n+1}}^{-1} z\rangle \geq 0.$$

Hence,

$$\langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} z\rangle \leq \langle e_{n+1}, -\exp_{x_{n+1}}^{-1} z\rangle. \tag{14}$$

Consider the geodesic triangle $\Delta(x_n x_{n+1} z)$. By (1), we have

$$d^2(x_{n+1}, x_n) + d^2(x_{n+1}, z) - 2\langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} z\rangle \leq d^2(x_n, z).$$

Combining with (14), one gets

$$\begin{aligned} d^2(x_{n+1}, z) &\leq d^2(x_n, z) - d^2(x_{n+1}, x_n) + 2\langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} z\rangle \\ &\leq d^2(x_n, z) - d^2(x_{n+1}, x_n) + 2\langle e_{n+1}, -\exp_{x_{n+1}}^{-1} z\rangle. \end{aligned} \tag{15}$$

This implies that

$$d^2(x_{n+1}, z) - d^2(x_n, z) \leq 2\langle e_{n+1}, -\exp_{x_{n+1}}^{-1} z\rangle \tag{16}$$

We will show that

$$d(x_{n+1}, z) - d(x_n, z) \leq 2||e_{n+1}|| \quad \text{for all } n. \tag{17}$$

Indeed, if $\exp_{x_{n+1}}^{-1} z = 0$, i.e. $d(x_{n+1}, z) = 0$, for some n, then (17) holds for that n.

Assume now that $\exp^{-1}_{x_{n+1}} z \neq 0$. It follows from (16) that

$$d^2(x_{n+1}, z) - d^2(x_n, z) \leq 2||e_{n+1}||.||\exp^{-1}_{x_{n+1}} z|| = 2||e_{n+1}||d(x_{n+1}, z).$$

Equivalently,

$$(d(x_{n+1}, z) - d(x_n, z)) \left(1 + \frac{d(x_n, z)}{d(x_{n+1}, z)}\right) \leq 2||e_{n+1}||.$$

This implies that

$$d(x_{n+1}, z) - d(x_n, z) \leq 2||e_{n+1}||,$$

that is, (17) holds. Then, by the assumption on $\{e_n\}$ and Lemma 2.5, there exists $\delta > 0$ such that

$$\lim_{n\to\infty} d(x_n, z) = \delta.$$

Moreover, from (15), one has

$$\begin{aligned} d^2(x_{n+1}, x_n) &\leq d^2(x_n, z) - d^2(x_{n+1}, z) + 2\langle e_{n+1}, -\exp^{-1}_{x_{n+1}}, z\rangle \\ &\leq d^2(x_n, z) - d^2(x_{n+1}, z) + 2||e_{n+1}||.d(x_{n+1}, z). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

This ends the proof. □

We now consider the case when e_{n+1} conforms to the following condition

$$||e_{n+1}|| \leq \eta_n d(x_{n+1}, x_n) \quad \text{with} \quad \sum_{n=0}^{\infty} \eta_n^2 < \infty. \tag{18}$$

For discussions on this condition, we refer the reader to [24] and references therein.

Theorem 3.5. *Let V be monotone on X and $\{x_n\}$ and $\{e_n\}$ be sequences generated by the algorithm (10) and (18) with $\liminf_{n\to\infty} \lambda_n > 0$. If X^* is weakly sharp, then $x_n \in X^*$ for all n sufficiently large.*

Proof. It is proved in [24], for a more general setting, that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

By (18), we also have that

$$\lim_{n\to\infty} ||e_n|| = 0.$$

Applying Proposition 3.3, we conclude that $x_n \in X^*$ for all n sufficiently large. □

Finally, we consider the case when $e_n = 0$ for all n. In this case, (10) reduces to the (exact) proximal point algorithm: let $x_0 \in X$ and for each $n \geq 0$,

$$\langle \lambda_n V(x_{n+1}) - \exp^{-1}_{x_{n+1}} x_n, \exp^{-1}_{x_{n+1}} y \rangle \geq 0, \quad \forall\, y \in X. \tag{19}$$

In this case, following the idea used in [10], one can obtain an upper bound for the number of iterations for which a sequence generated by (19) reaches the solution set of (2).

Theorem 3.6. *Let $\{x_n\}$ be a sequence generated by (19) with $\lambda_n \in [\theta, \infty)$ for some $\theta > 0$. If V is monotone and X^* is weakly sharp with modulus $\alpha > 0$, then $\{x_n\}$ converges to a point in X^* in at most $\kappa + 1$ iterations with*

$$\kappa \leq \frac{d^2(x_0, X^*)}{\alpha^2 \theta^2}.$$

Proof. Let $z \in X^*$. Using (15) with $e_{n+1} = 0$, we have

$$d^2(x_{n+1}, z) \leq d^2(x_n, z) - d^2(x_{n+1}, x_n).$$

Then, for $1 \leq N \in \mathbb{N}$, we have

$$\begin{aligned} d^2(x_0, z) &\geq d^2(x_1, z) + d^2(x_1, x_0) \\ &\geq d^2(x_2, z) + d^2(x_2, x_1) + d^2(x_1, x_0) \\ &\vdots \\ &\geq d^2(x_{N+1}, z) + \sum_{k=0}^{N} d^2(x_{k+1}, x_k) \\ &\geq \sum_{k=0}^{N} d^2(x_{k+1}, x_k) \end{aligned}$$

Thus, for all $N \geq 1$, one has

$$d^2(x_0, X^*) = \inf_{z \in X^*} d^2(x_0, z) \geq \sum_{k=0}^{N} d^2(x_{k+1}, x_k). \tag{20}$$

As in the proof of Theorem 3.4, it holds that

$$\lim_{n \to \infty} d(x_{n+1}, x_n) = 0.$$

Let κ be the smallest integer such that

$$d(x_{\kappa+1}, x_\kappa) < \alpha\theta. \tag{21}$$

Assume that $x_{\kappa+1} \notin X^*$, set $z_{\kappa+1} = P(x_{\kappa+1}, X^*)$. It follows from Proposition 3.2 and

(19) that

$$\begin{aligned}
\alpha d(x_{\kappa+1}, z_{\kappa+1}) &= \alpha d(x_{\kappa+1}, X^*) \\
&\leq -\langle V(x_{\kappa+1}), \exp^{-1}_{x_{\kappa+1}} z_{\kappa+1}\rangle \\
&\leq -\frac{1}{\lambda_\kappa}\langle \exp^{-1}_{x_\kappa+1} x_\kappa, \exp^{-1}_{x_{\kappa+1}} z_{\kappa+1}\rangle \\
&\leq \frac{1}{\lambda_\kappa}||\exp^{-1}_{x_\kappa+1} x_\kappa||.||\exp^{-1}_{x_{\kappa+1}} z_{\kappa+1}|| \\
&= \frac{1}{\lambda_\kappa} d(x_{\kappa+1}, x_\kappa).d(x_{\kappa+1}, z_{\kappa+1}).
\end{aligned}$$

Since $d(x_{\kappa+1}, z_{\kappa+1}) > 0$, using (21) and having in mind that $\lambda_n \geq \theta$ for all n, we have

$$\alpha \leq \frac{1}{\lambda_\kappa} d(x_{\kappa+1}, x_\kappa) < \frac{1}{\theta}.\alpha\theta = \alpha,$$

which is a contradiction. Thus, $x_{\kappa+1} \in X^*$. By (20),

$$d^2(x_0, X^*) \geq \sum_{i=0}^{\kappa-1} d^2(x_{i+1}, x_i) \geq \kappa\alpha^2\theta^2.$$

Then,

$$\kappa \leq \frac{d^2(x_0, X^*)}{\alpha^2\theta^2}.$$

This ends the proof. □

Remark 4. It follows from Theorem 3.6 that if the stepsizes are large enough, i.e., θ is large enough, then the PPA terminates after one iteration.

To conclude this paper, we present an example to illustrate our result.

Example 3.7. Let $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$ and $M = (\mathbb{R}_{++}, \langle\cdot,\cdot\rangle)$ be the Riemannian manifold with the Riemannian metric

$$\langle u, v\rangle := \frac{1}{x^2}uv, \quad \text{for } x \in M, \text{and} \quad u, v \in T_xM.$$

The Riemannian distance $d : M \times M \to \mathbb{R}_+$ is given by

$$d(x, y) = \left|\ln\left(\frac{x}{y}\right)\right| \quad \text{for all} \quad x, y \in M.$$

The sectional curvature of M is zero and it holds that M is a Hadamard manifold. For each $x \in M$, the tangent plane T_xM at x equals to $\mathbb{R}$. The unique geodesic γ starting from $x = \gamma(0) \in M$ with velocity $v = \gamma'(0) \in T_xM$ is defined by

$$\gamma(t) = xe^{\left(\frac{v}{x}\right)t}.$$

Thus,

$$\exp_x tv = xe^{\left(\frac{v}{x}\right)t}.$$

Moreover, for any $x, y \in M$, we have

$$y = \exp_x \left(d(x,y) \frac{\exp_x^{-1} y}{d(x,y)} \right) = xe^{\frac{\exp_x^{-1} y}{x d(x,y)} d(x,y)} = xe^{\frac{\exp_x^{-1} y}{x}}.$$

Hence, the inverse of exponential map is defined as

$$\exp_x^{-1} y = x \ln \left(\frac{y}{x} \right).$$

Let $X = [1, 2]$, then X is a closed convex subset of M. Let $\alpha > 0$ be a positive number. We consider the vector field $V : M \to TM$ defined by

$$V(x) = -\alpha x, \quad \text{for all } \ x \in M.$$

Let X^* be the solution set of the variational inequality (2). Then, $x \in X^*$ if and only if

$$\langle V(x), \exp_x^{-1} y \rangle \geq 0 \quad \text{for all } \ y \in S.$$

Equivalently,

$$\frac{1}{x^2}(-\alpha x).x \ln \left(\frac{y}{x} \right) \geq 0 \quad \text{for all } \ y \in [1, 2].$$

This is equivalent to $x = 2$. Thus, $X^* = \{2\}$. It is easy to see that $N_{X^*}(2) = \mathbb{R}$ and $T_X(2) = \mathbb{R}_-$ and

$$V(2) + [T_X(2) \cap N_{X^*}(2)]^\circ = [-2\alpha, +\infty).$$

Thus,

$$2\alpha \mathbb{B}_2 \subset V(2) + [T_X(2) \cap N_{X^*}(2)]^\circ,$$

i.e., X^* is weakly sharp with modulus 2α.

Moreover, for any $x, y \in M$, it holds that

$$\begin{aligned} \langle V(x), \exp_x^{-1} y \rangle &= \frac{1}{x^2}(-\alpha x).x \ln \left(\frac{y}{x} \right) \\ &= -\frac{1}{y^2}(-\alpha y).y \ln \left(\frac{x}{y} \right) = \langle V(y), -\exp_y^{-1} x \rangle. \end{aligned}$$

Thus, V is monotone on M.

	$\lambda_n = (n+1)/5(n+2)$		$\lambda_n = 1/4$		$\lambda_n = 1$	
x_0	1	1.5	1	1.5	1	1.5
x_1	1.105171	1.657756	1.284025	1.926038	2	2
x_2	1.262802	1.894203	1.648721	2	2	2
x_3	1.467167	2	2	2	2	2
x_4	1.721736	2	2	2	2	2
x_5	2	2	2	2	2	2
x_6	2	2	2	2	2	2

Table 1. Finite convergence for PPA

Now, we let $\alpha = 1$ and consider the proximal point algorithm (19). In this case, (19) is equivalent to

$$\begin{aligned} 0 &\le \frac{1}{x_{n+1}^2}\left(\lambda_n V(x_{n+1}) - \exp^{-1}_{x_{n+1}} x_n\right)\exp^{-1}_{x_{n+1}} y \\ &= \frac{1}{x_{n+1}^2}\left(-\lambda_n x_{n+1} - x_{n+1}\ln\left(\frac{x_n}{x_{n+1}}\right)\right) x_{n+1}\ln\left(\frac{y}{x_{n+1}}\right) \\ &= -\left[\lambda_n + \ln\left(\frac{x_n}{x_{n+1}}\right)\right]\ln\left(\frac{y}{x_{n+1}}\right), \quad \text{for all } \ y\in[1,2]. \end{aligned}$$

Thus,

$$\left[\lambda_n + \ln\left(\frac{x_n}{x_{n+1}}\right)\right]\ln\left(\frac{y}{x_{n+1}}\right) \le 0, \quad \text{for all } \ y\in[1,2].$$

Therefore,

$$x_{n+1} = \begin{cases} e^{\lambda_n} x_n & \text{if } \ e^{\lambda_n} x_n < 2, \\ 2 & \text{otherwise} \end{cases}.$$

The finite convergence results with different stepsizes $\{\lambda_n\}$ and different initial point x_0 are given in Table 1. One can see that the algorithm with suitable stepsize terminates after one iteration.

Remark 5. It is easy to see that the map V in Example 3.7 is not monotone in the Euclidean sense. Hence, we cannot apply existence results, for instances, [10,14], to get finite convergence of the proximal point algorithm for solving the corresponding variational inequality in the Euclidean setting.

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