On the intrinsic core of convex cones in real linear spaces

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Abstract

Convex cones play an important role in nonlinear analysis and optimization theory. In particular, specific normal cones and tangent cones are known to be convex cones, and it is a crucial fact that they are useful geometric objects for describing optimality conditions. As important applications (especially, in the fields of optimal control with PDE constraints, vector optimization and order theory) show, there are many examples of convex cones with an empty (topological as well as algebraic) interior. In such situations, generalized interiority notions can be useful. In this article, we present new representations and properties of the relative algebraic interior (also known as intrinsic core) of relatively solid, convex cones in real linear spaces (which are not necessarily endowed with a topology) of both finite and infinite dimension. For proving our main results, we are using new separation theorems where a relatively solid, convex set (cone) is involved. For the intrinsic core of the dual cone of a relatively solid, convex cone, we also state new representations that involve the lineality space of the given convex cone. To emphasize the importance of the derived results, some applications in vector optimization are given.

Keywords: Linear space, convex cone, relative algebraic interior, intrinsic core, separation theorem, lineality space, vector optimization, Pareto efficiency

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1 Introduction

In the last century, a lot of effort has been invested in deriving optimality conditions for solutions of scalar optimization problems and corresponding generalizations given by vector optimization problems. Clearly, the celebrated Karush-Kuhn-Tucker conditions and the Fritz-John conditions have influenced many researchers in this field. Due to the development of the field of convex analysis, certain useful geometric objects for describing optimality conditions has been proposed, such as tangent cones and corresponding dual objects, namely the normal cones (see, e.g., Jahn [16, Ch. 4], Khan, Tammer and Zălinescu [18, Ch. 4], Mordukhovich [22–24], Rockafellar [26], Zălinescu [29]). It is well-known that certain normal and tangent cones are convex cones. The notion of a convex cone (sometimes called "wedge", see Holmes [14, p. 17], and Aliprantis and Tourky [5, Sec. 1.1]) also plays a significant role in vector optimization and order theory (see, e.g., Jahn [17], Khan, Tammer and Zălinescu [18] and Zălinescu [29]). For a given convex cone $K \subseteq E$ (i.e., K is nonempty and $\mathbb{R}_+ \cdot K = K = K + K$) in a real linear space E, one can define a preorder relation $\leq_K$ on E by the well-known equivalence

$$x \leqq_K y \quad :\Longleftrightarrow \quad x \in y - K,$$

where $\leqq_K$ is reflexive (since $0 \in K$) and transitive (since K is convex).

In order to point out the role of convex cones in optimization theory, we collect some important examples below:

- certain normal cones (e.g. normal cone of convex analysis) and tangent cones (e.g., the sequential Clarke tangent cone; the sequential Bouligand tangent cone, which is also known as contingent cone, if the involved set is nonempty and convex);
- polar cone and dual cone of a nonempty set;
- recession cone and barrier cone of a nonempty, convex set;
- subspaces of linear spaces.

Assuming that E is actually a real linear topological space, in certain situations both the topological interior and the topological relative interior of a nonempty set $\Omega \subseteq E$ could be empty. For instance, in optimal control with PDE constraints, cones with empty topological (relative) interior (e.g., in L_p spaces) are of interest (see, e.g., Leugering and Schiel [20]). Generalized interiors such as the topological notions of quasi-interior and quasi-relative interior as well as the algebraic notions of algebraic interior (core) and relative algebraic interior (intrinsic core) are known to be helpful in certain situations (see, e.g., Borwein and Lewis [10], Borwein and Goebel [9] and Zălinescu [28, 30]). It is well-known that in a linear topological space the topological interior of Ω is a subset of the algebraic interior of Ω, and both coincide if Ω is convex and at least one of the following conditions is satisfied:

- the topological interior of Ω is nonempty (see Holmes [14, p. 59]);
- E is a Banach space and Ω is topological closed (see Barbu and Precupanu [8, Rem. 1.24]);

- E is separated and has finite dimension (see Barbu and Precupanu [8, Prop. 1.17].

Moreover, in a real linear topological space E, the topological relative interior of a set $\Omega \subseteq E$ (i.e., the topological interior of Ω with respect to the affine hull of Ω) is a subset of the intrinsic core of Ω (see, e.g., Zălinescu [28, p. 353]), and actually equality holds if Ω is convex and one of the following conditions is true:

- E is a Banach space and both Ω and its affine hull are topological closed (see Borwein and Goebel [9, Lem. 2.5]);
- E is separated and has finite dimension (see Barbu and Precupanu [8, Cor. 1.18]).

In this article, we completely concentrate on algebraic interiority notions, namely core and intrinsic core of a convex cone $K \subseteq E$ in a linear space E, where E must not necessarily be endowed with a topology. It is well-known that the core of K is always a subset of the intrinsic core of K. According to Holmes [14, p. 21], if E has finite dimension, then the intrinsic core of K is always nonempty while the core of K could be empty. For interesting works in this field, we refer the reader to Adán and Novo [1–4], Bagdasar and Popovici [6], Bao and Mordukhovich [7], Hernández, Jiménez and Novo [13], Holmes [14], Jahn [15, 17], Khan, Tammer and Zălinescu [18], Luc [21], Popovici [25], Werner [27], Zalinescu [28, 29], and references therein.

The outline of the article is as follows. At the beginning of Section 2, we recall important algebraic properties of convex sets and also more specifically for convex cones in linear spaces. In many results, we will deal with relatively solid, convex cones, and for proving some main theorems, we will work with separation techniques in linear spaces that are based on algebraic notions (for instance the support theorem by Holmes [14, p. 21] based on the intrinsic core notion). Thus, in Section 2.4, we present certain new versions of separation results for relatively solid, convex sets in linear spaces using algebraic notions. Moreover, we state two main auxiliary results in Proposition 2.11 and Proposition 2.13 which are using important facts from Adán and Novo [4, Prop. 2.2] and Popovici [25, Lem. 2.1], respectively.

In Section 3, we present new algebraic characterizations and properties of convex cones, in particular representations of the intrinsic core of a relatively solid, convex cone (see Theorems 3.1 and 3.5).

New algebraic characterizations and properties of dual cones of convex cones are stated in Section 4. Specifically, we present new representations for the intrinsic core of the dual cone of a relatively solid, convex cone (see Theorems 4.1 and 4.7).

In Section 5, we study vector optimization problems involving relatively solid, convex cones which are not necessarily pointed. By using our results derived in the previous sections, we show that certain monotone linear functions can be used to generate Pareto efficient and weakly Pareto efficient solutions of the considered problems.

This article concludes with a brief summary and an outlook to future work in Section 6.

2 Preliminaries in linear spaces

Throughout the article, let E be a real linear space, and let E' be its algebraic dual space, which is given by

$$E' = \{x' : E \to \mathbb{R} \mid x' \text{ is linear}\}.$$

It is convenient to define, for any two points x and $\overline{x}$ in E, the closed, the open, the half-open line segments by

$$[x, \overline{x}] := \{(1-\lambda)x + \lambda\overline{x} \mid \lambda \in [0,1]\}, \qquad (x, \overline{x}) := [x, \overline{x}] \setminus \{x, \overline{x}\},$$
$$[x, \overline{x}) := [x, \overline{x}] \setminus \{\overline{x}\}, \qquad (x, \overline{x}] := [x, \overline{x}] \setminus \{x\}.$$

Consider any set $\Omega \subseteq E$. The smallest affine subspace of E containing Ω is denoted by $\operatorname{aff}\Omega$ while the smallest linear subspace of E containing Ω is given by $\operatorname{span}\Omega$. Following Zălinescu [29, p. 2], the algebraic interior of Ω with respect to a linear space $M \subseteq E$ is defined by

$$\operatorname{aint}_M \Omega := \{x \in \Omega \mid \forall\, v \in M\ \exists\, \delta > 0 :\ x + [0, \delta] \cdot v \subseteq \Omega\}.$$

It is easy to check that the following properties for $\operatorname{aint}_M \Omega$ hold:

1° If $\operatorname{aint}_M \Omega \neq \emptyset$, then $M \subseteq \operatorname{aff}(\Omega - \Omega)$ (see Zălinescu [29, p. 2]).

2° If Ω is a convex set, then $\operatorname{aint}_M \Omega$ is convex as well.

Two special cases will be of interest (c.f. Holmes [14, pp. 7-8]), namely the algebraic interior (or the core) of Ω, which is given as

$$\operatorname{cor}\Omega := \operatorname{aint}_E \Omega,$$

and the relative algebraic interior (or the intrinsic core) of Ω, which is defined by

$$\operatorname{icor}\Omega := \operatorname{aint}_{\operatorname{aff}(\Omega-\Omega)} \Omega.$$

Using all linearly accessible points of Ω (c.f. Holmes [14, p. 9]), i.e., the set

$$\operatorname{lina}\Omega := \{x \in E \mid \exists\, \overline{x} \in \Omega \setminus \{x\} :\ [\overline{x}, x) \subseteq \Omega\},$$

we can define an algebraic closure of Ω by

$$\operatorname{acl}\Omega := \Omega \cup \operatorname{lina}\Omega.$$

As usual, the set $\Omega \subseteq E$ is said to be algebraically closed if $\operatorname{acl}\Omega = \Omega$; (algebraically) solid if $\operatorname{cor}\Omega \neq \emptyset$; relatively (algebraically) solid if $\operatorname{icor}\Omega \neq \emptyset$.

The following facts for sets in the linear space E are well-known (see, e.g., Zălinescu [29, pp. 2-3]):

Lemma 2.1 *Consider a nonempty set $\Omega \subseteq E$. The following assertions hold:*

1° *For any $\omega \in \Omega$, we have* $\operatorname{cor}\Omega \subseteq \operatorname{icor}\Omega \subseteq \Omega \subseteq \operatorname{acl}\Omega \subseteq \operatorname{aff}\Omega = \omega + \operatorname{aff}(\Omega - \Omega)$.

2° $x \in \operatorname{cor}\Omega$ *if and only if* $\operatorname{aff}\Omega = E$ *and* $x \in \operatorname{icor}\Omega$.

3° *If* Ω *is solid, then* $E = \operatorname{aff}(\Omega - \Omega) = \operatorname{aff}\Omega$ *and* $\operatorname{cor}\Omega = \operatorname{icor}\Omega$.

4° *If* $0 \in \Omega$*, then* $\operatorname{aff}\Omega = \operatorname{aff}(\Omega - \Omega) = \operatorname{span}(\Omega - \Omega) = \operatorname{span}\Omega$.

Let us define, for any nonempty set $\Omega \subseteq E$, the cone generated by the set Ω,

$$\operatorname{cone}\Omega := \{\lambda\omega \in E \mid \lambda \geq 0, \omega \in \Omega\}.$$

2.1 Algebraic properties of convex sets

The next lemma recalls useful known properties of $\operatorname{cor}\Omega$, $\operatorname{icor}\Omega$, $\operatorname{acl}\Omega$ and $\operatorname{aff}\Omega$ for any nonempty, convex set $\Omega \subseteq E$ (see, e.g., Adán and Novo [3, Prop. 3 and 4] and Zălinescu [29, p. 3]).

Lemma 2.2 *Consider a nonempty, convex set* $\Omega \subseteq E$*. Then, the following hold:*

1° $\operatorname{cor}\Omega = \{x \in \Omega \mid \operatorname{cone}(\Omega - x) = E\}$.

2° $\operatorname{icor}\Omega = \{x \in \Omega \mid \operatorname{cone}(\Omega - x)$ *is a linear subspace of* $E\}$
$= \{x \in \Omega \mid \operatorname{cone}(\Omega - x) = \operatorname{cone}(\Omega - \Omega)\}$.

3° *If* Ω *is relatively solid, then* $\operatorname{icor}(\operatorname{acl}\Omega) = \operatorname{icor}\Omega$ *and* $\operatorname{acl}(\operatorname{icor}\Omega) = \operatorname{acl}\Omega = \operatorname{acl}(\operatorname{acl}\Omega)$.

4° *For all* $\overline{x} \in \operatorname{icor}\Omega$ *and all* $x \in \operatorname{acl}\Omega$*, we have* $[\overline{x}, x) \subseteq \operatorname{icor}\Omega$.

Remark 2.3 *According to Holmes [14, p. 9], any finite dimensional convex set in a linear space has a nonempty intrinsic core. However, it is known that an infinite dimensional convex set in a linear space can have empty intrinsic core. In view of Lemma 2.1 (2°), for any nonempty, relatively solid set* $\Omega \subseteq E$ *in a linear space* E*, we have* $\operatorname{cor}\Omega \neq \emptyset$ *if and only if* $\operatorname{aff}\Omega = E$*. Thus, we get a theorem (in which* E *has finite dimension and* Ω *is convex) by Holmes [14, p. 9] as a corollary of the above facts. Moreover, due to Holmes' remark [14, p. 9] we also know that the relative solidness of* Ω *is essential (also under convexity assumption on* Ω*).*

According to Adán and Novo [4, Def. 1], for any nonempty set $\Omega \subseteq E$, the vectorial closure of Ω is defined as

$$\operatorname{vcl}\Omega := \{x \in E \mid \exists\, v \in E\ \forall\, \delta > 0\ \exists\, \delta' \in (0, \delta] :\ x + \delta' v \in \Omega\}.$$

Remark 2.4 *Notice, for any nonempty set* $\Omega \subseteq E$*, we have* $\Omega \subseteq \operatorname{acl}\Omega \subseteq \operatorname{vcl}\Omega$*, which means that the vectorial closure is a weaker closure of algebraic type (see Adán and Novo [3, p. 643]). As already mentioned by Adán and Novo [4, p. 517], the vectorial closure* $\operatorname{vcl}\Omega$ *is exactly the algebraic closure* $\operatorname{acl}\Omega$ *for any convex set* $\Omega \subseteq E$*. However, for a nonconvex set, this result may fail, as pointed out in [3, Ex. 1].*

Because of the equality $\operatorname{vcl}\Omega = \operatorname{acl}\Omega$ for any convex set $\Omega \subseteq E$, in our upcoming results we will only deal with the algebraic closure notion for convex cones in E.

2.2 Algebraic properties of convex cones

Assume that $\mathbb{R}_+$ denote the nonnegative real numbers and $\mathbb{R}_{++}$ the positive real numbers. Recall that a cone $K \subseteq E$ (i.e., K is nonempty and $0 \in K = \mathbb{R}_+ \cdot K$) is nontrivial if $\{0\} \neq K \neq E$; pointed if $K \cap (-K) = \{0\}$; convex if $K + K = K$.

In the next lemma, we recall some important properties for the intrinsic core of a convex cone $K \subseteq E$ (see, e.g., Adán and Novo [4, p. 517], [3, Prop. 6, (ii)] and Popovici [25, p. 105]).

Lemma 2.5 *Assume that $K \subseteq E$ is a convex cone. Then, we have:*

1° $(\operatorname{icor} K) \cup \{0\}$ *is a convex cone in* E.

2° $\operatorname{icor} K = \operatorname{acl} K + \operatorname{icor} K = K + \operatorname{icor} K = \operatorname{icor}(\operatorname{icor} K)$.

3° $\operatorname{aff}(K - K) = \operatorname{aff} K = \operatorname{span} K = \operatorname{cone}(K - K)$.

For deriving some representations of the intrinsic core of a convex cone $K \subseteq E$, we will use the notion of the lineality space of K,

$$l(K) := K \cap (-K).$$

Notice that $l(K)$ is the largest linear subspace contained in K, while, in contrast, $\operatorname{aff} K$ is the smallest linear subspace containing K. Thus, we have the following bounds given by linear subspaces, $l(K) \subseteq K \subseteq \operatorname{aff} K$. Moreover, K is a linear subspace of E if and only if $K = l(K)$ if and only if $K = \operatorname{aff} K$.

The next lemma states useful properties of the sets $K \setminus l(K)$ and $\operatorname{icor} K$.

Lemma 2.6 *Assume that $K \subseteq E$ is a convex cone. Then, the following hold:*

1° $K \setminus l(K)$ *is a convex set.*

2° *For all* $\overline{x} \in K \setminus l(K)$ *and all* $x \in l(K)$ *we have* $[\overline{x}, x) \subseteq K \setminus l(K)$.

3° *If* $K \neq l(K)$, *then* $K \subseteq \operatorname{acl}(K \setminus l(K))$.

4° *If* $K \neq l(K)$, *then* $\operatorname{icor} K \subseteq K \setminus l(K)$.

5° *If* $K = l(K)$, *then* $\operatorname{icor} K = \operatorname{aff} K = l(K) = K$.

6° $K \neq l(K)$ *if and only if* $(\operatorname{icor} K) \cap l(K) = \emptyset$ *if any only if* $0 \notin \operatorname{icor} K$.

Proof:

1° Take $k^1, k^2 \in K \setminus l(K)$ and $\alpha \in (0, 1)$. Clearly, $x := \alpha k^1 + (1 - \alpha)k^2 \in K$ by the convexity of K. Assuming by the contrary that $x \in l(K)$, i.e., $-x \in K$, then $-k^1 \in \frac{1-\alpha}{\alpha} k^2 + \frac{1}{\alpha} K \subseteq K$, a contradiction to $k^1 \notin -K$.

2° Consider $x \in l(K)$ and take some $\overline{x} \in K \setminus l(K)$. Since K is convex and $x, \overline{x} \in K$, it follows $[\overline{x}, x] \subseteq K$. Assume by the contrary there exists $\tilde{x} \in (\overline{x}, x)$ with $\tilde{x} \in l(K)$. More precisely, we have $\tilde{x} = \alpha\overline{x} + (1-\alpha)x$ for some $\alpha \in (0, 1)$. Then, since $l(K)$ is a linear subspace and $x, \tilde{x} \in l(K)$, we have $\overline{x} = \frac{1}{\alpha}\tilde{x} - \frac{1-\alpha}{\alpha} x \in l(K)$, a contradiction to $\overline{x} \in K \setminus l(K)$.

3° Follows immediately by the property in assertion 2° of this lemma.

4° Assume that $K \neq l(K)$. Take some $k \in \operatorname{icor} K \subseteq K$. Assuming by the contrary that $k \in -K$, by Lemmas 2.2 (2°) and 2.5 (3°), we have $\operatorname{aff} K = \operatorname{cone}(K - K) = \operatorname{cone}(K - k) \subseteq \operatorname{cone}(K + K) = \operatorname{cone} K = K$, hence $K = l(K)$, a contradiction.

5° First, notice that $K = l(K)$ implies $K = \operatorname{aff} K$. Moreover, since also $K = \operatorname{aff}(K - K)$ and K is a convex cone, the equality $K = \operatorname{icor} K$ can easily be derived.

6° If $K \neq l(K)$, then $(\operatorname{icor} K) \cap l(K) = \emptyset$ by assertion 4° of this lemma. Conversely, suppose $(\operatorname{icor} K) \cap l(K) = \emptyset$. Assuming by the contrary that $K = l(K)$, we get $\operatorname{icor} K = l(K) = K \neq \emptyset$ by assertion 5° of this lemma, which is a contradiction.

Now, assume that $0 \notin \operatorname{icor} K$. Suppose by the contrary that $(\operatorname{icor} K) \cap l(K) \neq \emptyset$. Clearly, then assertion 4° of this lemma ensures that $K = l(K)$, hence $\operatorname{icor} K = K$ by assertion 5° of this lemma. Since K is a cone, we have $0 \in K = \operatorname{icor} K$, a contradiction.

□

Also for the algebraic interior of the convex cone K, we get similar properties.

Lemma 2.7 *Assume that $K \subseteq E$ is a convex cone. Then, the following hold:*

1° *If $K \neq E$, then $\operatorname{cor} K \subseteq K \setminus l(K)$.*

2° *If $K = l(K) \neq E$, then $\operatorname{cor} K = \emptyset$.*

3° *$K \neq E$ if and only if $(\operatorname{cor} K) \cap l(K) = \emptyset$ if and only if $0 \notin \operatorname{cor} K$.*

Proof:

1° Assume that $K \neq E$. Take some $k \in \operatorname{cor} K \subseteq K$. Suppose by the contrary that $k \in -K$. Then, by Lemma 2.2 (1°) and since K is a convex cone, we have $E = \operatorname{cone}(K - k) \subseteq \operatorname{cone}(K + K) = \operatorname{cone} K = K$, which is a contradiction to $K \neq E$.

2° It is a direct consequence of assertion 1° in this lemma.

3° Assume that $K \neq E$. Then, by assertion 1° of this lemma, we get $(\operatorname{cor} K) \cap l(K) = \emptyset$. Conversely, suppose that $(\operatorname{cor} K) \cap l(K) = \emptyset$. Assuming by the contrary that $K = E$, we infer $K = E = l(K)$ and $K = E = \operatorname{cor} K$, which is a contradiction.

As a consequence of $0 \in \operatorname{cor} K$ if and only if $K = \operatorname{cone} K = E$ by Lemma 2.2 (1°), we get the remaining equivalence.

□

Remark 2.8 *Notice that assertion 1° and parts of 3° in Lemma 2.7 are already stated in Bagdasar and Popovici [6, Lem. 5 (1°, 2°)] for the case $K \neq l(K)$.*

Let us consider a convex cone $K \subseteq E$. Assume that

$$K' := \{y' \in E' \mid \forall\, k \in K:\ y'(k) \geq 0\}$$

is the (algebraic) dual cone of K, and define

$$K'_+ := \{y' \in E' \mid \forall\, k \in K \setminus \{0\} : \ y'(k) > 0\} \subseteq K'.$$

In particular, the following set

$$K'_\oplus := \{y' \in E' \mid \forall\, k \in K \setminus l(K) : \ y'(k) > 0\}$$

will be of special interest in this article.

Remark 2.9 *In the book by Jahn [17], the set K'_+ is called the quasi-interior of the algebraic dual cone K'. In view of some known results in this topic, for instance the result by Bot, Grad and Wanka [11, Prop. 2.1.1] (in a topological setting), the name quasi-interior of K' seems to be a good choice for the set K'_+.*

Notice that the set $K'_\oplus$ already appears in the literature, for instance in Luc [21, p. 7], or Khan, Tammer and Zălinescu [18, p. 40] (in a topological setting), but usually without giving a name for this object.

The next lemma collects some useful relationships between a convex cone K and its dual cone K' (see, e.g., Hernández, Jiménez and Novo [13, Lem. 3.8], and Jahn [17, Lem. 1.27]).

Lemma 2.10 *Assume that $K \subseteq E$ is a convex cone. Then, we have:*

1° *K' is always algebraically closed.*

2° *If K is relatively solid, then $K' = ((\text{icor}\, K) \cup \{0\})'$.*

3° *If K is solid, then K' is pointed.*

4° *If $K'_+ \neq \emptyset$, then K is pointed.*

The following convex cone

$$K'' := \{x \in E \mid \forall\, x' \in K' : \ x'(x) \geq 0\},$$

which contains the convex cone $K \subseteq E$, will play an important role in our article.

The next result is a consequence of Remark 2.4 and the result by Adán and Novo [4, Prop. 2.2], and states an important relationship between the cones K and K''.

Proposition 2.11 *If $K \subseteq E$ is a relatively solid, convex cone, then $\text{acl}\, K = K''$.*

It is convenient to introduce the following two sets

$$K''_+ := \{x \in E \mid \forall\, x' \in K' \setminus \{0\} : \ x'(x) > 0\};$$
$$K''_\oplus := \{x \in E \mid \forall\, x' \in K' \setminus l(K') : \ x'(x) > 0\}.$$

Remark 2.12 *Notice that $K''_+ \subseteq K''_\oplus \cup K'' \subseteq E$. The fact in Proposition 2.11 motivates the use of the name "algebraic (positive) bipolar cone" or "algebraic double dual cone" for the set K'' (in analogy to the definition of bipolar cones / double dual cones of convex cones in a topological setting). It is interesting to mention that Aliprantis and Tourky [5, Sec. 2.2] call the convex cone K as a "wedge", its dual cone K' as "dual wedge" and the set K'' as "double dual wedge" while the elements of $K \cap K''_+$ are called K-strictly positive.*

Using Popovici's interesting result [25, Lem. 2.1], we get the next proposition, which will play a key role for deriving representations of generalized interiors of K and K'.

Proposition 2.13 *Assume that $K \subseteq E$ is a relatively solid, convex cone. Then, the following assertions hold:*

1° *For any $k \in \operatorname{icor} K$,*

$$\operatorname{icor} K = \bigcup_{\alpha>0}(\alpha k + K) = \bigcup_{\alpha>0}(\alpha k + \operatorname{acl} K) = \bigcup_{\alpha>0}(\alpha k + K'').$$

2° *For any $k \in \operatorname{icor} K$,*

$$K'' = \operatorname{acl} K = \bigcap_{\beta<0}(\beta k + \operatorname{icor} K).$$

3° *For any $k \in \operatorname{icor} K$,*

$$\operatorname{aff} K = \bigcup_{\alpha>0}(\alpha k - K) = \bigcup_{\alpha>0}(\alpha k - \operatorname{acl} K) = \bigcup_{\alpha>0}(\alpha k - K'').$$

Proof:

1° The equality $\operatorname{icor} K = \bigcup_{\alpha>0}(\alpha k + K)$ is proven by Popovici [25, Lem. 2.1]. By taking a look on the proof in [25, Lem. 2.1], one can also derive $\operatorname{icor} K = \bigcup_{\alpha>0}(\alpha k + \operatorname{acl} K)$. The third equality follows directly by applying Proposition 2.11.

2° By Popovici [25, Lem. 2.1], we have $\operatorname{vcl} K = \bigcap_{\beta<0}(\beta k + \operatorname{icor} K)$. Thus, the result follows by using the equality $\operatorname{vcl} K = \operatorname{acl} K$ and the fact in Proposition 2.11.

3° We are going to prove $\operatorname{aff} K = \bigcup_{\alpha>0}(\alpha k - K)$. The inclusion "$\supseteq$" holds since $\{k\} \cup K \subseteq \operatorname{aff} K$ and $\operatorname{aff} K$ is a linear subspace of E. For proving the converse inclusion "$\subseteq$", let $x \in \operatorname{aff} K = \operatorname{aff}(K - K)$. Since $k \in \operatorname{icor} K$, there exists a real number $\varepsilon > 0$ such that $k + \varepsilon(-x) \in K$, which yields

$$x \in \frac{k}{\varepsilon} - \frac{1}{\varepsilon}K \subseteq \frac{k}{\varepsilon} - K \subseteq \bigcup_{\alpha>0}(\alpha k - K).$$

Notice that

$$\operatorname{aff} K = \bigcup_{\alpha>0}(\alpha k - K) \subseteq \bigcup_{\alpha>0}(\alpha k - \operatorname{acl} K) \subseteq \operatorname{aff} K.$$

The remaining third equality follows now by using Proposition 2.11.

□

2.3 Properties of the canonical embedding function

Let us define the second algebraic dual space of E by $E'' := (E')'$. The linear map $J_E : E \to E''$, defined, for any $x \in E$, by

$$J_E(x) : E' \to \mathbb{R},\ x' \mapsto J_E(x)(x') := x'(x),$$

is known as the canonical embedding.

Remark 2.14 *The map J_E is always linear and injective. As mentioned by Holmes [14, p. 3], J_E is surjective if and only if E has finite dimension.*

In order to present further properties of J_E in the next lemma, we need, for any set $\Omega \subseteq E$, the image of J_E over Ω,

$$J_E[\Omega] := \{J_E(x) \in E'' \mid x \in \Omega\}.$$

Proposition 2.15 *Assume that $K \subseteq E$ is a convex cone. Then,*

1° *We have* $\operatorname{acl}(K')' = (K')' \supseteq J_E[K''] \supseteq J_E[\operatorname{acl} K]$ *and* $(K')'_+ \supseteq J_E[K''_+]$ *as well as* $(K')'_\oplus \supseteq J_E[K''_\oplus]$.

Now, if E has finite dimension, then the following assertions hold:

2° $(K')' = J_E[K''] = J_E[\operatorname{acl} K] = \operatorname{acl} J_E[K] = \operatorname{acl} J_E[\operatorname{icor} K]$.

3° $\operatorname{icor}(K')' = \operatorname{icor} J_E[\operatorname{acl} K] = \operatorname{icor} J_E[K] = J_E[\operatorname{icor} K]$.

4° $(K')'_+ = J_E[K''_+]$ *and* $(K')'_\oplus = J_E[K''_\oplus]$.

Proof:

1° Clearly, the set $(K')'$ is algebraically closed, and we have

$$\begin{aligned}(K')' &= \{y'' \in E'' \mid \forall\, k' \in K' :\ y''(k') \geq 0\}\\ &\supseteq J_E[\{y \in E \mid \forall\, k' \in K' :\ k'(y) \geq 0\}] = J_E[K''].\end{aligned}$$

Moreover, since $\operatorname{acl} K \subseteq K''$ we have $J_E[K''] \supseteq J_E[\operatorname{acl} K]$.

The proofs of the inclusions $(K')'_+ \supseteq J_E[K''_+]$ and $(K')'_\oplus \supseteq J_E[K''_\oplus]$ are similar to the proof of $(K')' \supseteq J_E[K'']$.

2°, 3° At first, notice, if E has finite dimension, then the convex cone K is relatively solid by Remark 2.3. The equalities $(K')' = J_E[K''] = J_E[\operatorname{acl} K]$ follow by 1° of this lemma, by the surjectivity of J_E (since E has finite dimension), and by Proposition 2.11. In view of Hernández, Jiménez and Novo [13, Prop. 3.17], we have

$$J_E[\operatorname{icor} S] = \operatorname{icor} J_E[S] \quad \text{if } S \subseteq E \text{ is a relatively solid, convex set.} \tag{1}$$

Then, using (1) for $S := \operatorname{acl} K$, and Lemma 2.2 (3°), we have $J_E[\operatorname{icor} K] = J_E[\operatorname{icor}(\operatorname{acl} K)] = \operatorname{icor} J_E[\operatorname{acl} K] = \operatorname{icor}(K')'$. Due to (1) for $S := K$, we infer $J_E[\operatorname{icor} K] = \operatorname{icor} J_E[K]$. Now, we get $(K')' = \operatorname{acl}(K')' = \operatorname{acl} J_E[\operatorname{icor} K]$ by $J_E[\operatorname{icor} K] = \operatorname{icor}(K')'$ and Lemmas 2.2 (3°) and 2.10 (1°). Moreover, recalling that $J_E[\operatorname{icor} K] = \operatorname{icor} J_E[K]$ and applying Lemma 2.2 (3°) (notice that $J_E[K]$ is a relatively solid, convex set), we have $\operatorname{acl} J_E[\operatorname{icor} K] = \operatorname{acl}(\operatorname{icor} J_E[K]) = \operatorname{acl} J_E[K]$.

4° Since E has finite dimension, the canonical embedding J_E is surjective. Then, similar ideas as in the proof of 1° of this lemma show that $(K')'_+ = J_E[K''_+]$ and $(K')'_\oplus = J_E[K''_\oplus]$.

□

2.4 Separation theorems in linear spaces using algebraic notions

In order to prove some of our main theorems, we will apply algebraic versions of separation results for convex sets in linear spaces. The following result is well-known and can be found in a similar form in Holmes [14, p. 15], Jahn [15, 17, Th. 3.14], Kirsch, Warth and Werner [19, 1.8 Satz], and Werner [27, Cor. 3.1.10].

Proposition 2.16 *Assume that $\Omega^1, \Omega^2 \subseteq E$ are nonempty, convex sets, and Ω^1 is solid (i.e., $\operatorname{cor} \Omega^1 \neq \emptyset$). Then, $\Omega^2 \cap \operatorname{cor} \Omega^1 = \emptyset$ if and only if there is $x' \in E' \setminus \{0\}$ and a real number α with*

$$x'(\omega^2) \leq \alpha \leq x'(\omega^1) \qquad \text{for all } \omega^1 \in \operatorname{acl} \Omega^1 \text{ and } \omega^2 \in \operatorname{acl} \Omega^2; \tag{2}$$

$$\alpha < x'(\omega^1) \qquad \text{for all } \omega^1 \in \operatorname{cor} \Omega^1. \tag{3}$$

Remark 2.17 *We like to point out two important facts, which are not so known in the literature related to algebraic versions of separation theorems, but already used in the book by Werner [27, Cor. 3.1.10]:*

(*a*) *Consider two points $x, \overline{x} \in E$ and $x' \in E'$. Assume that, there is $\beta \in \mathbb{R}$ such that for all $\gamma \in (0, 1]$ we have $x'(\gamma x + (1-\gamma)\overline{x}) \leq \beta$. Then, observing*

$$\gamma x'(x) + (1-\gamma)x'(\overline{x}) = x'(\gamma x + (1-\gamma)\overline{x}) \leq \beta,$$

we get for $\gamma \to 0$ the condition $x'(\overline{x}) \leq \beta$, by using the continuity of the linear operations (addition and scalar multiplication) in the real linear topological space $\mathbb{R}$ (both $x'(x)$ and $x'(\overline{x})$ are fixed values in $\mathbb{R}$). This justifies to write the algebraic closures $\operatorname{acl} \Omega^1$ and $\operatorname{acl} \Omega^2$ instead of Ω^1 and Ω^2 in (2) of Proposition 2.16.

(*b*) *Take $x \in E$ and two linear maps $x', \overline{x}' \in E'$. Suppose that there is $\beta \in \mathbb{R}$ such that for all $\gamma \in (0, 1]$ we have $(\gamma x' + (1-\gamma)\overline{x}')(x) \leq \beta$. Now, due to*

$$\gamma x'(x) + (1-\gamma)\overline{x}'(x) = (\gamma x' + (1-\gamma)\overline{x}')(x) \leq \beta,$$

the limit $\gamma \to 0$ yields $\overline{x}'(x) \leq \beta$ (by using similar arguments as above).

Remark 2.18 *Separation theorems for two convex sets in a linear space, where one of them is assumed to be relatively solid, are already given in the literature, for instance by Adán and Novo [2–4], Holmes [14, p. 21], Khan, Tammer and Zălinescu [18, p. 259], and Werner [27, Th. 3.1.12]. In our article, we are in particular interested in separation results where a (relatively) solid convex cone is* *involved. Notice that the proof of Theorem 3.6 as well as of our main results in Theorems 3.1 and 4.1 are using separation techniques shown in Theorem 2.21 and Corollary 2.23.*

The next result is known, see Werner [27, Th. 3.1.12], but we give a different and more simple proof using the well-known support theorem by Holmes [14, p. 21].

Proposition 2.19 *Assume that $\Omega \subseteq E$ is a relatively solid, convex set, and $x \in E$. Then, $x \notin \operatorname{icor}\Omega$ if and only if there is $x' \in E' \setminus \{0\}$ and $\alpha \in \mathbb{R}$ with*

$$x'(x) \leq \alpha \leq x'(\omega) \qquad \text{for all } \omega \in \operatorname{acl}\Omega; \tag{4}$$

$$\alpha < x'(\omega) \qquad \text{for all } \omega \in \operatorname{icor}\Omega. \tag{5}$$

Proof: First, assume that $x \notin \operatorname{icor}\Omega$. By Holmes [14, p. 21], we get $x' \in E' \setminus \{0\}$ and $\alpha := x'(x)$ such that (5) holds. Due to the property $\operatorname{acl}(\operatorname{icor}\Omega) = \operatorname{acl}\Omega$ in view of Lemma 2.2 (3°), we conclude (4) taking into account Remark 2.17.

Conversely, the validity of both (4) and (5) for some $x' \in E' \setminus \{0\}$ implies $x \notin \operatorname{icor}\Omega$. □

The following separation property is well-known, however we present a simple proof that is based on the separation theorem for relatively solid, convex sets.

Corollary 2.20 *The dual space E' separates elements in the linear space E (i.e., two different elements in E can be separated by a hyperplane).*

Proof: Take two points $x, \overline{x} \in E$ with $x \neq \overline{x}$. Consider the convex set $\Omega := \{\overline{x}\}$. It is easy to see that $x \notin \operatorname{icor}\Omega = \{\overline{x}\}$. Then, applying Proposition 2.19, there are $x' \in E' \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that $x'(x) \leq \alpha < x'(\overline{x})$. □

The following theorem will play a key role for deriving our main results related to the intrinsic core of convex cones.

Theorem 2.21 *Assume that $K \subseteq E$ is a relatively solid, convex cone and $x \in \operatorname{acl} K$. Then, $x \notin \operatorname{icor} K$ if and only if there is $x' \in K' \setminus l(K')$ with*

$$x'(x) = 0 \leq x'(k) \qquad \text{for all } k \in \operatorname{acl} K; \tag{6}$$

$$0 < x'(k) \qquad \text{for all } k \in \operatorname{icor} K. \tag{7}$$

Proof: Assume that $x \notin \operatorname{icor} K$. By Proposition 2.19, there is $x' \in E' \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that (4) and (5) are satisfied.

Let us first show that $\alpha = 0$. Indeed, since $x \in \operatorname{acl} K$, $0 \in K$ and $\mathbb{R}_+ \cdot \{x\} \subseteq \operatorname{acl} K$, we get by (4) that $x'(x) \leq \alpha \leq 0 \leq x'(x)$, which means $x'(x) = \alpha = 0$.

Now, we show that $x' \in K' \setminus l(K')$. By (4) (with $\alpha = 0$ and $\Omega := K$) we directly get $x' \in K'$. Assuming by the contrary that $x' \in l(K')$, then $x'(k) = 0$ for all $k \in K$, a contradiction to (5) (with $\alpha = 0$ and $\Omega := K$).

We conclude that $x' \in K' \setminus l(K')$ satisfies (6) and (7).

Clearly, under the validity of (6) and (7) one has $x \notin \operatorname{icor} K$. □

Notice that for $x \in E \setminus (\operatorname{acl} K)$ the result given in Theorem 2.21 must not hold, as the following example shows.

Example 2.22 *Consider the linear space $E := \mathbb{R}^2$ and the relatively solid, convex cone*

$$K := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\} = \operatorname{icor} K = \operatorname{acl} K = l(K).$$

Notice that K is not solid and also not pointed. Consider the point $x := (-1, 0) \in E \setminus (\operatorname{acl} K)$. Assume by the contrary that there is $x' \in E' \setminus \{0\}$ such that

$$x'(x) = 0 < x'(k) \qquad \text{for all } k \in \operatorname{icor} K = K.$$

This directly contradicts the conditions $x'(0) = 0$ and $0 \in K$. Thus, the assumption $x \in \operatorname{acl} K$ in Theorem 2.21 is essential.

Corollary 2.23 *Assume that $K \subseteq E$ is a solid, convex cone and $x \in \operatorname{acl} K$. Then, $x \notin \operatorname{cor} K$ if and only if there is $x' \in K' \setminus \{0\}$ with*

$$\begin{aligned} x'(x) = 0 \leq x'(k) &\qquad \text{for all } k \in \operatorname{acl} K; \\ 0 < x'(k) &\qquad \text{for all } k \in \operatorname{cor} K. \end{aligned}$$

The case to separate an arbitrarily point $x \in E \setminus \operatorname{cor} K$ from the solid, convex cone $K \subseteq E$ is studied in the next result.

Theorem 2.24 *Assume that $K \subseteq E$ is a solid, convex cone and $x \in E$. Then, $x \notin \operatorname{cor} K$ if and only if there is $x' \in K' \setminus \{0\}$ with*

$$x'(x) \leq 0 \leq x'(k) \qquad \text{for all } k \in \operatorname{acl} K; \tag{8}$$

$$0 < x'(k) \qquad \text{for all } k \in \operatorname{cor} K. \tag{9}$$

Proof: Suppose that $x \notin \operatorname{cor} K$. We consider two cases:

Case 1: Assume that $x \in \operatorname{acl} K$. The result follows by Corollary 2.23.

Case 2: Assume that $x \in E \setminus \operatorname{acl} K$. Take some $\overline{k} \in \operatorname{cor} K$. It is not hard to check that there is $\tilde{x} \in (x, \overline{k})$ such that $\tilde{x} \in (\operatorname{acl} K) \setminus \operatorname{cor} K$. By Case 1, there is $x' \in K' \setminus \{0\}$ such that

$$x'(\tilde{x}) = 0 \leq x'(k) \qquad \text{for all } k \in \operatorname{acl} K; \tag{10}$$

$$0 < x'(k) \qquad \text{for all } k \in \operatorname{cor} K. \tag{11}$$

We now show that $x'(x) \leq 0$. Assume by the contrary that $x'(x) > 0$. Clearly, since $x'(\overline{k}) > 0$ by (11), then $x'(y) > 0$ for all $y \in [x, \overline{k}]$. Thus, in particular for $\tilde{x} \in (x, \overline{k})$ we have $x'(\tilde{x}) > 0$, a contradiction to (10).

We conclude that (8) and (9) hold.

Notice that the validity of both (8) and (9) for some $x' \in K' \setminus \{0\}$ implies $x \notin \operatorname{cor} K$. $\square$

3 New algebraic properties of convex cones

In this section, we derive new algebraic properties of convex cones. We start by presenting characterizations of the core and the intrinsic core of a convex cone in E in the next theorem.

Theorem 3.1 *Assume that $K \subseteq E$ is a convex cone. Then, the following hold:*

1° *If K is solid, then*

$$\operatorname{cor} K = \{y \in E \mid \forall\, y' \in K' \setminus \{0\} :\ y'(y) > 0\} = K''_{+} = \operatorname{cor} K''.$$

2° *If K is relatively solid and $K \subseteq \Omega \subseteq \operatorname{acl} K$, then*

$$\operatorname{icor} K = \{y \in \Omega \mid \forall\, y' \in K' \setminus l(K') :\ y'(y) > 0\} = K''_{\oplus} \cap \Omega = \operatorname{icor} K''.$$

Proof:

1° The equality $\operatorname{cor} K = K''_{+}$ is well-known (see, e.g., Jahn [17, Lem. 3.21, (b)]). The remaining equality $\operatorname{cor} K'' = \operatorname{cor} K$ is a consequence of Lemma 2.2 (3°) and Proposition 2.11.

2° We start by proving the equality $\operatorname{icor} K = K''_{\oplus} \cap \Omega$.

Consider $k \in \operatorname{icor} K$. Suppose by the contrary, it exists $y' \in K' \setminus l(K')$ with $y'(k) = 0$. Notice, since $y' \in K'$, for all $x \in K$, we have $y'(x) \geq 0$, and $y'(\overline{x}) > 0$ for some $\overline{x} \in K$. Since $k \in \operatorname{icor} K$ and $\overline{x} \in K \subseteq \operatorname{acl} K$, there exists $\delta > 0$ such that $k' := k - \delta\overline{x} \in K$. Thus, $y'(k') = y'(k) - \delta y'(\overline{x}) = -\delta y'(\overline{x}) < 0$, a contradiction to $y'(k') \geq 0$.

Now, assume $\operatorname{icor} K \neq \emptyset$. Take some $y \in \Omega$ which satisfies $y'(y) > 0$ for every $y' \in K' \setminus l(K')$. Assume by the contrary that $y \notin \operatorname{icor} K$. Then, by Theorem 2.21, there is $x' \in K' \setminus l(K')$ with $x'(y) = 0 \leq x'(k)$ for all $k \in K$, a contradiction.

The remaining equality $\operatorname{icor} K = \operatorname{icor} K''$ follows by Lemma 2.2 (3°) and Proposition 2.11.

□

In preparation of the next lemma, which states important properties of K', K'', K''_{+} and $K''_{\oplus}$, respectively, it is convenient to introduce, for any nonempty set $\Omega' \subseteq E'$,

$$(\Omega')^{\perp} := \{x \in E \mid \forall\, x' \in \Omega' :\ x'(x) = 0\}.$$

Lemma 3.2 *Assume that $K \subseteq E$ is a convex cone. The following assertions hold:*

1° *If K is relatively solid, then $K''_{\oplus} \neq \emptyset$.*

2° $K \subseteq K'' \subseteq \{x \in E \mid \forall\, x' \in l(K') :\ x'(x) = 0\} = (l(K'))^{\perp}$.

3° *If K is relatively solid and $K' = l(K')$, then $K = l(K) = \operatorname{acl} K = (K')^{\perp} = K''$.*

4° *If K' is not pointed, then $K''_{+} = \emptyset$.*

5° *$K' = l(K')$ if and only if $K''_{\oplus} = E$ if and only if $K''_{\oplus}$ is a cone.*

6° *$K' = \{0\}$ if and only if $K''_{+} = E$ if and only if $K'' = E$ if and only if K''_{+} is a cone.*

7° *If K is relatively solid, then $K' = \{0\}$ if and only if $K = E$.*

Proof:

1° By Theorem 3.1 (2°), we have $\emptyset \neq \operatorname{icor} K = K''_\oplus \cap K \subseteq K''_\oplus$.

2° Clearly, $K \subseteq K''$. Take some $x \in K''$. Then, in particular we have $x'(x) \geq 0$ for all $x' \in l(K') \subseteq K'$. Hence, $(-x')(x) \geq 0$ for all $x' \in l(K')$ as well, i.e., $x \in (l(K'))^\perp$.

3° By Theorem 3.1 (2°) and due to $K''_\oplus = E$ (see also 5° of this lemma), we have $\operatorname{icor} K = K''_\oplus \cap K = K$. In view of Lemma 2.6 (6°), we infer $K = l(K)$. Moreover, it is easy to see that we have $(K')^\perp = K''$, while by Proposition 2.11, we get $K'' = \operatorname{acl} K$. Thus, we conclude $\operatorname{acl} K \supseteq K = l(K) = \operatorname{aff} K \supseteq \operatorname{acl} K = K'' = (K')^\perp$, hence the assertion follows directly.

4° Suppose that K' is not pointed. Assuming by the contrary that $K''_+ \neq \emptyset$, then there exist $x \in K''_+$ and $x' \in l(K') \subseteq K'$ with $x'(x) > 0$. In contrast, in view of 2° of this lemma, we have $K''_+ \subseteq K'' \subseteq (l(K'))^\perp$, hence $x'(x) = 0$, a contradiction.

5° Obviously, $K' = l(K')$ yields $K''_\oplus = E$. Now, assume that $K''_\oplus = E$. Suppose by the contrary that there exists $x' \in K' \setminus l(K')$. Then, for any $x \in K''_\oplus = E$, we have $x'(x) > 0$. This in particular means that $x'(0) > 0$, a contradiction.

Clearly, $K''_\oplus = E$ is a cone. Conversely, assume that $K''_\oplus$ is a cone, which in particular means that $0 \in K''_\oplus$. Taking a look on the definition of $K''_\oplus$, we infer $K' = l(K)$.

6° The proof is quite similar to the proof of 5° by replacing $K''_\oplus$ by K''_+ as well as $l(K')$ by $\{0\}$. Furthermore, notice that $K''_+ = E$ implies $K'' = E$. Conversely, by $K'' = E$ we get $E = (K')^\perp$, hence $K' = \{0\}$ can easily be derived.

7° It is easy to observe by the definition of K' that $K = E$ implies $K' = \{0\}$ (actually without assuming the relative solidness of K). Now, assume that $K' = \{0\}$. Observing that $K' = l(K')$, in view of assertion 3° of this lemma, we conclude $K = K''$. Due to the assumption $K' = \{0\}$, we can also infer that $K'' = E$ by assertion 6° of this lemma. Thus, $K = E$ holds true.

□

Remark 3.3 *Notice that Lemma 3.2 (7°) extends a result (if $K \neq E$ is a solid, convex cone, then $K' \neq \{0\}$) derived by Holmes [14, p. 18].*

In order to state relationships between the cones K, K' and K'' in the next lemma, it is convenient to introduce the following sets:

$$\begin{aligned} C_{E'} &:= (K' \setminus l(K')) \cup \{0\}, \\ C_E &:= \{x \in E \mid \forall\, x' \in C_{E'} :\ x'(x) \geq 0\}, \\ D_E &:= \{x \in E \mid \forall\, x' \in l(K') \setminus \{0\} :\ x'(x) > 0\}. \end{aligned}$$

Lemma 3.4 *Assume that $K \subseteq E$ is a convex cone. The following assertions hold:*

1° *$C_{E'}$ is a pointed, convex cone in E'.*

2° *$K''_+ \subseteq K'' = C_E \cap (l(K'))^\perp \subseteq C_E$.*

3° $K''_+ = K''_\oplus \cap D_E \subseteq K''_\oplus = (C_E)_+ \subseteq C_E$.

4° *$K' \neq l(K')$ or $K' = \{0\}$ if and only if $K''_\oplus \subseteq K''$ if and only if $C_E \subseteq K''$.*

5° *If $K''_\oplus \neq \emptyset$, then $K' \neq l(K')$ if and only if $K''_\oplus \subseteq K'' \setminus l(K'')$.*

6° *If K is relatively solid, then K' is pointed if and only if $K''_+ \supseteq K''_\oplus$ if and only if $K''_+ \neq \emptyset$.*

Proof:

1° The proof of this assertion is easy by using Lemma 2.6 (1°).

2° The inclusion $K''_+ \subseteq K''$ is clear. Moreover, since $K' = C_{E'} \cup l(K')$, we get that $K'' = C_E \cap (l(K'))^\perp \subseteq C_E$.

3° If K' is pointed, then $K''_+ = K''_\oplus$ as well as $K''_\oplus \cap D_E = K''_\oplus \cap E = K''_\oplus$. Otherwise, if K' is not pointed, then $K''_+ = \emptyset$ in view of Lemma 3.2 (4°), as well as $K''_\oplus \cap D_E = K''_\oplus \cap \emptyset = \emptyset$ in view of the definition of D_E. We conclude that $K''_+ = K''_\oplus \cap D_E$. Moreover, $K''_\oplus = (C_E)_+$ and the other inclusions are obvious.

4° Assume that $K' \neq l(K')$ or $K' = \{0\}$. We prove that $C_E \subseteq K''$. Consider some $c \in C_E$. Now, we know that $x'(c) \geq 0$ for all $x' \in C_{E'}$. Thus, it is enough to show that $x'(c) \geq 0$ for all $x' \in l(K')$ in order to conclude $c \in K''$. Take some $\overline{x}' \in l(K')$. By Lemma 2.6 (3°) there is $\tilde{x}' \in K' \setminus l(K')$ such that $[\tilde{x}', \overline{x}') \subseteq K' \setminus l(K')$. Now, for any $\alpha \in (0, 1]$, we have $\alpha\tilde{x}'(c) + (1-\alpha)\overline{x}'(c) = (\alpha\tilde{x}' + (1-\alpha)\overline{x}')(c) \geq 0$, which yields for $\alpha \to 0$ as requested $\overline{x}'(c) \geq 0$ (see also Remark 2.17 (b)).

Now, suppose that $C_E \subseteq K''$. Due to assertion 3° of this lemma, we have $K''_\oplus = (C_E)_+ \subseteq C_E \subseteq K''$.

Finally, let $K''_\oplus \subseteq K''$ be satisfied. Assume by the contrary that $K' = l(K')$ and $K' \neq \{0\}$. By Lemma 3.2 (5°) we have $K''_\oplus = E$, hence $K'' = E$. Consequently, by Lemma 3.2 (6°), we get $K' = \{0\}$, which is a contradiction.

5° Assuming $K' \neq l(K')$, we get $K''_\oplus \subseteq K''$ by 4° of this lemma. More precisely, we have $K''_\oplus \subseteq K'' \setminus l(K'')$, since otherwise $x'(k) = 0$ for some $x' \in K' \setminus l(K')$ and $k \in K''_\oplus$, which is a contradiction.

Assume that $K''_\oplus \subseteq K'' \setminus l(K'') \subseteq K''$. Thus, in view of 4° of this lemma, we obtain $K' \neq l(K')$ or $K' = \{0\}$. Suppose by the contrary that $K' = \{0\}$. Then, $K'' = E = l(K'')$ by Lemma 3.2 (6°), which contradicts $K'' \setminus l(K'') \neq \emptyset$. Thus, we conclude $K' \neq l(K')$.

6° Assume that K is relatively solid. If K' is pointed, then $K''_+ = K''_\oplus \cap E = K''_\oplus$ by assertion 3°. Now, let $K''_+ \supseteq K''_\oplus$ be satisfied. Assuming by the contrary that K' is not pointed, then $K''_+ = \emptyset$ by Lemma 3.2 (4°), while $K''_\oplus \neq \emptyset$ by Lemma 3.2 (1°), a contradiction to $K''_+ \supseteq K''_\oplus$.

The second equivalence is a consequence of the first equivalence taking into account Lemma 3.2 (1°, 4°).

□

The next main theorem gives further characterizations of the core, the intrinsic core, the algebraic closure, and the affine hull of a relatively solid, convex cone in E.

Theorem 3.5 *Assume that $K \subseteq E$ is a relatively solid, convex cone. Then, the following assertions hold:*

1° *If either $K' \neq l(K')$ or $K' = \{0\}$, then*

$$\operatorname{icor} K = \{x \in E \mid \forall\, x' \in K' \setminus l(K') : \; x'(x) > 0\} = K''_{\oplus}.$$

2° *For any $k \in \operatorname{icor} K$,*

$$\operatorname{icor} K = \bigcup_{\alpha > 0} \{x \in E \mid \forall\, x' \in K' : \; x'(x) \geq \alpha x'(k)\}.$$

3° *If either $K' \neq l(K')$ or $K' = \{0\}$, then, for any $k \in \operatorname{icor} K$,*

$$\operatorname{icor} K = \bigcup_{\alpha > 0} \{x \in E \mid \forall\, x' \in K' \setminus l(K') : \; x'(x) \geq \alpha x'(k)\}.$$

4° *If either $K' \neq l(K')$ or $K' = \{0\}$, then, for any $k \in \operatorname{icor} K$,*

$$\operatorname{acl} K = \bigcap_{\beta < 0} \{x \in E \mid \forall\, x' \in K' \setminus l(K') : \; x'(x) > \beta x'(k)\}.$$

5° *For any $k \in \operatorname{icor} K$,*

$$\operatorname{aff} K = \bigcup_{\alpha > 0} \{x \in E \mid \forall\, x' \in K' : \; x'(x) \leq \alpha x'(k)\}.$$

Proof:

1° By Theorem 3.1 (2°) applied for $\Omega := \operatorname{acl} K$, Lemma 3.4 (4°), and Proposition 2.11, we have $\operatorname{icor} K = K''_{\oplus} \cap \operatorname{acl} K = K''_{\oplus} \cap K'' = K''_{\oplus}$.

2° Take some $k \in \operatorname{icor} K$. By Proposition 2.13 (1°), we infer

$$\begin{aligned}\operatorname{icor} K = \bigcup_{\alpha > 0} (\alpha k + K'') &= \bigcup_{\alpha > 0} (\alpha k + \{x \in E \mid \forall\, x' \in K' : \; x'(x) \geq 0\}) \\ &= \bigcup_{\alpha > 0} \{x \in E \mid \forall\, x' \in K' : \; x'(x) \geq \alpha x'(k)\}.\end{aligned}$$

3° Consider $k \in \operatorname{icor} K$ and assume that either $K' \neq l(K')$ or $K' = \{0\}$. By assertion 2° of this theorem, and due to $K' \setminus l(K') \subseteq K'$, we have

$$\begin{aligned}\operatorname{icor} K &= \bigcup_{\alpha > 0} \{x \in E \mid \forall\, x' \in K' : \; x'(x) \geq \alpha x'(k)\} \\ &\subseteq \bigcup_{\alpha > 0} \{x \in E \mid \forall\, x' \in K' \setminus l(K') : \; x'(x) \geq \alpha x'(k)\}. \qquad (12)\end{aligned}$$

Moreover, in view of assertion 1° of this theorem, for $k \in \operatorname{icor} K$, we have $x'(k) > 0$ for all $x' \in K' \setminus l(K')$. Thus, an upper set for all sets involved in (12) is given by $K''_{\oplus}$. Now, Lemma 3.2 (2°) and Lemma 3.4 (4°) yield

$K''_{\oplus} \subseteq K'' \subseteq (l(K'))^{\perp}$. We are going to show the reverse inclusion in (12). Consider some $x \in E$ and $\alpha > 0$ such that for all $x' \in K' \setminus l(K')$ we have $x'(x) \geq \alpha x'(k)$. It remains to show that $x'(x) \geq \alpha x'(k)$ for all $x' \in l(K')$ and $\alpha > 0$. The above analysis shows that $x, k \in (l(K'))^{\perp}$, hence, for any $x' \in l(K')$ and $\alpha > 0$, we have $x'(x) = 0 = \alpha x'(k)$.

Consequently, the inclusion "$\subseteq$" in (12) is actually an equality.

4° Take some $k \in \operatorname{icor} K$. Due to Proposition 2.13 (2°) and assertion 1° of this theorem, we have

$$\begin{aligned}\operatorname{acl} K &= \bigcap_{\beta<0} (\beta k + \operatorname{icor} K)\\ &= \bigcap_{\beta<0} (\beta k + \{x \in E \mid \forall\, x' \in K' \setminus l(K') :\ x'(x) > 0\})\\ &= \bigcap_{\beta<0} \{x \in E \mid \forall\, x' \in K' \setminus l(K') :\ x'(x) > \beta x'(k)\}.\end{aligned}$$

5° By Proposition 2.13 (3°), for any $k \in \operatorname{icor} K$, we have

$$\operatorname{aff} K = \bigcup_{\alpha>0} (\alpha k - K'') = \bigcup_{\alpha>0} \{x \in E \mid \forall\, x' \in K' :\ x'(x) \leq \alpha x'(k)\}.$$

□

Using the separation results given in Theorem 2.21, we can derive further properties for the convex cone K.

Theorem 3.6 *The following assertions hold:*

1°

$$E \setminus \{0\} = \{x \in E \mid \exists\, x' \in E' :\ x'(x) \neq 0\}.$$

2° *If $K \subseteq E$ is a relatively solid, algebraically closed, convex cone, then*

$$K \setminus l(K) = \{x \in K \mid \exists\, x' \in K' \setminus l(K') :\ x'(x) > 0\}.$$

Proof:

1° This result, which is a direct consequence of Corollary 2.20, is well-known.

2° The inclusion "$\supseteq$" is obvious. Let us show the reverse inclusion "$\subseteq$". Assume by the contrary that there is $x \in K \setminus l(K)$ such that $x'(x) = 0$ for every $x' \in K' \setminus l(K')$. We consider two cases:

Case 1: Let $x \in \operatorname{icor} K$. In view of Lemma 2.6 (6°), we have $0 \in K \setminus \operatorname{icor} K$. By the separation condition in Theorem 2.21, there is $x' \in K' \setminus l(K')$ with $x'(0) = 0 < x'(k)$ for all $k \in \operatorname{icor} K$. In particular, we get $0 < x'(x)$, which is a contradiction.

Case 2: Let $x \notin \operatorname{icor} K$. Since K is relatively solid, we can fix some $\overline{x} \in \operatorname{icor} K$. Now, observe that $v := -x - \overline{x} \in \operatorname{span}\{x, \overline{x}\} \subseteq \operatorname{aff} K$. We are going to show that there is $y \in (-x, \overline{x})$ with $y \in K \setminus \operatorname{icor} K$.

Since $\overline{x} \in \operatorname{icor} K$ and $v \in \operatorname{aff} K$, there is $\varepsilon > 0$ such that $\overline{x} + \varepsilon v \in K$. Clearly, since $-x \notin K$, we must have $\varepsilon \in (0, 1)$. Furthermore, due to $-x \notin K = \operatorname{acl} K$ and the convexity of K, there must be a $\overline{\varepsilon} \in (0, 1)$ with $y := \overline{x} + \overline{\varepsilon} v \in K$ and $\overline{x} + \delta v \notin K$ for all $\delta \in (\overline{\varepsilon}, 1]$. As requested we get that $y \in K \setminus \operatorname{icor} K$.

Now, by the separation condition in Theorem 2.21, there is $x' \in K' \setminus l(K')$ with $x'(y) = 0 < x'(k)$ for all $k \in \operatorname{icor} K$. Thus, in particular we have $x'(\overline{x}) > 0$. Moreover, from the above analysis, it is easy to verify that $\overline{x} = \alpha_1 x + \alpha_2 y \in \operatorname{span}\{x, y\}$ for $\alpha_1 := \frac{\overline{\varepsilon}}{1-\overline{\varepsilon}}$ and $\alpha_2 := \frac{1}{1-\overline{\varepsilon}}$. Recalling that $x'(x) = 0 = x'(y)$, we infer $0 < x'(\overline{x}) = \alpha_1 x'(x) + \alpha_2 x'(y) = 0$, which is a contradiction.

□

The algebraic closedness assumption of the relatively solid, convex cone in Theorem 3.6 (2°) is essential, as the next example shows.

Example 3.7 *Consider the lexicographic cone in a Euclidean space* $\mathbb{R}^2$*,*

$$K := \{k = (k_1, k_2) \in \mathbb{R}^2 \mid k_1 > 0 \} \cup [\{0\} \times \mathbb{R}_+].$$

It is easy to see that K is pointed (i.e., $l(K) = \{0\}$), convex and solid but not algebraically closed. Moreover, the corresponding dual cone can be stated as

$$K' = \{x' \in (\mathbb{R}^2)' \mid \exists\, v \in \mathbb{R}_+ \times \{0\} : \; x'(\cdot) = \langle v, \cdot \rangle\},$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product defined on $\mathbb{R}^2$. It is easy to observe that K' is pointed (i.e., $l(K') = \{0\}$) as well. Now, for the point $\overline{k} := (0, 1) \in K \setminus l(K)$, we have $x'(\overline{k}) = 0$ for all $x' \in K'$. Thus, the conclusion in Theorem 3.6 (2°) does not hold in this example, which shows that the algebraic closedness assumption on K is essential.

4 New algebraic properties of dual cones of convex cones

In this section, we derive new algebraic properties of dual cones of convex cones in linear spaces. We directly start by presenting one main theorem where important facts related to generalized interiors of the dual cone K' are given.

Theorem 4.1 *Assume that K is a convex cone. Then, the following hold:*

1°

$$\operatorname{cor} K' \subseteq \{y' \in E' \mid \forall\, y \in K \setminus \{0\} : \; y'(y) > 0\} = K'_+.$$

2° *If E has finite dimension, K' is solid and K is algebraically closed, then*

$$\operatorname{cor} K' = \{y' \in E' \mid \forall\, y \in K \setminus \{0\} : \; y'(y) > 0\} = K'_+.$$

3° *If K is relatively solid and algebraically closed, then*

$$\operatorname{icor} K' \subseteq \{y' \in E' \mid \forall\, y \in K \setminus l(K) : \; y'(y) > 0\} = K'_\oplus.$$

4° *If E has finite dimension and K is algebraically closed, then*

$$\operatorname{icor} K' = \{y' \in K' \mid \forall\, y \in K \setminus l(K): \; y'(y) > 0\} = K' \cap K'_{\oplus}.$$

Proof:

1° Assertion 1° is known, see Jahn [17, Lem. 1.25].

2° The inclusion "$\subseteq$" is given in 1°. Assume that K' is solid and K is algebraically closed. In order to show the reverse inclusion "$\supseteq$", let $x' \in E'$ such that $x'(k) > 0$ for all $k \in K \setminus \{0\}$. Clearly, we have $x' \in K'$. Assume by the contrary that $x' \notin \operatorname{cor} K'$. By Corollary 2.23, there is $x'' \in E'' \setminus \{0\}$ such that $x''(x') = 0 \leq x''(k')$ for all $k' \in K'$. Since E has finite dimension, in view of Remark 2.14, we know that the canonical embedding function J_E is surjective. Thus, there exists $\tilde{x} \in E$ such that $J_E(\tilde{x}) = x''$. Notice that $\tilde{x} \neq 0$ due to the fact that J_E is linear and $x'' \neq 0$. We infer $x'(\tilde{x}) = 0 \leq k'(\tilde{x})$ for all $k' \in K'$, which actually means that $\tilde{x} \in K''$. By Remark 2.3, since E has finite dimension, K is relatively solid. Then, applying Proposition 2.11, we obtain $\tilde{x} \in K'' = \operatorname{acl} K = K$. Consequently, we have $\tilde{x} \in K \setminus \{0\}$, hence $x'(\tilde{x}) > 0$ by our assumption that $x' \in K'_{+}$, a contradiction to $x'(\tilde{x}) = 0$.

3° Consider $x' \in \operatorname{icor} K' \subseteq K'$. Assume by the contrary that there is $k \in K \setminus l(K)$ such that $x'(k) \leq 0$. Because of $x' \in K'$, we actually have $x'(k) = 0$. In view of the Theorem 3.6 (2°), there is $y' \in K' \setminus l(K')$ such that $y'(k) > 0$. Since $x' \in \operatorname{icor} K'$ and $-y' \in \operatorname{aff} K' = \operatorname{aff}(K' - K')$, there exists $\varepsilon > 0$ such that $z' := x' + \varepsilon(-y') \in K'$. We conclude $z'(k) = x'(k) - \varepsilon y'(k) < 0$, which contradicts the condition that, for any $k \in K$, we have $z'(k) \geq 0$.

4° The inclusion "$\subseteq$" is given in 3°. In order to show the reverse inclusion "$\supseteq$", assume that K is algebraically closed and let $x' \in K'$ such that $x'(y) > 0$ for all $y \in K \setminus l(K)$. Notice that K and K' are relatively solid in by Remark 2.3 (since both E and E' have finite dimension). Suppose by the contrary that $x' \notin \operatorname{icor} K'$. Due to the separation condition in Theorem 2.21, there is $x'' \in E'' \setminus \{0\}$ such that $x''(x') = 0 \leq x''(k')$ for all $k' \in K'$; and $0 < x''(k')$ for all $k' \in \operatorname{icor} K'$. Since E has finite dimension, in view of Remark 2.14, we know that the canonical embedding function J_E is surjective. Thus, there exists $\tilde{x} \in E$ such that $J_E(\tilde{x}) = x''$. We infer

$$x'(\tilde{x}) = 0 \leq k'(\tilde{x}) \qquad \text{for all } k' \in K'; \tag{13}$$

$$0 < k'(\tilde{x}) \qquad \text{for all } k' \in \operatorname{icor} K'. \tag{14}$$

Due to Proposition 2.11 and condition (13), we have $\tilde{x} \in K'' = \operatorname{acl} K = K$. More precisely, (14) ensures that we have $\tilde{x} \notin l(K)$ (otherwise $k'(\tilde{x}) = 0$ for $k' \in \operatorname{icor} K'$, which is a contradiction). Thus, we get $\tilde{x} \in K \setminus l(K)$, hence $x'(\tilde{x}) > 0$ taking into account that $x' \in K'_{\oplus}$. This contradicts the fact that $x'(\tilde{x}) = 0$ given in (13).

□

Also in Theorem 4.1 (3°), the algebraic closedness assumption of the convex cone K is essential, as shown in the following example.

Example 4.2 *We consider again the simple Example 3.7 where the lexicographic cone in $\mathbb{R}^2$ is studied. Taking a look on K', one can see that*

$$\operatorname{icor} K' = \{x' \in (\mathbb{R}^2)' \mid \exists\, v \in \mathbb{R}_{++} \times \{0\} : x'(\cdot) = \langle v, \cdot \rangle\}.$$

Now, for the point $\overline{k} = (0,1) \in K \setminus l(K)$ we have $x'(\overline{k}) = 0$ for all $x' \in \operatorname{icor} K'$, while $K'_{\oplus} = \emptyset$. Consequently, $K'_{\oplus} \cap \operatorname{icor} K' = \emptyset$, which shows that the algebraic closedness assumption on K in Theorem 4.1 (3°) is essential.

Remark 4.3 *Notice, if E has finite dimension (hence also E'), both K and K' are relatively solid by Remark 2.3. We mention that some ideas in the proof of Adán and Novo [4, Prop. 2.3] (the proposition states that K is relatively solid if and only if K' is relatively solid) seem to be not valid by taking a closer look on Theorem 4.1 (4°) and the upcoming Theorems 4.7 (4°) and 4.8 (2°). Thus, it is an open question whether the fact stated in [4, Prop. 2.3] is true in the infinite dimensional case.*

In order to state the next lemma, where important properties of the sets K', K'_+ and $K'_{\oplus}$, respectively, are given, it is convenient to introduce, for any $\Omega \subseteq E$, the annihilator of Ω,

$$(\Omega)^{\perp} := \{x' \in E' \mid \forall\, x \in \Omega : \; x'(x) = 0\}.$$

Lemma 4.4 *Assume that $K \subseteq E$ is a convex cone. Then, the following hold:*

- 1° *If K is relatively solid and algebraically closed, and K' is relatively solid, then $K'_{\oplus} \neq \emptyset$.*
- 2° $K' \subseteq \{y' \in E' \mid \forall\, k \in l(K) : \; y'(k) = 0\} = (l(K))^{\perp} = (l(K))'$.
- 3° *If $K = l(K)$, then $K' = (K)^{\perp} = l(K')$.*
- 4° $K = l(K)$ *if and only if* $K'_{\oplus} = E'$ *if and only if* $K'_{\oplus}$ *is a cone.*
- 5° $K = \{0\}$ *if and only if* $K'_+ = E'$ *if and only if* $K' = E'$ *if and only if* K'_+ *is a cone.*

Proof:

- 1° This result follows by Theorem 4.1 (3°).
- 2° Consider some $x' \in K'$. Then, we have in particular $x'(k) \geq 0$ for all $k \in l(K) \subseteq K$. Hence, $-x'(k) = x'(-k) \geq 0$ for all $k \in l(K)$ as well. Thus, $K' \subseteq (l(K))^{\perp}$. The remaining equality $(l(K))^{\perp} = (l(K))'$ is easy to see.
- 3° Assume that $K = l(K)$. By assertion 2°, we have $K' \subseteq (l(K))^{\perp} = (l(K))' = K'$, hence $K' = (l(K))^{\perp} = (K)^{\perp}$. Moreover, it is easy to check that $(K)^{\perp} = l(K')$.
- 4° Obviously, $K = l(K)$ yields $K'_{\oplus} = E'$. Now, suppose that $K'_{\oplus} = E'$. Assuming by the contrary that there exists $\overline{k} \in K \setminus l(K)$. Then, in particular for $x' := 0 \in E' = K'_{\oplus}$, we have $x'(\overline{k}) > 0$, which is a contradiction.

  Clearly, $K'_{\oplus} = E'$ is a cone. Conversely, assume that $K'_{\oplus}$ is a cone, which in particular means that $0 \in K'_{\oplus}$. Thus, taking a look on the definition of $K'_{\oplus}$, we conclude $K = l(K)$.

5° The proof is analogous to the proof of 4° by replacing $K'_{\oplus}$ by K'_{+} as well as $l(K)$ by $\{0\}$. Furthermore, it is easy to see that $K'_{+} = E'$ implies $K' = E'$. Now, assume that $K' = E'$. Then, it is easy to check that $E' = (K)^{\perp}$. To prove $K = \{0\}$, suppose by the contrary that there is $\overline{k} \in K \setminus \{0\}$. Thus, for any $x' \in E'$, $x'(\overline{k}) = 0$, a contradiction to Theorem 3.6 (1°).

□

Remark 4.5 *Notice that in* 1° *of Lemma 4.4 the algebraical closedness assumption concerning* K *is essential, as Examples 3.7 and 4.2 show.*

In the next proposition, we will state some further properties of the dual cone of K and its generalized interiors. It is convenient to introduce the following set

$$C := (K \setminus l(K)) \cup \{0\} \subseteq K.$$

Now, we are able to analyze relationships between the dual cones K' and C', and the sets K'_{+}, $K'_{\oplus}$ and C'_{+}.

Lemma 4.6 *Assume that* $K \subseteq E$ *is a convex cone. Then, the following hold:*

1° *C is a pointed, convex cone in E.*

2° $K'_{+} \subseteq K' = C' \cap (l(K))' \subseteq C'$.

3° $K'_{+} = K'_{\oplus} \cap (l(K))'_{+} \subseteq K'_{\oplus} = C'_{+} \subseteq C'$

4° *$K \neq l(K)$ or $K = \{0\}$ if and only if $K'_{\oplus} \subseteq K'$ if and only if $C' \subseteq K'$.*

5° *If $K'_{\oplus} \neq \emptyset$, then $K \neq l(K)$ if and only if $K'_{\oplus} \subseteq K' \setminus l(K')$.*

6° *If K is relatively solid and algebraically closed, and K' is relatively solid, then K is pointed if and only if $K'_{+} \supseteq K'_{\oplus}$ if and only if $K'_{+} \neq \emptyset$.*

Proof:

1° The proof is straightforward by using Lemma 2.6.

2° Clearly, $K'_{+} \subseteq K'$. Moreover, since $K = C \cup l(K)$ we have $K' = C' \cap (l(K))' \subseteq C'$ by Jahn [17, Lem. 1.24].

3° If K is pointed, then $K'_{+} = K'_{\oplus}$ as well as $K'_{\oplus} \cap (\{0\})'_{+} = K'_{\oplus} \cap E = K'_{\oplus}$. Otherwise, if K is not pointed, then $K'_{+} = \emptyset$ in view of Lemma 2.10 (4°), as well as $K'_{\oplus} \cap (l(K))'_{+} = K'_{\oplus} \cap \emptyset = \emptyset$, again by Lemma 2.10 (4°) applied for the not pointed cone $l(K)$ in the role of K. We conclude that $K'_{+} = K'_{\oplus} \cap (l(K))'_{+}$. Moreover, $K'_{\oplus} = C'_{+}$ and the other inclusions are obvious.

4° Assume that $K \neq l(K)$ or $K = \{0\}$. We prove that $C' \subseteq K'$. Consider some $x' \in C'$. Now, we know that $x'(c) \geq 0$ for all $c \in C$. Thus, it is enough to show that $x'(k) \geq 0$ for all $k \in l(K)$ in order to conclude $x' \in K'$. Take some $\overline{k} \in l(K)$. By Lemma 2.6 (3°) there is $\tilde{k} \in K \setminus l(K)$ such that $[\tilde{k}, \overline{k}) \subseteq K \setminus l(K)$. Now, for any $\alpha \in (0, 1]$, we have $\alpha x'(\tilde{k}) + (1 - \alpha) x'(\overline{k}) = x'(\alpha \tilde{k} + (1 - \alpha)\overline{k}) \geq 0$, which yields for $\alpha \to 0$ as requested $x'(\overline{k}) \geq 0$ (see also Remark 2.17 (a)).

   Now, suppose that $C' \subseteq K'$. Due to assertion 3° of this lemma, we have $K'_{\oplus} = C'_{+} \subseteq C' \subseteq K'$.

Finally, let $K'_\oplus \subseteq K'$ be satisfied. Assume by the contrary that $K = l(K)$ and $K \neq \{0\}$. By Lemma 4.4 (4°) we have $K'_\oplus = E'$, hence $K' = E'$. Consequently, by Lemma 4.4 (5°), we get $K = \{0\}$, a contradiction.

5° By 4° of this lemma, we get that $K \neq l(K)$ implies $K'_\oplus \subseteq K'$. Actually we have $K'_\oplus \subseteq K' \setminus l(K')$, since otherwise $x'(k) = 0$ for some $k \in K \setminus l(K)$ and $x' \in K'_\oplus$, a contradiction.

Assume that $K'_\oplus \subseteq K' \setminus l(K') \subseteq K'$. Thus, in view of 4° of this lemma, we obtain $K = l(K)$ or $K = \{0\}$. However, $K = \{0\}$ implies $K' = E'$ and $l(K') = E'$ by Lemma 4.4 (5°), which contradicts $K' \setminus l(K') \neq \emptyset$.

6° Assume that K is relatively solid and algebraically closed, and K' is relatively solid. If K is pointed, then $K'_+ = K'_\oplus \cap (\{0\})'_+ = K'_\oplus$ by assertion 3°. Now, let $K'_+ \supseteq K'_\oplus$ be satisfied. Assuming by the contrary that K is not pointed, then $K'_+ = \emptyset$ by Lemma 2.10 (4°), while $K'_\oplus \neq \emptyset$ by Lemma 4.4 (1°), a contradiction to $K'_+ \supseteq K'_\oplus$.

The second equivalence is a consequence of the first equivalence taking into account Lemmas 4.4 (1°) and 2.10 (4°).

□

We are ready to state some more results for the intrinsic core of the dual cone. It should be mentioned that assertions 1° and 2° in the next Theorem 4.7 present representations of $\operatorname{icor} K'$ which are also true in the case that E has infinite dimension.

Theorem 4.7 *Assume that $K \subseteq E$ is a convex cone. Then, the following assertions hold:*

1° *For any $k' \in \operatorname{icor} K'$,*

$$\operatorname{icor} K' = \bigcup_{\alpha>0} \{x' \in E' \mid \forall\, k \in K : \; x'(k) \geq \alpha k'(k)\}.$$

2° *If K is relatively solid and algebraically closed, and either $K \neq l(K)$ or $K = \{0\}$, then, for any $k' \in \operatorname{icor} K'$, we have*

$$\operatorname{icor} K' = \left[\bigcup_{\alpha>0} \{x' \in E' \mid \forall\, k \in K \setminus l(K) : \; x'(k) \geq \alpha k'(k)\}\right] \subseteq K'_\oplus.$$

3° *For any $k' \in \operatorname{icor} K'$,*

$$\operatorname{aff} K' = \bigcup_{\alpha>0} \{x' \in E' \mid \forall\, k \in K : \; x'(k) \leq \alpha x'(k)\}.$$

Now, assume that E has finite dimension, K is algebraically closed, and either $K \neq l(K)$ or $K = \{0\}$. Then, we have:

4° *It holds that*

$$\operatorname{icor} K' = \{x' \in E' \mid \forall\, k \in K \setminus l(K) : \; x'(k) > 0\} = K'_\oplus.$$

5° *For any* $k' \in \operatorname{icor} K'$,

$$K' = \bigcap_{\beta<0} \{x' \in E' \mid \forall\, k \in K \setminus l(K) :\; x'(k) > \beta k'(k)\}.$$

Proof:

1° Take some $k' \in \operatorname{icor} K'$. By Proposition 2.13 (1°), we know that

$$\begin{aligned}\operatorname{icor} K' = \bigcup_{\alpha>0} (\alpha k' + K') &= \bigcup_{\alpha>0} (\alpha k' + \{x' \in E' \mid \forall\, k \in K :\; x'(k) \geq 0\}) \\ &= \bigcup_{\alpha>0} \{x' \in E' \mid \forall\, k \in K :\; x'(k) \geq \alpha k'(k)\}.\end{aligned}$$

2° Consider $k' \in \operatorname{icor} K'$ and assume that either $K \neq l(K)$ or $K = \{0\}$. By assertion 1° of this theorem and due to $K \setminus l(K) \subseteq K$, we have

$$\begin{aligned}\operatorname{icor} K' &= \bigcup_{\alpha>0} \{x' \in E' \mid \forall\, k \in K :\; x'(k) \geq \alpha k'(k)\} \\ &\subseteq \bigcup_{\alpha>0} \{x' \in E' \mid \forall\, k \in K \setminus l(K) :\; x'(k) \geq \alpha k'(k)\}. \qquad (15)\end{aligned}$$

Moreover, in view of Theorem 4.1 (3°), for $k \in K \setminus l(K)$, we have $k'(k) > 0$. Thus, an upper set for all sets involved in (15) is given by $K'_{\oplus}$. Now, Lemma 4.4 (2°) and Lemma 4.6 (4°) yield $K'_{\oplus} \subseteq K' \subseteq (l(K))^{\perp}$. We are going to show the reverse inclusion in (15). Consider some $x' \in E'$ and $\alpha > 0$ such that for all $k \in K \setminus l(K)$ we have $x'(k) \geq \alpha k'(k)$. It remains to show that $x'(k) \geq \alpha k'(k)$ for all $k \in l(K)$ and $\alpha > 0$. From the above analysis, we get that $x', k' \in (l(K))^{\perp}$. Thus, for any $k \in l(K)$ and $\alpha > 0$, we have $x'(k) = 0 = \alpha k'(k)$.

Therefore, we actually have equality in (15).

3° By Proposition 2.13 (3°), for any $k' \in \operatorname{icor} K'$, we have

$$\operatorname{aff} K' = \bigcup_{\alpha>0} (\alpha k' - K') = \bigcup_{\alpha>0} \{x' \in E' \mid \forall\, x \in K :\; x'(x) \leq \alpha x'(k)\}.$$

4° Follows directly by Theorem 4.1 (4°) and Lemma 4.6 (4°).

5° Take some $k' \in \operatorname{icor} K'$. Due to Lemma 2.10 (1°), Proposition 2.13 (2°) and assertion 4° of this theorem, we have

$$\begin{aligned}K' = \operatorname{acl} K' &= \bigcap_{\beta<0} (\beta k' + \operatorname{icor} K') \\ &= \bigcap_{\beta<0} (\beta k' + \{x' \in E' \mid \forall\, k \in K \setminus l(K) :\; x'(k) > 0\}) \\ &= \bigcap_{\beta<0} \{x' \in E' \mid \forall\, k \in K \setminus l(K) :\; x'(k) > \beta k'(k)\}.\end{aligned}$$

□

Further properties of the dual cone K' are given in the next theorem.

Theorem 4.8 *Suppose that $K \subseteq E$ is a convex cone. Then, the following hold:*

1° *Assume that K is solid. Then,*

$$K' \setminus \{0\} = \{x' \in E' \mid \forall\, k \in \operatorname{cor} K : \; x'(k) > 0\},$$

and the latter set is nonempty if and only if $K \neq l(K)$.

2° *Assume that K is relatively solid. Then,*

$$K' \setminus l(K') = \{x' \in E' \mid \forall\, k \in \operatorname{icor} K : \; x'(k) > 0\},$$

and the latter set is nonempty if and only if $K \neq l(K)$.

Proof:

1° First of all, notice that the quality in assertion 1° is a direct consequence of the corresponding equality in 2°. This is due to the facts that $\operatorname{cor} K \neq \emptyset$ implies $\operatorname{cor} K = \operatorname{icor} K$ by Lemma 2.1 (3°), and K' is pointed (i.e., $l(K') = \{0\}$) by Lemma 2.10 (3°).

Moreover, by Lemma 3.2 (7°) we know that $K \neq E$ if and only if $K' \neq \{0\}$. Notice, since K is solid, we infer that $K = E$ or $K \neq l(K)$ by Lemma 2.7 (2°). Thus, under the assumption $K' \neq \{0\}$, we get $K \neq E$, hence $K \neq l(K)$. Conversely, $K \neq l(K)$ implies $K \neq E$, which yields $K' \neq \{0\}$.

2° In order to show the inclusion "$\subseteq$", let $x' \in K' \setminus l(K')$. Assume by the contrary that there is $k \in \operatorname{icor} K$ such that $x'(k) \leq 0$, which actually means that $x'(k) = 0$. This directly contradicts Theorem 3.1 (2°).

Conversely, for proving "$\supseteq$", let $x' \in E'$ such that for all $k \in \operatorname{icor} K$ we have $x'(k) > 0$. By Lemma 2.2 (3°), we have $\operatorname{acl}(\operatorname{icor} K) = \operatorname{acl} K \supseteq K$. Thus, in view of Remark 2.17, we infer $x'(k) \geq 0$ for all $k \in K$, which yields $x' \in K'$. Moreover, since there is $k \in \operatorname{icor} K$ with $x'(k) > 0$, we have $-x' \notin K'$.

Finally, by Lemmas 3.2 (3°) and 4.4 (3°), we conclude $K' \neq l(K')$ if and only if $K \neq l(K)$.

□

Corollary 4.9 *Assume that $K \subseteq E$ is a convex cone. Then, we have:*

1° *If K is solid and $E \neq \{0\}$, then*

$$K' \setminus \{0\} = ((\operatorname{cor} K) \cup \{0\})'_{+}.$$

2° *If K is relatively solid and $K \neq \{0\}$, then*

$$K' \setminus l(K') = ((\operatorname{icor} K) \cup \{0\})'_{+}.$$

Proof:

1° Clearly, $((\operatorname{cor} K) \cup \{0\})'_{+} = \{x' \in E' \mid \forall\, k \in (\operatorname{cor} K) \setminus \{0\} :\ x'(k) > 0\}$ holds true. Hence, in view of Theorem 4.8 (1°) it is easy to see that $K' \setminus \{0\} \subseteq ((\operatorname{cor} K) \cup \{0\})'_{+}$. We consider two cases:

Case 1: In the case $K \neq E$ (which implies $E \neq \{0\}$), we have $0 \notin \operatorname{cor} K$ by Lemma 2.7 (3°), hence $(\operatorname{cor} K) \setminus \{0\} = \operatorname{cor} K$. Thus, we conclude the reverse inclusion.

Case 2: Assume that $K = E \neq \{0\}$. Notice that $K = E$ implies $\operatorname{cor} K = E$. Now, since $E \cup \{0\}$ is a not pointed, convex cone, we infer $(E \cup \{0\})'_{+} = \emptyset$ by Lemma 2.10 (4°). Thus, $K' \setminus \{0\} = \emptyset$ holds true as well.

2° Here we have $((\operatorname{icor} K) \cup \{0\})'_{+} = \{x' \in E' \mid \forall\, k \in (\operatorname{icor} K) \setminus \{0\} :\ x'(k) > 0\}$, and $K' \setminus l(K') \subseteq ((\operatorname{icor} K) \cup \{0\})'_{+}$ by Theorem 4.8 (2°). Let us consider two cases:

Case 1: In the case $K \neq l(K)$ (which implies $K \neq \{0\}$), we have $0 \notin \operatorname{icor} K$ by Lemma 2.6 (6°), hence $(\operatorname{icor} K) \setminus \{0\} = \operatorname{icor} K$. Therefore, the reverse inclusion holds.

Case 2: Assume that $K = l(K) \neq \{0\}$. By Lemma 2.6 (6°), we get $K = l(K) = \operatorname{icor} K$. Since $l(K) \cup \{0\}$ is a not pointed, convex cone, we infer $(l(K) \cup \{0\})'_{+} = \emptyset$ by Lemma 2.10 (4°). Thus, $K' \setminus l(K') = \emptyset$.

□

Remark 4.10 *Suppose that $K \neq \{0\}$ is a relatively solid, convex cone. By Lemma 2.10 (2°) one can also directly derive*

$$((\operatorname{icor} K) \cup \{0\})'_{+} \subseteq ((\operatorname{icor} K) \cup \{0\})' = K'.$$

It should be mentioned that Hernández, Jiménez and Novo [13, Rem. 3.14] observed the equality

$$K' = (K)^{\perp} \cup \left[((\operatorname{icor} K) \cup \{0\})'_{+} \cap K'\right].$$

Notice that $(K)^{\perp} = l(K')$. In Hernández, Jiménez and Novo [13, Cor. 3.15] the inclusions $K' \setminus \{0\} \subseteq ((\operatorname{cor} K) \cup \{0\})'_{+}$ and $K' \setminus l(K') \subseteq ((\operatorname{icor} K) \cup \{0\})'_{+}$ of our Corollary 4.9 are stated.

Corollary 4.11 *Assume that $K \subseteq E$ is a convex cone with $K \neq l(K)$. Then, the following assertions hold:*

1° *If K is solid, then*

$$\operatorname{cor} K' \subseteq K'_{+} \subseteq K' \setminus \{0\} = ((\operatorname{cor} K) \cup \{0\})'_{+}.$$

2° *If K is relatively solid and algebraically closed, then we have*

$$\operatorname{icor} K' \subseteq K'_{\oplus} \subseteq K' \setminus l(K') = ((\operatorname{icor} K) \cup \{0\})'_{+}.$$

Proof:

1° Follows by Corollary 4.9 (1°), Theorem 4.1 (1°) and the fact that $0 \notin K'_+$ for $K \neq \{0\}$, see Lemma 4.4 (5°).

2° This assertion is a consequence of Corollary 4.9 (2°), Theorem 4.1 (3°), Lemmas 4.4 (1°) and 4.6 (5°).

□

5 Application to vector optimization problems

Given two real linear spaces X and E, a nonempty feasible set $\Omega \subseteq X$, and a vector-valued objective function $f : X \to E$, we consider the following vector optimization problem:

$$\begin{cases} f(x) \to \min \text{ w.r.t. } K \\ x \in \Omega, \end{cases} \tag{$\mathcal{P}$}$$

where E is preordered by the convex cone $K \subseteq E$. More precisely, K induces on E a preorder relation $\leqq_K$ defined, for any two points $y, \overline{y} \in E$, by

$$y \leqq_K \overline{y} \quad :\Longleftrightarrow \quad y \in \overline{y} - K.$$

For notational convenience, we further define, for any two points $y, \overline{y} \in E$, the following three binary relations

$$\begin{aligned} y \leq^0_K \overline{y} \quad &:\Longleftrightarrow \quad y \in \overline{y} - K \setminus \{0\}, \\ y \leq_K \overline{y} \quad &:\Longleftrightarrow \quad y \in \overline{y} - K \setminus l(K), \\ y <_K \overline{y} \quad &:\Longleftrightarrow \quad y \in \overline{y} - (\operatorname{icor} K) \setminus l(K). \end{aligned}$$

Notice, in view of Lemma 2.6 (5°, 6°), we have

$$(\operatorname{icor} K) \setminus l(K) = \begin{cases} \operatorname{icor} K & \text{if } K \neq l(K), \\ \emptyset & \text{if } K = l(K). \end{cases}$$

Solutions of the problem ($\mathcal{P}$) are defined according to the next two definitions (see, e.g., Bagdasar and Popovici [6, Sec. 2.2], Jahn [17, Def. 4.1] and Luc [21, Def. 2.1]).

Definition 5.1 (Pareto efficiency) *A point $\overline{x} \in \Omega$ is said to be a Pareto efficient solution if for any $x \in \Omega$ the condition $f(x) \leqq_K f(\overline{x})$ implies $f(\overline{x}) \leqq_K f(x)$. The set of all Pareto efficient solutions of ($\mathcal{P}$) is denoted by*

$$\operatorname{Eff}(\Omega \mid f, K) := \{\overline{x} \in \Omega \mid \forall x \in \Omega : \, f(x) \leqq_K f(\overline{x}) \;\Rightarrow\; f(\overline{x}) \leqq_K f(x)\}.$$

The following assertions are well known:

1° $\operatorname{Eff}(\Omega \mid f, K) = \{\overline{x} \in \Omega \mid \nexists\, x \in \Omega : \, f(x) \leq_K f(\overline{x})\}$.

2° If $K = l(K)$, then $\operatorname{Eff}(\Omega \mid f, K) = \Omega$.

3° If K is pointed, then $\operatorname{Eff}(\Omega \mid f, K) = \{\overline{x} \in \Omega \mid \nexists\, x \in \Omega : \, f(x) \leq^0_K f(\overline{x})\}$.

To define a weaker solution concept for the problem $(\mathcal{P})$, the intrinsic core of the convex cone K will be used in the next definition.

Definition 5.2 (Weak Pareto efficiency) *A point $\overline{x} \in \Omega$ is said to be a weakly Pareto efficient solution if there is no $x \in \Omega$ such that $f(x) <_K f(\overline{x})$. The set of all weakly Pareto efficient solutions of $(\mathcal{P})$ is denoted by*

$$\mathrm{WEff}(\Omega \mid f, K) := \{\overline{x} \in \Omega \mid \nexists\, x \in \Omega: \ f(x) <_K f(\overline{x})\}.$$

By using Definitions 5.1 and 5.2 as well as Lemma 2.6 (4°, 5°, 6°), we get the following properties for sets of (weakly) Pareto efficient solutions:

1° If $K \neq l(K)$, then $\mathrm{Eff}(\Omega \mid f, K) \subseteq \mathrm{WEff}(\Omega \mid f, K)$.

2° If $K = l(K)$, then $\mathrm{Eff}(\Omega \mid f, K) = \Omega = \mathrm{WEff}(\Omega \mid f, K)$.

3° If $\mathrm{icor}\, K \subseteq l(K)$, then $\mathrm{WEff}(\Omega \mid f, K) = \Omega$.

Remark 5.3 *Some authors already defined weak solution concepts for vector optimization problems by using certain generalized interiority notions. For instance, Jahn [17, Def. 4.12] used the algebraic interior of K while Grad and Pop [12] used the quasi-interior of K, and Bao and Mordukhovich [7], Zălinescu [30], studied (beside other concepts) the quasi-relative interior of K in a topological setting. Assuming $K \neq l(K)$, then our Definition 5.2 is in accordance with the definition by Adán and Novo [3, Def. 5] and Bao and Mordukhovich [7, p. 303]. Following the "Intrinsic Relative Minimizer Concept" by Bao and Mordukhovich [7] (see also Mordukhovich [24, Def. 9.3]), the weak solution concept considered in Definition 5.2 could also be called "Intrinsic Relative Pareto Efficiency". For more details, we refer the reader to the above-mentioned works and the references therein.*

Let us consider some monotonicity concepts for real-valued functions (c.f. Jahn [17, Def. 5.1]). Given binary relations $\sim_E \in \{\leqq_K, \leq_K^0, \leq_K, <_K\}$ and $\sim_{\mathbb{R}} \in \{<, \leq\}$, a function $\varphi : E \to \mathbb{R}$ is said to be ($\sim_E$, $\sim_{\mathbb{R}}$)-increasing if for any $y, \overline{y} \in E$ with $y \sim_E \overline{y}$ we have $\varphi(y) \sim_{\mathbb{R}} \varphi(\overline{y})$.

Remark 5.4 *Every ($\leq_K^0$, $<$)-increasing function is ($\leqq_K$, $\leq$)-increasing as well. Assume that $K \neq l(K)$. Since* $\mathrm{icor}\, K \subseteq K \setminus l(K)$ *by Lemma 2.6 (4°), any ($\leq_K$, $<$)-increasing function is ($<_K$, $<$)-increasing as well.*

Lemma 5.5 *The following assertions hold:*

1° *Any $x' \in K'$ is ($\leqq_K$, $\leq$)-increasing.*

2° *Assume that K is relatively solid. Any $x' \in K' \setminus l(K')$ is ($<_K$, $<$)-increasing.*

3° *Any $x' \in K'_{\oplus}$ is ($\leq_K$, $<$)-increasing.*

4° *Any $x' \in K'_{+}$ is ($\leq_K^0$, $<$)-increasing.*

Proof:

1° See Jahn [17, Ex. 5.2 (a)].

2° In view of Theorem 4.8 (2°), we have $K \neq l(K)$. The equality given in Theorem 4.8 (2°) ensures now that, for any $y, \overline{y} \in E$ with $y <_K \overline{y}$, we have $x'(y) < x'(\overline{y})$. Consequently, x' is a $(<_K, <)$-increasing function.

3° By the definition of $K'_{\oplus}$, for any $y, \overline{y} \in E$ with $y \leq_K \overline{y}$, we have $x'(y) < x'(\overline{y})$, i.e., x' is a $(\leq_K, <)$-increasing function.

4° See Jahn [17, Ex. 5.2 (a)].

□

Next, we present some scalarazitaion results for the vector optimization problem $(\mathcal{P})$ by using increasing scalarization functions.

Lemma 5.6 *Consider a real-valued function $\varphi : E \to \mathbb{R}$. Then, the following assertions hold:*

1° *If φ is $(<_K, <)$-increasing, then* $\operatorname{argmin}_{x \in \Omega} (\varphi \circ f)(x) \subseteq \operatorname{WEff}(\Omega \mid f, K)$*.*

2° *If φ is $(\leq_K, <)$-increasing, then* $\operatorname{argmin}_{x \in \Omega} (\varphi \circ f)(x) \subseteq \operatorname{Eff}(\Omega \mid f, K)$*.*

3° *If φ is $(\leq_K, \leq)$-increasing, and* $\operatorname{argmin}_{x \in \Omega} (\varphi \circ f)(x) = \{\overline{x}\}$ *for some $\overline{x} \in \Omega$, then $\overline{x} \in \operatorname{Eff}(\Omega \mid f, K)$.*

Proof:

1° Consider any $\overline{x} \in \operatorname{argmin}_{x \in \Omega} (\varphi \circ f)(x)$. Assume the contrary holds, i.e., $\overline{x} \notin \operatorname{WEff}(\Omega \mid f, K)$. Then, there exists $y \in \Omega$ with $f(y) <_K f(\overline{x})$. If φ is $(<_K, <)$-increasing, then $\varphi(f(y)) < \varphi(f(\overline{x}))$, a contradiction to $(\varphi \circ f)(\overline{x}) \leq (\varphi \circ f)(y)$.

2° Let $\overline{x} \in \operatorname{argmin}_{x \in \Omega} (\varphi \circ f)(x)$. Assume the contrary holds, i.e., $\overline{x} \notin \operatorname{Eff}(\Omega \mid f, K)$. Thus, there exists $y \in \Omega$ with $f(y) \leq_K f(\overline{x})$. If φ is $(\leq_K, <)$-increasing, then $\varphi(f(y)) < \varphi(f(\overline{x}))$, a contradiction to $(\varphi \circ f)(\overline{x}) \leq (\varphi \circ f)(y)$.

3° Now, consider any $\overline{x} \in \Omega$ with $\{\overline{x}\} = \operatorname{argmin}_{x \in \Omega} (\varphi \circ f)(x)$. Suppose by the contrary that $\overline{x} \notin \operatorname{Eff}(\Omega \mid f, K)$. Then, there exists $y \in \Omega \setminus \{\overline{x}\}$ with $f(y) \leq_K f(\overline{x})$. If φ is $(\leq_K, \leq)$-increasing, then $\varphi(f(y)) \leq \varphi(f(\overline{x}))$, a contradiction to the fact that $\overline{x}$ is the unique minimizer of $\varphi \circ f$ on Ω.

□

Notice that the convex cone K considered in Lemma 5.6 is neither assumed to be pointed nor solid, in contrast to the known results by Jahn [17, Lem. 5.14 and 5.24].

For the linear scalarization case, we directly derive by combining Lemma 5.5 and Lemma 5.6 the following assertions:

Theorem 5.7 *The following assertions hold:*

1° *For any $x' \in K' \setminus l(K')$, we have* $\operatorname{argmin}_{x \in \Omega} (x' \circ f)(x) \subseteq \operatorname{WEff}(\Omega \mid f, K)$*.*

2° *For any $x' \in K'_{\oplus}$, we have* $\operatorname{argmin}_{x \in \Omega} (x' \circ f)(x) \subseteq \operatorname{Eff}(\Omega \mid f, K)$*.*

3° *For any* $x' \in K'$ *with* $\operatorname{argmin}_{x\in\Omega} (x' \circ f)(x) = \{\overline{x}\}$ *for some* $\overline{x} \in \Omega$*, we have* $\overline{x} \in \mathrm{Eff}(\Omega \mid f, K)$*.*

4° *Assume that* K *is pointed. For any* $x' \in K'_+$*, we have* $\operatorname{argmin}_{x\in\Omega} (x' \circ f)(x) \subseteq \mathrm{Eff}(\Omega \mid f, K)$*.*

Remark 5.8 *Notice that* $\operatorname{cor} K' \subseteq K'_+$ *by Theorem 4.1 (*$1°$*), and if* K *is relatively solid and algebraically closed, then* $\operatorname{icor} K' \subseteq K'_\oplus$ *by Theorem 4.1 (*$3°$*).*

It should be mentioned that assertion $1°$ *in Theorem 5.7 is exactly the result by Adan and Novo [3, Th. 2 (ii)] since*

$$\begin{aligned} K' \setminus l(K') &= \{x' \in E' \mid \forall\, k \in \operatorname{icor} K :\ x'(k) > 0\} \\ &= \{x' \in K' \setminus \{0\} \mid \exists\, k \in \operatorname{icor} K :\ x'(k) > 0\} \end{aligned}$$

for a relatively solid, convex cone $K \subseteq E$ *in view of Theorem 4.8 (*$2°$*) and the proof of Adán and Novo [3, Th. 2 (ii)]. In the case that* K *is solid (i.e.,* $\operatorname{cor} K = \operatorname{icor} K$ *and* K' *is pointed,* $l(K') = \{0\}$*), we recover from assertion* $1°$ *the result by Jahn [17, Th. 5.28]. Moreover, assertion* $3°$ *in Theorem 5.7 is comparable to Jahn [17, Th. 5.18 (a)] (however no pointedness of* K *is needed in our result) while assertion* $4°$ *in Theorem 5.7 is exactly the result by Jahn [17, Th. 5.18 (b)]. Thus, the novel result in Theorem 5.7 is given by assertion* $2°$*.*

6 Conclusions

Convex cones play a fundamental role in nonlinear analysis and optimization theory. In particular, tangent cones as well as normal cones have turned out to be important geometric objects for describing optimality conditions. This article contributed to the understanding of the algebraic interior (core) and the relative algebraic interior (intrinsic core) of convex cones in real liner spaces (which are not necessarily endowed with a topology). Using interesting facts from the field of analysis and optimization under an algebraic setting including specific separation theorems (where a relatively solid, convex set is involved), we derived new representations and properties of the intrinsic core of relatively solid, convex cones in linear spaces (see Sections 3 and 4). In particular, we were able to derive new representations of the intrinsic core of the dual cone of a relatively solid, convex cone in linear spaces.

In forthcoming works, we aim to point out relationships between generalized algebraic interiority notions and corresponding generalized topological interiority notions (such as quasi interiority and quasi-relative interiority notions). Moreover, we aim to extend the results derived in Section 5 for vector optimization problems involving relatively solid, convex cones which are not necessarily pointed.

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