

Tight compact extended relaxations for nonconvex quadratic programming problems with box constraints

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Abstract Cutting planes from the Boolean Quadric Polytope (BQP) can be used to reduce the optimality gap of the \mathcal{NP} -hard nonconvex quadratic program with box constraints (BoxQP). It is known that all cuts of the Chvátal-Gomory closure of the BQP are A -odd cycle inequalities. We obtain a compact extended relaxation of *all* A -odd cycle inequalities, which permits to optimize over the Chvátal-Gomory closure without repeated calls to separation algorithms. In a computational study, we verify the strength of this relaxation and show that we can provide very strong bounds for the BoxQP, even with a plain linear program.

Keywords Nonconvex quadratic programming · Linear relaxation · Chvátal-Gomory closure · Extended formulation

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1 Introduction

In their article *Globally solving nonconvex quadratic programming problems with box constraints via integer programming methods* [3], Bonami, Günlük, and Linderoth show how to solve the \mathcal{NP} -hard nonconvex quadratic program with box constraints, i.e.,

$$\begin{aligned} \min \quad & \frac{1}{2}x^T Qx + c^T x \\ \text{s.t.} \quad & l \leq x \leq u, \\ & Q \in \mathbb{R}^{n \times n}, Q \text{ symmetric,} \end{aligned} \tag{BoxQP}$$

effectively via linear programming techniques. Without loss of generality we assume $l = \mathbf{0}$ and $u = \mathbf{1}$, because l and u are finite. They first obtain a convex relaxation of the BoxQP with a linear objective function. This linearization induces nonlinear constraints, which are replaced by the so-called McCormick inequalities [8]. We denote the resulting *weak* linear relaxation of the BoxQP by $\text{LP}_{\mathcal{M}}$, see Section 4.

However, cutting planes from the Boolean Quadric Polytope

$$\text{BQP} = \text{conv} \left(\text{BQP}^{LP} \cap \mathbb{Z}^n \times \mathbb{Z}^{|E|} \right),$$

where

$$\text{BQP}^{LP} = \left\{ (x, X) \in \mathbb{R}^n \times \mathbb{R}^{|E|} \mid \min\{x_i, x_j\} \geq X_{ij} \geq \max\{0, x_i + x_j - 1\} \right. \\ \left. \forall \{i, j\} \in E \right\},$$

can be used to turn it into a very *strong* relaxation of the BoxQP. In particular, these efficient cuts are Chvátal-Gomory cuts

$$\alpha^T Ax \geq \lceil \alpha^T b \rceil, \quad \alpha \in \mathbb{R}_+^m.$$

Furthermore, they prove that all Chvátal-Gomory cuts for the BQP are $0 - \frac{1}{2}$ -Chvátal-Gomory cuts (i.e., $\alpha \in \{0, \frac{1}{2}\}^m$). Caprara and Fischetti [6] show that separating these cuts is \mathcal{NP} -hard in general. However, Koster, Zymolka and Kutschka [7] study ways to separate them effectively in practice. Fortunately for our purposes, all $0 - \frac{1}{2}$ -Chvátal-Gomory cuts of the BQP can be separated in polynomial time, as they are all dominated by the A -odd cycle inequalities, see [3].

2 A-Odd Cycle Inequalities for the BQP

Let N denote the set $\{1, \dots, n\}$ and

$$E := \{\{i, j\} \in N \times N \mid i \neq j, Q_{ij} \neq 0\}.$$

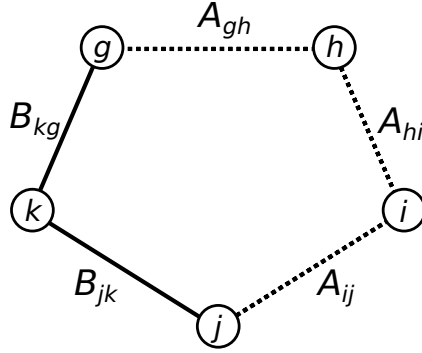


Fig. 1: An odd cycle with respect to $|E^A|$.

Notice that if the ordered pair $\{i, j\}$ is in E , then $\{j, i\} \in E$ because of the symmetry of Q . The McCormick inequalities, cf. [8], for the BQP^{LP} imply

$$0 \leq x_i \leq 1 \quad \forall i \in N,$$

since $0 \leq X_{ij} \leq x_i$ and $x_i + x_j - 1 \leq x_j$ for all $\{i, j\} \in E$. Moreover, if we combine the McCormick inequalities $X_{ij} \geq 0$ and $X_{ij} \geq x_i + x_j - 1$, we have

$$2X_{ij} - x_i - x_j + 1 \geq 0. \quad (A_{ij})$$

Analogously, adding up $x_i \geq X_{ij}$ and $x_j \geq X_{ij}$ yields

$$-2X_{ij} + x_i + x_j \geq 0. \quad (B_{ij})$$

To obtain additional cuts for the BQP, combinations of (A_{ij}) - and (B_{ij}) -inequalities can be useful. Let $E^A \subseteq E$ be the set of all $\{i, j\}$ for which we use inequality (A_{ij}) . Define the set \hat{E}^A as the set that contains an edge ij for every $\{i, j\} \in E^A$. The sets E^B and \hat{E}^B are defined analogously. We combine (A_{ij}) - and (B_{ij}) -inequalities such that $|E^A|$ is odd and $\hat{E}^A \cup \hat{E}^B$ is a simple cycle. Let

$$N^A \subseteq N : \text{vertices incident to exactly two edges in } \hat{E}^A,$$

$$N^B \subseteq N : \text{vertices incident to exactly two edges in } \hat{E}^B.$$

Remark 2.1 Adding inequality (A_{ij}) to (B_{jk}) eliminates variable x_j .

Example 2.2 Let $\{g, h\}, \{h, i\}, \{i, j\}, \{j, k\}, \{k, g\} \in E$. If we add up (A_{gh}) , (A_{hi}) , (A_{ij}) , (B_{jk}) , and (B_{kg}) , see Figure 1, we get

$$2(X_{gh} + X_{hi} + X_{ij} - X_{jk} - X_{kg}) - 2(x_h + x_i) + 2x_k + 3 \geq 0.$$

Subtracting 3 and dividing by 2 yields

$$X_{gh} + X_{hi} + X_{ij} - X_{jk} - X_{kg} - x_h - x_i + x_k \geq -\frac{3}{2}$$

and as all variables on the left hand side are integral, we are able to round up the fractional constant on the right hand side to

$$X_{gh} + X_{hi} + X_{ij} - X_{jk} - X_{kg} - x_h - x_i + x_k \geq -1.$$

In general, adding up inequalities for a simple cycle $\hat{E}^A \cup \hat{E}^B$ yields

$$2 \left(\sum_{\{i,j\} \in E^A} X_{ij} - \sum_{\{i,j\} \in E^B} X_{ij} - \sum_{i \in N^A} x_i + \sum_{i \in N^B} x_i \right) + |E^A| \geq 0.$$

Subtracting $|E^A|$ and dividing by 2 yields

$$\sum_{\{i,j\} \in E^A} X_{ij} - \sum_{\{i,j\} \in E^B} X_{ij} - \sum_{i \in N^A} x_i + \sum_{i \in N^B} x_i \geq -\frac{|E^A|}{2}.$$

In the case $|E^A|$ is odd, we can strengthen this inequality (cf. [3]) to

$$\sum_{\{i,j\} \in E^A} X_{ij} - \sum_{\{i,j\} \in E^B} X_{ij} - \sum_{i \in N^A} x_i + \sum_{i \in N^B} x_i \geq \left\lceil -\frac{|E^A|}{2} \right\rceil,$$

which is equivalent to

$$\sum_{\{i,j\} \in E^A} X_{ij} - \sum_{\{i,j\} \in E^B} X_{ij} - \sum_{i \in N^A} x_i + \sum_{i \in N^B} x_i \geq -\frac{|E^A|}{2} + \frac{1}{2},$$

and yields after another transformation

$$2 \left(\sum_{\{i,j\} \in E^A} \left(X_{ij} + \frac{1}{2} \right) - \sum_{\{i,j\} \in E^B} X_{ij} - \sum_{i \in N^A} x_i + \sum_{i \in N^B} x_i \right) \geq 1. \quad (1)$$

We call cycle inequality (1) *A-odd*, since $|E^A|$ must be odd, but the cycle given by $\hat{E}^A \cup \hat{E}^B$ may be of arbitrary parity.

3 Separation and Extended Formulation

For inequalities (A_{ij}) and (B_{ij}) , we define

$$w_{ij}^A = 2X_{ij} - x_i - x_j + 1, \quad (2)$$

$$w_{ij}^B = -2X_{ij} + x_i + x_j. \quad (3)$$

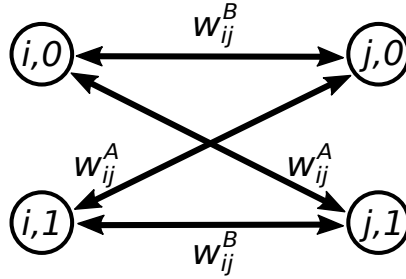
Notice that w_{ij}^A and w_{ij}^B do not depend on some particular \bar{x} or \bar{X} . They are variables restricted by the given equations and obviously nonnegative. We can interpret them as the slack of inequalities (A_{ij}) and (B_{ij}) .

The following construction is similar to the separation algorithm presented by Barahona, Jünger, and Reinelt [2]. Let $G = (V_G, E_G)$ be the simple graph with $V_G = N$ and $E_G = \{ij \mid \{i, j\}, \{j, i\} \in E\}$.

Fig. 2: Digraph F .

Consider the digraph $F = (V_F, A_F)$ with vertex set $V_F = \{0, 1\}$ and arc set $A_F = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ as given in Figure 2. The direct graph product $H = (V_{G.F}, A_{G.F})$ of G and F is given by the vertex set $V_{G.F} = N \times \{0, 1\}$ and the arc set $A_{G.F} = \{((i, r), (j, s)) : ij \in E_G \text{ and } (r, s) \in A_F\}$.

Now we assign arc variable w_{ij}^A to the arcs $((i, r), (j, 1-r)) \in A_{G.F}$ and w_{ij}^B to the arcs $((i, r), (j, r)) \in A_{G.F}$ for all $r \in \{0, 1\}$. Figure 3 shows the structure of H for a given edge $ij \in E_G$.

Fig. 3: Graph product of an edge ij and F .

Whenever we use an (A_{ij}) -inequality for an edge $ij \in E_G$, an arc with arc variable w_{ij}^A in the product graph H is used and the second index of a vertex in H changes from 0 to 1 or from 1 to 0. Otherwise, using a (B_{ij}) -inequality for an edge $ij \in E_G$ corresponds to an arc with arc variable w_{ij}^B in H and the second index does not change. We call a walk and a path, respectively, A -odd if the number of arcs ij with assigned arc variable w_{ij}^A is odd.

Lemma 3.1 *Every (u, r) - (v, s) -walk in H corresponds to an A -odd u - v -walk in G if and only if $r \neq s$.*

Proof Let $r, s \in \{0, 1\}$ with $r \neq s$. Then for every (u, r) - (v, s) -walk in H , the sum over all arc variables includes an odd number of variables w_{ij}^A since the second index does not change when using variables w_{ij}^B . The construction of variables w_{ij}^A that relate to inequalities (A_{ij}) yields that the corresponding u - v -walk in G is A -odd. Analogously, the corresponding u - v -walk in G is A -even if $r = s$.

Lemma 3.2 *Let $(u, r) \neq (v, s)$. If a shortest (u, r) - (v, s) -walk in H has weight l , then there exists a (u, r) - (v, s) -path in H of weight l .*

Proof Let P be a shortest A -odd (u, r) - (v, s) -walk in H with weight l . If P is a path, then there is nothing to show. Otherwise, there exists a closed walk through some vertex (w, t) in P with weight ≤ 0 . Since all the edge variables in H are nonnegative, this closed walk has weight 0. Notice that it is A -even, since for every walk starting and ending in vertex (w, t) , the second index alternates an even number of times between 0 and 1. Therefore, removing this walk from P does not change l or the parity of E^A . Removing all closed walks from P yields an A -odd (u, r) - (v, s) -path in H of weight l .

Lemma 3.3 *The weight of a shortest A -odd cycle in G is equal to the weight of a shortest $(i, 0)$ - $(i, 1)$ -path in H among all $i \in N$.*

Proof Notice first that the weight of a shortest $(i, 0)$ - $(i, 1)$ -path in H is equal to the weight of a shortest $(i, 1)$ - $(i, 0)$ -path in H because of arcs and arc weights being symmetric. Let P be a shortest $(i, 0)$ - $(i, 1)$ -path of all $(i, 0)$ - $(i, 1)$ -paths in H with $i \in N$. If the first index of all vertices except $(i, 0)$ and $(i, 1)$, which serve as start and end point, on P is different, then there is nothing to show. Otherwise, if for some j both vertices $(j, 0)$ and $(j, 1)$ lie on P , the subpath between $(j, 0)$ and $(j, 1)$ cannot have more weight than P as we do not have negative arc weights. Conversely, the subpath between $(j, 0)$ and $(j, 1)$ cannot have less weight than P by the assumption of P being one of the shortest of all $(i, 0)$ - $(i, 1)$ -paths in H with $i \in N$. Without loss of generality we can update P by a shortest $(j, 0)$ - $(j, 1)$ -path. Successively, we end up in the first case.

Lemma 3.4 *Given $(\bar{x}, \bar{X}) \in BQP^{LP}$. Then (\bar{x}, \bar{X}) violates an A -odd cycle inequality if and only if there exists a path P from $(i, 0)$ to $(i, 1)$ in H for some $i \in N$ of \bar{w} -weight less than 1.*

Proof Let $\hat{E}^A \cup \hat{E}^B$ be the edge set of a simple cycle C in G whose A -odd cycle inequality is violated by (\bar{x}, \bar{X}) . Then

$$2 \left(\sum_{\{i,j\} \in E^A} (\bar{X}_{ij} + \frac{1}{2}) - \sum_{\{i,j\} \in E^B} \bar{X}_{ij} - \sum_{i \in N^A} \bar{x}_i + \sum_{i \in N^B} \bar{x}_i \right) < 1.$$

by inequality (1). The left hand side can be transformed to

$$\sum_{\{i,j\} \in E^A} (2\bar{X}_{ij} + 1 - \bar{x}_i - \bar{x}_j) + \sum_{\{i,j\} \in E^B} (-2\bar{X}_{ij} + \bar{x}_i + \bar{x}_j),$$

since \bar{x}_j is eliminated for all edge pairs ij and jk , where one of these edges appears in \hat{E}^A and the other in \hat{E}^B , see Remark 2.1. This equals

$$\sum_{\{i,j\} \in E^A} \bar{w}_{ij}^A + \sum_{\{i,j\} \in E^B} \bar{w}_{ij}^B$$

and because $|E^A|$ is odd, there exist two paths from $(i, 0)$ to $(i, 1)$ and from $(i, 1)$ to $(i, 0)$ in H for all $i \in C$, that add up the same \bar{w}_{ij}^A and \bar{w}_{ij}^B as given above. Thus, the weight of each of these paths is less than 1.

For the converse, consider a path from $(i, 0)$ to $(i, 1)$ in H with weight less than 1. Analogously, there exists an A -odd cycle in G of equal weight.

With Lemmas 3.1, 3.2, 3.3, 3.4 and the separation algorithm presented in [2], we can state the following theorem:

Theorem 3.5 *For fixed (\bar{x}, \bar{X}) , the separation problem for the A -odd cycle inequalities of the BQP can be solved by computing the weight of a shortest odd path from $(i, 0)$ to $(i, 1)$ in H for every $i \in N$. If every and hence the shortest of these paths has weight at least 1, then (\bar{x}, \bar{X}) does not violate any A -odd cycle inequality.*

Theorem 3.5 allows us to solve the separation problem for the A -odd cycle inequalities of the BQP with a linear program. Ahuja et al. [1, Chapter 9.4] show how shortest path problems can be solved by a special case of the dual minimum cost flow problem. We apply their technique and consider for fixed $i \in N$ the LP

$$\begin{aligned} \max \quad & f_{i0i1} \\ \text{s.t.} \quad & f_{i0i0} = 0, \\ & f_{i0js} \leq f_{i0kt} + \bar{w}_{kj}^A \quad \forall \{k, j\} \in E, s, t \in \{0, 1\}, s \neq t, \\ & f_{i0js} \leq f_{i0kt} + \bar{w}_{kj}^B \quad \forall \{k, j\} \in E, s, t \in \{0, 1\}, s = t, \end{aligned}$$

with

$$\begin{aligned} \bar{w}_{kj}^A &= 2\bar{X}_{kj} - \bar{x}_k - \bar{x}_j + 1, \\ \bar{w}_{kj}^B &= -2\bar{X}_{kj} + \bar{x}_k + \bar{x}_j. \end{aligned}$$

If the objective value of a solution of this LP is greater or equal than 1 for every $i \in N$, then (\bar{x}, \bar{X}) fulfills all A -odd cycle inequalities of the BQP.

Using this idea, we obtain the following compact extended formulation that enforces all A -odd cycle inequalities. Notice that x and X as well as w are variables in contrast to what we have in the separation LP from above.

Theorem 3.6 *The linear system*

$$f_{irir} = 0 \quad \forall i \in N, r \in \{0, 1\}, \quad (4)$$

$$f_{irjs} \leq f_{irkt} + w_{kj}^A \quad \forall \{k, j\} \in E, i \in N, r, s, t \in \{0, 1\}, s \neq t, \quad (5)$$

$$f_{irjs} \leq f_{irkt} + w_{kj}^B \quad \forall \{k, j\} \in E, i \in N, r, s, t \in \{0, 1\}, s = t, \quad (6)$$

$$f_{i0i1} \geq 1 \quad \forall i \in N, \quad (7)$$

together with equations (2) and (3) is an extended formulation of the (potentially exponentially many) A -odd cycle inequalities of the BQP and therefore provides a relaxation for the BQP.

Proof Let $(\bar{x}, \bar{X}) \in \text{BQP}^{LP}$. Then the weights \bar{w}^A and \bar{w}^B are explicitly given by equations (2) and (3).

We first show that if inequalities (1) are fulfilled by (\bar{x}, \bar{X}) , then for every pair (i, r) and (j, s) in $V_{G.F}$ there exists \bar{f}_{irjs} such that $(\bar{x}, \bar{X}, \bar{f})$ is feasible for inequalities (4)–(7). Define \bar{f}_{irjs} as the weight of a shortest (i, r) - (j, s) -path in H if such a path exists. Otherwise, assign a large value to \bar{f}_{irjs} . Inequalities (4) are obviously fulfilled, as shortest paths from a vertex to itself have weight 0 in digraphs where all arc weights are nonnegative. Inequalities (5) and (6) express that the weight of a shortest (i, r) - (j, s) -path cannot exceed the weight of an (i, r) - (j, s) -path where the last arc is fixed, which is always true. Finally, Lemma 3.4 ensures that inequalities (7) are fulfilled.

Conversely, let $(\bar{x}, \bar{X}, \bar{f})$ be feasible for inequalities (4)–(7). Then $\bar{f}_{irir} = 0$ for every $i \in N$ and $r \in \{0, 1\}$ by equations (4), which is equal to the weight of a shortest path from vertex (i, r) in H to itself. Consider the case $(k, t) = (i, r)$ in inequalities (5) and (6). For every $\{i, j\} \in E$, variables \bar{f}_{irjs} are bounded from above by arc variables w_{ij}^A if $r \neq s$. Moreover, variables \bar{f}_{irjs} are bounded from above by arc variables w_{ij}^B if $r = s$. Taking those cases for inequalities (5) and (6) into account, where $(k, t) \neq (i, r)$, every variable \bar{f}_{irjs} is bounded from above by the weight of a shortest path from (i, r) to (j, s) in H . Thus, for every vertex pair (i, r) and (j, s) in H , the value \bar{f}_{irjs} is lower or equal than the weight of a shortest path from (i, r) to (j, s) in H . Since \bar{f}_{i0i1} is lower or equal than the weight of a shortest $(i, 0)$ - $(i, 1)$ -path in H and $\bar{f}_{i0i1} \geq 1$ for $i \in N$, every shortest $(i, 0)$ - $(i, 1)$ -path in H has weight at least 1. This holds for every $i \in N$ and therefore all A -odd cycle inequalities (1) are fulfilled by (\bar{x}, \bar{X}) , see Lemma 3.4.

Remark 3.7 *Our extended A -odd cycle formulation in Theorem 3.6 requires $4n^2$ additional variables f , whereas the w^A - and w^B -defining equations can be replaced by their definition in terms of x and X . In total, $8|E|n + n$ inequalities are added. Notice that \bar{f}_{irir} for all $i \in N$ and $r \in \{0, 1\}$ are just constant numbers.*

4 Numerical Experiments

Bonami, Günlük, and Linderoth [3] construct a weak linear relaxation of the BoxQP by linearizing the objective function and relaxing the nonlinear con-

straints. The resulting LP is given by

$$\begin{aligned}
\min \quad & \sum_{\{i,j\} \in E} Q_{ij} X_{ij} + \frac{1}{2} \sum_{i \in N} Q_{ii} Y_i + \sum_{i \in N} c_i x_i \\
\text{s.t.} \quad & x_i \geq Y_i \geq 2x_i - 1 & \forall i \in N, \\
& Y_i \geq 0 & \forall i \in N, \\
& x_i \geq X_{ij} & \forall \{i,j\} \in E, \\
& x_j \geq X_{ij} \geq x_i + x_j - 1 & \forall \{i,j\} \in E, \\
& X_{ij} \geq 0 & \forall \{i,j\} \in E.
\end{aligned} \tag{LP}_{\mathcal{M}}$$

Furthermore, they strengthen it to

$$\begin{aligned}
\min \quad & \sum_{\{i,j\} \in E} Q_{ij} X_{ij} + \frac{1}{2} \sum_{i \in N^-} Q_{ii} Y_i + \frac{1}{2} \sum_{i \in N^+} Q_{ii} x_i^2 + \sum_{i \in N} c_i x_i \\
\text{s.t.} \quad & x_i \geq Y_i & \forall i \in N^-, \\
& x_i \geq X_{ij} & \forall \{i,j\} \in E, \\
& x_j \geq X_{ij} \geq x_i + x_j - 1 & \forall \{i,j\} \in E, \\
& X_{ij} \geq 0 & \forall \{i,j\} \in E, \\
& 1 \geq x_i \geq 0 & \forall i \in N,
\end{aligned} \tag{QP}_{\mathcal{M}^2}$$

where $N^+ := \{i \in N \mid Q_{ii} \geq 0\}$ and $N^- := \{i \in N \mid Q_{ii} < 0\}$ partition N . Although $\text{QP}_{\mathcal{M}^2}$ has a convex quadratic objective and only linear constraints apart from that, solving the pure $\text{LP}_{\mathcal{M}}$ seems to be much faster in general. However, $\text{QP}_{\mathcal{M}^2}$ provides better lower bounds for the BoxQP.

In a computational study, Bonami, Günlük, and Linderoth [3] add $0 - \frac{1}{2}$ -Chvátal-Gomory cuts heuristically to $\text{LP}_{\mathcal{M}}$ and to $\text{QP}_{\mathcal{M}^2}$ in CPLEX for the 99 BoxQP test instances of Nemhauser and Vandenbussche [9], Burer and Vandenbussche [5], and Burer [4]. Our contribution is to compute the bounds that arise from *exact* A -odd cycle separation for the pure linear program $\text{LP}_{\mathcal{M}}$. To this end, we add the extended relaxation from Theorem 3.6 to the constraint set of $\text{LP}_{\mathcal{M}}$ and solve the resulting LP with CPLEX v. 12.8.0.0 on the same benchmark set. Adding the extended relaxation from Theorem 3.6 to $\text{QP}_{\mathcal{M}^2}$ gives a convex quadratic program with a large amount of variables and inequalities. Although solving these QPs in reasonable running time does not seem to be promising, we compute the solutions for some instances, i.e. all instances where $n \leq 40$ whose density is not too high.

Let d be the percentage of non-zeros in Q . An instance is called *sparse*, *medium*, or *dense*, if $d \leq 40\%$, $40\% < d \leq 60\%$, or $d > 60\%$, respectively. Moreover, we divide these classes further into *small* ($n \in \{20, 30, 40\}$), *medium* ($n \in \{50, 60, 70\}$), *large* ($n \in \{80, 90\}$), and *jumbo* ($n \in \{100, 125\}$).

The optimality gap is defined as

$$\text{gap}(z) := \left| \frac{z_{\text{BoxQP}} - z}{z} \right| \times 100,$$

| Density | $\text{gap}(z_{\mathcal{M}})$ | $\text{gap}(z_{\mathcal{M}^2})$ | $\text{gap}(z_{\mathcal{M}}^{\bullet})$ | $\text{gap}(z_{\mathcal{M}^2}^{\bullet})$ |
|---------|-------------------------------|---------------------------------|---|---|
| Sparse | 28.02 | 27.11 | 0.69 | 0.30 |
| Medium | 38.15 | 37.24 | 1.37 | 0.61 |
| Dense | 44.43 | 43.50 | 1.44 | – |

Table 1: Average optimality gap for instances with $n \in \{20, 30, 40\}$. Time limit exceeded when solving $\text{QP}_{\mathcal{M}^2}^{\bullet}$ for dense instances with $n = 40$.

where z_{BoxQP} is the optimal objective value of the BoxQP and z is the optimal objective value of the considered relaxation.

We denote the bounds arising from $\text{LP}_{\mathcal{M}}$ and $\text{QP}_{\mathcal{M}^2}$, respectively, by $z_{\mathcal{M}}$ and $z_{\mathcal{M}^2}$. For the $\text{LP}_{\mathcal{M}}$ strengthened with inequalities (2)–(7) we use the notation $\text{LP}_{\mathcal{M}}^{\bullet}$. The bound arising from an optimal solution of $\text{LP}_{\mathcal{M}}^{\bullet}$ is denoted by $z_{\mathcal{M}}^{\bullet}$. Analogously, we use the notation $z_{\mathcal{M}^2}^{\bullet}$ for the bound given by $\text{QP}_{\mathcal{M}^2}^{\bullet}$, which is the extension of $\text{QP}_{\mathcal{M}^2}$.

The set of small test instances is partitioned into 6 sparse, 9 medium dense, and 27 dense instances. Table 1 specifies how much of the optimality gap is closed by the A -odd cycle inequalities when adding them to $\text{LP}_{\mathcal{M}}^{\bullet}$ and $\text{QP}_{\mathcal{M}^2}^{\bullet}$, respectively.

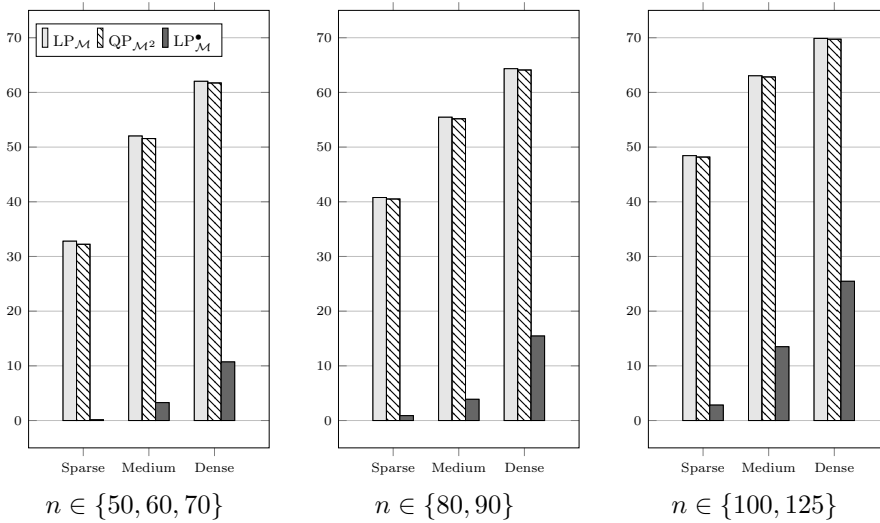


Fig. 4: Average optimality gap for classified instances.

The average optimality gap left by $\text{LP}_{\mathcal{M}}$, $\text{QP}_{\mathcal{M}^2}$, and $\text{LP}_{\mathcal{M}}^{\bullet}$, respectively, for all instances with $n \geq 50$ is visualized graphically in Figure 4. We obtain that the impact of the A -odd cycle inequalities increases when decreasing the

density of Q . Especially on sparse instances, the optimality gap is reduced tremendously by using the relaxation $LP_{\mathcal{M}}^{\bullet}$. For all test instances, bounds and optimality gaps are listed in the appendix.

5 Conclusion

We showed how to construct a tight compact extended relaxation for nonconvex QP with box constraints by enforcing the A -odd cycle inequalities for the BQP. Therefore, we are able to avoid executing a separation algorithm in multiple rounds. On a large benchmark set, our computational results illustrate how efficient it is to strengthen the weak linear relaxation $LP_{\mathcal{M}}$. Since our strengthened relaxation remains linear, it is applicable in practice.

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A Numerical Results

Table 2 is a modification of Table 9 from [3], which includes the columns z_{BoxQP} , $z_{\mathcal{M}}$, and $z_{\mathcal{M}^2}$. The bounds provided by the relaxations $\text{LP}_{\mathcal{M}}$ and $\text{QP}_{\mathcal{M}^2}$, respectively, where $0 - \frac{1}{2}$ -Chvátal-Gomory cuts of the BQP are added heuristically, are replaced by the values $z_{\mathcal{M}}^{\bullet}$ and $z_{\mathcal{M}^2}^{\bullet}$ that arise from exact separation. Notice that the bounds $z_{\mathcal{M}^2}^{\bullet}$ are at least as strong as the bounds $z_{\mathcal{M}}^{\bullet}$. However, computing $z_{\mathcal{M}^2}^{\bullet}$ was only possible in reasonable time for instances with $n \in \{20, 30\}$ and for half of the instances with $n = 40$.

The first number in the name of the test instance is equal to n , i.e., the number of variables of the BoxQP. Moreover, the second number expresses the density of Q . The third number enumerates different test instances that have similar parameters.

Table 2: Bounds for BoxQP relaxations. All values in columns $z_{\mathcal{M}}$, $z_{\mathcal{M}^2}$, and z_{BoxQP} were taken from [3].

| Name | $z_{\mathcal{M}}$ | $z_{\mathcal{M}^2}$ | $z_{\mathcal{M}}^{\bullet}$ | $z_{\mathcal{M}^2}^{\bullet}$ | z_{BoxQP} |
|---------------|-------------------|---------------------|-----------------------------|-------------------------------|--------------------|
| spar020-100-1 | -1,066.00 | -1,038.38 | -706.50 | -706.50 | -706.50 |
| spar020-100-2 | -1,289.00 | -1,258.38 | -880.25 | -867.14 | -856.50 |
| spar020-100-3 | -1,168.50 | -1,142.00 | -772.00 | -772.00 | -772.00 |
| spar030-060-1 | -1,454.75 | -1,430.00 | -730.06 | -714.21 | -706.00 |
| spar030-060-2 | -1,699.50 | -1,668.25 | -1,385.50 | -1,379.18 | -1,377.17 |
| spar030-060-3 | -2,047.00 | -2,006.50 | -1,323.56 | -1,305.97 | -1,293.50 |
| spar030-070-1 | -1,569.00 | -1,547.25 | -703.86 | -688.50 | -654.00 |
| spar030-070-2 | -1,940.25 | -1,888.25 | -1,321.75 | -1,315.82 | -1,313.00 |
| spar030-070-3 | -2,302.75 | -2,251.12 | -1,695.00 | -1,677.00 | -1,657.40 |
| spar030-080-1 | -2,107.50 | -2,072.00 | -988.93 | -967.73 | -952.73 |
| spar030-080-2 | -2,178.25 | -2,158.12 | -1,597.00 | -1,597.00 | -1,597.00 |
| spar030-080-3 | -2,403.50 | -2,376.25 | -1,813.50 | -1,809.78 | -1,809.78 |
| spar030-090-1 | -2,423.50 | -2,385.12 | -1,296.50 | -1,296.50 | -1,296.50 |
| spar030-090-2 | -2,667.00 | -2,622.75 | -1,478.00 | -1,470.64 | -1,466.84 |
| spar030-090-3 | -2,538.25 | -2,499.38 | -1,494.00 | -1,494.00 | -1,494.00 |
| spar030-100-1 | -2,602.00 | -2,541.50 | -1,235.38 | -1,227.38 | -1,227.12 |
| spar030-100-2 | -2,729.25 | -2,698.88 | -1,260.50 | -1,260.50 | -1,260.50 |
| spar030-100-3 | -2,751.75 | -2,703.75 | -1,541.50 | -1,524.07 | -1,511.05 |
| spar040-030-1 | -1,088.00 | -1,067.00 | -839.50 | -839.50 | -839.50 |
| spar040-030-2 | -1,635.00 | -1,617.75 | -1,431.50 | -1,429.36 | -1,429.00 |
| spar040-030-3 | -1,303.25 | -1,297.12 | -1,086.00 | -1,086.00 | -1,086.00 |
| spar040-040-1 | -1,606.25 | -1,575.50 | -856.82 | -847.93 | -837.00 |
| spar040-040-2 | -1,920.75 | -1,895.75 | -1,428.00 | -1,428.00 | -1,428.00 |
| spar040-040-3 | -2,039.75 | -2,017.25 | -1,193.00 | -1,179.26 | -1,173.50 |
| spar040-050-1 | -2,146.25 | -2,120.88 | -1,157.00 | -1,154.73 | -1,154.50 |
| spar040-050-2 | -2,357.25 | -2,334.88 | -1,435.50 | -1,432.04 | -1,430.98 |
| spar040-050-3 | -2,616.00 | -2,603.00 | -1,658.00 | -1,653.63 | -1,653.63 |
| spar040-060-1 | -2,872.00 | -2,817.88 | -1,390.40 | -1,365.00 | -1,322.67 |

| | | | | | |
|---------------|------------|------------|-----------|-----------|-----------|
| spar040-060-2 | -2,917.50 | -2,872.62 | -2,014.00 | -2,006.03 | -2,004.23 |
| spar040-060-3 | -3,434.00 | -3,386.12 | -2,454.50 | -2,454.50 | -2,454.50 |
| spar040-070-1 | -3,144.00 | -3,070.12 | -1,605.00 | - | -1,605.00 |
| spar040-070-2 | -3,369.25 | -3,323.00 | -1,867.50 | - | -1,867.50 |
| spar040-070-3 | -3,760.25 | -3,724.50 | -2,444.00 | - | -2,436.50 |
| spar040-080-1 | -3,846.50 | -3,788.62 | -1,838.50 | - | -1,838.50 |
| spar040-080-2 | -3,833.00 | -3,775.38 | -1,952.50 | - | -1,952.50 |
| spar040-080-3 | -4,361.50 | -4,311.12 | -2,561.50 | - | -2,545.50 |
| spar040-090-1 | -4,376.75 | -4,325.50 | -2,135.50 | - | -2,135.50 |
| spar040-090-2 | -4,357.75 | -4,304.38 | -2,123.29 | - | -2,113.00 |
| spar040-090-3 | -4,516.75 | -4,453.38 | -2,540.00 | - | -2,535.00 |
| spar040-100-1 | -5,009.75 | -4,932.12 | -2,487.50 | - | -2,476.38 |
| spar040-100-2 | -4,902.75 | -4,855.25 | -2,146.25 | - | -2,102.50 |
| spar040-100-3 | -5,075.75 | -5,017.25 | -2,192.17 | - | -1,866.07 |
| spar050-030-1 | -1,858.25 | -1,837.75 | -1,324.50 | - | -1,324.50 |
| spar050-030-2 | -2,334.00 | -2,324.62 | -1,669.00 | - | -1,668.00 |
| spar050-030-3 | -2,107.25 | -2,093.75 | -1,461.00 | - | -1,453.61 |
| spar050-040-1 | -2,632.00 | -2,580.62 | -1,411.00 | - | -1,411.00 |
| spar050-040-2 | -2,923.25 | -2,891.88 | -1,753.50 | - | -1,745.76 |
| spar050-040-3 | -3,273.50 | -3,236.00 | -2,094.50 | - | -2,094.50 |
| spar050-050-1 | -3,536.00 | -3,506.25 | -1,409.72 | - | -1,198.41 |
| spar050-050-2 | -3,500.50 | -3,467.12 | -1,776.81 | - | -1,776.00 |
| spar050-050-3 | -4,119.75 | -4,052.12 | -2,138.34 | - | -2,106.10 |
| spar060-020-1 | -1,757.25 | -1,745.50 | -1,212.00 | - | -1,212.00 |
| spar060-020-2 | -2,238.25 | -2,230.00 | -1,925.50 | - | -1,925.50 |
| spar060-020-3 | -2,098.75 | -2,081.00 | -1,483.00 | - | -1,483.00 |
| spar070-025-1 | -3,832.75 | -3,788.88 | -2,545.00 | - | -2,538.91 |
| spar070-025-2 | -3,248.00 | -3,232.88 | -1,888.50 | - | -1,888.00 |
| spar070-025-3 | -4,167.25 | -4,148.38 | -2,819.25 | - | -2,812.28 |
| spar070-050-1 | -7,210.75 | -7,151.12 | -3,356.00 | - | -3,252.50 |
| spar070-050-2 | -6,620.00 | -6,573.88 | -3,296.00 | - | -3,296.00 |
| spar070-050-3 | -7,522.00 | -7,473.88 | -4,306.50 | - | -4,306.50 |
| spar070-075-1 | -11,647.75 | -11,578.12 | -5,003.67 | - | -4,655.50 |
| spar070-075-2 | -10,884.75 | -10,793.38 | -4,504.92 | - | -3,865.15 |
| spar070-075-3 | -11,262.25 | -11,162.38 | -4,862.75 | - | -4,329.40 |
| spar080-025-1 | -4,840.75 | -4,829.12 | -3,157.00 | - | -3,157.00 |
| spar080-025-2 | -4,378.50 | -4,351.00 | -2,361.62 | - | -2,312.34 |
| spar080-025-3 | -5,130.25 | -5,102.88 | -3,101.00 | - | -3,090.88 |
| spar080-050-1 | -9,783.25 | -9,696.62 | -4,025.80 | - | -3,448.10 |
| spar080-050-2 | -9,270.00 | -9,205.50 | -4,450.50 | - | -4,449.20 |
| spar080-050-3 | -10,029.75 | -9,967.25 | -4,961.27 | - | -4,886.00 |

| | | | | | |
|---------------|------------|------------|------------|---|------------|
| spar080-075-1 | -15,250.75 | -15,154.75 | -6,601.92 | - | -5,896.00 |
| spar080-075-2 | -14,246.50 | -14,146.62 | -5,953.17 | - | -5,341.00 |
| spar080-075-3 | -14,961.50 | -14,860.88 | -6,584.00 | - | -5,980.50 |
| spar090-025-1 | -6,171.50 | -6,135.25 | -3,423.78 | - | -3,372.50 |
| spar090-025-2 | -6,015.00 | -5,978.38 | -3,550.65 | - | -3,500.29 |
| spar090-025-3 | -6,698.25 | -6,681.88 | -4,299.00 | - | -4,299.00 |
| spar090-050-1 | -12,584.00 | -12,522.38 | -5,468.90 | - | -5,152.00 |
| spar090-050-2 | -11,920.50 | -11,851.38 | -5,404.36 | - | -5,386.50 |
| spar090-050-3 | -12,514.00 | -12,452.50 | -6,230.59 | - | -6,151.00 |
| spar090-075-1 | -19,054.25 | -18,944.50 | -7,944.92 | - | -6,267.45 |
| spar090-075-2 | -18,245.50 | -18,132.50 | -7,334.75 | - | -5,647.50 |
| spar090-075-3 | -18,929.50 | -18,823.50 | -7,908.50 | - | -6,450.00 |
| spar100-025-1 | -7,660.75 | -7,611.38 | -4,116.48 | - | -4,027.50 |
| spar100-025-2 | -7,338.50 | -7,303.12 | -3,906.07 | - | -3,892.56 |
| spar100-025-3 | -7,942.25 | -7,894.75 | -4,459.25 | - | -4,453.50 |
| spar100-050-1 | -15,415.75 | -15,341.75 | -6,366.84 | - | -5,490.00 |
| spar100-050-2 | -14,920.50 | -14,814.62 | -6,504.93 | - | -5,866.00 |
| spar100-050-3 | -15,564.25 | -15,480.12 | -7,031.72 | - | -6,485.00 |
| spar100-075-1 | -23,387.50 | -23,277.12 | -9,551.75 | - | -7,384.20 |
| spar100-075-2 | -22,440.00 | -22,307.00 | -8,826.42 | - | -6,755.50 |
| spar100-075-3 | -23,243.50 | -23,109.62 | -9,614.25 | - | -7,554.00 |
| spar125-025-1 | -12,251.00 | -12,184.75 | -6,118.03 | - | -5,572.00 |
| spar125-025-2 | -12,732.00 | -12,662.62 | -6,401.29 | - | -6,156.06 |
| spar125-025-3 | -12,650.75 | -12,627.50 | -6,923.00 | - | -6,815.50 |
| spar125-050-1 | -24,993.00 | -24,880.25 | -10,879.42 | - | -9,308.38 |
| spar125-050-2 | -24,810.50 | -24,669.38 | -10,273.75 | - | -8,395.00 |
| spar125-050-3 | -24,424.50 | -24,308.00 | -10,032.50 | - | -8,343.91 |
| spar125-075-1 | -38,202.00 | -38,058.12 | -16,053.67 | - | -12,330.00 |
| spar125-075-2 | -37,466.75 | -37,341.38 | -15,088.58 | - | -10,382.47 |
| spar125-075-3 | -36,202.25 | -36,033.00 | -13,917.67 | - | -9,635.50 |

In Table 3, we list the optimality gap for every relaxation $LP_{\mathcal{M}}$, $QP_{\mathcal{M}^2}$, $LP_{\mathcal{M}}^{\bullet}$, and $QP_{\mathcal{M}^2}^{\bullet}$ on every test instance, except for those $QP_{\mathcal{M}^2}^{\bullet}$ that were not solved.

Table 3: Gap for BoxQP relaxations.

| Name | $\text{gap}(z_{\mathcal{M}})$ | $\text{gap}(z_{\mathcal{M}^2})$ | $\text{gap}(z_{\mathcal{M}}^{\bullet})$ | $\text{gap}(z_{\mathcal{M}^2}^{\bullet})$ |
|---------------|-------------------------------|---------------------------------|---|---|
| spar020-100-1 | 33.72 | 31.96 | 0.00 | 0.00 |
| spar020-100-2 | 33.55 | 31.94 | 2.70 | 1.23 |
| spar020-100-3 | 33.93 | 32.40 | 0.00 | 0.00 |
| spar030-060-1 | 51.47 | 50.63 | 3.30 | 1.15 |

| | | | | |
|---------------|-------|-------|-------|------|
| spar030-060-2 | 18.97 | 17.45 | 0.60 | 0.15 |
| spar030-060-3 | 36.81 | 35.53 | 2.27 | 0.95 |
| spar030-070-1 | 58.32 | 57.73 | 7.08 | 5.01 |
| spar030-070-2 | 32.33 | 30.46 | 0.66 | 0.21 |
| spar030-070-3 | 28.03 | 26.37 | 2.22 | 1.17 |
| spar030-080-1 | 54.79 | 54.02 | 3.66 | 1.55 |
| spar030-080-2 | 26.68 | 26.00 | 0.00 | 0.00 |
| spar030-080-3 | 24.70 | 23.84 | 0.21 | 0.00 |
| spar030-090-1 | 46.50 | 45.64 | 0.00 | 0.00 |
| spar030-090-2 | 45.00 | 44.07 | 0.76 | 0.26 |
| spar030-090-3 | 41.14 | 40.23 | 0.00 | 0.00 |
| spar030-100-1 | 52.84 | 51.72 | 0.67 | 0.02 |
| spar030-100-2 | 53.82 | 53.30 | 0.00 | 0.00 |
| spar030-100-3 | 45.09 | 44.11 | 1.98 | 0.85 |
| spar040-030-1 | 22.84 | 21.32 | 0.00 | 0.00 |
| spar040-030-2 | 12.60 | 11.67 | 0.17 | 0.03 |
| spar040-030-3 | 16.67 | 16.28 | 0.00 | 0.00 |
| spar040-040-1 | 47.89 | 46.87 | 2.31 | 1.29 |
| spar040-040-2 | 25.65 | 24.67 | 0.00 | 0.00 |
| spar040-040-3 | 42.47 | 41.83 | 1.63 | 0.49 |
| spar040-050-1 | 46.21 | 45.57 | 0.22 | 0.02 |
| spar040-050-2 | 39.29 | 38.71 | 0.31 | 0.07 |
| spar040-050-3 | 36.79 | 36.47 | 0.26 | 0.00 |
| spar040-060-1 | 53.95 | 53.06 | 4.87 | 3.10 |
| spar040-060-2 | 31.30 | 30.23 | 0.49 | 0.09 |
| spar040-060-3 | 28.52 | 27.51 | 0.00 | 0.00 |
| spar040-070-1 | 48.95 | 47.72 | 0.00 | – |
| spar040-070-2 | 44.57 | 43.80 | 0.00 | – |
| spar040-070-3 | 35.20 | 34.58 | 0.31 | – |
| spar040-080-1 | 52.20 | 51.47 | 0.00 | – |
| spar040-080-2 | 49.06 | 48.28 | 0.00 | – |
| spar040-080-3 | 41.64 | 40.96 | 0.62 | – |
| spar040-090-1 | 51.21 | 50.63 | 0.00 | – |
| spar040-090-2 | 51.51 | 50.91 | 0.48 | – |
| spar040-090-3 | 43.88 | 43.08 | 0.20 | – |
| spar040-100-1 | 50.57 | 49.79 | 0.45 | – |
| spar040-100-2 | 57.12 | 56.70 | 2.04 | – |
| spar040-100-3 | 63.24 | 62.81 | 14.88 | – |
| spar050-030-1 | 28.72 | 27.93 | 0.00 | – |
| spar050-030-2 | 28.53 | 28.25 | 0.06 | – |
| spar050-030-3 | 31.02 | 30.57 | 0.51 | – |

| | | | | |
|---------------|-------|-------|-------|---|
| spar050-040-1 | 46.39 | 45.32 | 0.00 | – |
| spar050-040-2 | 40.28 | 39.63 | 0.44 | – |
| spar050-040-3 | 36.02 | 35.28 | 0.00 | – |
| spar050-050-1 | 66.11 | 65.82 | 14.99 | – |
| spar050-050-2 | 49.26 | 48.78 | 0.05 | – |
| spar050-050-3 | 48.88 | 48.02 | 1.51 | – |
| spar060-020-1 | 31.03 | 30.56 | 0.00 | – |
| spar060-020-2 | 13.97 | 13.65 | 0.00 | – |
| spar060-020-3 | 29.34 | 28.74 | 0.00 | – |
| spar070-025-1 | 33.76 | 32.99 | 0.24 | – |
| spar070-025-2 | 41.87 | 41.60 | 0.03 | – |
| spar070-025-3 | 32.51 | 32.21 | 0.25 | – |
| spar070-050-1 | 54.89 | 54.52 | 3.08 | – |
| spar070-050-2 | 50.21 | 49.86 | 0.00 | – |
| spar070-050-3 | 42.75 | 42.38 | 0.00 | – |
| spar070-075-1 | 60.03 | 59.79 | 6.96 | – |
| spar070-075-2 | 64.49 | 64.19 | 14.20 | – |
| spar070-075-3 | 61.56 | 61.21 | 10.97 | – |
| spar080-025-1 | 34.78 | 34.63 | 0.00 | – |
| spar080-025-2 | 47.19 | 46.85 | 2.09 | – |
| spar080-025-3 | 39.75 | 39.43 | 0.33 | – |
| spar080-050-1 | 64.76 | 64.44 | 14.35 | – |
| spar080-050-2 | 52.00 | 51.67 | 0.03 | – |
| spar080-050-3 | 51.28 | 50.98 | 1.52 | – |
| spar080-075-1 | 61.34 | 61.09 | 10.69 | – |
| spar080-075-2 | 62.51 | 62.25 | 10.28 | – |
| spar080-075-3 | 60.03 | 59.76 | 9.17 | – |
| spar090-025-1 | 45.35 | 45.03 | 1.50 | – |
| spar090-025-2 | 41.81 | 41.45 | 1.42 | – |
| spar090-025-3 | 35.82 | 35.66 | 0.00 | – |
| spar090-050-1 | 59.06 | 58.86 | 5.79 | – |
| spar090-050-2 | 54.81 | 54.55 | 0.33 | – |
| spar090-050-3 | 50.85 | 50.60 | 1.28 | – |
| spar090-075-1 | 67.11 | 66.92 | 21.11 | – |
| spar090-075-2 | 69.05 | 68.85 | 23.00 | – |
| spar090-075-3 | 65.93 | 65.73 | 18.44 | – |
| spar100-025-1 | 47.43 | 47.09 | 2.16 | – |
| spar100-025-2 | 46.96 | 46.70 | 0.35 | – |
| spar100-025-3 | 43.93 | 43.59 | 0.13 | – |
| spar100-050-1 | 64.39 | 64.22 | 13.77 | – |
| spar100-050-2 | 60.68 | 60.40 | 9.82 | – |

| | | | | |
|---------------|-------|-------|-------|---|
| spar100-050-3 | 58.33 | 58.11 | 7.78 | — |
| spar100-075-1 | 68.43 | 68.28 | 22.69 | — |
| spar100-075-2 | 69.90 | 69.72 | 23.46 | — |
| spar100-075-3 | 67.50 | 67.31 | 21.43 | — |
| spar125-025-1 | 54.52 | 54.27 | 8.92 | — |
| spar125-025-2 | 51.65 | 51.38 | 3.83 | — |
| spar125-025-3 | 46.13 | 46.03 | 1.55 | — |
| spar125-050-1 | 62.76 | 62.59 | 14.44 | — |
| spar125-050-2 | 66.16 | 65.97 | 18.29 | — |
| spar125-050-3 | 65.84 | 65.67 | 16.83 | — |
| spar125-075-1 | 67.72 | 67.60 | 23.20 | — |
| spar125-075-2 | 72.29 | 72.20 | 31.19 | — |
| spar125-075-3 | 73.38 | 73.26 | 30.77 | — |