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Mixed-Integer Optimal Control under Minimum Dwell Time Constraints

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Abstract Tailored Mixed-Integer Optimal Control policies for real-world applications usually have to avoid very short successive changes of the active integer control. *Minimum dwell time* (MDT) constraints express this requirement and can be included into the *combinatorial integral approximation* decomposition, which solves mixed-integer optimal control problems (MIOCPs) by solving one continuous nonlinear program and one mixed-integer linear program (MILP). Within this work, we analyze the integrality gap of MIOCPs under MDT constraints by providing tight upper bounds on the MILP subproblem. We suggest different rounding schemes for constructing MDT feasible control solutions, e.g., we propose a modification of sum-up rounding. A numerical study supplements the theoretical results and compares objective values of integer feasible solutions with its relaxed solutions.

Keywords Mixed-integer linear programming · optimal control · discrete approximations · switched dynamic systems · approximation methods and heuristics · minimum dwell time constraints

Mathematics Subject Classification (2010) 49J15 · 49M25 · 49M27 · 90C11 · 93C30

1 Introduction

We consider mixed-integer optimal control problems (MIOCPs) on the fixed and finite time horizon $\mathscr{T} := [t_0, t_f] \subset \mathbb{R}$ of the following form

$$
\begin{aligned}
\inf_{\mathbf{x},\mathbf{v}} \quad & \Phi(\mathbf{x}(t_f)) \\
\text{s.t.} \quad \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{v}(t)) \qquad \text{for } t \in \mathscr{T}, && (1.1)\\
\mathbf{x}(t_0) &= \mathbf{x}_0, && (1.2)\\
\mathbf{v} &\in V_{\mathrm{MDT}} \subset V. && (1.3)
\end{aligned}
$$

The differential states $\mathbf{x} \in W^{1,\infty}(\mathscr{T}, R^{n_x})$ with fixed initial values $\mathbf{x}_0 \in \mathbb{R}^{n_x}$ are governed by the right-hand side ordinary differential equation (ODE) function $\mathbf{f} : \mathbb{R}^{n_x} \times \{v_1, \dots, v_{n_\omega}\} \to \mathbb{R}^{n_x}$, which is assumed to be continuous in the first argument. We further assume that there exists a solution $\mathbf{x}$ for the above problem. Let $V = L^\infty(\mathscr{T}, \{v_1, \dots, v_{n_\omega}\})$ and $\{v_1, \dots, v_{n_\omega}\} \subset \mathbb{R}^{n_v}$ so that the discrete control function $\mathbf{v} : \mathscr{T} \to \{v_1, \dots, v_{n_\omega}\}$ is assumed to be measurable and is further restricted by minimum dwell time (MDT) constraints represented by the subset V_{MDT}. We stress that this function takes values out of a finite set with cardinality $n_\omega \in \mathbb{N}$ and exclude the trivial case $n_\omega = 1$. We minimize $\Phi \in C^0(\mathbb{R}^{n_x}, \mathbb{R})$ over the end state, which in turn depends on the discrete control function $\mathbf{v}$.

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1.1 Related work

Optimal Control Problems with integer control choices have been investigated in many research articles during the last few decades [12, 14, 31, 40] since they naturally arise in a range of applications. In order to apply the obtained control policy in practice, additional switching constraints are usually needed, such as *minimum dwell time* requirements that describe the necessity of activating an integer control for at least a given minimal duration if at all and analogously for deactivation periods. Recent case studies of MIOCPs with MDT considerations can be found, e.g., for pesticide scheduling in agriculture [2], electric transmission lines [17], solar thermal climate systems [8] and hybrid electric vehicles [30]. MDT constraints attracted also a lot of attention as part of mixed-integer linear programs (MILPs), see [29] for a study of unit-commitment problems and [25] for a corresponding polytope investigation. For a recent work about Model Predictive Control under MDT constraints see [10].

There are MIOCP related problems and approaches that have been discussed in the literature. For example, the optimal control community has been successfully solving so-called bang-bang problems for decades. In contrast to the above problem formulation, the considered linear control problems are required to have the bang-bang property, i.e., the derivative of the Hamiltonian is strictly positive or strictly negative almost everywhere. The main challenge consists in guessing the correct switching order and numerically detecting the switching points. This approach does not work for problems that involve singular or path-constrained arcs. This can be overcome by using a discrete global maximum principle (see [15] for further references). Still, the indirect *first-optimize-then-discretize* approaches have some drawbacks compared to direct methods and it is not clear how combinatorial constraints such as MDT constraints could be incorporated. As part of *direct methods*, one common approach to tackle MIOCPs under MDT constraints is to consider two separate levels of optimization. At the upper level, a *mode insertion gradient* is usually evaluated in order to fix a sequence of active system modes with promising cost function value. At the lower level, the algorithm aims at minimizing the cost functional with respect to the *switching times* and continuous control input, if available. Such approaches can be found in [1,4,12,14,40] and are usually referred to as *transition-time optimization* or *variable time transformation* method. We remark such *switched systems* have also been intensively investigated with respect to stability under *average dwell time*, see [20]. Another approach to include MDT constraints into MIOCPs is to apply dynamic programming, see [9], which is, however, computationally expensive.

1.2 Results for the CIA decomposition

Similar to the two-stage optimization method, the *combinatorial integral approximation (CIA) decomposition* [31, 35], also known as *embedding transformation technique* [5], is based on the idea to solve the problem in several optimization steps. The first idea is to use the partial outer convexification method [31,34] that allows us to reformulate a problem with integer controls into an equivalent problem with a $\{0,1\}^{n_\omega}$-valued control function $\boldsymbol{\omega}$, which indicates that exactly one discrete realization $v_i \in \{v_1,\ldots,v_{n_\omega}\}$ is active for each time point. For this, we replace the Constraint (1.1) by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{n_\omega} \omega_i(t)\mathbf{f}(\mathbf{x}(t),\mathbf{v}_i) \quad \text{and} \quad \sum_{i=1}^{n_\omega} \omega_i(t) = 1, \qquad \text{for } t \in \mathscr{T}. \tag{1.4}$$

Second, discretizing the MIOCP in the spirit of *first-discretize-then-optimize* methods with, e.g., Direct Multiple Shooting [6] or Direct Collocation [37] results in a mixed-integer nonlinear program (MINLP). We introduce a time discretization by the ordered set $\mathscr{G}_N := \{t_0 < \ldots < t_N = t_f\}$ denoting a grid with N intervals and lengths $\Delta_j := t_j - t_{j-1}$, $\bar{\Delta} := \max_j \Delta_j$ as well as $\underline{\Delta} := \min_j \Delta_j$ for $j \in [N]$. The binary control functions $\omega_i(\cdot)$ are assumed be piecewise constant, changing values only on these grid points, so that $\omega(\cdot)$ can be uniquely represented by a matrix $\mathbf{w} \in \{0,1\}^{n_\omega \times N}$. This MINLP becomes a nonlinear program (NLP), if the binary controls are relaxed. We write $\mathbf{a} \in [0,1]^{n_\omega \times N}$ for denoting this relaxed value. After solving this NLP, the resulting $\mathbf{a}^\star$ is approximated in the CIA step - which is an MILP - with binary control values $\mathbf{w}$. The main idea of this CIA problem is to minimize the *integrality gap* $\theta(\mathbf{w})$, which is the accumulated control deviation between $\mathbf{a}^\star$ and $\mathbf{w}$, i.e., $\min_{\mathbf{w}} \theta(\mathbf{w})$, where

$$\theta(\mathbf{w}) := \max_{\substack{i=1,\ldots,n_\omega \\ k=1,\ldots,N}} \left| \sum_{j=1}^{k} (a^\star_{i,j} - w_{i,j})\Delta_j \right|.$$

This work is based on the CIA decomposition due to its advantages:

1. MINLPs fall generally into the class of *NP-hard* problems so that using approaches that bypass the direct solution of such problems is computationally favorable.
2. Convergence results have been proven for MIOCPs without MDT constraints in the sense that, under mild assumptions, the obtained solution was shown to be arbitrarily close to the optimal solution with grid length $\bar{\Delta}$ going to zero [26, 31, 34]. Moreover, the solution of the NLP represents a useful a priori lower bound on the objective, if solved to global optimality.[1].
3. An MILP enables the option to include intuitively a large variety of combinatorial constraints. Numerical case studies showed that the resulting feasible solution is close to the relaxed solution in case the applied combinatorial constraints are not too restrictive [7, 8].

For a generalization of the CIA decomposition to more optimization steps and MILP variants see [41] as well as to the PDE constraint case see [18, 19, 27]. Recently, an extension of the algorithm with the penalty alternating direction method has been made [16], which can be regarded as a feasibility pump [13] variant for MIOCPs.

Instead of minimizing $\theta(\mathbf{w})$ by means of solving an MILP, fast rounding algorithms can be applied, such as sum-up rounding (SUR) [31] and next forced rounding (NFR) [21], which generate binary solutions that still converge to the relaxed binary control solution with vanishing grid length. The SUR scheme computes for $j = 1, \ldots, N$

$$w_{i,j} := \begin{cases} 1, & \text{if } i = \underset{k=1,\ldots,n_\omega}{\operatorname{argmax}} \left\{ \sum_{l=1}^{j} a^\star_{k,l}\Delta_l - \sum_{l=1}^{j-1} w_{k,l}\Delta_l \right\} \quad \text{(break ties arbitrarily)}, \\ 0, & \text{else,} \end{cases} \qquad \text{for } i = 1, \ldots, n_\omega.$$

For defining the NFR algorithm, one needs for all $i = 1, \ldots, n_\omega$ and iteratively for $j = 1, \ldots, N$ the quantity

$$\mathcal{N}_j(i) := \begin{cases} \underset{k=j,\ldots,N}{\operatorname{argmin}} \{\sum_{l=1}^{k} a^\star_{i,l}\Delta_l - \sum_{l=1}^{j-1} w_{i,l}\Delta_l > \bar{\Delta}\}, & \text{if} \quad \sum_{l=1}^{N} a^\star_{i,l}\Delta_l - \sum_{l=1}^{j-1} w_{i,l}\Delta_l > \bar{\Delta}, \\ \infty, & \text{else.} \end{cases} \tag{1.5}$$

A control with index $i \in [n_\omega]$ on interval j is defined to be *next forced*, if and only if

$$\mathcal{N}_j(i) = \min_{k\in[n_\omega]} \mathcal{N}_j(k) \quad \text{and} \quad \mathcal{N}_j(i) < \infty. \tag{1.6}$$

Then, the NFR algorithm sets iteratively for $j = 1, \ldots, N$ the next forced control equal to one (break ties arbitrarily) and if there is no such control, the active control is chosen according to the SUR scheme. We summarize established integrality gap bounds for these schemes in the form of $\theta(\mathbf{w}) \le C(n_\omega)\bar{\Delta}$ in Table 1.

	Next-Forced Rounding (NFR)	Sum-Up Rounding (SUR)	SUR with vanishing constraints
$C(n_\omega) =$	1, see [21],	$\sum_{i=2}^{n_\omega} \frac{1}{i}$, see [24],	$\lfloor n_\omega/2 \rfloor$, see [28].

Table 1 So far known integrality gap bounds for binary control approximation algorithms. By vanishing constraints we refer to constraints of the form $\omega_i(t)g(x(t)) \le 0$, for $i = 1, \ldots, n_\omega$ where g is a smooth function.

1.3 Contribution

We conduct a theoretical analysis of the integrality gap in the presence of MDT constraints. In particular, we prove the upper bound

$$\min_{\mathbf{w}} \theta(\mathbf{w}) \le \tfrac{2n_\omega-3}{2n_\omega-2}(C_{UD} + \bar{\Delta}),$$

where C_{UD} denotes the maximum of the minimum up (MU) and minimum down (MD) duration of all controls. We show that this bound is tight for MU time constraints. As a consequence of this result, the tightest possible bound for the CIA problem is $C(n_\omega) = \frac{2n_\omega-3}{2n_\omega-2}$. The proof is constructive, as we introduce a generalization of the NFR scheme to the MDT setting. We present a rounding modification so that MD times can

[1] Solving an NLP to global optimality is in general computationally expensive. We are therefore content with a local solution constructed by a solver such as `IPOPT` [38], as we elaborate in the numerical results section.

be included explicitly allowing us to deduce an improved integrality gap bound for this case. In the same spirit, we modify SUR so that MDT requirements are satisfied by the obtained binary solution. We test our algorithms on the three tank problem from a benchmark library [33] and evaluate with increasingly restrictive MDT constraints how large the gap between the constructed integer feasible and relaxed solution becomes.

1.4 Outline

We give a problem definition of the MIOCP in Section 2 and describe the proposed CIA decomposition algorithm with (CIA) as subproblem in detail in Section 3. A generalization of NFR to the MDT setting is presented in Section 4. It provides the tools to derive upper bounds on the integrality gap in Section 5. The commonly used SUR scheme can be extended to include also MDT constraints, which we present in Section 6 together with a discussion on its integrality gap. Section 7 provides a numerical example and we finish this article with conclusions in Section 8.

1.5 Notations

Let $[n] := \{1,\dots,n\}, [n]_0 := \{0\} \cup [n]$, for $n \in \mathbb{N}$. We use Gauss' bracket notation, i.e., $\lfloor x \rfloor := \max\{k \in \mathbb{Z} \mid k \leq x\}, x \in \mathbb{R}$, and analogously for $\lceil x \rceil$.

2 Mixed-Integer Optimal Control Problem

We introduce our problem at hand in the discretized and (partial) outer convexified setting, but refer to [22, 31] and, more recently, to [23, 24] for extensive descriptions of partial outer convexification, the continuous MIOCP setting and its relation to the discretized problem.

2.1 Definition of Binary and Relaxed Controls

We base the following definitions on the grid $\mathscr{G}_N$ we already introduced in the previous section.

Definition 1 (Convex combination constraint (Conv) and Matrix sets W, A) *We express the requirement that the columnwise entries of a matrix $(m_{i,j}) \in [0,1]^{n_\omega \times N}$ sum up to one by*

$$\sum_{i \in [n_\omega]} m_{i,j} = 1, \qquad \textit{for } j \in [N], \tag{Conv}$$

and call it convex combination constraint (Conv) in the sequel. Based on this constraint, we define the sets

$$W := \left\{ \mathbf{w} \in \{0,1\}^{n_\omega \times N} \mid \mathbf{w} \textit{ fulfills (Conv)} \right\}, \quad A := Conv(W),$$

where we denote by $Conv(W)$ the convex hull of W.

We notice the geometric nature of W and A. They are the vertices respectively the set of faces of the N-fold iterated standard simplex without the origin and spanned by the n_ω unit vectors.

Definition 2 (Binary ω and relaxed control functions α) *Let the vector of binary control functions ω and its corresponding vector of relaxed control functions α be defined by their function space domains*

$$\begin{aligned}
\Omega &:= \{\omega : \mathscr{T} \to \{0,1\}^{n_\omega} \mid \omega(t) = \mathbf{w}_{\cdot,j}, \; \textit{for } t \in [t_{j-1}, t_j), \, t_{j-1}, t_j \in \mathscr{G}_N, \; j \in [N], \, \mathbf{w} \in W\}, \\
\mathscr{A} &:= \{\alpha : \mathscr{T} \to [0,1]^{n_\omega} \mid \alpha(t) = \mathbf{a}_{\cdot,j}, \; \textit{for } t \in [t_{j-1}, t_j), \, t_{j-1}, t_j \in \mathscr{G}_N, \; j \in [N], \, \mathbf{a} \in A\},
\end{aligned}$$

where $\mathbf{w}_{\cdot,j}$, respectively $\mathbf{a}_{\cdot,j}$, denotes the jth column of $\mathbf{w}$, respectively of $\mathbf{a}$.[2]

[2] We note that ω and α are unspecified on t_N. Since they are defined as L^∞ representatives of an equivalence class in $\mathscr{L}^\infty$, they can be unspecified on measure zero sets such as grid points of $\mathscr{G}_N$.

2.2 Optimal Control Problem Class

We take interest in the discretized binary control problem (DBCP) with MDT constraints and its naturally relaxed problem, the discretized relaxed control problem (DRCP), which arises from replacing ω by α.

Definition 3 (Problems (DBCP) and (DRCP)) *Let a fixed discretization grid $\mathscr{G}_N$ be given together with an MU time $C_U \geq 0$ and an MD time $C_D \geq 0$. We refer to the following general problem class as (**DBCP**)*

$$\begin{aligned}
\inf_{\mathbf{x},\,\omega} \quad & \Phi(\mathbf{x}(t_f)) \\
\text{s.t.} \quad & \textit{(discretized) outer convexified dynamics } (1.4), \\
& \textit{initial value condition for } \mathbf{x} \ (1.2), \\
& \textit{piecewise constant controls: } \omega \in \Omega, \\
& \textit{MDT constraints:} \\
& \quad w_{i,l} \geq w_{i,k+1} - w_{i,k}, \quad \textit{for } i \in [n_\omega], k \in [N-1], l \in \mathscr{J}_{k+1}(C_U), \qquad (2.1) \\
& 1 - w_{i,l} \geq w_{i,k} - w_{i,k+1}, \quad \textit{for } i \in [n_\omega], k \in [N-1], l \in \mathscr{J}_{k+1}(C_D), \qquad (2.2)
\end{aligned}$$

where we denote the intervals affected by the MDT $C_1 = C_U, C_D$ from interval $k \in [N]$ on with the set

$$\mathscr{J}_k(C_1) := \{k\} \cup \{ j \mid t_{j-1} \in \mathscr{G}_N \cap [t_{k-1}, t_{k-1} + C_1)\}.$$

We define a control $i \in [n_\omega]$ to be active on the interval starting with $t_{j-1} \in \mathscr{G}_N$, if and only if $\omega_i(t) = 1$ for $t \in [t_{j-1}, t_j)$ is true and the other way around for inactive controls. If a binary control is active after a switch on t_j, it has to stay active for a time period of at least C_U as required by (2.1)*, whereas* (2.2) *enforces the analogous case for deactivating a control. Finally, we define (**DRCP**) as the canonical relaxation of problem (DBCP) where we optimize over $\alpha \in \mathscr{A}$ instead of $\omega \in \Omega$.*

We remark that our study assumes no mode specific MU times $C_{i,U}$ or MD times $C_{i,D}$, but may include them by setting $C_U = \max_{i \in [n_\omega]} C_{i,U}$ and accordingly $C_D = \max_{i \in [n_\omega]} C_{i,D}$, even though this simplification may result in suboptimal solutions.

Without loss of generality, we omit in our problem definition further constraints and continuous controls $\mathbf{u} \in L^\infty(\mathscr{T}, \mathbb{R}^{n_u})$. See [32] for a discussion of extensions and [27] for PDE constraints.

3 Combinatorial Integral Approximation Decomposition

We recapitulate the CIA decomposition algorithm to solve MIOCPs in Figure 1. This approach is justified by a convergence result in the situation *without* MDT constraints establishing that the objective integer gap of (DBCP) compared to (DRCP) depends linearly on the integrated control deviation under certain regularity assumptions [26, 31].

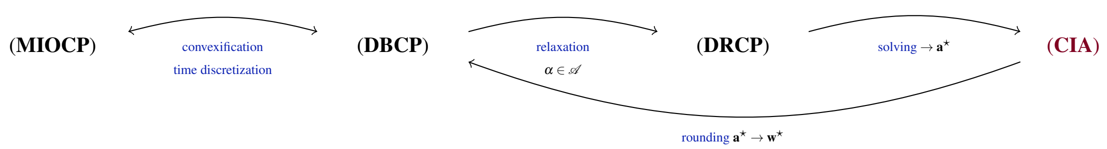

Fig. 1 Schematic representation of the CIA decomposition. MIOCPs can be equivalently reformulated into their partially outer convexified counterpart problem that is thereafter transformed into (DBCP) via a temporal discretization. Next, allowing a convex combination $\mathbf{a}$ in (Conv) yields the relaxed problem (DRCP). After solving this problem we obtain $\mathbf{a}^\star$, which are then approximated with binary values $\mathbf{w}^\star$ in the rounding problem (CIA). Finally, we use $\mathbf{w}^\star$ as fixed variables for solving (DBCP) as a continuous variable problem.

Thus, we are interested in an admissible ω represented by its corresponding value $\mathbf{w} \in W$ with integrality gap bounded by the grid length in the sense of

$$\min_{\mathbf{w} \in W} \max_{i \in [n_\omega], k \in [N]} \left| \sum_{j \in [k]} (a^\star_{i,j} - w_{i,j}) \Delta_j \right| \leq C(n_\omega) \bar{\Delta}, \qquad (3.1)$$

as already mentioned in the introduction, so that the solution of (DRCP) can be arbitrarily well approximated by an integer solution through refining the discretization grid [34]. The SUR scheme constructs solutions ω with this property; in fact, Kirches et al. [24] showed $C \in \mathcal{O}(\log(n_\omega))$ as conjectured in [34]. Hence, the integrality gap obtained by this rounding scheme is going to be arbitrarily small with arbitrarily small grid length, which is why SUR is often applied in the CIA decomposition as rounding step after solving (DRCP). Rather than using SUR, it has been proposed [35] to formulate the problem (3.1) as an MILP for an improved approximation and to be able to consider also different norms (such as the Manhattan norm) and combinatorial constraints. Therefore, we recall its definition and state its MDT variants.

Definition 4 (Problems (CIA), (CIA-U), (CIA-D), and (CIA-UD)) *Let $\mathbf{a}^\star \in A$ be the (local) optimal solution of (DRCP) and assumed to be given. Then, we define the problem (**CIA**) to be*

$$\begin{aligned} \theta^\star := \min_{\theta \in \mathbb{R}_{\geq 0}, \mathbf{w} \in W} \quad & \theta \\ \text{s.t.} \quad & \theta \geq \pm \sum_{l \in [j]} (a^\star_{i,l} - w_{i,l}) \Delta_l, \qquad \text{for } i \in [n_\omega],\ j \in [N]. \end{aligned}$$

The (CIA) problem with added MU time constraint (2.1) *from Definition 3 is in the remainder referred to as (**CIA-U**). Similarly, let us (CIA) with added MD time constraint* (2.2) *call (**CIA-D**) and (CIA) with both* (2.1) *and* (2.2) *(**CIA-UD**).*

Clearly, (CIA) is a reformulation of minimizing (3.1). In Section 5 we are going to elaborate upper bounds for (3.1) in the presence of MDT constraints by investigating its (CIA) variants. Since the lower bound on its objective is trivially zero (and can be reached), bounds always refer to upper bounds in this article.

With respect to the constraints (2.1) and (2.2) we stress that there are other, in many cases computationally more efficient, formulations of MDT constraints, e.g. in the spirit of extended formulations [25]. The latter may lead to relaxations that are less likely to deliver fractional solutions [25] and thus can be beneficial for including the constraints (2.1) and (2.2) into the NLP solving procedure. But, as this study does not focus on the NLP formulation, i.e., how $\mathbf{a}^\star$ is achieved, and since we propose to solve (CIA-UD) by means of tailored rounding heuristics or a branch and bound algorithm and not with a standard MILP solver exploiting extended formulations, we would benefit neither numerically nor theoretically from alternative MILP formulations and we therefore skip the presentation of these.

4 Dwell Time Next Forced Rounding

We introduced in (1.5)-(1.6) the NFR scheme as an algorithm that can compute approximations to solutions of (CIA) in $\mathcal{O}(n_\omega N^2)$ [21] and that constructs feasible solutions of (CIA) with objective no larger than $\bar{\Delta}$. In this section, we introduce dwell time next forced rounding (DNFR) as a generalization, aiming for a scheme that constructs MDT constraint feasible solutions and from which we derive bounds for the (CIA) objective and its MDT variants. Several definitions are needed for DNFR and we begin with a definition of sequences of intervals that are grouped into blocks in the presence of MDT constraints.

Definition 5 (Dwell time block interval sets) *Let an MDT $C_1 \geq 0$ be given. We define iteratively the dwell time invoked interval sets $\mathcal{J}_b$ and their last indices l_b for $b = 1, \dots, n_b$ and with $l_0 := 0$:*

$$\begin{aligned} \mathcal{J}_b &:= \{l_{b-1}+1\} \cup \{ j \mid t_{j-1} \in \mathcal{G}_N \cap [t_{l_{b-1}}, t_{l_{b-1}} + C_1)\}, \\ l_b &:= \max\{ j \mid j \in \mathcal{J}_b \}, \end{aligned}$$

where $n_b := \min\{ b \mid l_b = N\}$ represents the number of interval blocks.

In the following we will sometimes write loosely *block* instead of *dwell time block* for shortening our language. Next, we establish the lengths of dwell time blocks.

Definition 6 (Dwell time block length) *Let a family of dwell time block interval sets $\{\mathcal{J}_b\}_{b \in [n_b]}$ be given. We denote by $\mathcal{L}_b$ the length of dwell time block $b \in [n_b]$ and name the maximum, respectively minimum, length of all dwell time blocks $\overline{\mathcal{L}}$, respectively $\underline{\mathcal{L}}$, i.e.,*

$$\begin{aligned} \mathcal{L}_b &:= t_{l_b} - t_{l_{b-1}}, \qquad b \in [n_b], \\ \overline{\mathcal{L}} &:= \max_{b \in [n_b]} \mathcal{L}_b, \qquad \underline{\mathcal{L}} := \min_{b \in [n_b]} \mathcal{L}_b. \end{aligned}$$

By the definition of dwell time blocks, we see that $\mathscr{L}_b$ depends both on the time discretization $\mathscr{G}_N$ and on C_1. If $C_1 \leq \underline{\Delta}$, then, the blocks are in fact the grid intervals, i.e., $\mathscr{L}_j = \Delta_j,\ j \in [N]$ and $n_b = N$. As soon as $C_1 > \underline{\Delta}$ holds, there is at least one block b with length of two subsequent intervals $\mathscr{L}_b = \Delta_j + \Delta_{j+1},\ j \in [N-1]$. Overall, one recognizes that $\overline{\mathscr{L}}$ increases monotonically with increasing C_1 and obviously stops as soon as $C_1 > t_f - t_0$. The DNFR scheme relies crucially on the block dependent accumulated control deviation, which is why we introduce it as auxiliary variable in the next definition.

Definition 7 (Accumulated control deviation $\theta_{i,j}, \Theta_{i,b}, \gamma_{i,j}, \Gamma_{i,b}$) *Let* $\mathbf{a} \in A$ *and* $\mathbf{w} \in W$*. For controls* $i \in [n_\omega]$ *we define the accumulated control deviation on interval* $j \in [N]$ *as*

$$\theta_{i,j} := \sum_{l=1}^{j} (a_{i,l} - w_{i,l})\Delta_l, \qquad \gamma_{i,j} := \sum_{l=1}^{j} a_{i,l}\Delta_l - \sum_{l=1}^{j-1} w_{i,l}\Delta_l,$$

and further define $\theta_{i,0} := 0$ *for all* $i \in [n_\omega]$*. We introduce for blocks* $b \in [n_b]$ *the analogous variables*

$$\Theta_{i,b} := \theta_{i,l_b}, \qquad \Gamma_{i,b} := \Theta_{i,b-1} + \sum_{j \in \mathscr{J}_b} a_{i,j}\Delta_j.$$

In the sequel, we sometimes write forward control deviation for control i in order to distinguish $\gamma_{i,j}, \Gamma_{i,b}$ *from the (accumulated) control deviation* $\theta_{i,j}, \Theta_{i,b}$*.*

The defined quantities of Definitions 5 - 7 are illustrated in Figure 2.

Remark 1 If a small MDT $C_1 \leq \underline{\Delta}$ is given, $\theta_{i,j}$ equals trivially $\Theta_{i,j}$ and the same holds for $\gamma_{i,j}$ and $\Gamma_{i,j}$. Nevertheless, with an MDT of $C_1 > \underline{\Delta}$ one needs interval *and* block related variables to be able to clearly distinguish between both values.

Remark 2 Let $\mathbf{w} \in W$ and we denote by $\theta(\mathbf{w})$ its (CIA) objective value. With the above definition, we conclude $\theta(\mathbf{w}) = \max_{i \in [n_\omega], j \in [N]} |\theta_{i,j}|$. Generally, one notices that the maximum of the $|\theta_{i,j}|$ values must be assumed at an interval before a switch happens (i.e., $\mathbf{w}_{\cdot,j} \neq \mathbf{w}_{\cdot,j+1}$) or on the last interval, since $|\theta_{i,j}|$ increases monotonically with constant $\mathbf{w}_{\cdot,j}$ and increasing j. Hence, with constant $\mathbf{w}_{\cdot,j}$ on the dwell time blocks, we also have that $\theta(\mathbf{w}) = \max_{i \in [n_\omega], b \in [n_b]} |\Theta_{i,b}|$.

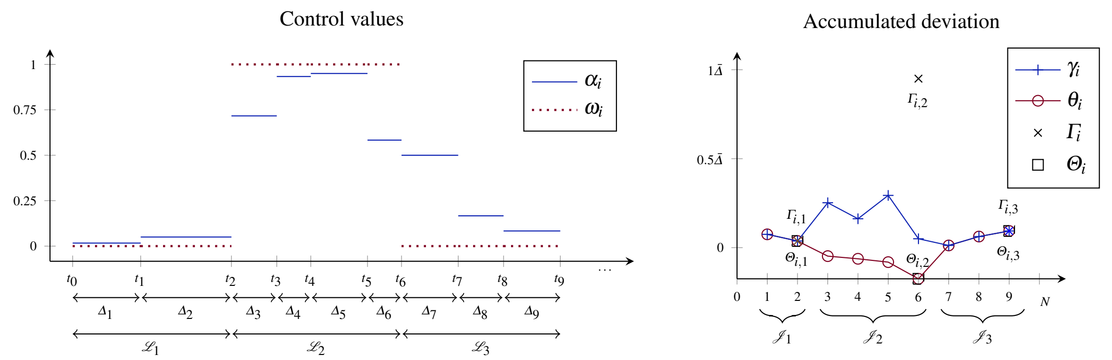

Fig. 2 Left: example binary and relaxed control values on nine intervals Δ_j, respectively three blocks $\mathscr{L}_b$. Right: corresponding accumulated control deviation. The forward control deviation with respect to intervals γ_i is greater or equal to θ_i with equality if the control i is inactive. Also, $\Gamma_{i,b}$ is greater or equal to $\Theta_{i,b}$, where the latter equals via definition the last θ_i before block b begins. Notice that $\Gamma_{i,2}$ sums up the weighted α values for the intervals 3-6, which results in a large value.

We introduced in (1.5)-(1.6) the concept of a *next-forced* control that depends on the maximum grid length $\bar{\Delta}$. We generalize this concept by using blocks and a generic rounding threshold factor $C_2 > 0$ instead of using always $C_2 = 1$ as in the NFR scheme. To this end we present a definition of different types of control variable activations that depend on prior variables choices and on $\mathbf{a} \in A$.

Definition 8 (Admissible, forced, and future forced activation) *Let the rounding threshold factor* $C_2 > 0$ *and* $\mathbf{a} \in A$ *be given. The choice* $w_{i,j} = 1$ *for* $i \in [n_\omega],\ j \in \mathscr{J}_b,\ b = 1, \ldots, n_b$ *is admissible, if*

$$\Gamma_{i,b} \geq -C_2\overline{\mathscr{L}} + \mathscr{L}_b$$

holds. Denote by W_a^b *the set of admissible controls for block b. Similarly, the choice* $w_{i,j} = 1$ *for* $i \in [n_\omega],\ j \in \mathscr{J}_b,\ b = 1,\dots,n_b$ *is forced, if*

$$\Gamma_{i,b} > C_2\overline{\mathscr{L}}$$

holds. We define a control $i \in W_a^b$ *on block b to be l-future forced, if*

$$\Theta_{i,b-1} + \sum_{k=b}^{l} \sum_{j\in\mathscr{J}_k} a_{i,j}\Delta_j > C_2\overline{\mathscr{L}}$$

holds, with the special case $l = b$ *meaning i is actually forced on b. If the above inequality holds for any* $l \le n_b$*, we call the control* $i \in W_a^b$ *on block b to be future forced, and group these controls into the set* W_f^b*.*

Definition 9 (Minimum down time forbidden control) *We introduce the constant* $\chi_D \in \{0,1\}$*. If the CIA problem involves an MD time constraint with* $C_D > \underline{\Delta}$*, we set* $\chi_D = 1$ *and otherwise* $\chi_D = 0$*. We define* $i_b^D,\ b = 3,\dots,n_\omega$*, as the index of the control that has been activated on block* $b-2$ *and deactivated on block* $b-1$ *- if such a control exists:*

$$\exists i \in [n_\omega]\ :\ w_{i,j} = 1, j \in \mathscr{J}_{b-2}\ \wedge\ w_{i,j} = 0, j \in \mathscr{J}_{b-1} \quad \Rightarrow \quad i_b^D := i.$$

Then, let $\mathscr{I}_b^D$ *denote the* χ_D *dependent set of the minimum down time forbidden control*

$$\mathscr{I}_b^D := \begin{cases} \{i_b^D\}, & \text{if } \chi_D = 1, \text{ and } b \ge 3,\\ \emptyset, & \text{otherwise.}\end{cases}$$

Note that $\mathscr{I}_b^D$ is either the empty set or contains exactly one control index. It may seem unintuitive to declare only one control per block as minimum down time forbidden, because a sufficiently large chosen MD time can comprise more than two intervals and therefore more than one control could be minimum down forbidden on certain blocks. However, in such a situation, where several controls are minimum down forbidden, it could happen that only one control may be allowed to be active, resulting in a large control deviation. Consequently, a fine granular definition would be critical for deriving bounds for (CIA-UD) using the DNFR scheme. We will specify such a worst case later in Example 1 as part of Appendix B and argue thereby why we tolerate at most one minimum down time forbidden control per block.

We illustrate the different control activation types of Definitions 8 and 9 in Figure 3.

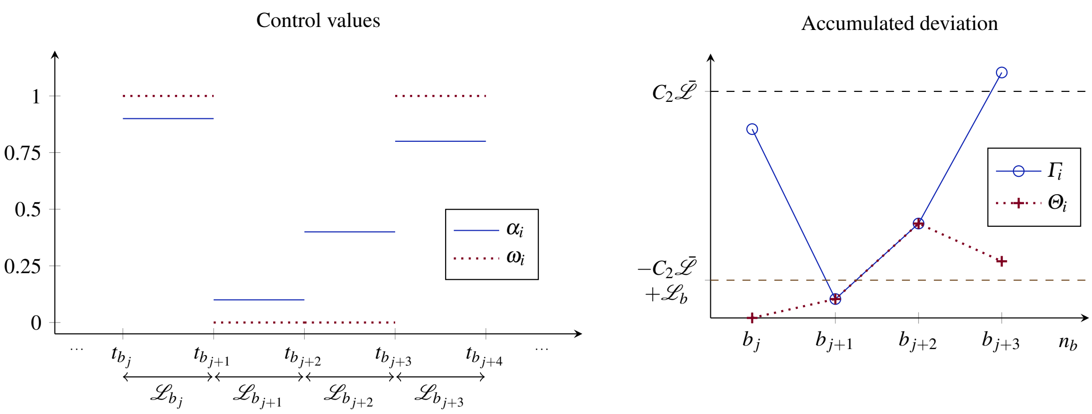

Fig. 3 Exemplary visualization of the defined quantities. Left: binary and relaxed control values on four blocks. Right: corresponding block accumulated control deviation. Control i is *admissible* on block b_j, not *admissible* on block b_{j+1}, *down time forbidden* and b_{j+3}-*future forced* on block b_{j+2} as well as *forced* on block b_{j+3}.

Finally, we use these control properties to declare the DNFR scheme in Algorithm 1. In contrast to the original NFR, we do not iterate over all intervals, but over all dwell time C_1 invoked blocks (line 2) and check on each block whether there is a forced control and activate it in this case (line 3-4). Otherwise, we test if there is an earliest future forced control and if so, we set it to be active (line 5-8). Else, the algorithm selects the control with the maximum forward control deviation (line 9-13), which represents a fallback to the classical SUR scheme. In case the MD mode is turned on by setting $\chi_D = 1$, we exclude the set $\mathscr{I}_b^D$ from our control selection task (line 3, 5, 11). This consideration of minimum down time forbidden controls is a further extension of the original NFR scheme.

Algorithm 1: Dwell time next forced rounding algorithm for (CIA-UD)

Input : Relaxed control values $\mathbf{a} \in A$, time increments Δ_j, $j \in [N]$, parameters C_1, C_2, χ_D.
Output: Feasible solution $\mathbf{w}^{\text{DNFR}}$ of (CIA-UD) with approximation quality depending on C_1, C_2, χ_D.

Initialize $\mathbf{w} = 0$.
for *all dwell time blocks* $b = 1, \dots n_b$ **do**
    if *there is a control* $i \in [n_\omega] \setminus \mathscr{I}_b^D$ *with forced activation* **then**
        Set $w_{i,j} = 1,\ j \in \mathscr{J}_b$.
    else if *it exists a future forced control, i.e.,* $W_f^b \setminus \mathscr{I}_b^D \neq \emptyset$ **then**
        Identify the control with the earliest future forced activation (break ties arbitrarily):
        $i = \arg\min \left\{ b(i) \in [n_b] \mid i \in W_a^b \setminus \mathscr{I}_b^D,\ i \text{ is } b(i)\text{-future forced} \right\}$.
        Set $w_{i,j} = 1,\ j \in \mathscr{J}_b$.
    else
        Find the admissible control with maximum control deviation (break ties arbitrarily):
        $i = \arg\max\{\Gamma_{i,b} \mid i \in W_a^b \setminus \mathscr{I}_b^D\}$.
        Set $w_{i,j} = 1,\ j \in \mathscr{J}_b$.
    end
end
return: $\mathbf{w}^{\text{DNFR}} = \mathbf{w}$;

5 Tight Bounds on the Integrality Gap for (CIA) with Dwell Time Constraints

We now consider the question of how large the objective function value $\theta^\star$ of (CIA) and its MDT variants can become. In other words we examine

$$\theta^{\max} := \max_{\mathbf{a}\in A} \min_{\mathbf{w}\in W} \max_{i\in[n_\omega], j\in[N]} |\theta_{i,j}| \quad \text{s.t.} \quad \text{MDT constraints (2.1), (2.2).}$$

It turns out that the DNFR scheme is particularly suitable for deriving these integrality gap bounds. We state approximation results for (CIA) by means of DNFR constructed solutions. These results are presented as two theorems in Section 5.1 with specified parameter choices for C_2 and χ_D. Afterwards, we deduce specific bounds for (CIA-U), (CIA-D) and (CIA-UD) and evaluate how tight they are in the Sections 5.2-5.3. We begin this section with the trivial upper bound

$$\theta^{\max} \leq \sum_{j\in[N]} \Delta_j = t_f - t_0$$

and another weak result in the following remark.

Remark 3 Neglecting for a moment MDT constraints, it is known from Minimax theory [39] that

$$\max_{\mathbf{a}\in A} \min_{\mathbf{w}\in W} \max_{i\in[n_\omega], j\in[N]} |\theta_{i,j}| \quad \leq \quad \min_{\mathbf{w}\in W} \max_{\mathbf{a}\in A} \max_{i\in[n_\omega], j\in[N]} |\theta_{i,j}|$$

holds. In the right hand side, we maximize over $\mathbf{a}$ for given $\mathbf{w}$ and check, which one of the latter leads to an overall minimum objective. In this way $\mathbf{a}$ manipulates the control deviation to be as large as possible. That means with given $\mathbf{w}$ it is possible to set $a_{i^{\min},j} = 1, j \in [N]$, where $i^{\min}$ is the control with smallest total accumulation $\sum_{j\in[N]} \mathbf{w}_{i,j}\Delta_j$ so that we obtain the (CIA) objective value $\theta^\star = \sum_{j\in[N]}(1 - \mathbf{w}_{i^{\min},j})\Delta_j$. With these arguments one may derive

$$\theta^{\max} \leq \left(N - \left\lfloor \frac{N}{n_\omega} \right\rfloor \right) \bar{\Delta}.$$

We omit the exact proof since this bound is generally weak as we will see later in this section.

5.1 Integrality Gap Results through Dwell Time Next Forced Rounding

We examine in Theorem 1 how large the control deviation can become as part of the DNFR algorithm during an MD time phase. Based on this result we derive in Theorem 2 that DNFR constructs (CIA) feasible solutions with objective bounds depending on the rounding threshold C_2 and whether down time forbidden controls are allowed, i.e., $\chi_D = 1$. The proofs and the corresponding lemmas are moved to Appendix A to enhance readability for readers interested in the results and algorithms.

Theorem 1 (Control accumulation of a *minimum down time forbidden* control) *Let* $\mathbf{a} \in A$, $(C_2, \chi_D) = \left(\frac{3}{2}, 1\right)$ *and* $C_1 \geq 0$ *be given and assume there is a minimum down time forbidden control* $i_D \in \mathscr{I}_b^D$ *on block* $b \geq 3$ *after Algorithm 1 was executed. Then, the forward control deviation satisfies*

$$\Gamma_{i_D,b} \leq \tfrac{3}{2}\overline{\mathscr{L}}.$$

Proof See Appendix A.2.

Theorem 2 (Rounding gap of solution constructed by DNFR) *Let* $\mathbf{a} \in A$ *and the following parameter settings be given:*

I. $(C_2, \chi_D) = \left(\frac{2n_\omega-3}{2n_\omega-2}, 0\right)$,
II. $(C_2, \chi_D) = \left(\frac{3}{2}, 1\right)$,

and $C_1 \geq 0$. *Then,* $\mathbf{w}^{DNFR}$ *obtained by Algorithm 1 is a feasible solution of (CIA) for both cases with approximation quality*

$$\theta(\mathbf{w}^{DNFR}) \;\leq\; C_2\overline{\mathscr{L}}.$$

Proof See Appendix A.3.

5.2 Implications for the Objectives of (CIA-U) and (CIA)

Theorem 2 states only generic approximation results for (CIA) with an MDT parameter C_1. We are going to assess the consequences for (CIA-U) by specifying C_1 and proving tightness of the resulting upper bound. Clearly, (CIA) is a special case of (CIA-U), where $C_U = 0$, so that results for (CIA-U) are inherited by (CIA).

Proposition 1 (Upper bound for (CIA-U)) *Let any MU time* $C_U \geq 0$, *grid* $\mathscr{G}_N$ *and* $\mathbf{a} \in A$ *be given. Then, for (CIA-U) holds:*

$$\theta^\star \leq \frac{2n_\omega - 3}{2n_\omega - 2}\left(C_U + \bar{\Delta}\right).$$

Proof We consider the DNFR scheme with $(C_1, C_2, \chi_D) = \left(C_U, \frac{2n_\omega-3}{2n_\omega-2}, 0\right)$. Then, $\mathbf{w}^{\text{DNFR}}$ is a feasible solution by Theorem 2 and by the property of DNFR to activate dwell time blocks of intervals with length at least $C_1 = C_U$. From the definition of block lengths we conclude $\overline{\mathscr{L}} < C_U + \bar{\Delta}$ so that the assertion follows directly from Theorem 2. □

In order to analyze the obtained bound with respect to tightness, we introduce a grid where the MDT C_1 overlaps the grid points by a small $\varepsilon > 0$. We determine the length of the resulting blocks in the following lemma.

Definition 1 (Minimal C_1-overlapping grid) Let us consider a non-degenerate MDT length, i.e., $C_1 > 0$, and let ε be $C_1 \gg \varepsilon > 0$. Let further a time horizon $[t_0, t_f]$ be given with length at least $4C_1$, i.e.,

$$t_f \geq t_0 + 4C_1.$$

We define a *minimal C_1-overlapping grid* $\mathscr{G}_N$ recursively as follows

$$\begin{aligned} t_1 &:= t_0 + C_1 - \varepsilon, \\ t_2 &:= t_1 + C_1, \\ t_j &:= \begin{cases} t_{j-1} + C_1 - \varepsilon, & \text{if } j \text{ odd}, \\ t_{j-1} + C_1, & \text{if } j \text{ even}, \end{cases} \qquad \text{for } j = 3, \ldots, N-1, \end{aligned}$$

where we set $N - 1 := \max\{j \,|\, t_j < t_f\}$, so that $\mathscr{G}_N$ consists of N intervals.

Lemma 1 (Length of blocks of a minimal C_1-overlapping grid) *The dwell time invoked blocks* $\mathscr{J}_b$, $b \in [n_b]$ *of a minimal* C_1*-overlapping grid as introduced in Definition 1 have the length*

$$\begin{aligned} \mathscr{L}_b &= 2C_1 - \varepsilon, \qquad && b \in [n_b - 1], \\ \mathscr{L}_{n_b} &= t_f - (t_0 + (n_b - 1)(2C_1 - \varepsilon)). \end{aligned}$$

Moreover, we have that

$$\bar{\Delta} = C_1, \qquad \overline{\mathscr{L}} = 2C_1 - \varepsilon.$$

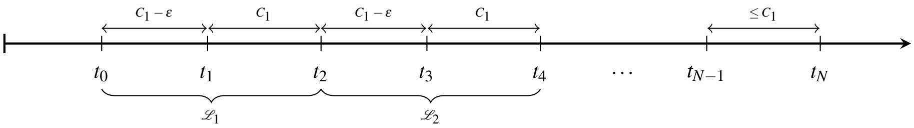

Fig. 4 Visualization of a *minimal* C_1*-overlapping grid*.

Proof We keep in mind Definition 5 from which we deduce $\mathscr{J}_1 = \{1,2\}$, because $t_0 + C_1$ (minimally) overlaps t_1. The next dwell time block begins at $t_2 = t_0 + 2C_1 - \varepsilon$ and consists again of two intervals. This argumentation can be extended to the first $n_b - 1$ blocks and by the definition of block lengths we conclude $\mathscr{L}_b = 2C_1 - \varepsilon$. The length of the last block $\mathscr{L}_{n_b}$ is directly computed by the definition of $N-1$ to be the last index of the grid point recursion before t_f. Finally, the definition of a *minimal* C_1*-overlapping grid* and the obtained block lengths implies

$$\bar{\Delta} = C_1, \qquad \overline{\mathscr{L}} = 2C_1 - \varepsilon.$$

□

With this preliminary work, we show that the deduced MU time bound is tight.

Proposition 2 (Tightness of the bound for (CIA-U)) *Let an MU time* $C_U \geq 0$ *and a grid* $\mathscr{G}_N$ *be given with*

$$t_f - t_0 \geq 2C_U(n_\omega - 1).$$

Then, the objective bound for (CIA-U) mentioned in Proposition 1 is the tightest possible bound.

Proof Let us first consider $C_U > 0$ and construct an example with the desired objective value by means of a *minimal* C_1*-overlapping grid*, where we set $C_1 = C_U$. The proposition assumes a time horizon length of at least $2C_1(n_\omega - 1)$ so that the grid induced by Lemma 1 consists of at least $n_b \geq n_\omega - 1$ blocks. Let $\mathbf{a} \in A$ be given as

$$(a_{i,j})_{i\in[n_\omega], j\in[N]} := \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2n_\omega-2} & \frac{1}{2n_\omega-2} & \frac{1}{n_\omega-1} & \cdots & \frac{1}{n_\omega-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \underbrace{\frac{1}{2n_\omega-2} \quad \frac{1}{2n_\omega-2}}_{j\in\mathscr{J}_1} & & \frac{1}{n_\omega-1} & \cdots & \frac{1}{n_\omega-1} \end{pmatrix}.$$

Consequently, all controls $i \in [n_\omega]$, $i \neq 1$, assume in $\mathbf{a}$ the same values on each interval. After the first block, we set control $i = 1$ to zero, while all other variables are set to $\frac{1}{n_\omega - 1}$ for the remaining intervals, i.e. blocks. Next, we discuss how the optimal solution of (CIA-U) on the first $n_\omega - 1$ blocks may be chosen. Let us calculate the control deviation if we were to activate a control $i = 2\dots n_\omega$ on the first block:

$$\begin{aligned} \Theta_{i,1} &= \left| \sum_{j\in\mathscr{J}_1} \frac{1}{2n_\omega - 2}\Delta_j - \mathscr{L}_1 \right| = \frac{2n_\omega - 3}{2n_\omega - 2}\mathscr{L}_1 = \frac{2n_\omega - 3}{2n_\omega - 2}(2C_U - \varepsilon) \\ &= \frac{2n_\omega - 3}{2n_\omega - 2}(C_U + \bar{\Delta} - \varepsilon). \end{aligned}$$

In the second and third equality we used Lemma 1. The values of the relaxed controls $\mathbf{a}$ for $i = 2\dots n_\omega$ and blocks $1,\dots,n_\omega - 1$ sum up to

$$\begin{aligned} \sum_{b\in[n_\omega-1]} \sum_{j\in\mathscr{J}_b} a_{i,j}\Delta_j &= \frac{1}{2n_\omega - 2}\mathscr{L}_1 + \sum_{b=2,\dots,n_\omega-1} \frac{1}{n_\omega - 1}\mathscr{L}_b \\ &= \frac{1}{2n_\omega - 2}(C_U + \bar{\Delta} - \varepsilon) + \sum_{b=2,\dots,n_\omega-1} \frac{1}{n_\omega - 1}(C_U + \bar{\Delta} - \varepsilon) \\ &= \frac{2n_\omega - 3}{2n_\omega - 2}(C_U + \bar{\Delta} - \varepsilon). \end{aligned}$$

Thus, there are $n_\omega - 1$ controls with this control accumulation on $n_\omega - 1$ blocks, but activating any of these controls on the first block yields already the same control deviation. Hence, the objective of (CIA-U) with this **a** is at least $\frac{2n_\omega-3}{2n_\omega-2}(C_U+\bar{\Delta}-\varepsilon)$, where ε is arbitrarily small. If we combine this result with Proposition 1, we get that $\frac{2n_\omega-3}{2n_\omega-2}(C_U+\bar{\Delta})$ is the tightest possible bound. Last, we argue for the degenerate case, $C_U=0$, that we can create an example with length of all blocks set to $\bar{\Delta}$ and obtain the same tight bound. □

Corollary 1 (Tight bound on the rounding gap for (CIA)) *Consider $\mathscr{G}_N$ and* **a** $\in A$*. The objective of (CIA) is bounded by*

$$\theta^\star \ \leq \ \frac{2n_\omega-3}{2n_\omega-2}\,\bar{\Delta}.$$

If $N \geq n_\omega-1$ holds, then this bound is tight.

Proof The bound follows from Proposition 1 with $C_U=0$ and if $N \geq n_\omega-1$, we are able to construct the same worst-case example as in the proof of Proposition 2, with intervals applied instead of blocks. □

5.3 Implications for the Objectives of (CIA-D) and (CIA-UD)

The bound obtained for (CIA-U) can be transferred in a straightforward manner to (CIA-D) by using $C_1=C_D$ as MDT in the DNFR scheme. However, we notice the increased number of degrees of freedom when dealing with MD times rather than MU times: only the down time forbidden control is fixed for a certain time duration in comparison with the MU time constraint situation where all controls are fixed due to the fixed active control. With this observation, we introduced in the DNFR scheme the min down mode $\chi_D=1$ and are going to deduce in the sequel an alternative upper bound compared to the one obtained by DNFR with $\chi_D=0$. As we are going to detect, this alternative bound is independent of n_ω but not always an improvement, so that we declare the minimum of both bounds as the upper bound in the following proposition.

Proposition 3 (Bounds on the objective of (CIA-D) and (CIA-UD)) *Consider any grid $\mathscr{G}_N$ and* **a** $\in A$*. Let MU and MD times $C_U, C_D \geq 0$ be given. Then*

1. *(CIA-D) is bounded by*
$$\theta^\star \leq \min\left\{\frac{3}{4}C_D+\frac{3}{2}\bar{\Delta},\frac{2n_\omega-3}{2n_\omega-2}\left(C_D+\bar{\Delta}\right)\right\}.$$
2. *(CIA-UD) is bounded by*
$$\theta^\star \leq \begin{cases} \frac{2n_\omega-3}{2n_\omega-2}\left(C_U+\bar{\Delta}\right), & \text{if } C_U \geq C_D,\\ \min\left\{\frac{3}{2}C_U+\frac{3}{2}\bar{\Delta},\frac{2n_\omega-3}{2n_\omega-2}\left(C_D+\bar{\Delta}\right)\right\}, & \text{if } C_D > C_U > C_D/2,\\ \min\left\{\frac{3}{4}C_D+\frac{3}{2}\bar{\Delta},\frac{2n_\omega-3}{2n_\omega-2}\left(C_D+\bar{\Delta}\right)\right\}, & \text{if } C_D/2 \geq C_U.\end{cases}$$

Proof Generally, if $C_D > C_U$ or if MU constraints are absent, we may apply the DNFR scheme with $(C_1,C_2,\chi_D)=\left(C_D,\frac{2n_\omega-3}{2n_\omega-2},0\right)$, which constructs feasible solutions for (CIA-D) respectively (CIA-UD) with objective bound $\frac{2n_\omega-3}{2n_\omega-2}(C_D+\bar{\Delta})$. We are left with the other case $\chi_D=1$:

1. If we set $C_1=\frac{1}{2}C_D$, we have $\overline{\mathscr{L}} < \frac{1}{2}C_D+\bar{\Delta}$. With this MDT and the choice $\chi_D=1$ the DNFR scheme constructs a feasible solution for (CIA-D). Then, by virtue of Theorem 2, case *II.*, we deduce with $C_2=\frac{3}{2}$ the bound $\frac{3}{4}C_D+\frac{3}{2}\bar{\Delta}$.
2. (a) If $C_U \geq C_D$ is given, we can set $C_1=C_U$ and all block lengths are at least as large as those of the MD time C_D. Therefore, the solution constructed by DNFR with $\chi_D=0$ and $C_2=\frac{2n_\omega-3}{2n_\omega-2}$ fulfills both the MU and MD time constraint.
   (b) We set $\chi_D=1$ and $C_1=C_U$, $C_2=\frac{3}{2}$, when $C_D > C_U > C_D/2$ is given. By this choice, the solution of DNFR fulfills a MD time of $2C_U$ because of
   $$2\underline{\mathscr{L}} > 2C_U > 2C_D/2 = C_D.$$
   Furthermore, by setting $C_1=C_U$ it is clear that $\mathbf{w}^{\mathrm{DNFR}}$ does not violate the MU time.
   (c) $C_D/2 \geq C_U$: DNFR with down time configuration $\chi_D=1$ and $C_1=C_D/2 \geq C_U$, $C_2=\frac{3}{2}$ can be executed without violating the MU time constraint. □

Since tightness results for the problems (CIA-D) and (CIA-UD) are not as straightforward obtained as for the problem (CIA-U), we move the discussion on the quality of the bounds obtained in Proposition 3 to the Appendix B.

6 Sum-Up Rounding in the Dwell Time Context

SUR is computationally less expensive ($\mathcal{O}(n_\omega N)$) than the DNFR scheme executed on intervals. But, on the other hand, the last section showed DNFR constructs solutions for (CIA) with a maximum integrality gap that is less than the one of solutions constructed by SUR for $n_\omega \geq 3$ (equality for the case $n_\omega = 2$). Since the SUR scheme is very often used for finding approximative solutions of (CIA), but does not necessarily fulfill MDT constraints, we discuss in this section a canonical extension of the algorithm to this setting.

6.1 Dwell Time Sum-Up Rounding (DSUR)

We introduce the concept of a *currently activated* control and dwell time blocks that depend on the initial interval and the MDT duration C_1.

Definition 10 (Initial interval dwell time block index sets) *Let an MDT $C_1 \geq 0$ be given. We define for all intervals $k \in [N]$ the initial interval dependent dwell time index sets to be*

$$\mathscr{J}_k^{SUR}(C_1) := \{k\} \cup \{j \mid t_{j-1} \in \mathscr{G}_N \cap [t_{k-1}, t_{k-1} + C_1)\}.$$

Definition 11 (Currently activated control) *We call a control index i currently activated at interval $j = 2, \ldots, N$, if*

$$w_{i,j-1} = 1$$

holds. Otherwise, or if $j = 1$, we call the binary control i currently deactivated.

In contrast to the DNFR scheme we are now interested in considering intervals individually and work independently of the constant χ_D. Hence, grouping of minimum down time forbidden controls for each interval into sets $\mathscr{I}_j^{\mathrm{SUR}}$ is necessary.

Definition 12 (SUR minimum down time forbidden control set) *Let a MDT $C_D \geq 0$ be given. We define the set of down time forbidden controls $\mathscr{I}_j^{SUR} \subset [n_\omega]$ on interval $j \in [N]$ as follows:*

$$\mathscr{I}_j^{SUR} := \{i \in [n_\omega] \mid \exists k < j : \; t_{j-1} \leq t_{k-1} + C_D, t_{k-1} \in \mathscr{G}_N \;\wedge\; w_{i,k} = 1\}.$$

We say $i \in [n_\omega]$ is MD time admissible on $j \in [N]$, if $i \notin \mathscr{I}_j^{SUR}$ holds.

Note that the above definition assumes implicitly fixed control variables for previous intervals $[N] \ni k < j$. We have $\mathscr{I}_1^{\mathrm{SUR}} = \emptyset$, because there are no down time forbidden controls on the first interval. Moreover, the set $\mathscr{I}_j^{\mathrm{SUR}}$ may contain several controls, but at most $n_\omega - 1$.

Next, we give a definition of the dwell time sum-up rounding (DSUR) scheme in Algorithm 2. It iterates over all intervals $j \in [N]$ and selects initially the interval representing the beginning of the time horizon, where a *currently activated* control does not yet exist. The control dependent MDT C_i is updated in line 3 for each iteration inside the `while` loop so that C_i equals the maximum of MU time C_U and MD time C_D for a currently activated control, and otherwise it is set to the MU time C_U. The algorithm sets $C_i = C_U$ for all controls in the first `while` iteration. Next, in line 4, one searches for the *MD time admissible* control $i^\star$ with maximum forward control deviation on the upcoming intervals covering the dwell time $C_{i^\star}$. If it is the *currently activated* one, we fix this control to be active also on the current selected interval j and increase the interval index (line 5-6). Else, the control is activated on the whole dwell time block represented by its interval indices $\mathscr{J}_j^{\mathrm{SUR}}(C_{i^\star})$ and the interval index is updated accordingly (line 8-9). Lastly, DSUR updates the set of *down time forbidden* controls for the next iteration in line 11. The algorithm stops as soon as the control choice has been made for the last interval N.

Clearly, $\mathbf{w}^{\mathrm{DSUR}}$ is a feasible solution for (CIA), because exactly one control is active per interval. It is also feasible for (CIA-U), because whenever a currently deactivated control is activated, it dwells on active for at least the duration C_U (line 8-9). The solution also satisfies MD time constraints by the definition of $\mathscr{I}_j^{\mathrm{SUR}}$ and altogether $\mathbf{w}^{\mathrm{DSUR}}$ is a feasible solution for (CIA-UD).

We transferred the main idea from the original SUR scheme to the problem setting with MDT constraints by selecting in each iteration the control with maximum forward control deviation. In the presence of MU time requirements, we need to calculate this forward accumulation for the set of next intervals with total length at least C_U. For a given MD time larger than the MU time, Algorithm 2 compares the forward accumulation with length at least C_D of the *currently activated* control with the ones of other controls with length at least C_U. The idea behind this approach is to prevent a situation in which a control gets deactivated, but is going to accumulate a large control deviation during its *down time forbidden* period.

Algorithm 2: DSUR algorithm for (CIA-UD)

Input : Relaxed control values $\mathbf{a} \in A$, time increments Δ_j, $j \in [N]$, MU time C_U, MD time C_D.
Output: Feasible solution $\mathbf{w}^{\mathrm{DSUR}}$ of (CIA-UD).

Initialize $\mathbf{w} = 0$, $j = 1$ and $\mathscr{I}_j^{\mathrm{SUR}} = \emptyset$.
while $j \leq N$ **do**
  Set $C_{i_a} = \max\{C_U, C_D\}$ for the *currently activated* control i_a, and set $C_i = C_U$ for all other controls $i \neq i_a$.
  Find the control with maximum deviation (break ties arbitrarily):
  $i^\star = \arg\max\{\theta_{i,j-1} + \sum_{l \in \mathscr{I}_j^{\mathrm{SUR}}(C_i)} a_{i,l}\Delta_l \mid i \in [n_\omega] \setminus \mathscr{I}_j^{\mathrm{SUR}}\}$
  if $i^\star = i_a$ **then**
    Set $w_{i^\star,j} = 1$ and update $j = j+1$.
  else
    Set $w_{i^\star,l} = 1$, $l \in \mathscr{I}_j^{\mathrm{SUR}}(C_{i^\star,1})$;
    Update $j = \max\{l \mid l \in \mathscr{I}_j^{\mathrm{SUR}}(C_{i^\star,1})\} + 1$.
  end
  Update the set of down time forbidden controls $\mathscr{I}_j^{\mathrm{SUR}}$.
end
return: $\mathbf{w}^{\mathrm{DSUR}} = \mathbf{w}$;

Remark 4 (Run time of DSUR) Algorithm 2 is in $\mathscr{O}(n_\omega N^2)$, since it sums up, on each interval and for all controls, the relaxed control values $\mathbf{a}$ on the next dwell time induced intervals.

6.2 Rounding Gap Bounds for Dwell Time Sum-Up Rounding

Kirches et al. [24] have proven the tightest possible bound on the integrality gap for the original SUR. From this we can derive implications for DSUR in the absence of MD conditions.

Theorem 3 (Tight bound for SUR integrality gap, cf. Theorem 7.1, [24]) *Let $\mathbf{w}^{SUR}$ be constructed from $\mathbf{a} \in A$ by means of SUR for an equidistant discretization of $[t_0, t_f]$ and let denote by $\theta(\mathbf{w}^{SUR})$ its (CIA) objective value. Then, the rounding gap is bounded by*

$$\theta(\mathbf{w}^{SUR}) \leq \bar{\Delta} \sum_{i=2}^{n_\omega} \frac{1}{i},$$

which is the tightest possible upper bound.

Corollary 2 (Tight bound for DSUR integrality gap without MD times) *Let $C_D < \underline{\Delta}$ and an MU time $C_U > 0$ be given. Let the time horizon $[t_0, t_f]$ with a minimal C_U-overlapping grid be discretized and let $\mathbf{w}^{DSUR}$ be constructed from $\mathbf{a} \in A$ by means of DSUR. Then, the rounding gap $\theta(\mathbf{w}^{DSUR})$ of its (CIA) objective value is bounded by*

$$\theta(\mathbf{w}^{DSUR}) \leq (C_U + \bar{\Delta}) \sum_{i=2}^{n_\omega} \frac{1}{i},$$

which is the tightest possible upper bound.

Proof As in the proof of Theorem 3 in [24] a dynamic programming argument can be applied, here with equidistant block length of $(C_U + \bar{\Delta} - \varepsilon)$ as derived in Lemma 1. With a time horizon length of $n_\omega(C_U + \bar{\Delta} - \varepsilon)$ we may analogously to the proof of Theorem 3 construct an example indicating that the bound is tight as follows.

$$a_{i,j} = \begin{cases} 0, & \text{if } 2i+1 \leq j \leq N, \\ 1/(n_\omega + 1 - j/2), & \text{if } j \text{ is even}, \\ 1/(n_\omega + 1 - (j+1)/2), & \text{if } j \text{ is odd}, \end{cases} \qquad 1 \leq j \leq N = 2n_\omega.$$

The DSUR scheme constructs for this example a solution that switches directly after each block with length $(C_U + \bar{\Delta} - \varepsilon)$. Moreover, the controls $i \in [n_\omega - 1]$ are active each on block i so that the last control n_ω accumulates the asserted rounding gap until the end of block $n_\omega - 1$. □

Remark 5 (DSUR as generalization of SUR) The last corollary implicitly states that DSUR can be seen as a generalization of the original SUR algorithm, since it reduces to the latter for a negligible MDT $C_U, C_D \leq \underline{\Delta}$.

Theorem 3 allows no direct conclusion for the case with absent MU times and an active MD time $C_D > \underline{\Delta}$. At least, it is possible to provide worst-case examples for $\mathbf{a}$ in order to get a clue of how large the upper bound can be for the DSUR rounding gap without MU times. This expresses the following proposition.

Proposition 4 (Rounding gap for DSUR without MU times) *Consider an inactive MU time constraint, i.e., $C_U \leq \underline{\Delta}$ and an equidistant grid $\mathscr{G}_N$. We assume for the MD time*

$$C_D > (2(n_\omega - 1) - 1)\bar{\Delta}. \tag{6.1}$$

Let for the grid hold

$$N \geq (n_\omega - 1)(1 + M_D) + \lceil M_D/2 \rceil - 1, \tag{6.2}$$

where M_D denotes the number of minimum down time intervals constructed by C_D, i.e. $M_D := \lceil C_D/\bar{\Delta} \rceil$. Then, there is an $\mathbf{a} \in A$ that yields a (CIA-D) objective value $\theta(\mathbf{w}^{DSUR})$ of $\mathbf{w}^{DSUR}$ constructed by DSUR with

$$\theta(\mathbf{w}^{DSUR}) \geq \left(\frac{M_D}{2} + (n_\omega - 2) \right) \bar{\Delta}. \tag{6.3}$$

Proof See Appendix C.

Remark 6 (Rounding gap for DSUR with MU and MD constraints) Generally, when the problem setting involves both MU and MD time constraints, i.e., $C_D, C_U > \underline{\Delta}$, the worst-case rounding gap constructed by the DSUR scheme is at least the maximum of the bounds obtained in Corollary 2 and in Proposition 4.

7 Computational Experiments

We consider a three tank flow system problem with three controlling modes in order to evaluate the integrality gap in practice and to test the proposed rounding methods. It models the dynamics of an upper, middle and lower level tank, connected to each other with pipes. The goal is to minimize the deviation of certain fluid levels k_2, k_4 in the middle, respectively lower, level tank. This problem type was discussed in a variety of publications for the optimal control of constrained switched systems [11,36] and is taken from the benchmark `https://mintOC.de` library [33]. The problem reads

$$\begin{aligned}
\min_{\mathbf{x},\,\omega} \quad & \int_0^T k_1(x_2(s) - k_2)^2 + k_3(x_3(s) - k_4)^2 \,\mathrm{d}s \\
\text{s.t.}\ \dot{x}_1(t) &= -\sqrt{x_1(t)} + c_1\omega_1(t) + c_2\omega_2(t) - \omega_3(t)\sqrt{c_3 x_1(t)}, && \text{for a.e. } t \in [0,T], \\
\dot{x}_2(t) &= \sqrt{x_1(t)} - \sqrt{x_2(t)}, && \text{for a.e. } t \in [0,T], \\
\dot{x}_3(t) &= \sqrt{x_2(t)} - \sqrt{x_3(t)} + \omega_3(t)\sqrt{c_3 x_1(t)}, && \text{for a.e. } t \in [0,T], \\
\mathbf{x}(0) &= \mathbf{x_0}, \\
1 &= \sum_{i=1}^{3} \omega_i(t), \quad \omega(t) \in \{0,1\}^3, && \text{for a.e. } t \in [0,T].
\end{aligned} \tag{P}$$

The additional parameters are

$$k := (2,3,1,3)^T, \qquad c := (1,2,0.8)^T, \qquad \mathbf{x_0} := (2,2,2)^T, \qquad T := 12.$$

Furthermore, we add the MU and MD time constraints (2.1)-(2.2) to the three tank problem with varying C_U and C_D parameters. We leave open the question of the regularity of the differential states $\mathbf{x}$, but we assume that there exists a unique solution that is continuous.

We solve this test problem with the CIA decomposition. We applied Direct Multiple Shooting for temporal discretization with varying number of grid intervals N together with a fourth order Runge-Kutta scheme for obtaining the differential state's evolution and thus the objective value.[3] We used `CasADi` v3.4.5 [3] to

[3] When applying the fourth order Runge-Kutta scheme we need to have for the differential states that $\mathbf{x} \in C^5(\mathscr{T}, R^{n_x})$ to generate a fourth order error term. In the introduction we required only a continuous $\mathbf{x} \in W^{1,\infty}(\mathscr{T}, R^{n_x})$; however, we could assume stronger regularity due to piecewise continuously differentiable control functions from Definition 2. Also, the Runge-Kutta scheme in the context of Direct Multiple Shooting is applied on the piecewise continuously differentiable right-hand side of the ODE, where the control function changes its values only at grid points. Nevertheless, our presented algorithms are independent of the chosen numerical integration scheme and one may choose a more accurate scheme according to the dynamical system at hand.

parse the NLP with efficient derivative calculation of Jacobians and Hessians to the solver `IPOPT` 3.12.3 [38]. Here, the helper classes `Opti stack` are useful, as they allow a compact syntax for NLP modeling. For finding the optimal solution of the resulting (CIA) problem and its MDT variants we used the branch and bound solver of the open source software package *pycombina*[4] [8]. We published the `python` source code of solving (P) via the CIA decomposition online[5].

We stress that the obtained feasible solutions for (P) via the CIA decomposition are in general not global optimal solutions. In fact, the Problem (P) appears to be nonconvex so that `IPOPT` may construct a local solution, just like the rounding via (CIA) may do. Nevertheless, finding a global optimal solution is computational expensive as argued in the introduction.

Figure 5 depicts the state and control trajectories with relaxed (DRCP) and binary (DBCP) control values and a required MU time of $C_U = 0.3$. We remark that the binary solution's objective value under MU time constraints is about 1.3% larger than the one obtained by the relaxed solution and about 1.2% larger than the binary solution's objective value without MU time constraints.

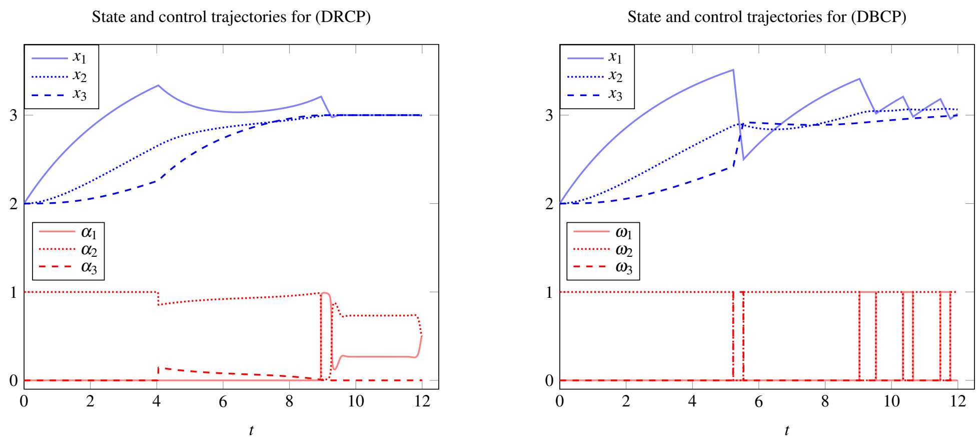

Fig. 5 Differential state and control trajectories for the test problem (P) with relaxed binary controls, i.e. for problem (DRCP), on the left and approximated binary controls, i.e. for problem (DBCP), on the right with MU time $C_U = 0.3$ and a temporal discretization with $N = 1280$ intervals. The objective value for (DRCP) amounts to $\Phi = 8.776$, while the one of (DBCP) is $\Phi = 8.888$.

We illustrate in Figure 6 the effect of an increasing MU time on the objective values of (CIA-U) and (DBCP). As expected, the finer the discretization grid and the smaller the required MDT time, the better the objective values of both problems. A small MU time results in a weak restriction for (DBCP) so that its objective value is close to the one of (DRCP), which is $\Phi = 8.776$. But, from $C_U = 0.7$ on, a refinement of the grid cannot compensate the MU time restriction anymore and the (DBCP) objective value is about 25% larger than (DRCP) in this case. Interestingly, this objective value increases hardly in $C_U \geq 0.7$, but it even decreases slightly after $C_U = 0.7$ before increasing again and staying constant from $C_U \approx 2.0$ on. We observe a few outlier instances, e.g., $N = 20$ with $C_U = 1.2$ or $N = 40$ with $C_U \in \{0.4, 0.5\}$, where the objective value appears to be unexpectedly large. This can be explained by the coarse grid choices, which in turn stresses the importance of a fine time discretization regarding the stability of the obtained solution for (DBCP).

On the other hand, (CIA-U) objective's value increases roughly linearly in C_U on fine grids until it reaches $\theta^\star \approx 0.87$, which seems to be the maximum value for P in this setting. Thus, while small values of the (CIA-U)'s objective correspond to promising objective values of (DBCP), the relationship of (CIA-U) and (DBCP) appears to be quite uncorrelated from $C_U \geq 0.7$ on. We computed similar results for (P) with MD time constraints (not shown). We also tested whether including the relaxed MDT constraints into the NLP or not has a significant impact on the solution - this was not the case.

We analyze the performance of DNFR and DSUR for both MU and MD time constraints and with respect to $\theta^\star$ in Figure 7. The obtained solutions are compared with the global minima for (CIA-UD) from

[4] see `https://github.com/adbuerger/pycombina`

[5] see `https://mintoc.de/index.php/Three_Tank_multimode_problem_(python/casadi)`

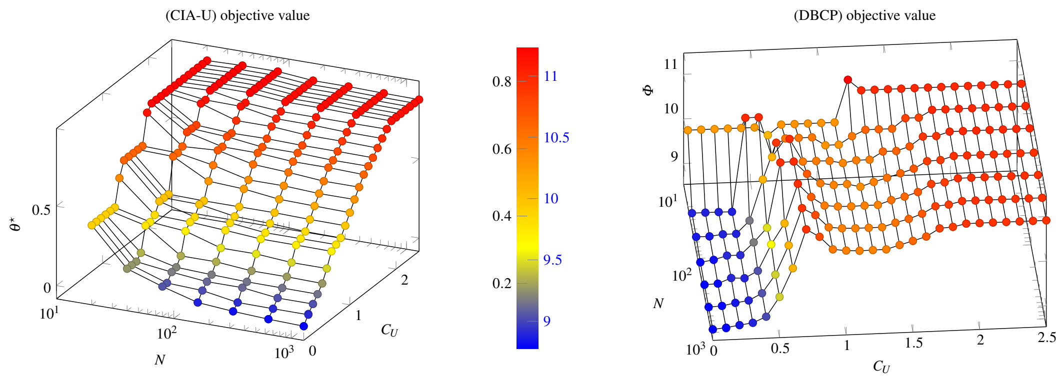

Fig. 6 Objective values of (CIA-U) and (DBCP) based on the test problem (P) and different control discretizations N and MU time durations C_U.

pycombina. We observe that DNFR seems to perform better for MU time constraints, while DSUR performs better for the instances with MD time requirements. We plotted also the theoretical upper bound (UB) from Propositions 1 and 3, which are here $\frac{3}{4}(C_1+\bar{\Delta})$, $C_1 = C_U, C_D$. As already observed for Figure 6, the minima of (CIA-U) and (CIA-D) do hardly increase, if at all, for large MDTs and therefore diverge compared with the theoretical upper bound. We explain this behavior by the problem specific given relaxed values, which induce here an objective value of $\theta^\star \approx 0.87$ for (CIA-U) and (CIA-D) even if no switches are used in the binary solution.

We show also the upper bound derived for DSUR with MU time constraints from Corollary 2, i.e. $\frac{5}{6}(C_U+\bar{\Delta})$, and the lower bound for the upper bound for DSUR with MD time constraints by Proposition 4, i.e. $\frac{1}{2}(C_D+\bar{\Delta})$. While the solutions constructed by DSUR may violate the upper bounds for (CIA-U) and (CIA-D), as happening for the MU time case, the bounds for DSUR are not violated. We observe that the (CIA-D) objective values are not only by far smaller than their upper bound, but even smaller or equal to the DSUR bound by Proposition 4.

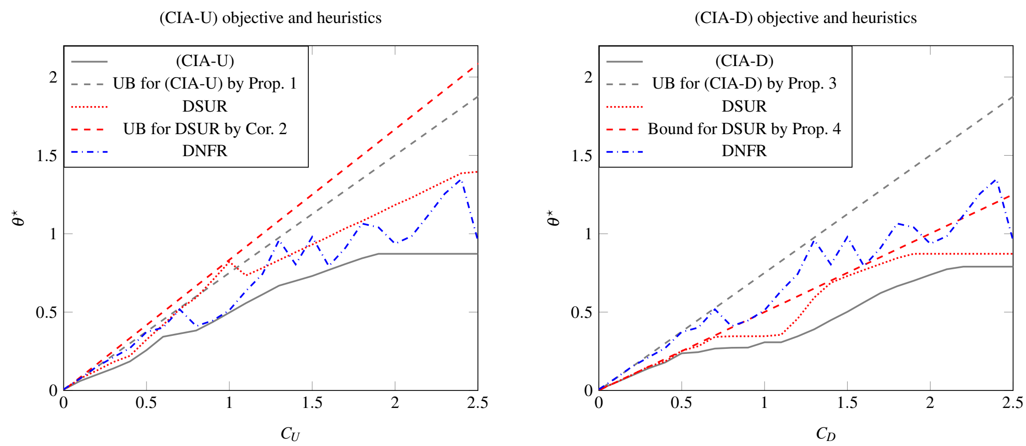

Fig. 7 CIA objective function values θ for test problem (P) with time discretization $N = 1280$ and varying MU time C_U (left), respectively varying MD time C_D (right). The optimal solutions denoted with (CIA-U), respectively (CIA-D), are obtained via pycombina's branch and bound algorithm and compared with the solutions constructed by DNFR and DSUR. We also show the upper bound (UB) for (CIA-U) respectively (CIA-D) from Propositions 1-3 and the bounds derived for DSUR from Corollary 2 and Proposition 4. We note that although Proposition 4 derives only a lower bound for the upper bound of DSUR with MD time constraints, this bound is not violated by the computational results.

We implemented DNFR and DSUR in Python 3.6 as additional solvers for pycombina. The execution of the heuristics took for all instances not more than 0.02 seconds on a workstation with 4 Intel i5-4210U CPUs (1.7 GHz) and 7.7 GB RAM. We conclude that the heuristics can be used to quickly generate robust solutions in terms of competitive objective values. They might also be useful for initializing pycombina's branch and bound algorithm with a good upper bound. However, a numerical study is needed for verifying the added benefit, which could be elaborated in future work.

8 Conclusions

In this article, we have derived tight bounds for the integrality gap of the CIA decomposition applied to MIOCPs under MDT constraints. The presented proofs are constructive and take advantage of the introduced DNFR scheme. Numerical experiments show that the CIA decomposition performs notably well in terms of objective quality for problems with small MDT requirements, since the deviation from its relaxed solution is negligible. For more restrictive dwell time constraints, the algorithm may provide feasible solutions that differ a lot from the relaxed solution. Hence, the constructed solution quality might be low compared to the optimal solution. Nevertheless, when considering the runtime of MINLP solvers, the CIA decomposition solution is computationally inexpensive and can be easily assessed with the relaxed solution for its quality. We have extended the SUR scheme so that it constructs dwell time feasible solutions and tested the proposed algorithm on a benchmark problem. The resulting (CIA) problem solutions of this and the DNFR scheme are close to the optimal ones. Hence, we propose that the DNFR or DSUR scheme may be beneficial in the context of huge (CIA) problems or in the setting of model predictive control, where a branch and bound algorithm struggles to find the optimal solution efficiently.

A Proof of Theorems 1 and 2

A.1 Lemmata for DNFR Approximation Results

We present a series of lemmata, which will be needed later in the proofs to Theorem 1 and Theorem 2.

Lemma 2 (Family of dwell time block sets) *The family of dwell time block interval sets $\{\mathscr{J}_b\}_{b\in[n_b]}$ as defined in Definition 5 is a partition of the set of all interval indices $[N]$.*

Proof This follows directly from the definition of dwell time block interval sets. □

Lemma 3 (Block accumulated control deviation properties) *For all $b \in [n_b]$ we have that*

$$\sum_{i\in[n_\omega]} \Gamma_{i,b} = \mathscr{L}_b, \qquad \sum_{i\in[n_\omega]} \Theta_{i,b} = 0.$$

Proof Let us derive an auxiliary result for $j \in [N]$:

$$\sum_{i\in[n_\omega]} \theta_{i,j} = \sum_{i\in[n_\omega]} \sum_{l\in[j]} (a_{i,l} - w_{i,l})\Delta_l \overset{\text{(Conv)}}{=} \sum_{l\in[j]} \Delta_l - \sum_{l\in[j]} \Delta_l = 0. \tag{A.1}$$

We use this and rearrange the sums in order to proof the first assertion:

$$\sum_{i\in[n_\omega]} \Gamma_{i,b} = \sum_{i\in[n_\omega]} \left(\theta_{i,l_{b-1}} + \sum_{j\in\mathscr{J}_b} a_{i,j}\Delta_j \right) = 0 + \sum_{i\in[n_\omega]} \sum_{j\in\mathscr{J}_b} a_{i,j}\Delta_j = \sum_{j\in\mathscr{J}_b} \sum_{i\in[n_\omega]} a_{i,j}\Delta_j \overset{\text{(Conv)}}{=} \sum_{j\in\mathscr{J}_b} \Delta_j$$
$$= \mathscr{L}_b.$$

The auxiliary result is also useful for the second statement:

$$\sum_{i\in[n_\omega]} \Theta_{i,b} = \sum_{i\in[n_\omega]} \theta_{i,l_{b-1}} \overset{\text{(A.1)}}{=} 0.$$

□

Lemma 4 (Accumulated difference of Γ and Θ over active controls) *Let $b_1, b_2 \in [n_b]$ and we define S_{b_1,b_2} as the set of active controls between b_1 and b_2:*

$$S_{b_1,b_2} := \{i \in [n_\omega] \mid \exists b:\ b_1 < b < b_2 \text{ with } w_{i,j} = 1,\ \forall j \in \mathscr{J}_b\}.$$

Then, we have

$$\sum_{i\in S_{b_1,b_2}} \left(\Gamma_{i,b_2} - \Theta_{i,b_1}\right) \le \overline{\mathscr{L}}.$$

Proof Using Definition 7 of Γ, Θ and rearranging sums yields

$$\begin{aligned}\sum_{i\in S_{b_1,b_2}}\left(\Gamma_{i,b_2}-\Theta_{i,b_1}\right) &= \sum_{i\in S_{b_1,b_2}}\left(\sum_{b=b_1+1}^{b_2}\sum_{j\in\mathscr{J}_b} a_{i,j}\Delta_j - \sum_{b=b_1+1}^{b_2-1}\sum_{j\in\mathscr{J}_b} w_{i,j}\Delta_j\right)\\ &= \sum_{b=b_1+1}^{b_2}\sum_{j\in\mathscr{J}_b}\Delta_j \underbrace{\sum_{i\in S_{b_1,b_2}} a_{i,j}}_{\leq 1} - \sum_{b=b_1+1}^{b_2-1}\sum_{j\in\mathscr{J}_b}\Delta_j \underbrace{\sum_{i\in S_{b_1,b_2}} w_{i,j}}_{=1}\\ &\leq \sum_{b=b_1+1}^{b_2}\sum_{j\in\mathscr{J}_b}\Delta_j - \sum_{b=b_1+1}^{b_2-1}\sum_{j\in\mathscr{J}_b}\Delta_j\\ &= \mathscr{L}_{b_2} \leq \overline{\mathscr{L}}.\end{aligned}$$

□

Note that S_{b_1,b_2} is trivially the empty set, if $b_2 \leq b_1+1$, but the result remains true in this case. We employ the concept of S_{b_1,b_2} for a contradiction in the proofs of Theorem 1 and Theorem 2.

Lemma 5 (Control with negative Γ value has not been future forced) *Let (C_1, C_2, χ_D) be given and assume that the forward control deviation of a control $i \in [n_\omega]$ and a block $b_2 \geq 2$ after executing Algorithm 1 satisfies:*

$$\Gamma_{i,b_2} \leq C_2\overline{\mathscr{L}} - \overline{\mathscr{L}}, \qquad \textit{and} \qquad \Gamma_{i,b_2} < 0.$$

Then, there is an earlier activation of i on some block $b_1 < b_2$ and this activation has not been b_2-future forced on b_1.

Proof Note that $\Gamma_{i,b}$ is monotonically increasing in b for deactivated controls i. We conclude from this and $\Gamma_{i,b_2} < 0$ that there is an earlier activation of i on some block $b_1 < b_2$. We take a closer look on the forward control deviation on block b_2:

$$C_2\overline{\mathscr{L}} - \overline{\mathscr{L}} \geq \Gamma_{i,b_2} = \sum_{k=1}^{b_2}\sum_{j\in\mathscr{J}_k} a_{i,j}\Delta_j - \sum_{k=1}^{b_1}\sum_{j\in\mathscr{J}_k} w_{i,j}\Delta_j = \sum_{k=1}^{b_2}\sum_{j\in\mathscr{J}_k} a_{i,j}\Delta_j - \sum_{k=1}^{b_1-1}\sum_{j\in\mathscr{J}_k} w_{i,j}\Delta_j - \mathscr{L}_{b_1},$$

and rearranging terms implies

$$\sum_{k=1}^{b_2}\sum_{j\in\mathscr{J}_k} a_{i,j}\Delta_j - \sum_{k=1}^{b_1-1}\sum_{j\in\mathscr{J}_k} w_{i,j}\Delta_j \leq C_2\overline{\mathscr{L}} - \overline{\mathscr{L}} + \mathscr{L}_{b_1} \leq C_2\overline{\mathscr{L}}.$$

The last inequality shows us that i has been not b_2-future forced on b_1. □

A.2 Proof of Theorem 1

Proof We proceed via induction.
Base case: We consider the first b on which a down time forbidden control $i_D \in [n_\omega]$ appears and let us assume

$$\Gamma_{i_D,b} > \tfrac{3}{2}\overline{\mathscr{L}} \tag{A.2}$$

holds and we prove that it results in a contradiction. It follows from Lemma 3

$$\tfrac{3}{2}\overline{\mathscr{L}} \geq \mathscr{L}_b = \sum_{i\neq i_D}\Gamma_{i,b} + \Gamma_{i_D,b},$$

so that there must be a control $i_1 \neq i_D$ with negative forward control deviation on b:

$$\exists\, i_1 \neq i_D :\ \Gamma_{i_1,b} < 0.$$

We apply Lemma 5 to the last inequality: i_1 has not been b-future forced on its last activation and we denote the block of this activation with b_1. In other words, we know that there is at least one block b_1 and one control i_1 that was not b-future forced on b_1 and still was activated on b_1. Now, we denote by i_1 the control of this property with the last activation before b. By this definition, we observe that all controls being activated after b_1 would become forced until b. We notate

$$F_{b_1,b} := \{i \in [n_\omega] \mid \exists k(i):\ b_1 < k(i) \leq b \text{ on which } i \text{ is forced or } b\text{-future forced}\}.$$

In particular, we have $i_D \in F_{b_1,b}$. For $i \in F_{b_1,b}\backslash\{i_D\}$ we conclude

$$\Gamma_{i,b} = \Theta_{i,b-1} + \sum_{j\in\mathscr{J}_b} a_{i,j}\Delta_j = \sum_{k=1}^{b}\sum_{j\in\mathscr{J}_k} a_{i,j}\Delta_j - \sum_{k=1}^{k(i)-1}\sum_{j\in\mathscr{J}_k} w_{i,j}\Delta_j > \tfrac{3}{2}\overline{\mathscr{L}},$$

and therefore

$$\Gamma_{i,b} > \tfrac{3}{2}\overline{\mathscr{L}} - \mathscr{L}_{k(i)}, \qquad i \in F_{b_1,b}\backslash\{i_D\}. \tag{A.3}$$

The last inequality holds, since control i was last activated at block $k(i)$. For block b_1 we know that i_1 has been chosen, despite not being b-future forced. We use this observation and our assumption of $b > b_1$ being the first block with a down time forbidden control to conclude that all controls out of $F_{b_1,b}$ have been *inadmissible* on b_1. Hence, it results for $i \in F_{b_1,b}$

$$\Gamma_{i,b_1} < -\tfrac{3}{2}\overline{\mathscr{L}} + \mathscr{L}_{b_1}, \qquad \Rightarrow \qquad \Theta_{i,b_1} < -\tfrac{3}{2}\overline{\mathscr{L}} + \mathscr{L}_{b_1}. \tag{A.4}$$

We sum up the inequalities (A.2) and (A.3) over $F_{b_1,b}$ and similarly for (A.4), which yields

$$\sum_{i\in F_{b_1,b}} \Gamma_{i,b} \;>\; \tfrac{3}{2}\overline{\mathscr{L}} + (|F_{b_1,b}|-1)\left(\tfrac{3}{2}\overline{\mathscr{L}} - \mathscr{L}_{b_2}\right), \tag{A.5}$$

$$\sum_{i\in F_{b_1,b}} \Theta_{i,b_1} \;<\; |F_{b_1,b}|\left(-\tfrac{3}{2}\overline{\mathscr{L}} + \mathscr{L}_{b_1}\right), \tag{A.6}$$

where we set $b_2 := \arg\min\{\mathscr{L}_k \mid b_1 < k \leq b\}$ and notate with $|F_{b_1,b}|$ the cardinality of $F_{b_1,b}$. Subtracting (A.6) from (A.5) results in

$$\begin{aligned}\sum_{i\in F_{b_1,b}} \left(\Gamma_{i,b} - \Theta_{i,b_1}\right) \;&>\; \tfrac{3}{2}\overline{\mathscr{L}} + (|F_{b_1,b}|-1)\left(\tfrac{3}{2}\overline{\mathscr{L}} - \mathscr{L}_{b_2}\right) - |F_{b_1,b}|(-\tfrac{3}{2}\overline{\mathscr{L}} + \mathscr{L}_{b_1})\\ &= \tfrac{3}{2}\overline{\mathscr{L}} + (2|F_{b_1,b}|-1)\tfrac{3}{2}\overline{\mathscr{L}} - (|F_{b_1,b}|-1)\mathscr{L}_{b_2} - |F_{b_1,b}|\mathscr{L}_{b_1}\\ &\geq \tfrac{3}{2}\overline{\mathscr{L}} + (2|F_{b_1,b}|-1)\tfrac{1}{2}\overline{\mathscr{L}}\\ &> \overline{\mathscr{L}}.\end{aligned} \tag{A.7}$$

We used $\mathscr{L}_{b_1}, \mathscr{L}_{b_2} \leq \overline{\mathscr{L}}$ in the second inequality. We finish our calculations by considering the property of $F_{b_1,b}$ comprising all control activations between time blocks b_1+1 and $b-1$, therefore we can apply Lemma 4 with $F_{b_1,b} = S_{b_1,b}$ and obtain

$$\sum_{i\in F_{b_1,b}} \left(\Gamma_{i,b} - \Theta_{i,b_1}\right) \leq \overline{\mathscr{L}}, \qquad\qquad ↯ \tag{A.8}$$

which is a contradiction to inequality (A.7).

Step case: Let the assertion hold until any block $b-1 \in [n_b]$ and we prove that then the statement holds for b. Again, we consider $i_D \in [n_\omega]$ and assume

$$\Gamma_{i_D,b} > \tfrac{3}{2}\overline{\mathscr{L}} \tag{A.9}$$

holds and we prove that it results in a contradiction. With a similar argumentation as in the base case we deduce that there is a control i_1 that has not been b-future forced on block $b_1 < b$ and reuse the definition of $F_{b_1,b}$. Thus, inequality (A.3) still holds. Now, we distinguish between two cases why the controls out of $F_{b_1,b}$ have not been activated on b_1. If all controls $i \in F_{b_1,b}$ have been *inadmissible* on b_1, we can argue as in the base case. Hence, we focus on the other case: there is an $i_2 \in F_{b_1,b}$, which was *down time forbidden* on b_1 and all other controls $i \in F_{b_1,b}\backslash\{i_2\}$ were inadmissible. By the induction hypothesis and the previously derived inequality (A.4) we have

$$\Theta_{i_2,b_1} \leq \tfrac{3}{2}\overline{\mathscr{L}}, \qquad\qquad \Theta_{i,b_1} < -\tfrac{3}{2}\overline{\mathscr{L}} + \mathscr{L}_{b_1}, \quad i \in F_{b_1,b}\backslash\{i_2\}.$$

Summing up these inequalities over $F_{b_1,b}$ results therefore in

$$\sum_{i\in F_{b_1,b}} \Theta_{i,b_1} \;<\; \tfrac{3}{2}\overline{\mathscr{L}} + |F_{b_1,b}-1|\left(-\tfrac{3}{2}\overline{\mathscr{L}} + \mathscr{L}_{b_1}\right). \tag{A.10}$$

Next, we argue that $F_{b_1,b}$ contains at least two controls: the case $b_1 = b-1$ is not possible, since i_D is via assumption forced and down time forbidden on b, hence admissible on $b-1$. Therefore, $b_1 \leq b-2$ and there is a control $i \neq i_D,\ i \in F_{b_1,b}$, which is activated on $b-1$. Altogether we have $|F_{b_1,b}| \geq 2$. With this observation we subtract inequality (A.10) from (A.5):

$$\begin{aligned}\sum_{i\in F_{b_1,b}} \left(\Gamma_{i,b} - \Theta_{i,b_1}\right) \;&>\; \tfrac{3}{2}\overline{\mathscr{L}} + (|F_{b_1,b}|-1)\left(\tfrac{3}{2}\overline{\mathscr{L}} - \mathscr{L}_{b_2}\right)\\ &\quad - \left(\tfrac{3}{2}\overline{\mathscr{L}} + (|F_{b_1,b}|-1)(-\tfrac{3}{2}\overline{\mathscr{L}} + \mathscr{L}_{b_1})\right)\\ &= 3(|F_{b_1,b}|-1)\overline{\mathscr{L}} - (|F_{b_1,b}|-1)\mathscr{L}_{b_2} - |F_{b_1,b}-1|\mathscr{L}_{b_1}\\ &\geq (|F_{b_1,b}|-1)\overline{\mathscr{L}}\\ &\geq \overline{\mathscr{L}}.\end{aligned}$$

Notice that $|F_{b_1,b}| \geq 2$ is used in the last inequality. Finally, we build again on Lemma 4, where it is justified to set $F_{b_1,b} = S_{b_1,b}$. Thus, the above inequality is a contradiction to the inequality from the lemma and we have shown that the assertion holds for all $b \in [n_b]$ on which a down time forbidden control exists. □

A.3 Proof of Theorem 2

Proof The assertion can be shown for the parameter choices I. and II. in a very similar way, which is why we prove both cases in parallel. Since the algorithm activates for each block $b \in [n_\omega]$ either a *forced*, or *future forced*, or *admissible* control and the family of blocks is a partition $[N]$ by Lemma 2, exactly one control is activated on each interval $j \in [N]$ and therefore the (Conv) constraint satisfied. Hence, DNFR guarantees feasibility of $\mathbf{w}^{\mathrm{DNFR}}$. If down time forbidden controls are neglected, i.e., $\chi_D = 0$, $\mathbf{w}^{\mathrm{DNFR}}$ yields an objective value with at most the claimed upper bound by the definition of admissible and forced activation. The same holds for the choice $\chi_D = 1$, since the control deviation does not become greater than the claimed upper bound during a MD time phase by Theorem 1. Therefore, we need only to prove that DNFR always provides a solution. For this, we show that for each interval there is 1.) at least one admissible control and 2.) at most one forced control.

1.) We prove by contradiction that there exists at least one admissible control: Let us assume there is no admissible activation for block $b \in [n_b]$ and distinguish between the cases:

I. With $C_2 = \frac{2n_\omega - 3}{2n_\omega - 2}$ we assume

$$\Gamma_{i,b} < -\frac{2n_\omega - 3}{2n_\omega - 2}\overline{\mathscr{L}} + \mathscr{L}_b, \qquad i \in [n_\omega],$$

and we prove that it results in a contradiction. It follows from summing up all controls and from Lemma 3:

$$\mathscr{L}_b = \sum_{i \in [n_\omega]} \Gamma_{i,b} < n_\omega \left(-\frac{2n_\omega - 3}{2n_\omega - 2}\overline{\mathscr{L}} + \mathscr{L}_b \right) = -n_\omega \frac{2n_\omega - 3}{2n_\omega - 2}\overline{\mathscr{L}} + n_\omega \mathscr{L}_b,$$

and subtracting $n_\omega \mathscr{L}_b$ from the right hand side yields

$$(1-n_\omega)\overline{\mathscr{L}} \le (1-n_\omega)\mathscr{L}_b < -n_\omega \frac{2n_\omega - 3}{2n_\omega - 2}\overline{\mathscr{L}} = -n_\omega \overline{\mathscr{L}} + \frac{n_\omega}{2(n_\omega - 1)}\overline{\mathscr{L}} \overset{n_\omega \ge 2}{\le} (1-n_\omega)\overline{\mathscr{L}}. \qquad ↯$$

II. If there is no down time forbidden control on b, we can proceed as in I. Otherwise, we may have one control i_D that is down time forbidden. We assume all other controls are *inadmissible*, i.e.,

$$\Gamma_{i,b} < -\tfrac{3}{2}\overline{\mathscr{L}} + \mathscr{L}_b, \qquad i \in [n_\omega],\ i \ne i_D,$$

and we prove that it results in a contradiction. Hence, again by Lemma 3

$$\mathscr{L}_b = \sum_{i \in [n_\omega]} \Gamma_{i,b} = \sum_{i \ne i_D} \Gamma_{i,b} + \Gamma_{i_D,b} < (n_\omega - 1)(-\tfrac{3}{2}\overline{\mathscr{L}} + \mathscr{L}_b) + \Gamma_{i_D,b},$$

and therefore

$$\tfrac{3}{2}(n_\omega - 1)\overline{\mathscr{L}} - (n_\omega - 2)\mathscr{L}_b \le \tfrac{3}{2}\overline{\mathscr{L}} < \Gamma_{i_D,b}. \qquad ↯$$

The last inequality is a contradiction to Theorem 1.

We conclude that there must be an admissible activation for all blocks and thereby for all intervals.

2.) If there were more than one forced control at a time step, the algorithm would be ambiguous at line 3-4. Moreover, DNFR would provide, in this case, a solution that does not satisfy the upper bound on the objective. Therefore, we prove that this case is impossible and we do so again by contradiction. Assume that there $b \in [n_b]$ is the block with smallest index on which at least two controls i_1, i_2 are being forced, i.e.,

$$I. \quad \Gamma_{i_1,b}, \Gamma_{i_2,b} > \tfrac{2n_\omega - 3}{2n_\omega - 2}\overline{\mathscr{L}}, \qquad II. \quad \Gamma_{i_1,b}, \Gamma_{i_2,b} > \tfrac{3}{2}\overline{\mathscr{L}}. \tag{A.11}$$

In the proof of Theorem 1 we have already shown how to obtain a contradiction with only one forward control deviation $\Gamma_{i,b}$ greater than the rounding threshold. So, case II. is settled with this theorem and we focus on case I. for which we proceed very similarly as in the proof of Theorem 1. Let us first apply Lemma 3:

$$\overline{\mathscr{L}} \ge \mathscr{L}_b = \sum_{i \in [n_\omega]} \Gamma_{i,b} = \sum_{\substack{i \in [n_\omega], \\ i \ne i_1, i_2}} \Gamma_{i,b} + \sum_{i = i_1, i_2} \Gamma_{i,b} > \sum_{\substack{i \in [n_\omega], \\ i \ne i_1, i_2}} \Gamma_{i,b} + 2\frac{2n_\omega - 3}{2n_\omega - 2}\overline{\mathscr{L}}.$$

Hence, we have

$$\sum_{i \in [n_\omega], i \ne i_1, i_2} \Gamma_{i,b} < \overline{\mathscr{L}} - 2\,\frac{2n_\omega - 3}{2n_\omega - 2}\overline{\mathscr{L}} = -\frac{2n_\omega - 4}{2n_\omega - 2}\overline{\mathscr{L}},$$

which implies that there is at least one control i_3 such that

$$\Gamma_{i_3,b} < -\frac{1}{n_\omega - 2}\frac{2n_\omega - 4}{2n_\omega - 2}\overline{\mathscr{L}} = -\frac{2}{2n_\omega - 2}\overline{\mathscr{L}}.$$

Then, by Lemma 5, there is an earlier activation of i_3 on some block $b_3 < b$ and this activation has not been b-future forced on b_3. Let i_3 denote the control of this property with the last activation before b. This definition implies that all controls that are active between b_3 and b become forced until b. We use again the notation

$$F_{b_3,b} := \{ i \in [n_\omega] \mid \exists k(i) :\ b_3 < k(i) \le b \text{ on which } i \text{ is forced or } b\text{-future forced.} \}.$$

In particular, we find $i_1, i_2 \in F_{b_3,b}$. For $i \in F_{b_3,b} \setminus \{i_1, i_2\}$, we apply the definition of $F_{b_3,b}$ and Γ:

$$\Gamma_{i,b} = \Theta_{i,b-1} + \sum_{j \in \mathscr{J}_b} a_{i,j}\Delta_j = \sum_{k=1}^{b}\sum_{j \in \mathscr{J}_k} a_{i,j}\Delta_j - \sum_{k=1}^{k(i)}\sum_{j \in \mathscr{J}_k} w_{i,j}\Delta_j.$$

Since control i was last activated on block $k(i)$ and b- future forced on $k(i)$, we have

$$\Gamma_{i,b} > \frac{2n_\omega - 3}{2n_\omega - 2}\overline{\mathscr{L}} - \mathscr{L}_{k(i)}, \qquad i \in F_{j_3,j} \setminus \{i_1, i_2\}. \tag{A.12}$$

For block b_3 we know that i_3 has been chosen, even though not being b-future forced. That implies i_3 was selected on b_3 because all controls out of $F_{b_3,b}$ were not admissible at this block. Hence, it results for $i \in F_{b_3,b}$

$$\Gamma_{i,b_3} < -\frac{2n_\omega - 3}{2n_\omega - 2}\overline{\mathscr{L}} + \mathscr{L}_{b_3}, \qquad \Rightarrow \qquad \Theta_{i,b_3} < -\frac{2n_\omega - 3}{2n_\omega - 2}\overline{\mathscr{L}} + \mathscr{L}_{b_3}. \tag{A.13}$$

Now, we consider the sum of inequalities (A.12), (A.11) and sum up (A.13) over $F_{b_3,b}$, which yields

$$\sum_{i\in F_{b_3,b}} \Gamma_{i,b} \;>\; 2\frac{2n_\omega-3}{2n_\omega-2}\overline{\mathscr{L}}+(|F_{b_3,b}|-2)\left(\frac{2n_\omega-3}{2n_\omega-2}\overline{\mathscr{L}}-\mathscr{L}_{b_2}\right), \tag{A.14}$$

$$\sum_{i\in F_{b_3,b_3}} \Theta_{i,b_3} \;<\; |F_{b_3,b}|\left(-\frac{2n_\omega-3}{2n_\omega-2}\overline{\mathscr{L}}+\mathscr{L}_{b_3}\right), \tag{A.15}$$

where $b_2 := \arg\min\{\mathscr{L}_k \mid b_3 < k \leq b\}$. Substracting (A.15) from (A.14), we obtain

$$\begin{aligned}
\sum_{i\in F_{b_3,b}} \left(\Gamma_{i,b}-\Theta_{i,b_3}\right) \;&>\; 2\frac{2n_\omega-3}{2n_\omega-2}\overline{\mathscr{L}}+(|F_{b_3,b}|-2)\left(\frac{2n_\omega-3}{2n_\omega-2}\overline{\mathscr{L}}-\mathscr{L}_{b_2}\right)\\
&\quad -|F_{b_3,b}|\left(-\frac{2n_\omega-3}{2n_\omega-2}\overline{\mathscr{L}}+\mathscr{L}_{b_3}\right)\\
&= 2|F_{b_3,b}|\frac{2n_\omega-3}{2n_\omega-2}\overline{\mathscr{L}}-(|F_{b_3,b}|-2)\mathscr{L}_{b_2}-|F_{b_3,b}|\,\mathscr{L}_{b_3}\\
&\geq \overline{\mathscr{L}}\left(2|F_{b_3,b}|\frac{2n_\omega-3}{2n_\omega-2}-2|F_{b_3,b}|+2\right) && \text{(A.16)}\\
&= \overline{\mathscr{L}}\left(2-\frac{|F_{b_3,b}|}{n_\omega-1}\right)\\
&\geq \overline{\mathscr{L}}. && \text{(A.17)}
\end{aligned}$$

In (A.16) we used $\mathscr{L}_{b_2},\mathscr{L}_{b_3}\leq\overline{\mathscr{L}}$, while the last inequality holds due to $|F_{b_3,b}|\leq n_\omega-1$. As in the proof of Theorem 1, we invoke now Lemma 4 with $F_{b_3,b}=S_{b_1,b}$ in order to raise a contradiction to inequality (A.17). Overall, we have shown that there is at most one forced activation per block and thereby per interval. This completes the proof. □

Remark 7 The proceeding in the proof of Theorem 2 shows us, on closer inspection, that DNFR provides a solution with control deviation bounded by $C_2\overline{\mathscr{L}}$ for the absence of MD time constraints, i.e., $\chi_D=0$, and any chosen rounding threshold $C_2\geq\frac{2n_\omega-3}{2n_\omega-2}$ and any block length parameter $C_1\geq 0$. This implies the previously known result by NFR [21], $\theta(\mathbf{w}^{\mathrm{NFR}})\leq\bar{\Delta}$, evolves as special case of DNFR with $C_1=0$, and $C_2=1$.

B Discussion on the Tightness of the obtained Minimum Down Time Bounds

Proposition 5 (Tightness of the bound for (CIA-D)) *Let us assume the MD time constraint is active, i.e., $C_D>\underline{\Delta}$ is given. Then, the following is true:*

1. *The bound for (CIA-D) stated in Proposition 3 can not be improved by the DNFR scheme with $\chi_D=1$ for $n_\omega\geq 3$.*
2. *The bound for (CIA-D) is tight up to at most the constant $\frac{1}{4}C_D+\bar{\Delta}$.*

Proof The assumption of an active MD time constraint ensures that the bound cannot be improved by the bound for MU times from Proposition 2. We use again the concept of a *minimal C_1-overlapping grid*, here with $C_1=C_D/2$.

1. We want to prove that the DNFR scheme with $\chi_D=1$ and $C_2<\frac{3}{2}$ may provide solutions with a (CIA-D) objective greater than $C_2\overline{\mathscr{L}}$. Let us consider first $C_2\leq\frac{3}{2}-\varepsilon_1$, with $0<\varepsilon_1\leq 0.5$. We present example values for $\mathbf{a}\in A$ with a time horizon length of at least $12C_1$, so that at least six blocks with length $\overline{\mathscr{L}}$ exist by Lemma 1. Let $0<\varepsilon_2<\varepsilon_1$ be small and let the relaxed control values $\mathbf{a}$ be given as

$$(a_{i,b})_{i\in[n_\omega],b\in[n_b]}=\begin{pmatrix}
1 & 0.5-\varepsilon_1+\varepsilon_2 & 1-\varepsilon_2 & 2\varepsilon_1-2\varepsilon_2 & 0.5 & 0.5 & \cdots & 0.5\\
0 & 0.5+\varepsilon_1-\varepsilon_2 & 0 & 1-2\varepsilon_1+2\varepsilon_2 & 0.5 & 0.5 & \cdots & 0.5\\
0 & 0 & \varepsilon_2 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}.$$

With this example values we discuss the thereby constructed DNFR solution as well its objective quality.
- First block: i_1 is 2-future forced and activated.
- Second block: Both i_1 and i_2 are 4-future forced. The DNFR algorithm breaks ties arbitrarily, so that activating i_2 is legitimate.
- Third block: i_1 is *down time forbidden*, while i_2 is not *admissible*. DNFR activates therefore i_3.
- Fourth block: i_1 is activated, since it is *forced*.
- Fifth block: We have in the meantime $\Theta_{i_1,4}=(0.5+\varepsilon_1-2\varepsilon_2)\overline{\mathscr{L}}$ and $\Theta_{i_2,4}=(0.5-\varepsilon_1+\varepsilon_2)\overline{\mathscr{L}}$. Since ε_2 satisfies $\varepsilon_2<\varepsilon_1$, both controls are 6-future forced on the fifth block. Let DNFR activate i_2.
- Sixth block: i_1 is still *down time forbidden* and can not be active, which implies

$$\Theta_{i_1,6}=(0.5+\varepsilon_1-2\varepsilon_2+1)\overline{\mathscr{L}}>(\tfrac{3}{2}-\varepsilon_1)\overline{\mathscr{L}}=C_2\overline{\mathscr{L}},$$

so that the proposed control deviation bound is not fulfilled.

Finally, if $\varepsilon_1>0.5$, thus $C_2<1$, we can construct a similar example for which the control i_1 is already forced on the first block and the control deviation is again greater than $C_2\overline{\mathscr{L}}$.

2. The MD time constraints are equivalent to MU time constraints with $C_U = C_D$ for a problem with only two controls $n_\omega = 2$. Proposition 2 provides an example for this case, where $\theta \geq \frac{1}{2}(C_U + \bar{\Delta})$ holds. This example can be also applied for more than two controls by setting the relaxed control values $a_{i,b}$ to zero, for $i > 2$. Then, the difference to the upper bound $\frac{3}{4}C_D + \frac{3}{2}\bar{\Delta}$ from Proposition 3 is the one stated in the assertion.

□

Proposition 5 tells us the DNFR scheme with $C_2 = \frac{3}{2}$ and $\chi_D = 1$ can not be improved. The following example motivates why we have chosen the set of the *down time forbidden control* in such a way that the active control can only be changed after a duration of $C_D/2$ at the earliest. If it is possible to switch already after one interval Δ_j, DNFR may construct greedy solutions with a large control deviation at long MD times C_D. The following example shows the reason for this.

Example 1 Let $n_\omega = 3$ and a rounding threshold $C_2 \geq \frac{2n_\omega - 3}{2n_\omega - 2}$ be given. We alter DNFR in the following way: instead over blocks we iterate forward over all intervals. We keep for a given MD time $C_1 = C_D$ the threshold $C_2\overline{\mathscr{L}}$ for forced, future forced and admissible activation. In order to construct feasible solutions for (CIA-D), we extend the definition of $\mathscr{I}_b^D$ by letting all controls to be *down time forbidden* that are inactive and were active in the previous period of length C_D. Next, we are going to construct exemplary relaxed values for this modified DNFR scheme with large control deviation. We first introduce recursively the indices

$$j_i := \min\left\{ j \in [N] \mid \textstyle\sum_{l=j_{i-1}}^{j} \Delta_l > C_2\overline{\mathscr{L}} \right\}, \quad i = 1, 2, 3,$$

where $j_0 := 1$. Let the relaxed values be given as follows

$$(a_{i,j})_{i\in[n_\omega], j\in[N]} = \begin{pmatrix} 1 \cdots & \overbrace{1}^{j_1} & 0 \cdots & \overbrace{0}^{j_2} & 0 \cdots & \overbrace{0}^{j_3} & \cdots \\ 0 \cdots & 0 & 1 \cdots & 1 & 0 \cdots & 0 & \cdots \\ 0 \cdots & 0 & 0 \cdots & 0 & 1 \cdots & 1 & \cdots \end{pmatrix}.$$

Then, the modified DNFR may construct the following solution:

$$(w_{i,j})_{i\in[n_\omega], j\in[N]} = \begin{pmatrix} 1\ 0\ 0 \cdots & \overbrace{0}^{j_4} & \cdots \\ 0\ 1\ 0 \cdots & 0 & \cdots \\ 0\ 0\ 1 \cdots & 1 & \cdots \end{pmatrix},$$

where $j_4 := \min\{j \in [N] \mid \sum_{l\in[j]} \Delta_l < C_D + \Delta_1\}$ is the last index before the MD phase of i_1 ends. At first, i_1 is earliest future forced, but not anymore after being active on $j = 1$. Then, i_2 is activated on the second interval before i_3 is the earliest future forced control and needs to stay active until j_4, since the other controls are down time forbidden. We notice that with small Δ_1, Δ_2 it can result $j_4 \leq j_2$ and therefore

$$|\theta_{i_3,j_4}| = |\sum_{l=3}^{j_4} (0-1)\Delta_l| \leq |C_D + \bar{\Delta} - \Delta_2| \leq C_D + \bar{\Delta} - \underline{\Delta}.$$

If we compare the term on the right inequality side with the bound from Proposition 3, i.e., $\frac{2n_\omega - 3}{2n_\omega - 2}(C_D + \bar{\Delta}) = \frac{3}{4}(C_D + \bar{\Delta})$, we conclude that the latter is less only if $C_D < 4\underline{\Delta} - \bar{\Delta}$. Since $\underline{\Delta}$ can be arbitrarily small and we assumed C_D big compared with the grid length, the modified DNFR scheme would construct no improving bounds. Similar "greedy" examples can be constructed for $n_\omega > 3$ and block lengths greater than $\underline{\Delta}$.

Some comments on these tightness properties are in order.

Remark 8 (Quality of bound for (CIA-D)) The MD time configuration of DNFR, i.e., $\chi_D = 1$, yields smaller upper bounds compared with the DNFR algorithm with MU time configuration, i.e. $\chi_D = 0$ and $C_1 = C_D$, only for instances with more than three controls and a large MD time C_D compared with the grid length $\bar{\Delta}$. In fact, we conjecture the upper bound for any n_ω to be $\theta^{\max} = \frac{1}{2}C_D + \bar{\Delta}$, therefore only slightly greater than the one for $n_\omega = 2$. With this threshold, there would be no forced control until the first down time forbidden control appears and we postulate that active controls that become forced without activation during the next C_D time duration may stay active without other controls becoming forced. Of course, this argumentation justifies no proof - Proposition 5 together with Example 1 states that a generic solution fulfilling this bound can not be found via the DNFR scheme and it is presumably hard, if not even impossible, to construct it by another polynomial time algorithm.

Remark 9 (Quality of bound for (CIA-UD)) The integrality gap bound for (CIA-UD) as stated in Proposition 3 is tight for $C_U \geq C_D$ by the result of Proposition 2. For $C_U < C_D$, the bound is not necessarily tight, but it is again difficult to prove tight bounds due to the problem's combinatorial structure.

Remark 10 If we deal with an MDT C_1 that begins and ends exactly on the grid points, the upper bounds become $\frac{2n_\omega - 3}{2n_\omega - 2}C_U$ for (CIA-U), $\frac{3}{4}C_D$ for (CIA-D), and accordingly reduced for (CIA-UD).

C Proof of Proposition 4

Proof Using (6.1) and the definition of M_D, we obtain $M_D \geq 2(n_\omega - 1)$. Notice that even if $C_D/\bar{\Delta} \notin \mathbb{N}$, we still find for the cardinality of the dwell time index sets $|\mathscr{J}_k^{\mathrm{SUR}}(C_D)| = M_D \in \mathbb{N}$ for $k \leq N - M_D$ because we deal with an equidistant grid. Hence, we calculate the forward control deviation of the currently activated control in the DSUR algorithm (line 3) on the next M_D intervals.

We prove the claim by proving the following claims: For any $n_\omega \geq 2$, C_D and N fulfilling (6.1) and (6.2), there is an $\mathbf{a} \in A$ with

$$a_{i,j} = 0, \qquad \text{for} \quad i = 2, \ldots, n_\omega,\ j = 1, \ldots, i-1, \tag{C.1}$$

resulting in a constructed $\mathbf{w}^{\text{DSUR}}$ with

$$w_{i,j}^{\text{DSUR}} = 0, \qquad \text{for} \quad i = 2, \ldots, n_\omega, \; j = 1, \ldots, i-1, \tag{C.2}$$

and

$$\theta_{2,j} = \left(\frac{M_D}{2} + (n_\omega - 2)\right)\bar{\Delta}, \quad j = (n_\omega - 1)(1 + M_D) + \lceil M_D/2 \rceil - M_D. \tag{C.3}$$

This implies the Claim 6.3 by the definition of the objective value of (CIA-D). We proceed via induction.

Base case:
$n_\omega = 2$: By assumption we have $M_D \geq 2\bar{\Delta}$ and thus a nontrivial MD time. We construct an $\mathbf{a} \in A$ on $N = (1+M_D) + \lceil M_D/2 \rceil - 1$ intervals. If the Claim C.3 is true for this $\mathbf{a}$, it does also hold for $N \geq (1+M_D) + \lceil M_D/2 \rceil - 1$ because we can extend $\mathbf{a}$ by inserting arbitrary unit vector columns after the last column without affecting Claim C.3. We consider

$$(a_{i,j})_{i\in[n_\omega], j\in[N]} := \begin{cases} \begin{pmatrix} 1 & 0 \cdots 0 & 1 \cdots 1 \\ 0 & \underbrace{1 \cdots 1}_{j \in J_1} & \underbrace{0 \cdots 0}_{j \in J_2} \end{pmatrix}, & \begin{array}{l} J_1 = \{2, \ldots, M_D/2 + 1\}, \\ J_2 = \{M_D/2 + 2, \ldots, N\}, \end{array} \quad \text{if } M_D \text{ even}, \\ \begin{pmatrix} 1 & 0 \cdots 0 & 0.5 & 1 \cdots 1 \\ 0 & \underbrace{1 \cdots 1}_{j \in J_1} & 0.5 & \underbrace{0 \cdots 0}_{j \in J_2} \end{pmatrix}, & \begin{array}{l} J_1 = \{2, \ldots, \lceil M_D/2 \rceil\}, \\ J_2 = \{\lceil M_D/2 \rceil + 2, \ldots, N\}, \end{array} \quad \text{if } M_D \text{ odd}. \end{cases}$$

Because of $a_{2,1} = 0$, C.1 is true. The DSUR algorithm activates the first control on interval $j = 1$. Then, $i = 1$ is the *currently activated* control. Assuming $i = 1$ is active until $2 \leq k-1 \leq M_D/2$ for M_D being even, respectively $2 \leq k-1 \leq \lceil M_D/2 \rceil$ for M_D being odd, its dwell time block index set is $\mathscr{J}_k^{\text{SUR}}(C_D) = \{k, \ldots, k + M_D - 1\}$ and its forward control deviation on interval k as given in line 4 of DSUR amounts to

$$\theta_{1,k-1} + \sum_{l \in \mathscr{J}_k^{\text{SUR}}(C_D)} a_{1,l}\bar{\Delta} = -(k-2)\bar{\Delta} + (M_D/2 + (k-2))\,\bar{\Delta} = \frac{M_D}{2}\bar{\Delta}.$$

On the other hand, the forward control deviation for $i = 2$ on these intervals k amounts to

$$\gamma_{2,k} = \theta_{2,k-1} + a_{2,k}\bar{\Delta} = \begin{cases} (k-2)\bar{\Delta} + 0.5\bar{\Delta} = M_D/2, & \text{if } M_D \text{ odd and } k-1 = \lceil M_D/2 \rceil, \\ (k-2)\bar{\Delta} + 1\bar{\Delta} = (k-1)\bar{\Delta} \leq M_D/2, & \text{else}. \end{cases}$$

We observe that the forward control deviation for control $i = 1$ for all these intervals k is greater or equal to the one of $i = 2$ and we let DSUR deliberately choose $i = 1$ to be active in case of equality. Hence, $w_{1,j}^{\text{DSUR}} = 1$, for $j \in [N]$. This implies the control $i = 2$ stays inactive and in particular (C.2) is true. Combining this with the above forward control deviation for $i = 2$ yields

$$\theta_{2,1+\lceil M_D/2 \rceil} = \frac{M_D}{2}\bar{\Delta},$$

which settles the Claim (C.3) for $n_\omega = 2$.

Inductive step: We show that, if the claim holds for $n_\omega - 1$, then it is also true for n_ω.
Let $\mathbf{a}^{(n_\omega-1)} \in [0,1]^{(n_\omega-1)\times((n_\omega-2)(1+M_D)+\lceil M_D/2 \rceil-1)}$ denote a matrix for which DSUR constructs a $\mathbf{w}^{\text{DSUR}}$ that satisfies the Claims (C.1)-(C.3) for $n_\omega - 1$. We construct an $\mathbf{a} \in A$ on $N = (n_\omega - 1)(1 + M_D) + \lceil M_D/2 \rceil - 1$ intervals and with n_ω controls. We can argue similarly to the base case that we can neglect the case $N > (n_\omega - 1)(1 + M_D) + \lceil M_D/2 \rceil - 1$. Let $\mathbf{I}_k$ denote the identity matrix of dimension $k \times k$ and let $\mathbf{0}_k$ denote the zero matrix of dimension $k \times n$, where n is specified by the dimension of the block matrix below the zero matrix. We consider the following matrix

$$(a_{i,j})_{i\in[n_\omega], j\in[N]} := \left(\mathbf{I}_{n_\omega} \left| \begin{array}{c} \mathbf{I}_{n_\omega-1} \\ \hline 0 \cdots 0 \end{array} \right| \begin{array}{c} \mathbf{0}_{n_\omega-1} \\ \hline \underbrace{1 \cdots 1}_{j \in J} \end{array} \left| \begin{array}{c} \mathbf{a}^{(n_\omega-1)} \\ \hline 0 \cdots 0 \end{array} \right.\right), \qquad J = \{2n_\omega, \ldots, M_D + 1\},$$

where the third block of columns may be nonexistent, if $2n_\omega > M_D + 1$. The first two blocks of columns, however, are well-defined due to $M_D \geq 2(n_\omega - 1)$ by (6.1) and thus $2n_\omega - 1 \leq M_D + 1$. The above matrix is defined on N intervals, with N chosen as above, since we add $M_D + 1$ intervals to the existing $(n_\omega - 2)(1 + M_D) + \lceil M_D/2 \rceil - 1$ intervals from $\mathbf{a}^{(n_\omega-1)}$. At first, we see that (C.1) is satisfied by $\mathbf{a}$. Second, we claim that DSUR constructs the following $\mathbf{w}^{\text{DSUR}} \in W$:

$$(w_{i,j}^{\text{DSUR}})_{i\in[n_\omega], j\in[N]} := \left(\mathbf{I}_{n_\omega} \left| \begin{array}{c} \mathbf{0}_{n_\omega-1} \\ \hline \underbrace{1 \cdots 1}_{j \in J} \end{array} \right| \begin{array}{c} \mathbf{w}^{\text{DSUR},(n_\omega-1)} \\ \hline 0 \cdots 0 \end{array}\right), \qquad J = \{n_\omega + 1, \ldots, M_D + 1\},$$

where $\mathbf{w}^{\text{DSUR},(n_\omega-1)}$ denotes the obtained solution of DSUR for $\mathbf{a}^{(n_\omega-1)}$. We first justify this value for the intervals $k = 1, \ldots, n_\omega$:

- $k = 1$: DSUR selects the control $i = 1$ because of $a_{1,1} = 1$.
- $k = 2$: The control $i = 1$ is currently activated with a forward control deviation of $\bar{\Delta}$, calculated on the next M_D intervals. The forward control deviation for control $i = 2$ amounts to $\gamma_{2,2} = \theta_{2,1} + a_{2,2}\bar{\Delta} = 0 + \bar{\Delta}$. Therefore, DSUR may set the control $i = 2$ to be active.
- $k = 3$: We use the induction hypothesis for $\mathbf{a}^{(n_\omega-1)}$ and Claim (C.1) that yields $a_{2,M_D+2}^{(n_\omega-1)} = 0$. Thus, the forward control deviation of control $i = 2$ is $\bar{\Delta}$, which is the same for $i = 3$. We let DSUR deliberately set the control $i = 3$ to be active.

– $k = 4,\dots,n_\omega$: We argue analogously to the case $k = 3$.

Hence, (C.2) is established. After control $i = n_\omega$ has been activated on interval $k = n_\omega$, all other controls are *down time forbidden* until interval $M_D + 1$. Thus, control $i = n_\omega$ stays to be active up to and including interval $M_D + 1$. Because the controls $i = 1,\dots,n_\omega - 1$ are only once active until interval $M_D + 1$, but $\sum_{k\in[M_D+1]} a_{i,k}\bar{\Delta} = 2\bar{\Delta}$, we conclude $\theta_{i,M_D+1} = \bar{\Delta}$. This justifies why DSUR constructs $\mathbf{w}^{\mathrm{DSUR},(n_\omega-1)}$ after interval $M_D + 1$:

– The controls $i = 2,\dots,n_\omega - 1$ are *down time forbidden* on the intervals $k = (M_D+1)+1,\dots,(M_D+1)+i-1$, but are not activated in $\mathbf{w}^{\mathrm{DSUR},(n_\omega-1)}$ on these intervals according to the induction hypothesis (C.2) anyway.
– The control deviation for control n_ω is negative, i.e., $\theta_{n_\omega,k} = -(M_D + 1 - n_\omega)\bar{\Delta}$ for $k \geq M_D + 1$ so that control n_ω is not activated after interval $M_D + 1$.
– All other controls $1,\dots,n_\omega - 1$ start with the same control deviation $\theta_{i,M_D+1} = \bar{\Delta}$, when DSUR iterates on interval $M_D + 2$. Thus, DSUR constructs the same $\mathbf{w}$ from $\mathbf{a}^{(n_\omega-1)}$ as it would construct from $\mathbf{a}^{(n_\omega-1)}$ starting with the first interval and $\theta_{i,0} = 0$. This implies by the induction hypothesis DSUR generates $\mathbf{w}^{\mathrm{DSUR},(n_\omega-1)}$.

The induction hypothesis regarding (C.3) implies for $\mathbf{w}^{\mathrm{DSUR},(n_\omega-1)}$

$$\theta_{2,j} = \left(\frac{M_D}{2} + ((n_\omega - 1) - 2)\right)\bar{\Delta}, \quad j = ((n_\omega - 1) - 1)(1 + M_D) + \lceil M_D/2 \rceil - M_D.$$

We argued that this control deviation value is increased in $\mathbf{w}^{\mathrm{DSUR}}$ by $\bar{\Delta}$ and before the choice $\mathbf{w}^{\mathrm{DSUR},(n_\omega-1)}$ there exist $M_D + 1$ columns in $\mathbf{w}^{\mathrm{DSUR}}$. So, (C.3) is also true for n_ω. □

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