# An analysis of the superiorization method via the principle of concentration of measure

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**Abstract**

The superiorization methodology is intended to work with input data of constrained minimization problems, i.e., a target function and a constraints set. However, it is based on an antipodal way of thinking to the thinking that leads constrained minimization methods. Instead of adapting unconstrained minimization algorithms to handling constraints, it adapts feasibility-seeking algorithms to reduce (not necessarily minimize) target function values. This is done while retaining the feasibility-seeking nature of the algorithm and without paying a high computational price. A guarantee that the local target function

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reduction steps properly accumulate to a global target function value reduction is still missing in spite of an ever-growing body of publications that supply evidence of the success of the superiorization method in various problems. We propose an analysis based on the principle of concentration of measure that attempts to alleviate the guarantee question of the superiorization method.

**Keywords**: Superiorization, perturbation resilience, feasibility-seeking algorithm, target function reduction, concentration of measure, superiorization matrix, linear superiorization, Hilbert-Schmidt norm, random matrix.

# 1 Introduction

**The superiorization method studied in this paper.** Let $\mathcal{H}$ be a $J$-dimensional Hilbert space, i.e., the Euclidean space $E^J$ with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and consider the convex feasibility problem (CFP) which is to find a point in the nonempty intersection $C$ of a finite number $C_1, C_2, \ldots, C_I$ of closed convex sets in $\mathcal{H}$. Let $(A_t)_{t=1}^{\infty}$ be a sequence of operators $A_t : \mathcal{H} \to \mathcal{H}$ that gives rise to an iterative process which, starting from an initial $x_0 \in \mathcal{H}$, generates a sequence $(x_n)_n \subset \mathcal{H}$ by

$$x_{n+1} := A_{n+1}(x_n), \qquad n = 0, 1, 2, \ldots. \tag{1}$$

Further, assume that any sequence $(x_n)_n$, generated by this process converges, for any initial $x_0 \in \mathcal{H}$, to some point $x_\infty \in C$. An algorithm[1] that employs such a process is called a 'feasibility-seeking algorithm' and will be, henceforth, referred to as a 'basic algorithm'.

Now, consider an iterative process that uses the same algorithmic operators $(A_t)_{t=1}^{\infty}$ but perturbs the iterates and generates another sequence $(x'_n)_n \subset \mathcal{H}$ by

$$x'_0 = x_0, \quad x'_{n+1} := A_{n+1}(x'_n + \beta_n v_n), \qquad n = 0, 1, 2, \ldots, \tag{2}$$

where $v_n \in \mathcal{H}$ and $\beta_n$ are real numbers so that $\|v_n\| \leq M$, are bounded by some $M$, and $\beta_n \geq 0$, for all $n \geq 0$, and $\sum_{n=0}^{\infty} \beta_n < +\infty$. Assume

[1] As common, we use the terms algorithm or algorithmic structure for the iterative processes studied here although no termination criteria are present and only the asymptotic behavior of these processes is studied.

that any sequence $(x'_n)_n$, generated by this process, converges to some point $x'_\infty \in C$. An algorithm that employs such a process is called a 'superiorized version of the basic algorithm'. Modifications of this superiorized version of the basic algorithm have been developed, see, e.g., the Appendix, entitled: "The algorithmic evolution of superiorization" in [12], however, our current investigation focuses solely on the above formulation.

The superiorization method (SM) considered here looks at basic algorithms of the form (1) that are resilient to perturbations as those that appear in (2) and aims at using inexpensive such perturbations in order to reach (i.e., asymptotically converge to) a feasible point in $C$ that is superior with respect to some given target function. These notions are made precise in the next sections.

**Readings.** To a novice on the SM and perturbation resilience of algorithms we recommend to read first the recent reviews in [11, 25, 26]. Current work on superiorization can be appreciated from the continuously updated Internet page [10]. For a recent description of previous work that is related to superiorization but is not included in [10], such as the works of Sidky and Pan, e.g., [33], we direct the reader to [14, Section 3]. The SNARK14 software package [20], with its in-built capability to superiorize iterative algorithms to improve their performance, can be helpful to practitioners. Naturally, there is variability among the bibliography items of [10] in their degree of relevance to the superiorization methodology and perturbation resilience of algorithms. In some, superiorization does not appear in the title, abstract or introduction but only inside the work, e.g., [38, Subsection 6.2.1: Optimization vs. Superiorization].

**A word about the history.** The terms and notions "superiorization" and "perturbation resilience", in the present context, first appeared in the 2009 paper of Davidi, Herman and Censor [21] which followed its 2007 forerunner by Butnariu, Davidi, Herman and Kazantsev [5]. The ideas have some of their roots in the 2006 and 2008 papers of Butnariu, Reich and Zaslavski [6, 7]. All these culminated in Ran Davidi's 2010 PhD dissertation [19] and the many papers since then cited in [10].

**The guarantee problem of the SM.** The SM interlaces into a feasibility-seeking basic algorithm target function reduction steps. These steps cause the target function to reach lower values locally, prior to performing the next feasibility-seeking iterations. A mathematical guarantee has not been found to date that the overall process of the superiorized version of the basic algorithm will not only retain its feasibility-seeking nature but also preserve

globally the target function reductions. We call this fundamental question of the SM "the guarantee problem of the SM" which is: "under which conditions one can guarantee that a superiorized version of a bounded perturbation resilient feasibility-seeking algorithm converges to a feasible point that has target function value smaller or equal to that of a point to which this algorithm would have converged if no perturbations were applied – everything else being equal."

Numerous works that are cited in [10] show that this global function reduction of the SM occurs in practice in many real-world applications. But until the guarantee problem of the SM is answered one wonders if the SM is just a successful heuristic or if there is a mathematical foundation for the accumulating reports on its performance success? Therefore, answering the guarantee problem of the SM is an intriguing issue, which to our knowledge, has not been discussed in the literature in any way.

**Concentration of measure.** Concentration of measure (about a median) is a principle that is applied in measure theory, probability and combinatorics, and has consequences for other fields such as Banach space theory. Informally, it states that "A random variable that depends in a Lipschitz way on many independent variables (but not too much on any of them) is essentially constant", [37].

The concentration of measure phenomenon was put forth in the early 1970s by Vitali Milman in his works on the local theory of Banach spaces, extending an idea going back to the work of Paul Lévy, as noted in [24]. It was further developed in the works of Milman and Gromov, Maurey, Pisier, Schechtman, Talagrand [37], Ledoux [28], and others.

**Contribution and structure of this paper.** We offer an analysis of the guarantee problem of the SM via the principle of concentration of measure. This approach raises though some further questions but it is a first step toward explaining why the SM works. In Section 2 we elaborate on the SM while in Section 3 we describe it in detail and offer a layout of the situation in "matrix" form via an infinite lower triangular matrix called the superiorization matrix. In Section 4 we present a brief primer on the principle of concentration of measure with which we intend to analyze the behavior of the SM. The special case of linear superiorization (LinSup) is discussed in Section 5. A pathway to the nonlinear case is discussed in Section 6, followed by some concluding remarks in Section 7. Technical results that support and enable the analysis are presented in the Appendices A.1–A.7 at the end of the paper.

# 2 Background of the superiorization methodology

**The superiorization methodology**. To answer in a succinct manner the question "what is the superiorization methodology?" the next three paragraphs are quoted from our preface to the special issue "Superiorization: Theory and Applications" [15]:

"The superiorization methodology (SM) is used for improving the efficacy of iterative algorithms whose convergence is resilient to certain kinds of perturbations. Such perturbations are designed to 'force' the perturbed algorithm to produce more useful results for the intended application than the ones that are produced by the original iterative algorithm. The perturbed algorithm is called the 'superiorized version' of the original unperturbed algorithm. When the original algorithm is computationally efficient and useful in terms of the application at hand and if the perturbations are simple and not expensive to calculate, then the advantage of this method is that, for essentially the computational cost of the original algorithm, we are able to get something more desirable by steering its iterates according to the designed perturbations. This is a very general principle that has been used successfully in some important practical applications, especially for inverse problems such as image reconstruction from projections, intensity-modulated radiation therapy and nondestructive testing, and awaits to be implemented and tested in additional fields.

An important case is when the original algorithm is 'feasibility-seeking' (in the sense that it strives to find some point that is compatible with a family of constraints) and the perturbations that are introduced into the original iterative algorithm aim at reducing (not necessarily minimizing) a given merit function. In this case, superiorization has a unique place in optimization theory and practice. Many constrained optimization methods are based on methods for unconstrained optimization that are adapted to deal with constraints. Such is, for example, the class of projected gradient methods wherein the unconstrained minimization inner step 'leads' the process and a projection onto the whole constraint set (the feasible set) is performed after each minimization step in order to regain feasibility. This projection onto the entire constraints set is in itself a non-trivial optimization problem and the need to solve it in every iteration hinders projected gradient methods and restricts their efficiency only to feasible sets that are 'simple to project

onto.' Barrier or penalty methods likewise are based on unconstrained optimization combined with various 'add-on's that guarantee that the constraints are preserved. Regularization methods embed the constraints into a 'regularized' objective function and proceed with unconstrained solution methods for the new regularized objective function.

In contrast to these approaches, the superiorization methodology can be viewed as an antipodal way of thinking. Instead of adapting unconstrained minimization algorithms to handling constraints, it adapts feasibility-seeking algorithms to reduce merit function values. This is done while retaining the feasibility-seeking nature of the algorithm and without paying a high computational price. Furthermore, general-purpose approaches have been developed for automatically superiorizing iterative algorithms for large classes of constraints sets and merit functions; these provide algorithms for many application tasks." (end of quote.)

**Usefulness of the approach.** The usefulness of the SM relies on two features: (i) **Computational**: feasibility-seeking is logically a less-demanding task than seeking a constrained minimization point in a feasible set. Therefore, letting efficient feasibility-seeking algorithms "lead" the algorithmic effort and modifying them with inexpensive add-ons works well in practice. (ii) **Applicational**: in some significant real-world applications the choice of a target function is exogenous to the modeling and data collection which give rise to the constraints. In such situations the limited confidence in the usefulness of a chosen target function leads often to the recognition that, from the application-at-hand point of view, there is no need, neither a justification, to search for an exact constrained minimum[2]. For obtaining "good results", evaluated by how well they serve the task of the application at hand, it is often enough to find a feasible point that has reduced (not necessarily minimal) target function value.

**Weak superiorization and strong superiorization**. It is worthwhile to note here that there are two research directions in the general area of the superiorization methodology. One is the direction when only bounded perturbation resilience is used and the constraints are assumed to be consistent

[2]Some support for this reasoning may be borrowed from the American scientist and Noble-laureate Herbert Simon who was in favor of "satisficing" rather than "maximizing". Satisficing is a decision-making strategy that aims for a satisfactory or adequate result, rather than the optimal solution. This is because aiming for the optimal solution may necessitate needless expenditure of time, energy and resources. The term "satisfice" was coined by Herbert Simon in 1956 [35], see: https://en.wikipedia.org/wiki/Satisficing.

(having nonempty intersection). Then, one treats the "superiorized version" of the original unperturbed basic algorithm actually as a recursion formula that produces an infinite sequence of iterates, and convergence questions are meant in their asymptotic nature. This is the framework in which we work in this paper. The second direction does not assume consistency of the constraints but uses instead a proximity function that "measures" the violation of the constraints. Instead of seeking asymptotic feasibility, it looks at $\varepsilon$-compatibility with $C$ and uses the notion of "strong perturbation resilience", see [26, Subsection II.C] where this direction has been initiated. The same core "superiorized version" of the original unperturbed algorithm might be investigated in each of these directions, but the second is the more useful one for practical applications, whereas the first makes only asymptotic statements. The terms "weak superiorization" and "strong superiorization" were proposed as a nomenclature for the first and second directions, respectively, in [17, Section 6] and [11]. We do not discuss here the latter, therefore, whenever we say superiorization in the sequel we mean weak superiorization.

# 3 The guarantee problem of the superiorization methodology

In order to consider basic algorithms of the form (1) that are resilient to perturbations as those that appear in (2) formally, the following definition is used, see, e.g., [13, Definition 1], where it was formulated for a single algorithmic operator, i.e., $A_t = A$ for all $t \geq 0$.

**Definition 1** ***Bounded Perturbation Resilience (BPR)*** *Given a sequence of operators $A_t : \mathcal{H} \to \mathcal{H}$, for all $t \geq 0$, an algorithm as in (1) is said to be* `bounded perturbations resilient` *if the following holds: If the algorithm (1) generates sequences $(x_n)_n$ that converge to points in $C$ for all $x_0 \in \mathcal{H}$, then any sequence $(x'_n)_n$, generated by (2) where the vector sequence $(v_n)_n$ is bounded, $\beta_n \geq 0$ for all $n \geq 0$, and $\sum_{n=0}^{\infty} \beta_n < +\infty$, also converges to a point in $C$ for any $x'_0 \in \mathcal{H}$.*

These notions appear in earlier papers on the SM, see, e.g., [13, 21, 14, 26].

In addition to the basic algorithm and its superiorized version we consider in the SM a target function $\phi : \Delta \to E$, whose domain $\Delta \subseteq E^J$ contains the feasible set $C$, and we adopt the convention that a point in $\Delta$ for which the

value of $\phi$ is smaller is considered *superior* to a point in $\Delta$ for which the value of $\phi$ is larger. The essential idea of the SM is to make use of the perturbations of (2) to transform a perturbation resilient algorithm that seeks a feasible solution (the basic algorithm) into its superiorized version whose outputs are equally good from the point of view of feasibility-seeking, but are superior (not necessarily optimal) with respect to the target function $\phi$.

The SM, which works well in numerous numerical applications (consult [10]), consists of choosing the perturbation vectors $v_n$ in (2) as directions of nonascent of $\phi$ in the superiorized version of the basic algorithm. With the above information we formulate the guarantee problem of the SM.

**Problem 2** ***The guarantee problem of weak superiorization***

*The* `guarantee problem of the weak superiorization method`*, discussed here, is the following question: Can we provably guarantee, maybe under some assumptions, that for a given nonempty constraints set $C$ of a CFP and a target function $\phi : \Delta \to E$ such that $C \subseteq \Delta$ we will have $\phi(x'_\infty) \leq \phi(x_\infty)$ for the limits $x_\infty$ and $x'_\infty$ of sequences $(x_n)_n$ and $(x'_n)_n$ generated by the basic algorithm (1) and its superiorized version (2), respectively, both initiated at the same starting point $x_0 = x'_0$?*

To the best of our knowledge, this has not been answered in any way. The only result in this direction is the attempt to investigate the behavior of a superiorized version of a basic feasibility-seeking algorithm done by us in [17]. The main result there (Theorem 4.1 in [17]) establishes a mathematical basis for the behavior of the SM when dealing with input data of constrained minimization problems, i.e., a target function and a constraints set. In particular, a feasible region that is the intersection of finitely many closed convex constraint sets is assumed. The dynamic string-averaging projection (DSAP) method, with variable strings and variable weights, is playing there the role of a feasibility-seeking algorithm, which is indeed bounded perturbations resilient. The bounded perturbations resilience of the DSAP method has been proved in [16] and it is worthwhile to note that the DSAP is an algorithmic scheme that includes several well-known specific feasibility-seeking algorithms as special cases. These include, but are not limited to, the sequential Kaczmarz projections method and the simultaneous Cimmino projections method, see, e.g., [9].

Theorem 4.1 in [17] says that any sequence, generated by the superiorized version of a DSAP algorithm (Algorithm 4.1. there), will not only converge

to a feasible point, a fact which is due to the bounded perturbations resilience of the DSAP method, but, additionally, that exactly one of two alternatives must hold. Either its limit point will solve the constrained minimization problem of the same data, or that the sequence is strictly Fejér monotone with respect to (i.e., gets strictly closer to the points of) a subset of the solution set of the constrained minimization problem of the same data. But Fejér monotonicity, even if strict, does not yield convergence to a point in the set with respect to which the sequence is strictly Fejér monotone. So, this result shows that one gets closer to a subset of the solution set of the constrained minimization problem but it falls short of proving the convergence toward such a set.

The superiorization method uses input data consisting of a constraints set $C$ which is the intersection of several individual sets $C_1, C_2, \ldots, C_I$ and a target function $\phi$. Feasibility-seeking with a sequential projections basic algorithm will lead asymptotically to a feasible point $x^*$. Perturbations via interlaced local moves in the negative gradient direction will not prevent the process from converging to a feasible point if the basic feasibility-seeking algorithm is bounded perturbations resilient. Convergence of the superiorized algorithm to any superior feasible point is the subject of the "guarantee problem of SM" discussed in this paper. Any superior feasible point has a target function value $\phi$ that is lower than that of the feasible point $x^*$ which is reached (asymptotically) by the same basic feasibility-seeking algorithm without any interlaced perturbations – everything else in the implementation, such as relaxation parameters, initialization point, ordering of the individual sets that are projected on, etc. – being equal.

## 3.1 A layout of the SM as a matrix of elements

We will use the following definitions.

**Definition 3** *Let $A : \mathcal{H} \to \mathcal{H}$ be an operator and let $D \subset \mathcal{H}$.*

*(i) The operator $A$ is called* `nonexpansive` *on $D$ if*

$$\|A(x) - A(y)\| \leq \|x - y\|, \text{ for all } x, y \in D. \tag{3}$$

*(ii) The operator $A$ is called* `monotone` *on $D$ if*

$$\langle y - x, Ay - Ax \rangle \geq 0, \text{ for all } x, y \in D. \tag{4}$$

These definitions describe the action of $A$ on a pair $x, y$ compared with the original pair: Nonexpansive operators do not make the pair "further apart", while monotone operators "do not rotate it in more than 90 degrees." A linear orthogonal projection is nonexpansive and monotone, any linear operator with norm $\leq 1$ is nonexpansive while any linear operator whose symmetric part is positive definite is monotone. Also, the nearest point projection on a closed convex set is nonexpansive and monotone, see, e.g., [2, Example 20.12]. To facilitate our analysis we define an infinite lower triangular matrix of elements of $\mathcal{H}$ and name it *the Superiorization Matrix*.

**Definition 4** ***(The Superiorization Matrix)*** *Let $(A_t)_{t=1}^{\infty}$, $(x_n)_n$, $(v_n)_n$, and $(\beta_n)_n$ be as in the previous section above. Define $(\mathcal{M}(n,k))_{n=0,k=0}^{\infty,\infty}$, an infinite lower triangular matrix of elements of $\mathcal{H}$ as follows:*

*(1) In the upper left corner define an arbitrary vector in $\mathcal{M}(0,0) := x_0 = x'_0 \in \mathcal{H}$.*

*(2) Construct the $n$-th row from the $(n-1)$-th row by applying $A_n$, in each column $k$, to the entry above it in the $(n-1)$-th row in that column:*

$$\mathcal{M}(n,k) := A_n(\mathcal{M}(n-1,k)), \qquad k = 0, 1, \ldots, n. \tag{5}$$

*(3) Additionally, add for each $n \geq 0$, the $(n, n+1)$-th entry which is obtained from the $(n,n)$-th entry by adding to it $\beta_n v_n$:*

$$\mathcal{M}(n, n+1) := \mathcal{M}(n,n) + \beta_n v_n. \tag{6}$$

The superiorization matrix can be described in the following form:

$$\begin{bmatrix}
 & 0 & 1 & 2 & \cdots & n \\
0 & x_0 = x'_0 & x_0{+}\beta_0 v_0 & & & \\
1 & x_1{=}\, A_1(x_0) & x'_1{=}\, A_1(x_0{+}\beta_0 v_0) & x'_1{+}\beta_1 v_1 & & \\
2 & x_2{=}\, A_2(x_1) & A_2(x'_1) & x'_2{=}\, A_2(x'_1{+}\beta_1 v_1) & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & x'_{n-1}{+}\beta_{n-1} v_{n-1} \\
n & x_n{=}\, A_n(x_{n-1}) & \cdots & \cdots & \cdots & x'_n{=}\, A_n(x'_{n-1}{+}\beta_{n-1} v_{n-1}) \\
\vdots & \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\
 & x_{\infty,0}{=}\, x_\infty \in C & x_{\infty,1} \in C & x_{\infty,2} \in C & \cdots & x_{\infty,n} \in C
\end{bmatrix}. \tag{7}$$

The upper-most row and left-hand side column include the column and row indices of the matrix, respectively. The bottom-row is not part of the

matrix either but depicts the limits of the sequences of each column. Only the first $n$ columns are depicted but the matrix has infinitely many columns as well as infinitely many rows. The sequence in the 0-th column is generated by the basic algorithm, thus, converges to $x_{\infty,0} = x_\infty \in C$, while the sequence of the main diagonal elements of the matrix are the iterates generated by the superiorized version of the basic algorithm which, therefore, converges to $x'_\infty$.

This matrix representation of the SM is new and has never been published before. We consider it an indispensable tool in analyzing the progress of iterative sequences generated by the SM.

**Lemma 5** *The Superiorization Matrix of Definition 4 has the following properties:*

*(i) For all* $n \geq 0$, $\mathcal{M}(n, 0) = x_n$.

*(ii) The infinite sequence of the elements in the* $k$*-th column, for each* $k \geq 0$, *converges to a point* $x_{\infty,k} \in C$. *Observe that* $x_{\infty,0} = x_\infty$.

*(iii) The diagonal elements of the matrix are* $\mathcal{M}(n, n) = x'_n$, *thus, if bounded perturbations resilience holds then the infinite sequence of the elements along the main diagonal of the superiorization matrix will converge to* $x'_\infty$.

*(iv) For a target function* $\phi$ *whose domain contains the feasible set* $C$

$$\phi(x_n) - \phi(x'_n) = \sum_{k=1}^{n} \phi(\mathcal{M}(n, k-1)) - \phi(\mathcal{M}(n, k)), \text{ For all } n \geq 0. \quad (8)$$

**Proof.** (i) This follows from the definition. It means that the infinite sequence of all elements in the 0-th column constitute the sequence $(x_n)_n$ generated by the basic (feasibility-seeking) algorithm.

(ii) This is so because in each column only a finite number of initial elements are perturbed and from one point onward the operators are applied without further perturbations. Recall that we assumed that any sequence $(x_n)_n$, generated by the basic algorithm (1), converges to a point in $C$, for any initial point.

(iii) This follows by induction since $\mathcal{M}(0, 1) = x_0 + \beta_0 v_0$; $\mathcal{M}(1, 1) = A_1(x_0 + \beta_0 v_0) = x'_1$; $\mathcal{M}(1, 2) = x'_1 + \beta_1 v_1$; $\mathcal{M}(2, 2) = A_2(x'_1 + \beta_1 v_1) = x'_2$, and so on.

(iv) This follows from $\phi(x_n) = \phi(\mathcal{M}(n, 0))$ and $\phi(x'_n) = \phi(\mathcal{M}(n, n))$ by going along the $n$-th row of the matrix. ■

The following lemma states that if a basic algorithm like (1) always converges to a point in $C$ and if the operators $A_t : \mathcal{H} \to \mathcal{H}$, for all $t \geq 0$, are nonexpansive, then the superiorized version of the basic algorithm also converges to a point in $C$.

**Lemma 6** *Let $(A_t)_{t=1}^{\infty}$ be a sequence of operators $A_t : \mathcal{H} \to \mathcal{H}$ that gives rise to an iterative process which, starting from any initial $x_0 \in \mathcal{H}$, generates a sequence $(x_n)_n \subset \mathcal{H}$ by (1) and assume that any sequence $(x_n)_n$, generated by this process, converges to some point $x_\infty \in C$. If the operators $(A_t)_{t=1}^{\infty}$ are nonexpansive and if $v_n \in \mathcal{H}$ and $\beta_n$ are real numbers so that $\|v_n\| \leq M$, $\beta_n \geq 0$, for all $n \geq 0$, and $\sum_{n=0}^{\infty} \beta_n < +\infty$ then the algorithm (1) is bounded perturbations resilient.*

**Proof.** In each column, from some row downward, only consecutive applications of the operators $A_t$ occur. Therefore, since any sequence generated by the basic algorithm always converges, every column $k$ converges to some limit $x_{\infty,k} \in C$. Dividing all $v_n$ by their norms we can assume, without loss of generality, that $\|v_n\| = 1$, for all $n \geq 0$. Thus, looking at the main diagonal entries, we have

$$\|\mathcal{M}(k, k+1) - \mathcal{M}(k, k)\| = \|\beta_k v_k\| = \beta_k, \tag{9}$$

and, due to the nonepansiveness of the operators, all distances between any other pair of horizontally-neighboring entries in the $s$-th and $(s+1)$-th columns are smaller or equal $\beta_s$, hence, we have also that neighboring column limits are close, i.e., $\|x_{\infty,s+1} - x_{\infty,s}\| \leq \beta_s$ for all $s \geq 0$. Therefore, the sequence $(x_{\infty,k})_k$ of all limits of the columns is a Cauchy sequence of elements in $C$, which will converge to some $x_{\infty,\infty} \in C$. To show that the latter is the limit of $(x_n)_n$, observe that for every row $n$ and for all $k < n$, we have

$$\|x'_n - \mathcal{M}(n, k)\| = \|\mathcal{M}(n, n) - \mathcal{M}(n, k)\| \leq \sum_{s=k}^{n} \beta_s \leq \sum_{s=k}^{\infty} \beta_s. \tag{10}$$

Since $\lim_{n\to\infty} \mathcal{M}(n, k) = x_{\infty,k}$, the distance of any limit point of $(x'_n)$ from $x_{\infty,k}$ must be smaller or equal $\sum_{s=k}^{\infty} \beta_s$, hence, has distance smaller or equal $2\sum_{s=k}^{\infty} \beta_s$ from $x_{\infty,\infty}$. But $\lim_{k\to\infty} (2\sum_{s=k}^{\infty} \beta_s) = 0$ which yields $\lim_{n\to\infty} x'_n = x_{\infty,\infty}$. ■

This lemma should be compared with Theorem 1 of [26]. The latter makes more assumptions and proves strong perturbation resilience, not only bounded perturbation resilience. So, these two results complement each other.

# 4 Concentration of measure

The phenomenon of *concentration of measure* is the fact that, in some important cases of random variables, it turns out that with almost full probability the random variable is very close to its expectation, aka mean. For example, a classical case of concentration of measure is the Law of Large Numbers, see, e.g., [31], combined with the Central Limit Theorem of probability theory, which describe how with almost full probability a *sum of many independent random variables* is concentrated near its mean. Moreover, the distribution of the sum is almost normal. The literature on this topic is wide and varied, see, e.g., [22, 8, 30].

To explain the principle of concentration of measure in a manner that is appropriate for our needs, we focus on a case, featuring in high-dimensional Euclidean spaces, i.e., $E^N$ with the Euclidean norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ – the $N$-dimensional real Hilbert space. There, for uniform probability in its unit sphere $S^{N-1} := \{u \in E^N \,|\, \|u\| = 1\}$, almost the whole mass concentrates near the equator. In other words, for randomly given two vectors, with almost full probability the angle between them is near 90°.

More precisely, fix a unit vector $u_0 \in S^{N-1}$, e.g., $u_0 = (1, 0, \ldots, 0)$. For $u \in S^{N-1}$, let $\alpha$ be the latitude relative to $u_0$, i.e., the angle between $u$ and the hyperplane orthogonal to $u_0$, so, $-\frac{1}{2}\pi \leq \alpha \leq \frac{1}{2}\pi$. The uniform measure in $S^{N-1}$ can be "disintegrated" along $\alpha$, to levels $\alpha = const.$ which are translates of $\cos\alpha \cdot S^{N-2}$. Therefore, if we denote by $d\omega = d\omega_{N-2}$ the uniform (say, normalized to be probability) measure on $S^{N-2}$ then the uniform measure on $S^{N-1}$ will be

$$K \cdot (\cos\alpha)^{N-2}\, d\omega\, d\alpha, \tag{11}$$

where $K$ is a normalizing constant. For $N$ large, $(\cos\alpha)^{N-2}$ has a steep peak near $\alpha = 0$, thus, almost the whole mass is concentrated there. Indeed, for $\alpha$ small, which will, thus, be the significant case,

$$(\cos\alpha)^{N-2} \approx (1 - \frac{1}{2}\alpha^2)^{N-2} \approx \exp(-\frac{1}{2}N\alpha^2), \tag{12}$$

i.e., the distribution of $\alpha$, for big $N$, is very close to normal distribution with standard deviation $1/\sqrt{N}$.

The concentration of measure principle may be derived also in an alternative way, where the uniform distribution on the sphere $S^{N-1}$ is treated, very usefully, as follows. Take the distribution on $x \in E^N$ with coordinates i.i.d. (independent identically distributed) $\sim \mathcal{N} = \mathcal{N}(0,1)$, i.e., distributed as standard normal – with mean 0 and standard deviation 1 (that is, according to $(1/\sqrt{2\pi})\exp(-\frac{1}{2}x^2)\,dx$.)

As is well-known, this distribution in $E^N$ is invariant under any orthogonal self-map of $E^N$. This means that $u = x/\|x\|$ will be distributed uniformly on $S^{N-1}$. So, we have here a vehicle to get this uniform distribution. This also implies that $\langle x, a\rangle \sim \mathcal{N}$, for any fixed unit vector $a$.

If one considers $\sum_{i=1}^N \eta(x_i)$, with any function $\eta$, then the distribution of that sum will lose the orthogonal symmetry, but since the $\eta(x_i)$, $i = 1, 2, \ldots, N$, are still independent, the Law of Large Numbers and the Central Limit Theorem still apply. Thus, the distribution of the sum is concentrated near its expectation.

This applies, in particular, to $\|x\|_2 = \left(\sum_{i=1}^N x_i^2\right)^{1/2}$. Its expectation is $N$, since the expectation of a single $x_i^2$ is the variance which is equal to the square of the standard deviation, thus, equal to 1. And we recapture the main assertion above: As $\|x\|/\sqrt{N}$ is near 1 with almost full probability, the distribution of $\langle u, a\rangle$, for $u$ uniform on $S^{N-1}$, (e.g., our $u = x/\|x\|$), is very near $(1/\sqrt{N}) \cdot \langle x, a\rangle$ – the latter standard normal. In particular, $\langle u, a\rangle$ is very unlikely to be different from zero more than in an order of magnitude of $1/\sqrt{N}$.

We shall make use of some facts, in spirit of concentration of measure, which arise in high dimensional Euclidean (i.e., real Hilbert) $E^N$, which are derived in Appendix A at the end of this paper.

# 5 The case of linear superiorization (LinSup)

Linear superiorization (LinSup) was investigated in [12, 18] where a linear setting is considered. The operators of the basic algorithm are projections on half-spaces, thus, involve linear projections on hyperplanes plus constants, and the target function $\phi$ is linear, i.e., $\phi(x) := \langle c, x\rangle + a$ where $c$ is a given vector and $a$ is a given real constant.

In the superiorization matrix $\mathcal{M}$ (Definition 4) setting, these operators act on the pairs along the neighboring $i$-th and $(i+1)$-th columns, in particular, these operators are rotating and stretching/shrinking the "increments" $\Delta_{k,i}$ defined by

$$\Delta_{k,i} := \mathcal{M}(k, i+1) - \mathcal{M}(k, i). \tag{13}$$

To handle this, the idea is to treat the operators as a random sample. Since what the operators do to increments does not depend on the constant part, we characterize the operators by the unit vector $u$ orthogonal to the bounding hyperplane of each half-space, and assume that these vectors are a sample from a uniform distribution on $S^{N-1}$.

Then, by the principle of concentration of measure, with almost full probability, $u$ will be almost orthogonal to the increment in question, indeed making angle $\frac{1}{2}\pi + \alpha$ where $\alpha$ is distributed in almost a normal distribution with standard deviation $1/\sqrt{N}$. The hyperplane orthogonal to $u$, onto which $A_n$ projects, will make that small angle $\alpha$ with the increment, thus, the projection of that increment – the increment in the next row – is rotated in that small angle $\alpha$ (and has almost the same length.)

In other words, by the principle of concentration of measure in high-dimensional spaces that we speak of, if one has an instance of our operator acting on a vector (in our case - an increment) $y$, it would be a very unexpected "anomaly" not to find $u$ and $y$ to be almost orthogonal – making an angle $\frac{1}{2}\pi + \alpha$ with $\alpha$ small as above, thus, to have the hyperplane orthogonal to $u$ making that small angle $\alpha$ with $y$. All these arguments are true provided that we are justified to use our probabilistic model (i.e., with $u$ distributed uniformly).

Thus, the application of the linear operator $A_k$ in the passage from the $(k-1)$-th to the $k$-th row downward along the neighboring $i$-th and $(i+1)$-th columns, the increment $\Delta_{k-1,i}$ becomes

$$A_k \Delta_{k-1,i} = A_k(\mathcal{M}(k-1, i+1) - A_k(\mathcal{M}(k-1, i)) = \mathcal{M}(k, i+1) - \mathcal{M}(k, i) = \Delta_{k,i}, \tag{14}$$

in fact adding to it an "alteration" which is, with almost full probability, normed relatively $O(1/\sqrt{N})$ of it.

In adding these alterations when moving from the $i$-th row (where the increment was $\beta_i v_i$) to the $n$-th row where we would use (8), one may, with almost full probability, use Conclusion 10 in Appendix A.1 below, to find that the relative accumulated alteration is $O(\sqrt{n-i}/\sqrt{N})$.

Yet, as long as that relative accumulated alteration does not approach 1, we can be sure that the increment at the $n$-th row has less than 90° angle with the original "good" direction $v_i$. Thus, the pair will be "good" (i.e., $\phi$ will decrease along it), since, $\phi$ being affine, the direction of decrease does not depend on the point in space, and we will be done.

So, we should be safe, with almost full probability, as long as $n$ (the number of steps the algorithm has taken before being stopped) does not approach $N$. Then we may very well expect to find that $\phi(x'_\infty) \leq \phi(x_\infty)$.

We conjecture that such considerations should give us more than the desired inequality $\phi(x'_\infty) \leq \phi(x_\infty)$. We should be able to estimate quantitatively how much $\phi(x'_\infty)$ is less than $\phi(x_\infty)$ (with almost full probability), but we are unable to do so at this time.

# 6 The nonlinear case: A potential pathway

## 6.1 A multi-dimensional "mean-value" fact

Let $X$, $Y$ be real Banach spaces and let $F : U \subset X \to Y$ be a $C^1$ function from an open subset $U$ in $X$ to $Y$. Let $x_0$ and $x_1$ be two points in $U$, such that the line-segment connecting them is contained in $U$. Write $w := x_1 - x_0$.

Then,

$$F(x_1) - F(x_0) = \int_0^1 \frac{d}{dt} F(x_0 + tw)\, dt = \int_0^1 DF(x_0 + tw)_w\, dt. \tag{15}$$

Thus, the vector $F(x_1) - F(x_0)$ belongs to the closed convex hull of the set of values of the operator $DF$ that is the derivative operator of $F$ computed at the points $x$ on the segment connecting $x_0$ and $x_1$, and applied to $w$ denoted by $DF(x)_w$.

This means that in order to bound an "increment" $F(x_1) - F(x_0)$, in reference to $w = x_1 - x_0$, we may as well, for $C^1$ functions, bound the value $DF(x)_w$ that the operator $DF(x)$ takes on $w$ for $x$ along the line-segment connecting $x_0$ and $x_1$.

## 6.2 Computing the derivative of the projection on a convex set

Differentiability of the metric projection operator onto a convex set has been studied in the literature, see, e.g., [34] and references therein. We develop this here in a self-contained manner suitable to our needs. Let $C$ be a closed convex subset of a Hilbert space $\mathcal{H}$ and let $P$ be the nearest-point (metric) projection operator onto $C$. We wish to compute the derivative operator $DP(x)$. For that we assume that $C$ has smooth boundary $\partial C$ (in the general case $\partial C$ might be approximated by a smooth one) and assume that $x \notin C$.

Often in the literature one investigates conditions for such a projection to be differentiable, in one or another sense, for general convex $C$, which is not always the case and is a subtle question, e.g., [32]. Here we concentrate on computing the formula for the operator derivative. We do not detail here justifications from Differential Geometry.

Let $x$ be a point in the complement of $C$, and let $\bar{x} := P(x)$ be the point on the (assumed smooth) boundary $\partial C$ of $C$, at which $C$ has a tangent (affine) hyperplane $\bar{x} + H$, which is the translation of some (linear) hyperplane $H$, which, of course, depends on $x$. Since $\bar{x}$ is the nearest point to $x$ in $C$, we have the orthogonality relation $x - P(x) \perp H$. Also, $d(x) := \|x - P(x)\|$ is the distance from $x$ to $C$, and along the line-segment connecting $x$ to $P(x)$, $P$ is constant, equal to $P(x)$. Therefore, the operator derivative $DP(x)_w$ vanishes on the line through $x$ and $P(x)$, i.e., for all $w \in R(x - P(x))$.

We still have to compute $DP(x)$ on the orthogonal complement hyperplane $H$. Let $\mathcal{C} := \{x \in \mathcal{H} \mid d(x) = c\}$ be the "hypersurface" of points at constant distance $c > 0$ from $C$, passing through $x$.

**Claim 7** *The tangent hyperplane to $\mathcal{C}$ at $x$ is $H$.*

**Proof.** Indeed, for every $x$ on the hypersurface $\mathcal{C}$ we have $d^2(x) = \langle x - P(x), x - P(x) \rangle = c^2$. Differentiating this, we find for any $w$ in the tangent hyperplane to $\mathcal{C}$ at $x$, that $\langle w - DP(x)_w, x - P(x) \rangle = 0$. This means that $(w - DP(x)_w) \in H$. But the image of $P$ is contained in the boundary of $C$, hence $DP(x)_w \in H$, for all $w$, and we find that if $w$ is in the tangent hyperplane to $\mathcal{C}$ then $w \in H$, which proves the claim. ■

So, our task of finding $DP(x)_w$ for $w \in H$ boils down to computing the operator derivative, from $H$ to itself, of $P|_{\mathcal{C}}$, the restriction of $P$ to $\mathcal{C}$.

Looking at the inverse mapping $Q$ of $P|_{\mathcal{C}}$ we see that on points $y$ at the boundary of $C$, $Q(y) = y + d \cdot \vec{n}(y)$, where $\vec{n}(y)$ is the outer unit normal to the boundary of $C$ at $y$, and $d$ is the constant value of the distance on $\mathcal{C}$.

But the operator derivative of $\vec{n}$ is, by definition, the curvature operator $\kappa$ from $H$ to itself, which is a positive-definite symmetric operator, with principal axes and eigenvalues that are the directions and values of principal curvatures, respectively, see, e.g., [29, Chapters 1 and 7]. In extreme (limiting) cases these are 0 for flat and $\infty$ for an angle. Thus, $DQ(y) = \mathbf{1} + d \cdot \kappa$, $\mathbf{1}$ denoting the identity operator, and for the inverse $DP(x) = (\mathbf{1} + d \cdot \kappa)^{-1}$ on $H$. All the above leads to, and proves, the following lemma.

**Lemma 8** *The operator derivative of $P(x)$ at some $x \notin C$, in the case of smooth $C$, is a positive-definite symmetric operator, equal to 0 on $R(x-P(x))$ and equal to $(\mathbf{1}+d\cdot\kappa)^{-1}$ on $H = R(x-P(x))^{\perp}$, $\kappa$ being the curvature operator for $\partial C$ at $\bar{x} = P(x)$. Thus, $DP(x)$ is between 0 and 1.*

By the way, this immediately implies, by our "Mean-Value" Fact in Subsection 6.1 that, in the smooth $\partial C$ case, (otherwise one may approximate $C$ by a smooth) $P$ is nonexpansive and monotone. This is a well-known fact, that is usually proved in the literature in other ways. See, e.g., [1, Fact 1.5] for nonexpansivness and [2, Example 20.12] for monotonicity of $P$, respectively.

## 6.3 Toward the nonlinear case

For the nonlinear case the situation is more complicated. Here the operators $A_n$ are projections onto convex sets. Recall the superiorization matrix (Subsection 3.1). To compare $\phi(x_n) = \phi(\mathcal{M}(n,0))$ with $\phi(x'_n) = \phi(\mathcal{M}(n,n))$, we add, as in (8), the "increments" at the $(n,i)$-th and $(n,i+1)$-th entry, these coming from moving along the columns by applying the $A_n$ operators and then applying a $\phi$ value reduction step.

Our task is basically to assess increments. By the "Mean-Value" Fact in Subsection 6.1, we may instead assess the result of operator derivatives $DP(x)$ acting successively, and then $\nabla\phi$, on the original difference $\beta_i v_i$ that we had between the $(i,i+1)$-th and the $(i,i)$-th entries. Indeed, the summands in (8) are their integrals as in (15).

By Subsection 6.2, this cascade of $DP(x)$'s operates as follows: each of them first projects its argument $w$ onto $H$ ($H = (x - P(x))^{\perp}$, is, of course,

a function of $x$). By our above principle of concentration of measure, $w$ is very unlikely not to be almost orthogonal to the normal of $H$, i.e., to form a small angle $\alpha$ with $H$, where $\alpha$ is distributed almost $\sim \mathcal{N}(0, 1/\sqrt{N})$ ($N$ is the dimension of the Euclidean space $E^N$). But, contrary to the linear case, the projected part is then subjected to the action of $(\mathbf{1} + d \cdot \kappa)^{-1}$. Indeed, in the linear case the curvature operator $\kappa$ is always equal 0 and $(\mathbf{1} + d \cdot \kappa)^{-1} = \mathbf{1}$.

In order to achieve our goal to have "good" increments along the $n$-th row, it would be good if the result of applying successively the cascade of operators on $\beta_i v_i$ makes an angle smaller or equal 90° with $-\nabla\phi(x)$ *computed at the final point* $x$ (while we chose the $v_i$ in some way to be OK at the initial point). That might be hampered both by the deviations caused by the operator derivatives – the $\alpha$ and the effect of $\kappa$, and by the change in $\nabla\phi$ between the initial and final points. We address these issues, in the light of "concentration of measure" conclusions of Appendices A.2, A.6 and A.7 that are at the end of the paper. Specifically, we try to bound, for our path down the column of the superiorization matrix (7),

(1) How much the vector is rotated by the $DP$'s – the effects both of $\alpha$ and $\kappa$,

(2) By how much its norm has decreased, and

(3) How much the place to compute $\nabla\phi$ "moved" from the initial to the final point in $\mathcal{H}$.

First, by Lemma 10 the distance between $x$ (where we chose $v_i$) and the result of applying the cascade of $P$'s to it (and where we should compute $\nabla\phi$) is supposed to be near the square root of the sum of the distances $d = \|x - P(x)\|$ along the way from the $i$-th to the $n$-th stage. Thus, it is small in the final stages when the $d$'s are small (indeed, they are converging to 0).

As for the effect of the $\alpha$ alteration by the projections, the situation is as in the linear case – we should be safe as long as $n$ does not approach the dimension $N$ of the Euclidean space $E^N$.

For the accumulated terms $(\mathbf{1} + d \cdot \kappa)^{-1}$ along the path (in what follows we denote by $k$ indices along the path, i.e., $k \in$ path) denote the eigenvalues (here, also the singular values) of the encountered $\kappa_k$ (the curvature operator in the hyperplane $H$), i.e., the relevant principal curvatures, by $(\kappa_\ell^{(k)})_\ell$, for $\ell = 1, 2, \ldots, N-1$. Then those of $(\mathbf{1} + d \cdot \kappa)^{-1}$ are $\left((1 + d_k \cdot \kappa_\ell^{(k)})^{-1}\right)_\ell$, so that, by Conclusion 14, and using the $\|\cdot\|_p^{(\pi)}$ norm of Appendix A.1 below, for $(N-1)$-dimensional vectors, their product is expected to multiply the

norm of the vector they act upon by

$$\prod_{k\in\text{path}} \left\| \left( (1 + d_k \cdot \kappa_\ell^{(k)})^{-1} \right)_\ell \right\|_2^{(\pi)}, \tag{16}$$

still with relative deviation of the order of at most $O(\sqrt{n-i}/\sqrt{N})$.

By Conclusion 17 in Appendix A.7, they are expected to rotate the direction of the vector, i.e., shift the normalized vector, by

$$\sqrt{2\left(1 - \prod_{k\in\text{path}} \frac{\left\| \left( (1 + d_k \cdot \kappa_\ell^{(k)})^{-1} \right)_\ell \right\|_1^{(\pi)}}{\left\| \left( (1 + d_k \cdot \kappa_\ell^{(k)})^{-1} \right)_\ell \right\|_2^{(\pi)}}\right)}. \tag{17}$$

with relative deviation of the order of at most $O(\sqrt{n-i}/\sqrt{N})$.

Observe that $\|\cdot\|_1^{(\pi)} \leq \|\cdot\|_2^{(\pi)}$ (cf. Appendix A.1) and, by Remark 18, the value of (17) is always $\leq \sqrt{2}$, meaning angle of rotation $\leq 90°$. Indeed, in many cases it will be much less than 90°. For example, for vectors $\left( (1 + d_k \cdot \kappa_\ell^{(k)})^{-1} \right)_\ell$ with equal (resp. almost equal) entries (in our case – either "spherical" curvature or when the $d_k \cdot \kappa$ are small), the $\|\cdot\|_2^{(\pi)}$ norm will be equal (resp. almost equal) to the $\|\cdot\|_1^{(\pi)}$ norm, hence the terms in the product in (17) will be near 1.

Both (16) and (17) refer to the $(N-1)$-dimensional vectors $v = ((1 + d_k \cdot \kappa_\ell^{(k)})^{-1})_\ell$, having entries in $(0, 1]$. In (16), which controls how much the norm was reduced, we have the product of $\|v\|_2^{(\pi)}$. In (17), which controls how much the direction was rotated, we have the square root of twice 1 minus the product of $\|v\|_1^{(\pi)}/\|v\|_2^{(\pi)}$.

**Proposition 9** *For an $(N-1)$-dimensional vector $v = (v_\ell)_\ell$ with components $v_\ell \in (0, 1]$, we have*

$$\left( \|v\|_2^{(\pi)} \right)^2 \leq \|v\|_1^{(\pi)} \leq \frac{1}{2} \left( \left( \|v\|_2^{(\pi)} \right)^2 + 1 \right). \tag{18}$$

**Proof.** Since $v_\ell \in (0, 1]$, one has $v_\ell^2 \leq v_\ell$. Averaging, we get $\left( \|v\|_2^{(\pi)} \right)^2 \leq \|v\|_1^{(\pi)}$. Also, by definition of $\|v\|_2^{(\pi)}$ for $(N-1)$-dimensional vectors, see

Appendix A.1,

$$\left(\|v\|_2^{(\pi)}\right)^2 = \frac{1}{N-1}\sum_{\ell=1}^{N-1} v_\ell^2 = 1 - \frac{1}{N-1}\sum_{\ell=1}^{N-1}(1-v_\ell^2) \tag{19}$$

$$= 1 - \frac{1}{N-1}\sum_{\ell=1}^{N-1}(1-v_\ell)(1+v_\ell) \geq 1 - 2\frac{1}{N-1}\sum_{\ell=1}^{N-1}(1-v_\ell)$$

$$= 2\frac{1}{N-1}\sum_{\ell=1}^{N-1} v_\ell - 1 = 2\|v\|_1^{(\pi)} - 1. \tag{20}$$

Hence, $\|v\|_1^{(\pi)} \leq \frac{1}{2}(\|v\|_2^{(\pi)})^2 + 1)$, which completes the proof. ■

As a consequence of this proposition we have,

$$\frac{\|v\|_1^{(\pi)}}{\|v\|_2^{(\pi)}} \geq \frac{(\|v\|_2^{(\pi)})^2}{\|v\|_2^{(\pi)}} = \|v\|_2^{(\pi)},$$

$$\frac{\|v\|_1^{(\pi)}}{\|v\|_2^{(\pi)}} \leq \frac{1}{2}\frac{(\|v\|_2^{(\pi)})^2 + 1}{\|v\|_2^{(\pi)}} = \frac{1}{2}\left(\|v\|_2^{(\pi)} + \frac{1}{\|v\|_2^{(\pi)}}\right), \tag{21}$$

So, there is here a *"balancing effect"* – if the angle of rotation becomes close to 90° in (17), then the norm will be reduced considerably in (16). Thus, when $i$ is such that $d_i$ times a "typical" curvature $\kappa$ (loosely, the ratio between $d$ and a "typical" radius of the $C_i$) is still considerably larger than 1 (maybe while in the early columns of the superiorization matrix with $i$ small), then, by (16), the cascade of $DP$ will reduce the norm hugely, hence, anyway applying $\nabla\phi$ then will give a negligible result.

On the other hand, when we reach a stage where $d_i, d_{i+1}, \ldots, d_n$ are small, both the possible rotation and the distance traveled are controlled. But of course, then the decrease of the $\beta_k$ should also be taken into account. For big $i$, thus small $\beta_i$, the contribution might again be negligible. This shows that the main contribution in (8) seems to come from intermediate terms.

As said above, the angle of rotation, both by the $\alpha$ and by the $\kappa$ seems to be controlled, as long as the number of steps $n$ does not approach the vector space dimension $N$. If conditions are imposed on the target function $\phi$ then point (3) above could also be tackled, in view of the preceeding paragraph, bringing our analysis closer to conclusion.

# 7 Concluding comments

We explored here the fundamental open problem of the superiorization method which is the question under what conditions one can guarantee that a superiorized version of a bounded perturbation resilient feasibility-seeking algorithm converges to a feasible point that has target function value smaller or equal to that of a point to which this algorithm would have converged if no perturbations were applied – everything else being equal.

The success of the superiorization method in many real-world applications, as witnessed in [10], made this an important question. However, in the absence of a conclusive deterministic argument, we applied here the probabilistic principle of concentration of measure. For linear superiorization (LinSup) this approach works quite well whereas our analysis aimed at using it for a general nonlinear situation is still less conclusive.

# A Some concentration of measure facts in a high-dimensional $E^N$

## A.1 The probability $L^p$ norms of vectors

For a vector $x \in E^N$, and $1 \leq p < \infty$, denote by $\| \cdot \|_p^{(\pi)}$ ($\pi$ stands for "probability space") its $L^p$ norm when the set of indices $\{1, 2, \ldots, N\}$ is made into a uniform probability space, giving each index a weight $1/N$, namely

$$\|x\|_p^{(\pi)} := \left( \frac{1}{N} \sum_{j=1}^{N} |x_j|^p \right)^{1/p}, \tag{22}$$

see, e.g., [36]. As with any probability measure, always $\| \cdot \|_p^{(\pi)}$ increases with $p$.

For $x_1, x_2, \ldots, x_N$ i.i.d. $\sim \mathcal{N}$, $\left( \|x\|_p^{(\pi)} \right)^p$ is an average: its expectation $\mathbb{E}$ will be the same as the expectation of $|x|^p$ for $x$ a scalar distributed $\sim \mathcal{N}$:

$$\mathbb{E}\left[|x|^p\right] = \frac{1}{\sqrt{2\pi}} \int |x|^p \exp(-\tfrac{1}{2}x^2)\, dx, \tag{23}$$

but its standard deviation will be $1/\sqrt{N}$ that of $|x|^p$ for a scalar $\sim \mathcal{N}$:

$$\frac{1}{\sqrt{N}} \frac{1}{\sqrt{2\pi}} \int \left(|x|^p - \mathbb{E}\left[|y|^p\right]\right)^2 \exp(-\tfrac{1}{2}x^2)\, dx. \tag{24}$$

Thus, $\|x\|_p^{(\pi)}$ is highly concentrated around the, not depending on $N$, $\left(\mathbb{E}\left[|x|^p\right]\right)^{1/p}$ with degree of concentration $O(1/\sqrt{N})$.

One may conclude, loosely speaking, that in any case, these $\|\cdot\|_p^{(\pi)}$ norms, having not depending on $N$ means, are expected to be $O(1)$, for all $N$.

## A.2 The norm of the sum of vectors with given norms

Suppose we are given $M$ vectors $y_1, y_2, \ldots, y_M$ of known norms $d_1, d_2, \ldots . d_M$ in $E^N$. What should we expect the norm of their sum to be?

This can be answered: take the direction of each of them distributed uniformly on $S^{N-1}$, *even conditioned on fixed valued for the others*. In other words, take them independent, each with direction distributed uniformly. This can be constructed by taking random $M$ vectors in $E^N$ (that is, a random $M \times N$ matrix), with entries i.i.d. $\sim \mathcal{N}$, dividing them by $\sqrt{N}$, then by their norm (now highly concentrated near 1) and multiplying them by $d_1, d_2, \ldots, d_M$, respectively.

The sum $\sum_{i=1}^M y_i$, if we ignore the division by the norm, is $1/\sqrt{N}$ times the random matrix applied to the vector $(d_1, d_2, \ldots, d_M)$. But the distribution of the random matrix is invariant with respect to any transformation which is orthogonal with respect to the Hilbert-Schmidt norm – the square root of the sum of squares of the entries (i.e., $\|T\|_{HS} := \sqrt{\mathrm{tr}(T' \cdot T)} = \sqrt{\mathrm{tr}(T \cdot T')}$, $T'$ denoting the transpose and tr standing for the trace, see, e.g., [4]). In particular, the distribution of the sum is the same as that of $1/\sqrt{N}$ times $\sqrt{d_1^2 + d_2^2 + \cdots + d_M^2}$ times the random matrix applied to $(1, 0, \ldots, 0)$, which is, of course, distributed with independent $\sim \mathcal{N}$ entries, thus, with norm concentrated near $\sqrt{N}$. (With relative deviation $O(1/\sqrt{N})$.) This leads to the following conclusion.

**Conclusion 10** *For $M$ vectors $y_1, y_2, \ldots, y_M$ of known norms $d_1, d_2, \ldots . d_M$, in $E^N$ we have that $\|\sum_{i=1}^M y_i\|$ is near $\sqrt{d_1^2 + d_2^2 + \cdots + d_M^2}$ with almost full probability (With relative deviation $O(1/\sqrt{N})$.)*

## A.3 The accumulation of given distances on the unit sphere

As in the previous Appendix A.2, we seek to find what should we expect the norm of a sum of $M$ vectors of given norms $d_1, d_2, \ldots . d_M$ to be. But here the vectors are *the differences between consecutive elements in a sequence of points on the unit sphere* $S^{N-1} \subset E^N$. Denote by $\omega_{N-1}$ the normalized to be probability (i.e., of total mass 1) uniform measure on $S^{N-1}$.

**Remark 11** *By symmetry, for* $x = (x_1, x_2, \ldots, x_N) \in S^{N-1}$, $\int x_k^2 \, d\omega_{N-1}$ *is the same for all* $k$. *Of course, their sum is* $\int 1 \, d\omega_{N-1} = 1$. *Therefore,*

$$\int x_k^2 \, d\omega_{N-1} = \frac{1}{N}, \qquad k = 1, 2, \ldots, N. \tag{25}$$

*Hence, for a polynomial of degree* $\leq 2$ *on* $E^n$*:*

$$p(x) = \langle Qx, x \rangle + 2\langle a, x \rangle + \gamma, \tag{26}$$

*where* $Q$ *is a symmetric* $N \times N$ *matrix,* $a \in E^N$ *and* $\gamma \in E$, *we will have*

$$\int p(x) \, d\omega_{N-1} = \frac{1}{N} \text{ tr } Q + \gamma. \tag{27}$$

Note that, for some fixed $0 \leq d \leq 2$, the set of points in $S^{N-1}$ of distance $d$ from some fixed vector $u \in S^{N-1}$ is the $(N-2)$-sphere $\subset S^{N-1}$, $\Sigma(u, d)$ given by

$$\Sigma(u, d) := (1 - d^2/2)u + d\sqrt{1 - d^2/4} \cdot {S^{N-2}}_{u^\perp}, \tag{28}$$

where ${S^{N-2}}_{u^\perp}$ stands for the unit sphere in the hyperplane prependicular to $u$. In our scenario, one performs a *Markov chain*, see, e.g., [3]. Starting from a point $u_0$ on $S^{N-1}$, and moving to a point $u_1 \in \Sigma(u_0, d_1)$ uniformly distributed there. Then, from that $u_1$, to a point $u_2 \in \Sigma(u_1, d_2)$ uniformly distributed there, and so on, until one ends with $u_M$. We would like to find $\mathbb{E}\left[\|u_M - u_0\|\right]$.

If we denote by $\mathcal{L}_d$ the operator mapping a function $p$ on $S^{N-1}$ to the function whose value at a vector $u \in S^{N-1}$ is the average of $p$ on $\Sigma(u, d)$, then $\mathcal{L}_{d_k}(p)$ evaluated at $u$ is the expectation of $p$ at the point to which $u$ moved in the $k$-th step above. Hence, in the above Markov chain, the expectation of $p(u_M)$ is

$$\mathcal{L}_{d_M} \mathcal{L}_{d_{M-1}} \cdots \mathcal{L}_{d_1} p(u_0). \tag{29}$$

Thus, what we are interested in is

$$\mathbb{E}\left[\|u_M - u_0\|\right] = \mathcal{L}_{d_M}\mathcal{L}_{d_{M-1}}\cdots\mathcal{L}_{d_1}(\|x-u_0\|^2)(u_0). \tag{30}$$

So, let us calculate $\mathcal{L}_d(p)$ for polynomials of degree $\leq 2$ as in (26). In performing the calculation, assume $u = (1, 0, \ldots, 0)$. For $x = (x_1, x_2, \ldots, x_N) \in E^N$ write $y = (x_2, x_3, \ldots, x_N) \in E^{N-1}$. In (26) write $a = (a_1, b)$ where $b = (a_2, a_3, \ldots a_N) \in E^{N-1}$ and

$$Q = \begin{pmatrix} \eta & c' \\ c & Q' \end{pmatrix}, \tag{31}$$

where $Q'$ is a symmetric $(N-1)\times(N-1)$ matrix, $c \in E^{N-1}$ and $\eta \in E$. Note that for our $u = (1, 0, \ldots, 0)$, $a_1 = \langle a, u\rangle$, $\eta = \langle Qu, u\rangle$ and tr $Q' =$ tr $Q - \eta =$ tr $Q - \langle Qu, u\rangle$.

Then, for $p$ as in in (26),

$$p(x) = \eta x_1^2 + 2x_1\langle c, y\rangle + \langle Q'y, y\rangle + 2a_1x_1 + 2\langle b, y\rangle + \gamma. \tag{32}$$

Hence, taking account of (28) for $u = (1, 0, \ldots, 0)$, and using (25),

$$\begin{aligned}
&(\mathcal{L}_d p)(u) = (\mathcal{L}p)(1, 0, \ldots, 0) \\
&= \left(1 - \frac{d^2}{2}\right)^2 \eta + \frac{1}{N-1}d^2\left(1 - \frac{d^2}{4}\right)\text{tr } Q' + 2\left(1 - \frac{d^2}{2}\right)a_1 + \gamma \\
&= \left(1 - \frac{d^2}{2}\right)^2 \langle Qu, u\rangle + \frac{1}{N-1}d^2\left(1 - \frac{d^2}{4}\right)(\text{tr}\, Q - \langle Qu, u\rangle) \\
&+ 2\left(1 - \frac{d^2}{2}\right)\langle a, u\rangle + \gamma,
\end{aligned} \tag{33}$$

which, by symmetry, will hold for any $u \in S^{N-1}$. In particular, we find, as should be expected, that

$$\begin{aligned}
&\int (\mathcal{L}_d(p))(x)\, d\omega_{N-1} \\
&= \frac{1}{N}\left(1 - \frac{d^2}{2}\right)^2 \text{tr } Q + \frac{1}{N-1}d^2\left(1 - \frac{d^2}{4}\right)\left(1 - \frac{1}{N}\right)\text{tr } Q + \gamma \\
&= \frac{1}{N}\,\text{tr } Q + \gamma = \int p(x)\, d\omega_{N-1}.
\end{aligned} \tag{34}$$

We are interested, for some fixed $u \in S^{N-1}$, in

$$p(x) = \|x - u\|^2 = 2(1 - \langle u, x\rangle). \tag{35}$$

Then there is no $Q$ term, so one has

$$\Big(\mathcal{L}_d\big(2(1 - \langle u, x\rangle)\big)\Big)(u) = 2\left(1 - \left(1 - \frac{d^2}{2}\right)\langle u, x\rangle\right). \tag{36}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\left[\|u_M - u_0\|^2\right] = \left(\mathcal{L}_{d_M}\mathcal{L}_{d_{M-1}}\cdots\mathcal{L}_{d_1}(\|x - u_0\|^2)\right)(u_0) \\
&= 2\left(1 - \left(\mathcal{L}_{d_M}\mathcal{L}_{d_{M-1}}\cdots\mathcal{L}_{d_1}(\langle u_0, x\rangle)\right)\big|_{x=u_0}\right) \\
&2\left(1 - \prod_{i=1}^{M}\left(1 - \frac{d_i^2}{2}\right)\langle u_0, x\rangle\right)\Bigg|_{x=u_0} = 2\left(1 - \prod_{i=1}^{M}\left(1 - \frac{d_i^2}{2}\right)\right).
\end{aligned} \tag{37}$$

This is $O\left(M \cdot \left(\|(d_1, d_2, \ldots, d_M)\|_2^{(\pi)}\right)^2\right)$. We also assess the standard deviation, which is

$$= 2\sqrt{\mathcal{L}_{d_M}\mathcal{L}_{d_{M-1}}\cdots\mathcal{L}_{d_1}(\langle u_0, x\rangle^2)\big|_{x=u_0} - \left(\prod_{i=1}^{M}\left(1 - \frac{d_i^2}{2}\right)\right)^2}. \tag{38}$$

Here $p(x) = \langle a, x\rangle^2$, so there is only the $Q$ term with $Q(x) := \langle a, x\rangle^2$. Then $\operatorname{tr} Q = \|a\|^2$, and we find

$$\begin{aligned}
&\Big(\mathcal{L}_d\big(\langle a, x\rangle^2\big)\Big)(u) \\
&= \left(1 - \frac{d^2}{2}\right)^2\langle a, u\rangle^2 + \frac{1}{N-1}d^2\left(1 - \frac{d^2}{4}\right)\left(\|a\|^2 - \langle a, u\rangle^2\right) \\
&= \left(1 - \frac{N}{N-1}d^2\left(1 - \frac{d^2}{4}\right)\right)\langle a, u\rangle^2 + \frac{1}{N-1}d^2\left(1 - \frac{d^2}{4}\right)\|a\|^2.
\end{aligned} \tag{39}$$

Consequently, for $a = u_0$ (note $\|u_0\|^2 = 1$)),

$$\begin{aligned}
&\left(\mathcal{L}_{d_M}\mathcal{L}_{d_{M-1}}\cdots\mathcal{L}_{d_1}(\langle u_0, x\rangle^2)\right)\Big|_{x=u_0} - \left(\prod_{i=1}^{M}\left(1-\frac{d_i^2}{2}\right)\right)^2 \\
&= -\left(\prod_{i=1}^{M}\left(1-\frac{d_i^2}{2}\right)\right)^2 + \prod_{i=1}^{M}\left(1-\frac{N}{N-1}d_i^2\left(1-\frac{d_i^2}{4}\right)\right) \\
&+ \frac{1}{N-1}\left[d_1^2\left(1-\frac{d_1^2}{4}\right) + d_2^2\left(1-\frac{d_2^2}{4}\right)\left(1-\frac{N}{N-1}d_1^2\left(1-\frac{d_1^2}{4}\right)\right)\right. \\
&+ d_3^2\left(1-\frac{d_3^2}{4}\right)\left(1-\frac{N}{N-1}d_2^2\left(1-\frac{d_2^2}{4}\right)\right)\left(1-\frac{N}{N-1}d_1^2\left(1-\frac{d_1^2}{4}\right)\right) \\
&\left.+\cdots+\ d_M^2\left(1-\frac{d_M^2}{4}\right)\prod_{i=1}^{M}\left(1-\frac{N}{N-1}d_i^2\left(1-\frac{d_i^2}{4}\right)\right)\right]. \qquad (40)
\end{aligned}$$

This is $O\left((M/N)\cdot\left(\|(d_1,d_2,\ldots,d_M)\|_4^{(\pi)}\right)^4\right)$, since the constant terms and the terms with $d_k^2$ cancel, and the terms which do not cancel are coefficiented by $O(1/N)$. Therefore, twice its square root, the standard deviation, will be $O\left(\sqrt{M}/\sqrt{N}\left(\|(d_1,d_2,\ldots,d_M)\|_4^{(\pi)}\right)^2\right)$, making the relative deviation $O(1/\sqrt{MN})$. This leads to the following conclusion.

**Conclusion 12** *The square of the norm of the sum of $M$ vectors of given norms $d_1, d_2, \ldots . d_M$, which are differences between consecutive elements in a sequence of points on the unit sphere $S^{N-1} \subset E^N$, modeled by the above Markov chain, is with almost full probability, near*

$$2\left(1-\prod_{i=1}^{M}\left(1-\frac{d_i^2}{2}\right)\right). \qquad (41)$$

*(With relative deviation $O(1/\sqrt{MN})$.)*

## A.4 A reminder: Polar decomposition and singular values of a matrix

As is well-known, see, e.g., [27], every fixed $N \times N$ matrix $T$ can be uniquely written as $T = UA$ with $U$ orthogonal and $A$ symmetric positive semidefinite

(take $A = \sqrt{T' \cdot T}$, then for every vector $x$, $\|Tx\|=\|Ax\|$, so the map $Ax \mapsto Tx$ is norm-preserving, i.e., orthogonal), and also uniquely written as $T = A_1 U_1$ with $U_1$ orthogonal and $A_1$ symmetric positive semidefinite (take $A_1 = \sqrt{T \cdot T'}$).

The *singular values* of $T$ are defined as the eigenvalues of its positive semidefinite part in the above decomposition. (It does not matter from which side: $T \cdot T'$ and $T' \cdot T$ have the same eigenvalues. Note that if $T$ is invertible they are similar: $T'^{-1}(T \cdot T')T$.)

Since any positive semidefinite matrix with eigenvalues $s_1, s_2, \ldots, s_N$ is of the form

$$U' \cdot \mathrm{diag}(s_1, s_2, \ldots, s_N) \cdot U \tag{42}$$

with $U$ orthogonal (diag denotes a diagonal matrix), we find that the general form of a matrix with singular values $s_1, s_2, \ldots, s_N$ is

$$T = U_1 \cdot \mathrm{diag}(s_1, s_2, \ldots, s_N) \cdot U_2, \qquad U_1 \text{ and } U_2 \text{ orthogonal.} \tag{43}$$

## A.5 Square matrix with entries independently $\sim \mathcal{N}$ and the uniform distribution on orthogonals

Take a random $N \times N$ matrix $Y$ with entries $Y_{i,j}$ i.i.d. $\sim \mathcal{N}$. If we polarly decompose the random $Y$ as per Appendix A.4, from either side, then the orthogonal part will be distributed uniformly (i.e., by Haar's measure) on the orthogonal group. This follows from the fact that, by the symmetries of the above distribution of $Y$, it is invariant under multiplying the random matrix on the right or left by a fixed orthogonal matrix. So, we have here a vehicle to get this uniform distribution. For a general excellent text on random matrices consult [23].

For the positive semidefinite part we have to check, say, $Y' \cdot Y$ for our random matrix $Y$. But if $u$ is any vector then, by the symmetries of the distribution of the random $Y$, $Yu$ is distributed like $\|u\|$ times $Y \cdot (1, 0, \ldots .0)$ – i.i.d. $\sim \mathcal{N}$ entries, thus, with norm concentrated near $\|u\| \cdot \sqrt{N}$, with relative deviation $O(1/\sqrt{N})$. But, *all the entries of* $Y' \cdot Y$ *being discernible from* $\langle Y' \cdot Yu, u \rangle$ *if we take as* $u$ *elements of the standard basis* $e_i = (0, \ldots, 0, 1, 0 \ldots, 0)$ *and sums of two of these*, we obtain the following conclusion.

**Conclusion 13** $(1/N)Y' \cdot Y$ *(and likewise* $(1/N)Y \cdot Y'$*) is concentrated near* $\mathbf{1}$ *(*$\mathbf{1}$ *denotes the identity matrix), with relative deviation* $O(1/N)$*.*

In other words, the random $Y$ is, with almost full probability, very near $\sqrt{N}$ times an orthogonal matrix. Indeed. to check how orthogonal $(1/\sqrt{N})Y$ is, note that the amount it distorts the inner product between unit vectors $u$ and $v$ is

$$(1/N)\langle Yu, Yv\rangle - \langle u, v\rangle = \langle((1/N)Y' \cdot Y - \mathbf{1})u, v\rangle = O(1/N). \tag{44}$$

## A.6 The action of a linear operator in a high-dimensional space

Consider an $N \times N$ matrix $T$ with given singular values $s_1, s_2, \ldots, s_N$ as in (43). Let $T$ act on a unit vector $u$ with direction uniformly distributed over $S^{N-1}$. By (43) this is distributed, up to an orthogonal "rotation" of the space, the same as $S = \mathrm{diag}(s_1, s_2, \ldots, s_N)$ acting on such a vector.

But by Section 4, that would be almost as $S$ applied to $(1/\sqrt{N})x$, $x$ with coordinates i.i.d. $\sim \mathcal{N}$, which is, of course, a vector with independent coordinates but the $j$-th coordinate distributed as $(1/\sqrt{N})s_j$ times $\mathcal{N}$.

Now, similarly to what we had in Section 4, the square of the norm of $S \cdot (1/\sqrt{N})x$, which is $(1/N)\sum_{j=1}^{N} s_j^2 x_j^2$ has mean

$$(1/N)\sum_{j=1}^{N} s_j^2 = \left(\|(s_1, s_2, \ldots, s_N)\|_2^{(\pi)}\right)^2, \tag{45}$$

around which it is concentrated – its standard deviation being

$$\sigma \cdot \sqrt{(1/N^2)\sum_{j=1}^{N} s_j^4} = (1/\sqrt{N})\sigma \cdot \left(\|(s_1, s_2, \ldots, s_N)\|_4^{(\pi)}\right)^2, \tag{46}$$

where $\sigma$ is the standard deviation for $x^2$ when $x \sim \mathcal{N}$, namely,

$$\sigma = \sqrt{\frac{1}{\sqrt{2\pi}}\int (x^2-1)^2 \exp(-\tfrac{1}{2}x^2)\, dx} = \sqrt{2}. \tag{47}$$

By Appendix A.1, the relative deviation is, thus, expected, with almost full probability, to be $O(1/\sqrt{N})$. Note that since $T = U_1 \cdot \mathrm{diag}(s_1, s_2, \ldots, s_N) \cdot U_2$, the value around which the norm of $T$ applied to a uniformly distributed unit vector is concentrated is

$$\|(s_1, s_2, \ldots, s_N)\|_2^{(\pi)} = (1/\sqrt{N})\|S\|_{HS} = (1/\sqrt{N})\|T\|_{HS}. \tag{48}$$

Dividing $T$ by that value, we get a $T$ with $(1/\sqrt{N})\|T\|_{HS} = 1$ which, with almost full probability, will approximately preserve the norm. How "orthogonal" will it be? Let us see how $S$ distorts the inner product between $(1/\sqrt{N})x$ and $(1/\sqrt{N})y$, all $2N$ coordinates of $x$ and $y$ i.i.d. $\sim \mathcal{N}$. The mean of the square of the difference

$$\langle S(1/\sqrt{N})x, S(1/\sqrt{N})y\rangle - \langle (1/\sqrt{N})x, (1/\sqrt{N})y\rangle \tag{49}$$

is

$$\begin{aligned}
&(1/N^2)\mathbb{E}\left(\sum_{j=1}^{N} s_j^2 x_j y_j - \sum_{j=1}^{N} x_j y_j\right)^2 = (1/N^2)\mathbb{E}\left(\sum_{j=1}^{N}(s_j^2-1)x_j y_j\right)^2 \\
&= (1/N)\left((1/N)\sum_{j=1}^{N}(s_j^4 - 2s_j^2 + 1)\right) \\
&= (1/N)\left(\left(\|(s_1, s_2, \ldots, s_N)\|_4^{(\pi)}\right)^4 - 2\left(\|(s_1, s_2, \ldots, s_N)\|_2^{(\pi)}\right)^2 + 1\right) \\
&= (1/N)\left(\left(\|(s_1, s_2, \ldots, s_N)\|_4^{(\pi)}\right)^4 - 1\right). \qquad (50)
\end{aligned}$$

Consequently, $T$ is orthogonal, with almost full probability, up to $O(1/\sqrt{N})$. This leads to the following conclusion.

**Conclusion 14** *An $N\times N$ matrix $T$ with given singular values $s_1, s_2, \ldots, s_N$, acting on a high-dimensional $E^N$, would be expected to act, with almost full probability, as*

$$\|(s_1, s_2, \ldots, s_N)\|_2^{(\pi)} = (1/\sqrt{N})\|T\|_{HS} \tag{51}$$

*times an orthogonal matrix, up to a relative deviation $O(1/\sqrt{N})$.*

**Remark 15** *Now we address a seeming mystery raised by Conclusion 14. That conclusion seems to require that $(1/\sqrt{N})$ times the Hilbert-Schmidt norm of the product of two matrices with singular values $(s_1, s_2, \ldots, s_N)$ and $(s'_1, s'_2 \ldots, s'_N)$, respectively, be equal to the product of the same for the factors, i.e., to $\|(s_1, s_2, \ldots, s_N)\|_2^{(\pi)} \cdot \|(s_1, s_2, \ldots, s_N)\|_2^{(\pi)}$, up to relative deviation $O(1/\sqrt{N})$. Is that so?*

*Note that, by (43), the HS-norm of the product is that of $SUS'$ where $S = \mathrm{diag}(s_1, s_2, \ldots, s_N)$, $S' = \mathrm{diag}(s'_1, s'_2 \ldots, s'_N)$ and $U$ is orthogonal. So, if, up*

*to an* $O(1/\sqrt{N})$ *relative deviation, we model* $U$ *as* $(1/\sqrt{N})Y$, $Y = (Y_{i,j})_{i,j}$ *as in Appendix A.5, then* $SUS' = \left((1/\sqrt{N})s_i Y_{i,j} s'_j\right)_{i,j}$. *The square of* $(1/\sqrt{N})$ *times its HS-norm is* $(1/N^2)\sum_{i,j=1,1}^{N,N} s_i^2 Y_{i,j}^2 s_j'^2$, *with mean indeed equal to the square of* $\|(s_1, s_2, \ldots, s_N)\|_2^{(\pi)} \cdot \|(s'_1, s'_2 \ldots, s'_N)\|_2^{(\pi)}$, *and with standard deviation* $\sigma \cdot 1/N$ *times the square of* $\|(s_1, s_2, \ldots, s_N)\|_4^{(\pi)} \cdot \|(s'_1, s'_2 \ldots, s'_N)\|_4^{(\pi)}$.

## A.7 The rotation effected by an operator and by a product of operators in a high-dimensional space

Let $T$ be an an $N \times N$ matrix, and consider *the amount of rotation between* $v$ *and* $Tv$. The square of the distance between these vectors, both normalized to norm 1 will be

$$\begin{aligned}\left\| \frac{Tv}{\|Tv\|} - \frac{v}{\|v\|} \right\|^2 &= \left\langle \frac{Tv}{\|Tv\|} - \frac{v}{\|v\|}, \frac{Tv}{\|Tv\|} - \frac{v}{\|v\|} \right\rangle \\ =2 - \frac{\langle Tv, v\rangle + \langle v, Tv\rangle}{\|Tv\|\|v\|} &= 2\left(1 - \frac{\langle T^{(sym)}v, v\rangle}{\|Tv\|\|v\|}\right), \end{aligned} \tag{52}$$

where $T^{(sym)} := \frac{1}{2}(T + T')$ is the *symmetric part* of $T$. Note that $\operatorname{tr} T^{(sym)} = \operatorname{tr} T$. So, we are led to investigate the inner product $\langle Ax, x\rangle$ for $A$ symmetric. Let $(s_1, s_2, \ldots, s_N)$ be its eigenvalues, then $A = U'SU$ where $S = \operatorname{diag}(s_1, s_2, \ldots, s_N)$ and $U$ orthogonal. As we did above, we take $v = (1/\sqrt{N})x$, and $x$ with coordinates i.i.d. $\sim \mathcal{N}$. Then

$$\left\langle A(1/\sqrt{N})x, (1/\sqrt{N})x \right\rangle = (1/N)\langle U'SUx, x\rangle = (1/N)\langle SUx, Ux\rangle. \tag{53}$$

But, $Ux$ being distributed like $x$, this will have the same distribution as

$$(1/N)\langle Sx, x\rangle = (1/N)\sum_{j=1}^{N} s_j x_j^2, \tag{54}$$

which has mean $(1/N)\sum_{j=1}^{N} s_j = (1/N)\operatorname{tr} A$ and $(1/\sqrt{N})\sigma\|(s_1, \ldots, s_N)\|_2^{(\pi)}$ is its standard deviation. Of course, if $A$ is positive semidefinite then the $s_\ell \geq 0$ and the above mean is $\|(s_1, s_2, \ldots, s_N)\|_1^{(\pi)}$. This leads to the following conclusion.

**Conclusion 16** *For $T$ with symmetric part with eigenvalues $(s_1, s_2, \ldots, s_N)$, the square of the distance between $v$ and $Tv$, both normalized to norm 1, is, with almost full probability, near (with deviation $O(1/\sqrt{N})$)*

$$2\left(1 - \frac{(1/N)\,\mathrm{tr}\ T}{(1/\sqrt{N})\|T\|_{HS}}\right) = 2\left(1 - \frac{(1/N)\,\mathrm{tr}\ T}{\|(s_1, s_2, \ldots, s_N)\|_2^{(\pi)}}\right), \tag{55}$$

*which, if the symmetric part of $T$ is positive-semidefinite, is equal to*

$$2\left(1 - \frac{\|(s_1, s_2, \ldots, s_N)\|_1^{(\pi)}}{\|(s_1, s_2, \ldots, s_N)\|_2^{(\pi)}}\right). \tag{56}$$

The next discussion will lead to a conclusion about a product $A_M A_{M-1} \cdots A_1$ of a sequence of *symmetric* operators. Consider a symmetric $A = U' \,\mathrm{diag}(s_1, s_2, \ldots, s_N)U$ with given $s_1, s_2, \ldots, s_N$. Take $U$ uniformly distributed on the orthogonal group, which we model up to a relative deviation $O(1/\sqrt{N})$ by $1/\sqrt{N} \cdot Y$, $Y = (Y_{i,j})_{i,j}$ as in Appendix A.5. Then

$$A = USU' \approx \left((1/N)\sum_{k=1}^{N} Y_{k,i} s_k Y_{k,j}\right)_{i,j}. \tag{57}$$

Consequently,

$$\mathbb{E}[A] \approx (1/N)\left(\sum_{k=1}^{N} s_k\right) \cdot \mathbf{1} = (1/N)\,\mathrm{tr}\ A \cdot \mathbf{1}. \tag{58}$$

But here we cannot say, as we did in previous cases, that, with high probability, $A$ would be near that average – indeed they cannot be "near" since the eigenvalues of the average are all $(1/N)\,\mathrm{tr}\ A$ while those of $A$ are with full probability $s_1, s_2, \ldots, s_N$.

To apply the considerations of Appendix A.3, where one relies on a Markov chain employing uniform distribution on spheres, we inquire what is the distribution of $Av_0$, and of the difference vector $\left(\frac{Av_0}{\|Av_0\|} - \frac{v_0}{\|v_0\|}\right)$ for a fixed $v_0$, with $A$ random as in (57) above. To fix matters, assume $v_0 = (1, 0, \ldots, 0)$. As above, we have $Av_0 = U'SUv_0$. where $S := \mathrm{diag}(s_1, s_2, \ldots, s_N)$. Or, with $U$ replaced by $1/\sqrt{N} \cdot Y$, $Av_0 \approx (1/N)Y'SYv_0$. Write $Y$ as $(w, Z)$ where $w$ is the $N \times 1$ matrix which is the first column of $Y$, and $Z$ is the

$N \times (N-1)$ matrix of the other columns. Then, with $v_0 = (1, 0, \ldots, 0)$, $Yv_0 = w$, and

$$(1/N)Y'SYv_0 = (1/N)\begin{pmatrix} w' \\ Z' \end{pmatrix} Sw = (1/N)\begin{pmatrix} w'Sw \\ Z'Sw \end{pmatrix}. \tag{59}$$

Note that the random $Z$ and $w$ are *independent*. $Z$ is an $N \times (N-1)$ matrix with entries i.i.d. $\sim \mathcal{N}$, and by the symmetries of this distribution (as in Appendices A.2 and A.5), $(1/N)Z'Sw$ is distributed like $(1/N)\|Sw\|$ times an $(N-1)$ vector with entries i.i.d. $\sim \mathcal{N}$ – near $(1/\sqrt{N})\|Sw\|$ times a vector uniformly distributed on $S^{N-2}$. And, as in Appendix A.6, $(1/\sqrt{N})\|Sw\|$ is concentrated near $\|(s_1, s_2, \ldots, s_N)\|_2^{(\pi)}$. As for $(1/N)w'Sw$ – it is just (54) – its value is concentrated near $(1/N)\,\mathrm{tr}\,A$, which if $A$ is positive-semidefinite is equal to $\|(s_1, s_2, \ldots, s_N)\|_1^{(\pi)}$.

To conclude, the value our random $A$ gives to $(1, 0, \ldots, 0)$ is a vector with first coordinate near $(1/N)\,\mathrm{tr}\,A$ – which if $A$ is positive-semidefinite is $\|(s_1, s_2, \ldots, s_N)\|_1^{(\pi)}$, and other coordinates forming a vector near the product of $\|(s_1, s_2, \ldots, s_N)\|_2^{(\pi)}$ with a vector uniformly distributed on $S^{N-2}$. Its norm is $\|(s_1, s_2, \ldots, s_N)\|_2^{(\pi)}$ up to a deviation $O(1/N)$, and one obtains values agreeing with the above for $\langle Ax, x\rangle$ and the square of the distance between $v$ and $Av$, both normalized.

In particular, for $A$ symmetric, employing uniform distribution on spheres in the Markov chain as in Appendix A.3 and Conclusion 12 is vindicated. Therefore, for a product of a sequence of symmetric operators $A_M A_{M-1} \cdots A_1$, we may apply Conclusion 12 to obtain the following conclusion.

**Conclusion 17** *For a product $A_M A_{M-1} \cdots A_1$, of a sequence of* symmetric *operators $A_i$ with given eigenvalues $(s_1^{(i)}, s_2^{(i)}, \ldots, s_N^{(i)})$, the square of the distance between $v$ and $A_M A_{M-1} \cdots A_1 v$, both normalized to norm 1, is, with almost full probability, near (with deviation $O(\sqrt{M}/\sqrt{N})$)*

$$2\left(1 - \prod_{i=1}^{M} \frac{(1/N)\,\mathrm{tr}\;A_i}{(1/\sqrt{N})\|A_i\|_{HS}}\right) = 2\left(1 - \prod_{i=1}^{M} \frac{(1/N)\,\mathrm{tr}\,A_i}{\|(s_1^{(i)}, s_2^{(i)}, \ldots, s_N^{(i)})\|_2^{(\pi)}}\right), \tag{60}$$

*which, if for all $i$, $A_i$ is positive semidefinite, is equal to*

$$2\left(1 - \prod_{i=1}^{M} \frac{\|(s_1^{(i)}, s_2^{(i)}, \ldots, s_N^{(i)})\|_1^{(\pi)}}{\|(s_1^{(i)}, s_2^{(i)}, \ldots, s_N^{(i)})\|_2^{(\pi)}}\right). \tag{61}$$

**Remark 18** *Note that* if the $A_i$ are *positive semidefinite, the value (61) around which the square of the distance between the points on $S^{N-1}$ is concentrated, is $\leq 2$, that is, the distance is $\leq \sqrt{2}$ and the angle between the vectors is $\leq 90^\circ$.*

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**Conflict of Interest**. The authors declare that they have no conflict of interest.

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