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Continuous Selections of Solutions for Locally Lipschitzian Equations

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Abstract This paper answers in affirmative the long-standing question of nonlinear analysis, concerning the existence of a continuous single-valued local selection of the right inverse to a locally Lipschitzian mapping. Moreover, we develop a much more general result, providing conditions for the existence of a continuous single-valued selection not only locally, but rather on any given ball centered at the point in question. Finally, by driving the radius of this ball to infinity, we obtain the global inverse function theorem, essentially implying the well known Hadamard's theorem on a global homeomorphism for smooth mappings, and the more general Pourciau's theorem for locally Lipschitzian mappings.

Keywords Nonlinear equation · Locally Lipschitzian mapping · Clarke's generalized Jacobian · Inverse function theorem · Continuous selection of solutions · Hadamard theorem

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1 Introduction

The celebrated Clarke's inverse function theorem [1, Theorem 7.1.1] states that if all linear operators in the generalized Jacobian of a locally Lipschitzian mapping from an arithmetic space to itself are nonsingular, then this mapping is locally a Lipschitzian homeomorphism. This is a direct generalization of the classical inverse function theorem from smooth to locally Lipschitzian mappings.

For a locally Lipschitzian mapping from one arithmetic space to another one of a possibly smaller dimension, it was established by Pourciau in [2] that if all linear operators in the generalized Jacobian are surjective, then this mapping is locally open (or covering, or interior, in Pourciau's terminology).

The formal statements of these results can be found in Section 2 below.

The result by Pourciau implies that under its assumptions, the right inverse to the mapping at hand is locally nonempty-valued. About ten years ago, the first author of this paper came up with the question whether this right inverse necessarily possesses locally continuous single-valued selections (see [3] for a discussion of related issues). This question was admitted with interest by experts in the field, though, to the best of our knowledge, no much progress has been achieved in answering it so far. A positive answer would have had useful applications, say, to local solvability of control problems with mixed constraints (i.e., those involving both control and state variables; see [4]), among other areas. The attention to this question was attracted again recently by Dontchev (see [5]), which indicated that it remains both important and unsolved. In addition, Dontchev is also interested in existence of a calm continuous selection.

All these issues are addressed in full in Section 4 below, where the stated questions are answered in affirmative in Theorem 4.1. In the smooth case, the corresponding results are immediate consequences of the classical inverse function theorem; see, e.g., [6, Theorem 1F.6, Exercise 1F.8].

Moreover, in Section 5 we develop a much more general result, namely, Theorem 5.1, which we call semilocal inverse function theorem; it provides conditions for the existence of a continuous single-valued selection not only on *some* ball around the image of the point in question, but rather on a *given* ball.

Furthermore, by infinitely increasing the radius of this ball, in Section 6 we establish Theorem 6.1, which is the global inverse function theorem. In case of a mapping from an arithmetic space to itself, the latter result essentially implies the well known Hadamard's theorem on a global homeomorphism for smooth mappings (see, e.g., [7, 5.3.9]), and the more general Pourciau's theorem [8] for locally Lipschitzian mappings. In case of a smooth mapping, Theorem 6.1 is a corollary of global inverse function theorem from [9], where a global implicit function theorem was also obtained.

2 Formal Problem Setting

All arithmetic spaces appearing in this paper are equipped with Euclidian inner products $\langle \cdot, \cdot \rangle$ and the corresponding norms $| \cdot |$. Closed ball centered at $x \in \mathbb{R}^n$ and of radius $r \geq 0$ will be denoted by $B^n(x, r)$, i.e.,

$$B^n(x, r) := \{u \in \mathbb{R}^n : |u - x| \leq r\}.$$

For a set $S \subset \mathbb{R}^n$, let $\operatorname{conv} S$ stand for the convex hull of S, and $\operatorname{int} S$ stand for the interior of S. For any $x, x' \in \mathbb{R}^n$ we write $[x, x']$ for the line segment connecting x and x', i.e. $[x, x'] := \{\lambda x + (1 - \lambda)x' : \lambda \in [0, 1]\}$.

For a set-valued mapping $S : \Sigma \rightrightarrows \mathbb{R}^s$ defined on some set $\Sigma \subset \mathbb{R}^n$, let $\operatorname{gph}(S)$ stand for its graph, i.e. $\operatorname{gph}(S) = \{(\sigma, y) : \sigma \in \Sigma, y \in S(\sigma)\}$.

For a mapping $f : \mathbb{R}^n \to \mathbb{R}^s$ which is Lipschitz-continuous near a point $x \in \mathbb{R}^n$, by $\partial f(x)$ we denote its Clarke's generalized Jacobian at x [1, Definition 2.6.1]. For any $x, x' \in \mathbb{R}^n$ we further define

$$\partial f([x, x']) := \operatorname{conv} \bigcup_{u \in [x, x']} \partial f(u).$$

We first give a formal statement of Clarke's inverse function theorem established in [10] (see also [1, Theorem 7.1.1]); it deals with the case when $s = n$.

Theorem 2.1 *For a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ which is Lipschitz-continuous near $\bar{x} \in \mathbb{R}^n$, if all linear operators $A \in \partial f(\bar{x})$ are nonsingular, then there exists $r > 0$ such that $f(\bar{x}) \in \operatorname{int} f(B^n(\bar{x}, r))$ and the mapping $x \mapsto f(x) : B^n(x, r) \to f(B^n(x, r))$ is a Lipschitzian homeomorphism, i.e., it is one-to-one, and the uniquely defined local inverse mapping $f^{-1} : f(B^n(x, r)) \to B^n(x, r)$ is Lipschitz-continuous.*

For the case when s is possibly smaller than n, the following covering result was obtained in [2].

Theorem 2.2 *For a mapping $f : \mathbb{R}^n \to \mathbb{R}^s$ Lipschitz-continuous near $\bar{x} \in \mathbb{R}^n$, if all linear operators $A \in \partial f(\bar{x})$ are surjective, then for every $r > 0$ it holds that $f(\bar{x}) \in \operatorname{int} f(B^n(\bar{x}, r))$.*

When $n \geq s$, it makes sense to consider the (generally set-valued) right inverse mapping $f^{-1} : \mathbb{R}^s \rightrightarrows \mathbb{R}^n$ defined as

$$f^{-1}(y) := \{x \in \mathbb{R}^n : f(x) = y\}.$$

Under the assumptions of Theorem 2.2, f^{-1} evidently possesses near $f(\bar{x})$ a single-valued selection which is continuous at the point $f(\bar{x})$, i.e., there exists $r > 0$ and a mapping $x(\cdot) : B^s(f(\bar{x}), r) \to \mathbb{R}^n$ such that $f(x(y)) = y$ for all $y \in B^s(f(\bar{x}), r)$ and $x(\cdot)$ is continuous at $f(\bar{x})$. Indeed, for any $y \in \mathbb{R}^n$

close enough to $f(\bar{x})$, one can define such $x(y)$ as any (global) solution of the optimization problem

$$\text{mimimize} \quad \|x - \bar{x}\| \quad \text{subject to} \quad f(x) = y.$$

The main question stated in Section 1 is much more involved: under the assumptions of Theorem 2.2, is it possible to find a single-valued selection of f^{-1}, such that it would be continuous on an entire neighborhood of $f(\bar{x})$ rather than just at $f(\bar{x})$? This question will be answered in Theorem 4.1 below.

The results surveyed so far are local by nature. Getting back for a moment to the case when $s = n$, let us now recall the main result in [8], providing conditions ensuring that f is a global homeomorphism.

Theorem 2.3 *For a locally Lipschitzian mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, if for some $k > 0$ is holds that*

$$A \text{ is invertible}, \quad \|A^{-1}\| \leq \frac{1}{k} \quad \forall\, A \in \partial f(x),\ \forall\, x \in \mathbb{R}^n, \tag{1}$$

then f is a homeomorphism.

Under a more restrictive assumption that f is continuously differentiable, this result recovers Hadamard's theorem (see, e.g., [7, 5.3.9]) with the generalized Jacobian in (1) boiling down to the (single-valued) classical one. In Theorem 6.1, we will derive possible extension of these results to the case when s can be smaller than n. Moreover, a quantitatively sharper semilocal result will be derived in Theorem 5.1. As mentioned above, global inverse and implicit functions theorems for smooth mappings in the case when $s \leq n$ were obtained in [9].

3 Preliminaries

Let $L(\mathbb{R}^n, \mathbb{R}^s)$ be the linear space of linear operators $A : \mathbb{R}^n \to \mathbb{R}^s$, and let $SL(\mathbb{R}^n, \mathbb{R}^s)$ stand for its subset consisting of surjective operators. Define the norm in $L(\mathbb{R}^n, \mathbb{R}^s)$ in a standard way:

$$\|A\| := \max_{x \in S^{n-1}} |Ax|,$$

where S^{n-1} stands for the unit sphere in $\mathbb{R}^n$, i.e.,

$$S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}.$$

Closed ball centered at $A \in L(\mathbb{R}^n, \mathbb{R}^s)$ and of radius $\varepsilon \geq 0$ will be denoted by $B(A, \varepsilon)$.

Following [11, Definition 1.51], for any $A \in L(\mathbb{R}^n, \mathbb{R}^s)$ set

$$\operatorname{cov}(A) := \max\{\alpha \geq 0 : B^s(0, \alpha) \subset AB^n(0, 1)\}$$

(cov is for "covering"). It can be easily seen that the function $\mathrm{cov} : L(\mathbb{R}^n, \mathbb{R}^s) \to \mathbb{R}_+$ defined this way is continuous, $\mathrm{cov}(A) > 0$ if and only if $A \in SL(\mathbb{R}^n, \mathbb{R}^s)$, and

$$\mathrm{cov}(A) = \frac{1}{\|A^*(AA^*)^{-1}\|} \quad \forall\, A \in SL(\mathbb{R}^n, \mathbb{R}^s). \tag{2}$$

Consider any compact set $\mathcal{A} \subset L(\mathbb{R}^n, \mathbb{R}^s)$ such that

$$\mathcal{A} \subset SL(\mathbb{R}^n, \mathbb{R}^s). \tag{3}$$

Define

$$\mathrm{cov}(\mathcal{A}) := \min\{\mathrm{cov}(A) :\ A \in \mathcal{A}\}.$$

Since the function $\mathrm{cov}(\cdot)$ is continuous and positive-valued on $SL(\mathbb{R}^n, \mathbb{R}^s)$, by compactness of $\mathcal{A}$ we have that minimum in the right-hand side of this expression is attained, and $\mathrm{cov}(\mathcal{A}) > 0$.

The following lemma plays an important role in subsequent developments (cf. [10, Lemma 3] or [1, Section 7.1, Lemma 1]).

Lemma 3.1 *If a compact convex set $\mathcal{A} \subset L(\mathbb{R}^n, \mathbb{R}^s)$ satisfies* (3)*, then there exists a continuous mapping $\xi : S^{s-1} \to S^{n-1}$ such that*

$$\langle A\xi(v),\, v\rangle \geq \mathrm{cov}(\mathcal{A}) \quad \forall\, v \in S^{s-1},\ \forall\, A \in \mathcal{A}. \tag{4}$$

Proof For every $v \in S^{s-1}$ define the set

$$\mathcal{A}^*v := \{A^*v :\ A \in \mathcal{A}\}.$$

Evidently, this set is compact and convex. Moreover, $0 \notin \mathcal{A}^*v$, since (3) implies that A^* is injective, and hence, $A^*v \neq 0$ for all $A \in \mathcal{A}$.

Let $\eta(v)$ be the smallest by norm point in $\mathcal{A}^*v$, i.e.

$$\eta(v) \in \mathcal{A}^*v, \quad |\eta(v)| \leq |u| \quad \forall\, u \in \mathcal{A}^*v.$$

It is known (see, e.g., [12, Section 2.8]) that such $\eta(v)$ exists, is unique, and depends continuously on v. Observe also that according to the argument above, $\eta(v) \neq 0$ for all $v \in S^{s-1}$. Moreover, applying the classical optimality conditions for a problem of minimizing a smooth convex function $\|\cdot\|^2$ over a convex set $\mathcal{A}^*v$, we readily obtain the equality

$$|\eta(v)|^2 = \inf_{u \in \mathcal{A}^*v} \langle \eta(v),\, u\rangle \tag{5}$$

for all $v \in S^{s-1}$.

Set

$$\xi(v) := \frac{\eta(v)}{|\eta(v)|}, \quad v \in S^{s-1}.$$

We next show that thus defined mapping $\xi : S^{s-1} \to S^{n-1}$ satisfies all the needed requirements.

Continuity of $\xi(\cdot)$ follows by continuity of $\eta(\cdot)$ and the property $\eta(v) \neq 0$ for all $v \in S^{s-1}$. Furthermore, for any $v \in S^{s-1}$ and $A \in \mathcal{A}$ we have

$$\langle A\xi(v),\, v\rangle = \frac{\langle \eta(v),\, A^*v\rangle}{|\eta(v)|} \geq \frac{|\eta(v)|^2}{|\eta(v)|} = |\eta(v)|, \tag{6}$$

where the inequality is by (5). Moreover, for every $A \in \mathcal{A}$ it holds that

$$\begin{aligned} |A^*v| &= \frac{|A^*v||A^*(AA^*)^{-1}v|}{|A^*(AA^*)^{-1}v|} \\ &\geq \frac{\langle A^*v,\, A^*(AA^*)^{-1}v\rangle}{|A^*(AA^*)^{-1}v|} \\ &= \frac{1}{|A^*(AA^*)^{-1}v|} \\ &= \operatorname{cov}(A) \\ &\geq \operatorname{cov}(\mathcal{A}), \end{aligned}$$

where the last equality is by (2). Since $\eta(v) \in \mathcal{A}^*v$, we then have $|\eta(v)| \geq \operatorname{cov}(\mathcal{A})$. Combining this inequality with (6), we finally obtain (4). □

Another key tool needed below is the following result established in [14, Theorem 3].

Theorem 3.1 *Let $(X,\, \rho)$ be a complete metric space, and let $U : X \to \mathbb{R}_+$ be a lower semicontinuous function satisfying the Caristi-like condition with constant k, i.e.,*

$$\begin{gathered} \forall\, x \in X \text{ such that } U(x) > 0 \\ \exists\, x' \in X,\ x' \neq x,\ \text{satisfying } U(x') + k\rho(x,\, x') \leq U(x), \end{gathered} \tag{7}$$

then for every $x_0 \in X$ there exists $x \in X$ such that $U(x) = 0$ and $\rho(x_0,\, x) \leq U(x_0)/k$.

The relations of this theorem with other known variational principles of nonlinear analysis was studied in [15,16].

4 Local Inverse Function Theorem

Theorem 4.1 *For a mapping $f : \mathbb{R}^n \to \mathbb{R}^s$ Lipschitz-continuous near $\bar{x} \in \mathbb{R}^n$, if*

$$\partial f(\bar{x}) \subset SL(\mathbb{R}^n,\, \mathbb{R}^s), \tag{8}$$

then for any $k \in (0,\, \operatorname{cov}(\partial f(\bar{x})))$ there exists $r > 0$ and a continuous mapping $x : B^s(f(\bar{x}),\, r) \to \mathbb{R}^n$ such that

$$f(x(y)) = y, \quad |x(y) - \bar{x}| \leq \frac{|y - f(\bar{x})|}{k} \quad \forall\, y \in B^s(f(\bar{x}),\, r).$$

This theorem is an immediate corollary of a more general Theorem 5.1 below. Nevertheless, we provide here a full direct proof of Theorem 4.1, as it is much simpler than that of Theorem 5.1, and quite instructive.

Proof Without loss of generality, we assume for simplicity that $\bar{x} = 0$ and $f(\bar{x}) = 0$.

We start with some auxiliary constructions. Specifically, we define positive reals ε, d, a, r, a set $\mathcal{A} \subset L(\mathbb{R}^n, \mathbb{R}^s)$, a metric space (X, ρ), and a function $U : X \to \mathbb{R}$ as follows.

Employing (8), compactness of $\partial f(0)$, continuity of $\operatorname{cov}(\cdot)$, and inequality $k < \operatorname{cov}(\partial f(0))$, we get the existence $\varepsilon > 0$ such that the set

$$\mathcal{A} := \bigcup_{A \in \partial f(0)} B(A, \varepsilon)$$

satisfies the inequality

$$\operatorname{cov}(\mathcal{A}) > k. \tag{9}$$

In particular, thus defined $\mathcal{A}$ satisfies (3), and it is evidently compact and convex. Therefore, according to Lemma 3.1 there exists a continuous mapping $\xi : S^{s-1} \to S^{n-1}$ satisfying (4). At the same time, since the set-valued mapping $x \mapsto \partial f(x)$ is upper semicontinuous [1, Proposition 2.6.2], there exists $d > 0$ such that

$$\partial f(x) \subset \mathcal{A} \quad \forall\, x \in B^n(0, d). \tag{10}$$

Set

$$a := \max_{A \in \mathcal{A}} \|A\|, \quad r := kd. \tag{11}$$

Denote by X the set of all continuous mappings $x : B^s(0, r) \to \mathbb{R}^n$ such that

$$|f(x(y)) - y| + k|x(y)| \le |y| \quad \forall\, y \in B^s(0, r). \tag{12}$$

This set is nonempty since it contains the mapping $x(\cdot) \equiv 0$. A metric over X is defined in a standard way as follows:

$$\rho(u, w) := \max_{y \in B^s(0, r)} |u(y) - w(y)|, \quad u, w \in X.$$

Finally, define a function $U : X \to \mathbb{R}$ by

$$U(x) := \max_{y \in B^s(0, r)} |f(x(y)) - y|.$$

If there exists a mapping $x \in X$ such that $U(x) = 0$, this mapping x obviously satisfies all the needed requirements (recall (12)). In order to prove the existence of such mapping we will make use of Theorem 3.1. Observe that the space (X, ρ) defined above is evidently complete, while the function U has nonnegative values. Moreover, since f is continuous on $B^n(0, d)$ the function U is continuous (see, e.g., [12, Proposition 2.4.26] [13, Proposition 4.4.]). It remains to show that U satisfies the Caristi-like condition (7), and this is what we do in the rest of the proof.

Take any $x \in X$ such that $U(x) > 0$. We need to construct $x' \in X$, $x' \neq x$, satisfying the inequality in the second line of (7). The construction is introduced and justified in the following six steps.

1. Take any $t > 0$ such that

$$kt \leq 1, \quad (a^2 - k^2)t \leq 2(\mathrm{cov}(\mathcal{A}) - k). \tag{13}$$

The existence of such t follows from (9). For all $y \in B^s(0, r)$ such that $f(x(y)) \neq y$ set

$$v(y) := -\frac{f(x(y)) - y}{|f(x(y)) - y|}.$$

Define the mappings δ, $x' : B^s(0, r) \to \mathbb{R}^n$ as follows:

$$\delta(y) := \begin{cases} |f(x(y)) - y|\xi(v(y)) & \text{if } f(x(y)) \neq y, \\ 0 & \text{otherwise,} \end{cases} \qquad x'(y) := x(y) + t\delta(y).$$

Thus defined mapping x' is evidently continuous, and moreover, $x' \neq x$, as $U(x) > 0$ implies the existence of $y \in B^s(0, r)$ such that $\delta(y) \neq 0$.

2. We first prove that

$$x(y),\, x'(y) \in B^n(0, d) \quad \forall\, y \in B^s(0, r). \tag{14}$$

Indeed, take any $y \in B^s(0, r)$. Then

$$|x(y)| \leq \frac{|y|}{k} \leq \frac{r}{k} = d,$$

where the first inequality is by (12), while the equality is by the definition of r in (11). This proves (14) regarding x.

Consider separately the two cases: $f(x(y)) = y$ and $f(x(y)) \neq y$. In the fist case, $x'(y) = x(y) \in B^n(0, d)$. In the second case,

$$\begin{aligned} |x'(y)| &\leq |x(y)| + t|\delta(y)| \\ &\leq \frac{|y| - |f(x(y)) - y|}{k} + t|\delta(y)| \\ &\leq \frac{|y| - |f(x(y)) - y|}{k} + t|f(x(y)) - y| \\ &= \frac{|y|}{k} + \frac{kt - 1}{k}|f(x(y)) - y| \\ &\leq \frac{|y|}{k} \leq \frac{r}{k} = d, \end{aligned}$$

where the second inequality is by (12), the next-to-the-last inequality is by the first relation in (13), while the last equality is again by the definition of r in (11). This completes the proof of (14).

3. Our next step is to prove that for every $y \in B^s(0, r)$ there exists $A(y) \in \mathcal{A}$ such that

$$f(x'(y)) - f(x(y)) = tA(y)\delta(y). \tag{15}$$

Indeed, by the mean-value theorem (see, e.g., [2, Theorem 3.1] or [1, Proposition 2.6.5]) there exists $A(y) \in \partial f([x(y),\, x'(y)])$ such that

$$f(x'(y)) - f(x(y)) = A(y)(x'(y) - x(y)) = tA(y)\delta(y).$$

From (14) it follows that $[x'(y),\, x(y)] \subset B^n(0,\, d)$. But then by (10) and convexity of $\mathcal{A}$ we have that $\partial f([x(y),\, x'(y)]) \subset \mathcal{A}$. Hence, $A(y) \in \mathcal{A}$.

4. We now demonstrate that

$$|f(x'(y)) - y| \le (1 - kt)|f(x(y)) - y| \quad \forall\, y \in B^s(0,\, r). \tag{16}$$

If $f(x(y)) = y$, then $x'(y) = x(y)$, and hence, $f(x'(y)) = y$, implying the inequality in (16). Suppose now that $f(x(y)) \ne y$. Then

$$\begin{aligned}
|f(x'(y)) - y|^2 &= |f(x(y)) - y + f(x'(y)) - f(x(y))|^2 \\
&= |tA(y)\delta(y) + (f(x(y)) - y)|^2 \\
&= |f(x(y)) - y|^2 |tA(y)\xi(v(y)) - v(y)|^2 \\
&= |f(x(y)) - y|^2 (t^2 |A(y)\xi(v(y))|^2 - 2t\langle A(y)\xi(v(y)),\, v(y)\rangle + 1) \\
&\le |f(x(y)) - y|^2 (a^2 t^2 - 2t \operatorname{cov}(\mathcal{A}) + 1) \\
&\le |f(x(y)) - y|^2 (1 - kt)^2,
\end{aligned}$$

where the second equality is by (15), the third equality is by the definitions of δ and v, the next-to-the-last inequality is by (4) and (11), and the last is by (13). This gives (16).

5. Our next goal is to show that $x' \in X$, and in order to do this, it remains to verify that

$$|f(x'(y)) - y| + k|x'(y)| \le |y| \quad \forall\, y \in B^s(0,\, r).$$

Fix an arbitrary $y \in B^s(0,\, r)$. If $f(x(y)) = y$, this inequality holds trivially because again $x'(y) = x(y)$. Suppose now that $f(x(y)) \ne y$. Then by the definition of $x'(y)$

$$\begin{aligned}
|f(x'(y)) - y| + k|x'(y)| &\le |f(x'(y)) - y| + k|x(y)| + kt|\delta(y)| \\
&\le (1 - kt)|f(x(y)) - y| + k|x(y)| + kt|\delta(y)| \\
&= (1 - kt)|f(x(y)) - y| + k|x(y)| + kt|f(x(y)) - y| \\
&= |f(x(y)) - y| + k|x(y)| \\
&\le |y|,
\end{aligned}$$

where the second inequality is by (16), while the last is by (12). Therefore, $x' \in X$.

6. It remains to prove that x' satisfies the inequality in the second line of (7).

For any $y \in B^s(0,\, r)$ we have

$$|f(x'(y)) - y| \le (1 - kt)|f(x(y)) - y| = |f(x(y)) - y| - k|x'(y) - x(y)|, \tag{17}$$

where the inequality is by (16), while the equality follows by the relations $|f(x(y)) - y| = |\delta(y)|$ and $x'(y) - x(y) = t\delta(y)$ implied by the definitions of the mappings δ and x'. These relations also imply that

$$\begin{aligned}\max_{y \in B^s(0,\, r)}(|f(x(y)) - y| - k|x'(y) - x(y)|) &= (1 - kt) \max_{y \in B^s(0,\, r)} |f(x(y)) - y| \\ &= \max_{y \in B^s(0,\, r)} |f(x(y)) - y| \\ &\quad -k \max_{y \in B^s(0,\, r)} t|f(x(y)) - y| \\ &= \max_{y \in B^s(0,\, r)} |f(x(y)) - y| \\ &\quad -k \max_{y \in B^s(0,\, r)} |x'(y) - x(y)|.\end{aligned}$$

Combining this with (17) we obtain the needed inequality $U(x') + k\rho(x,\, x') \leq U(x)$. □

5 Semilocal Inverse Function Theorem

Theorem 5.1 *For a locally Lipschitzian mapping $f : \mathbb{R}^n \to \mathbb{R}^s$ if for some $\bar{x} \in \mathbb{R}^n$, $d > 0$, and $k > 0$ it holds that*

$$\min_{x \in B^n(\bar{x},\, d)} \operatorname{cov}(\partial f(x)) > k, \tag{18}$$

then there exists a continuous mapping $x : B^s(f(\bar{x}),\, kd) \to \mathbb{R}^n$ such that

$$f(x(y)) = y, \quad |x(y) - \bar{x}| \leq \frac{|y - f(\bar{x})|}{k} \quad \forall\, y \in B^s(f(\bar{x}),\, kd).$$

Note that in (18), the minimum in the left-hand side is attained. Indeed, since the set-valued mapping $x \mapsto \partial f(x)$ is upper semicontinuous and compact-valued, and its domain $B^n(\bar{x},\, d)$ is compact, its graph is compact. Hence, the continuous function $(x,\, A) \mapsto \operatorname{cov}(A) : \mathbb{R}^n \times L(\mathbb{R}^n,\, \mathbb{R}^s) \to \mathbb{R}$ attains its minimum on this graph at some point $(\widehat{x},\, \widehat{A})$, and then $\widehat{x}$ is evidently a minimizer of $\operatorname{cov}(\partial f(\cdot))$ on $B^n(\bar{x},\, d)$. The same reasoning applies in Lemma 5.2 and Theorem 6.1 below.

The proof of Theorem 5.1 relies on the following auxiliary statements. In these statements, continuity of a set-valued mapping with nonempty compact values is understood in Hausdorff sense.

The first fact is a parametric counterpart of Lemma 3.1.

Lemma 5.1 *Let Σ be a topological space, and assume that a set-valued mapping $\mathcal{A} : \Sigma \rightrightarrows L(\mathbb{R}^n,\, \mathbb{R}^s)$ with nonempty, compact, and convex values is continuous, and*

$$\inf_{\sigma \in \Sigma} \operatorname{cov}(\mathcal{A}(\sigma)) > 0.$$

Then there exists a continuous mapping $\xi : S^{s-1} \times \Sigma \to S^{n-1}$ such that

$$\langle A\xi(v,\, \sigma),\, v\rangle \geq \inf_{\sigma \in \Sigma} \operatorname{cov}(\mathcal{A}(\sigma)) \quad \forall\, v \in S^{s-1},\, \forall\, A \in \mathcal{A}(\sigma),\, \forall\, \sigma \in \Sigma.$$

Proof For each pair $(v, \sigma) \in S^{s-1} \times \Sigma$ set

$$\mathcal{A}(\sigma)^* v := \{A^* v : A \in \mathcal{A}(\sigma)\},$$

and define $\eta(v, \sigma)$ as the smallest by norm point in $\mathcal{A}(\sigma)^* v$. Repeating the subsequent argument from the proof of Lemma 3.1 with evident modifications, we come to the needed conclusion. □

The next lemma is concerned with outer approximation of the generalized Jacobian by continuous set-valued mappings.

Lemma 5.2 *For a locally Lipschitzian mapping $f : \mathbb{R}^n \to \mathbb{R}^s$ if for some $\bar{x} \in \mathbb{R}^n$, $R \geq 0$, and $k > 0$ it holds that*

$$\min_{x \in B^n(\bar{x},\, R)} \mathrm{cov}(\partial f(x)) > k, \tag{19}$$

then there exists a set-valued mapping $\mathcal{A} : B^n(\bar{x}, R) \rightrightarrows L(\mathbb{R}^n, \mathbb{R}^s)$ with compact and convex values, and such that

$$\partial f(x) \subset \mathrm{int}\, \mathcal{A}(x) \quad \forall\, x \in B^n(\bar{x}, R),$$

$\mathcal{A}$ is continuous, and

$$\min_{x \in B^n(\bar{x},\, R)} \mathrm{cov}(\mathcal{A}(x)) > k.$$

Proof Let again for simplicity $\bar{x} = 0$.

Recall again that the set-valued mapping $x \mapsto \partial f(x)$ is upper semicontinuous [1, Proposition 2.6.2]. Hence, according to [17, Theorem 2], there exists a sequence of set-valued mappings $\mathcal{A}_j : B^n(0, R) \rightrightarrows L(\mathbb{R}^n, \mathbb{R}^s)$ with compact and convex values, such that for every $x \in B^n(0, R)$ and every j it holds that

$$\mathcal{A}_{j+1}(x) \subset \mathcal{A}_j(x), \tag{20}$$

$$\partial f(x) \subset \mathrm{int}\, \mathcal{A}_j(x), \tag{21}$$

$$\partial f(x) = \lim_{j \to \infty} \mathcal{A}_j(x) = \bigcap_{j=1}^{\infty} \mathcal{A}_j(x), \tag{22}$$

and all mappings A_j are continuous.

We next show that there exists j such that

$$\mathrm{cov}(A) > k \quad \forall\, A \in \mathcal{A}_j(x),\ \forall\, x \in B^n(0, R). \tag{23}$$

Suppose the contrary: for each j there exist $x_j \in B^n(0, R)$ and $A_j \in \mathcal{A}_j(x_j)$ such that $\mathrm{cov}(A_j) \leq k$. Taking into account continuity of $\mathcal{A}_j$ and property (20), without loss of generality we may assume that sequences $\{x_j\}$ and $\{A_j\}$ converge to some $\widehat{x} \in B^n(0, R)$ and $\widehat{A} \in L(\mathbb{R}^n, \mathbb{R}^s)$, respectively. Then

$$\mathrm{cov}(\widehat{A}) \leq k. \tag{24}$$

Now we need to show that $\widehat{A} \in \partial f(\widehat{x})$. Take any j and $i \geq j$. Since $A_i \in \mathcal{A}_i(x_i)$, by (20) we have that $A_i \in \mathcal{A}_j(x_i)$. Passing to the limit as $i \to \infty$, by continuity of $\mathcal{A}_j$ we conclude that $\widehat{A} \in \mathcal{A}_j(\widehat{x})$, and since j is arbitrary, this yields the inclusion $\widehat{A} \in \bigcap_{j=1}^{\infty} \mathcal{A}_j(\widehat{x})$. Therefore, according to (22), the needed inclusion $\widehat{A} \in \partial f(\widehat{x})$ holds. Taking into account (19), the latter implies that $\operatorname{cov}(\widehat{A}) > k$, which contradicts (24). This yields the existence of j satisfying (23).

From (21) and (23), and from compactness of values of $\mathcal{A}_j$ and continuity, it follows that the mapping $\mathcal{A} := \mathcal{A}_j$ satisfies all the needed requirements. □

Combining Lemmas 5.1 and 5.2, and using Theorem 3.1, we now derive the following key result on continuous extension of the inverse function from a given ball to a larger one.

Lemma 5.3 *For a locally Lipschitzian mapping $f : \mathbb{R}^n \to \mathbb{R}^s$, assume that (18) holds for some $\bar{x} \in \mathbb{R}^n$, $d \geq 0$, and $k > 0$. Let $r_1 \geq 0$, $r_2 > 0$, and a continuous mapping $x_1 : B^s(\bar{x}, r_1) \to \mathbb{R}^n$ be such that $r_1 < r_2 \leq kd$ and*

$$f(x_1(y)) = y, \quad |x_1(y) - \bar{x}| \leq \frac{|y - f(\bar{x})|}{k} \quad \forall\, y \in B^s(f(\bar{x}), r_1). \tag{25}$$

Then there exists a continuous mapping $x_2 : B^s(f(\bar{x}), r_2) \to \mathbb{R}^n$ such that

$$x_2(y) = x_1(y) \quad \forall\, y \in B^s(f(\bar{x}), r_1), \tag{26}$$

$$f(x_2(y)) = y, \quad |x_2(y) - \bar{x}| \leq \frac{|y - f(\bar{x})|}{k} \quad \forall\, y \in B^s(f(\bar{x}), r_2). \tag{27}$$

Proof We again assume for simplicity that $\bar{x} = 0$ and $f(\bar{x}) = 0$, and as in the proof of Theorem 4.1, we start with some auxiliary constructions. Define positive reals R, α, ε, a, a set $\Sigma \subset \mathbb{R}^n$, mappings $\xi : S^{s-1} \times \Sigma \to \mathbb{R}$ and $\mathcal{A} : \Sigma \rightrightarrows L(\mathbb{R}^n, \mathbb{R}^s)$, a metric space (X, ρ), and a function $U : X \to \mathbb{R}$ as follows.

Set

$$R := \frac{r_2}{k}, \quad \Sigma := B^n(0, R).$$

By Lemma 5.2, there exists a set-valued mapping $\mathcal{A} : \Sigma \rightrightarrows L(\mathbb{R}^n, \mathbb{R}^s)$ with compact and convex values, and such that

$$\partial f(\sigma) \subset \operatorname{int} \mathcal{A}(\sigma) \quad \forall\, \sigma \in \Sigma, \tag{28}$$

$\mathcal{A}$ is continuous, and

$$\alpha := \min_{\sigma \in \Sigma} \operatorname{cov}(\mathcal{A}(\sigma)) > k.$$

We claim that there exists $\varepsilon > 0$ such that

$$\partial f(\sigma') \subset \mathcal{A}(\sigma) \quad \forall\, \sigma \in \Sigma,\ \forall\, \sigma' \in B^n(\sigma, \varepsilon) \cap \Sigma. \tag{29}$$

Indeed, suppose the contrary: there exist sequences $\{\sigma_j\}$, $\{\sigma'_j\} \subset \Sigma$ and $\{A_j\} \subset L(\mathbb{R}^n, \mathbb{R}^s)$ such that $|\sigma_j - \sigma'_j| \to 0$ as $j \to \infty$, $A_j \in \partial f(\sigma'_j)$, and $A_j \notin \mathcal{A}(\sigma_j)$ for all j. Since Σ is compact, while $\partial f(\cdot)$ is upper semicontinuous and has compact values, by passing to subsequences, if necessary, we may assume that there exist $\sigma' \in \Sigma$ and $A \in \partial f(\sigma')$ such that $\{\sigma'_j\} \to \sigma'$ and

$\{A_j\} \to A$. Then it holds that $\{\sigma_j\} \to \sigma'$ as well, and therefore, $\{(\sigma_j, A_j)\} \to (\sigma', A)$, $(\sigma_j, A_j) \notin \mathrm{gph}(\mathrm{int}\,\mathcal{A})$ for all j, while $(\sigma', A) \in \mathrm{gph}(\mathrm{int}\,\mathcal{A})$, since $A \in \partial f(\sigma')$ and $\partial f(\sigma') \subset \mathrm{int}\,\mathcal{A}(\sigma')$. Therefore, $\mathrm{gph}(\mathrm{int}\,\mathcal{A})$ cannot be an open set in $\Sigma \times L(\mathbb{R}^n, \mathbb{R}^s)$. However, since $\mathcal{A}$ has compact convex values with nonempty interior, and is continuous, $\mathrm{gph}(\mathrm{int}\,\mathcal{A})$ must be open (see, e.g., [18, Theorem 1.2.4], [19, Theorem 1.3.8]). This contradiction yields the existence of a needed ε.

Furthermore, according to Lemma 5.1, there exists a continuous mapping $\xi : S^{s-1} \times \Sigma \to S^{n-1}$ such that

$$\langle A\xi(v, \sigma), v\rangle \geq \alpha \quad \forall\, v \in S^{s-1},\ \forall\, A \in \mathcal{A}(\sigma),\ \forall\, \sigma \in \Sigma. \tag{30}$$

Set

$$a := \max\{\|A\| :\ A \in \mathcal{A}(\sigma),\ \sigma \in \Sigma\}. \tag{31}$$

Extend the mapping $x_1(\cdot)$ to the layer $B^s(0, r_2) \setminus B^s(0, r_1)$ by setting

$$x_1(y) := x_1\left(r_1 \frac{y}{|y|}\right), \quad y \in B^s(0, r_2) \setminus B^s(0, r_1). \tag{32}$$

Thus re-defined mapping $x_1 : B^s(0, r_2) \to \mathbb{R}^n$ is evidently continuous.

Let X be the set of all continuous mappings $x : B^s(0, r_2) \to \mathbb{R}^n$ such that

$$|f(x(y)) - y| + k|x(y) - x_1(y)| \leq |f(x_1(y)) - y| \quad \forall\, y \in B^s(0, r_2). \tag{33}$$

This set is nonempty since it contains $x_1(\cdot)$. Define a metric over X in a standard way as

$$\rho(u, w) := \max_{y \in B^s(0, r_2)} |u(y) - w(y)|, \quad u, w \in X.$$

Finally, define a function $U : X \to \mathbb{R}$ by

$$U(x) := \max_{y \in B^s(0, r_2)} |f(x(y)) - y|.$$

For a reader's convenience, we next list the properties of the metric space (X, ρ) and the function U, to be used below:

– (X, ρ) is complete, and U is continuous. The completeness of (X, ρ) is obvious, the continuity of U follows, for example, from [12, Proposition 2.4.26], [13, Proposition 4.4.].
– For any mapping $x \in X$, by (33) it holds that

$$x(y) = x_1(y) \quad \forall\, y \in B^s(0, r_1), \tag{34}$$

i.e., x is a continuous extension of x_1 to $B^s(0, r_2)$.

– For any mapping $x \in X$ it holds that

$$|x(y)| \leq \frac{|y|}{k} \quad \forall\, y \in B^s(0,\, r_2). \tag{35}$$

This is because for every $y \in B^s(0,\, r_1)$ it holds that $x(y) = x_1(y)$ and $|x_1(y)| \leq |y|/k$ (see (25)), while for $y \in B^s(0,\, r_2) \setminus B^s(0, r_1)$ we have

$$\begin{aligned} |x(y)| &\leq |x(y) - x_1(y)| + |x_1(y)| \\ &\leq \frac{|f(x_1(y)) - y|}{k} + |x_1(y)| \\ &= \frac{1}{k}\left| f\left(x_1\left(r_1\frac{y}{|y|}\right)\right) - y\right| + \left|x_1\left(r_1\frac{y}{|y|}\right)\right| \\ &= \frac{1}{k}\left| r_1\frac{y}{|y|} - y\right| + \left|x_1\left(r_1\frac{y}{|y|}\right)\right| \\ &\leq \frac{1}{k}\left| r_1\frac{y}{|y|} - y\right| + \frac{1}{k}\left|r_1\frac{y}{|y|}\right| \\ &= \frac{|y|}{k}, \end{aligned}$$

where the second inequality is by (33), the first equality is by (32), and the second equality and the last inequality are by (25).

– For any mapping $x \in X$ it holds that

$$x(y) \in \Sigma \quad \forall\, y \in B^s(0,\, r_2),$$

since (35) implies $|x(y)| \leq |y|/k \leq r_2/k = R$.

We need to show that U satisfies the Caristi-like condition (7). Take any $x \in X$ such that $U(x) > 0$. We now construct $x' \in X$, $x' \neq x$, satisfying the inequality in the second line of (7).

1. Take any $t > 0$ such that

$$kt \leq 1, \quad tU(x) < \varepsilon, \quad (a^2 - k^2)t \leq 2(\alpha - k) \tag{36}$$

The existence of such t is evident since $\alpha > k$ and $\varepsilon > 0$.

For all $y \in B^s(0,\, r)$ such that $f(x(y)) \neq y$ set

$$v(y) := -\frac{f(x(y)) - y}{|f(x(y)) - y|}.$$

Define the mappings $\delta,\, x' : B^s(0,\, r) \to \mathbb{R}^n$ as follows:

$$\delta(y) := \begin{cases} |f(x(y)) - y|\xi(v(y),\, x(y)) & \text{if } f(x(y)) \neq y, \\ 0 & \text{otherwise,} \end{cases} \qquad x'(y) := x(y) + t\delta(y). \tag{37}$$

Thus defined mapping x' is evidently continuous, and moreover, $x' \neq x$, as $U(x) > 0$ implies the existence of $y \in B^s(0,\, r)$ such that $\delta(y) \neq 0$.

2. We now prove that

$$x'(y) \in \Sigma \quad \forall\, y \in B^s(0,\, r_2). \tag{38}$$

Consider separately the two cases: $f(x(y)) = y$ and $f(x(y)) \neq y$. In the fist case $x'(y) = x(y) \in \Sigma$. In the second case,

$$\begin{aligned}
|x'(y)| &\leq |x(y)| + t|\delta(y)| \\
&\leq \frac{|f(x_1(y)) - y| - |f(x(y)) - y|}{k} + t|\delta(y)| \\
&= \frac{|f(x_1(y)) - y| - |f(x(y)) - y|}{k} + t|f(x(y)) - y| \\
&\leq \frac{|f(x_1(y)) - y|}{k} \\
&= \frac{1}{k}\left| f\left(x_1\left(r_1\frac{y}{|y|}\right)\right) - y\right| \\
&= \frac{1}{k}\left| r_1\frac{y}{|y|} - y\right| \\
&\leq \frac{|y|}{k} \leq R,
\end{aligned}$$

where the second inequality is by (33), the first equality is by (37), the third inequality is by the first inequality in (36), the next-to-the-last equality is by (32), the last equality is by (25), and the next-to-the-last inequality is because of the inequality $|y| \geq r_1$ following from the relations $f(x(y)) \neq y$, (25), and (34). This completes the proof of (38).

3. We next prove that for every $y \in B^s(0,\, r_2)$ there exists $A(y) \in \mathcal{A}(x(y))$ such that

$$f(x'(y)) - f(x(y)) = tA(y)\delta(y). \tag{39}$$

Indeed, by the mean-value theorem (see, e.g., [2, Theorem 3.1] or [1, Proposition 2.6.5]) there exists $A(y) \in \partial f([x(y),\, x'(y)])$ such that

$$f(x'(y)) - f(x(y)) = A(y)(x'(y) - x(y)) = tA(y)\delta(y).$$

Employing (37), and the second inequality in (36), we obtain

$$|x'(y) - x(y)| = t|\delta(y)| = t|y - f(x(y))| \leq tU(x) < \varepsilon.$$

Then from (29) (applied with $\sigma = x$, $\sigma' = x'$) and from convexity of $\mathcal{A}(x(y))$ we get that $\partial f([x(y),\, x'(y)]) \subset \mathcal{A}(x(y))$. and hence, $A(y) \in \mathcal{A}(x(y))$.

4. We now demonstrate that

$$|f(x'(y)) - y| \leq (1 - kt)|f(x(y)) - y| \quad \forall\, y \in B^s(0,\, r_2). \tag{40}$$

If $f(x(y)) = y$, then $x'(y) = x(y)$, and hence, $f(x'(y)) = y$, implying the inequality in (40). Suppose now that $f(x(y)) \neq y$. Then

$$\begin{aligned} |f(x'(y)) - y|^2 &= |f(x(y)) - y + f(x'(y)) - f(x(y))|^2 \\ &= |tA(y)\delta(y) + (f(x(y)) - y)|^2 \\ &= |f(x(y)) - y|^2 |tA(y)\xi(v(y),\, x(y)) - v(y)|^2 \\ &= |f(x(y)) - y|^2 (t^2 |A(y)\xi(v(y),\, x(y))|^2 \\ &\quad -2t\langle A(y)\xi(v(y),\, x(y)),\, v(y)\rangle + 1) \\ &\leq |f(x(y)) - y|^2 (a^2 t^2 - 2\alpha t + 1) \\ &\leq |f(x(y)) - y|^2 (1 - kt)^2, \end{aligned}$$

where the second equality is by (39), the next-to-the-last inequality is by (30) and (31), and the last inequality is by the third inequality in (36). This gives (40).

5. We now need to show that $x' \in X$, and in order to do this, it remains to verify that

$$|f(x'(y)) - y| + k|x'(y) - x_1(y)| \leq |f(x_1(y)) - y| \quad \forall\, y \in B^s(0,\, r_2).$$

If $f(x(y)) = y$, this inequality holds trivially because again $x'(y) = x(y)$. Suppose now that $f(x(y)) \neq y$. Then by the definition of $x'(y)$

$$\begin{aligned} |f(x'(y)) - y| + k|x'(y) - x_1(y)| &\leq |f(x'(y)) - y| \\ &\quad + k|x'(y) - x(y)| + k|x(y) - x_1(y)| \\ &= |f(x'(y)) - y| + kt|\delta(y)| + k|x(y) - x_1(y)| \\ &\leq (1 - kt)|f(x(y)) - y| + kt|f(x(y)) - y| \\ &\quad + k|x(y) - x_1(y)| \\ &= |f(x(y)) - y| + k|x(y) - x_1(y)| \\ &\leq |f(x_1(y)) - y|, \end{aligned}$$

where the first inequality is by (37) and (40), while the last is by (33). Therefore, $x' \in X$.

6. We finally prove that x' satisfies the inequality in the second line of (7).

For any $y \in B^s(0,\, r_2)$ we have

$$|f(x'(y)) - y| \leq (1 - kt)|f(x(y)) - y| = |f(x(y)) - y| - k|x'(y) - x(y)|. \quad (41)$$

where the inequality is by (40), while the equality follows by the relations $|f(x(y)) - y| = |\delta(y)|$ and $x'(y) - x(y) = t\delta(y)$ implied by the definitions of the mappings δ and x'. These relations also imply that

$$\begin{aligned}\max_{y\in B^s(0,\,r_2)}(|f(x(y))-y|-k|x'(y)-x(y)|) &= (1-kt)\max_{y\in B^s(0,\,r_2)}|f(x(y))-y| \\ &= \max_{y\in B^s(0,\,r_2)}|f(x(y))-y| \\ &\quad -k\max_{y\in B^s(0,\,r_2)}(t|f(x(y))-y|) \\ &= \max_{y\in B^s(0,\,r_2)}|f(x(y))-y| \\ &\quad -k\max_{y\in B^s(0,\,r_2)}|x'(y)-x(y)|.\end{aligned}$$

Combining this with (41) we obtain the needed inequality $U(x')+k\rho(x,\,x')\le U(x)$.

We have thus demonstrated that U satisfies the Caristi-like condition (7). It remains to apply Theorem 3.1 in order to conclude that there exists $x_2\in X$ such that $U(x_2)=0$. By the definition of U, the equality $U(x_2)=0$ implies that

$$f(x_2(y))=y\quad \forall\, y\in B^s(0,\,r_2),$$

yielding the first relation in (27). Furthermore the inclusion $x_2\in X$ implies that x_2 is continuous and

$$x_2(y)=x_1(y)\quad \forall\, y\in B^s(0,\,r_1)$$

according to (34), yielding (26), and

$$|x_2(y)|\le\frac{|y|}{k}\quad \forall\, y\in B^s(f(\bar{x}),\,r_2)$$

according to (35), yielding the second relation in (27). Therefore, the x_2 satisfies all the needed requirements. □

Proof of Theorem 5.1 Applying Lemma 5.3 with $r_1:=0$, $x_1(f(\bar{x})):=\bar{x}$, and $r_2:=kd$, we get the existence of a continuous mapping $x_2:B^s(f(\bar{x}),\,kd)\to\mathbb{R}^n$ such that

$$f(x_2(y))=y,\quad |x_2(y)-\bar{x}|\le\frac{|y-f(\bar{x})|}{k}\quad \forall\, y\in B^s(f(\bar{x}),\,kd).$$

It remains to take $x(\cdot):=x_2(\cdot)$. □

6 Global Inverse Function Theorem

Theorem 6.1 *For a locally Lipschitzian mapping $f:\mathbb{R}^n\to\mathbb{R}^s$ if for some $k>0$ it holds that*

$$\inf_{x\in\mathbb{R}^n}\operatorname{cov}(\partial f(x))>k, \tag{42}$$

then for any $\bar{x}\in\mathbb{R}^n$ there exists a continuous mapping $x:\mathbb{R}^s\to\mathbb{R}^n$ such that

$$f(x(y))=y,\quad |x(y)-\bar{x}|\le\frac{|y-f(\bar{x})|}{k}\quad \forall\, y\in\mathbb{R}^s.$$

Proof Under the assumption (42), from Lemma 5.3 we obtain the existence of a sequence of continuous mappings $x_j : B^s(f(\bar{x}), j) \to \mathbb{R}^n$ such that for all $j = 0, 1, \ldots$

$$f(x_j(y)) = y, \ \ |x_j(y) - \bar{x}| \leq \frac{|y - f(\bar{x})|}{k}, \ \ x_{j+1}(y) = x_j(y) \quad \forall\, y \in B^s(f(\bar{x}), j).$$

Therefore, the mapping $x : \mathbb{R}^s \to \mathbb{R}^n$, defined as

$$x(y) := x_j(y), \quad y \in B(f(\bar{x}), j), \ j = 0, 1, \ldots,$$

is well-defined, and satisfies all the needed requirements. □

Remark 6.1 Theorem 6.1 can be viewed as a complement of Theorem 5.1. With such a complement at hand, one can present both results as a single statement by formally allowing $+\infty$ to be taken as d in Theorem 5.1.

Remark 6.2 Theorem 6.1 certainly implies that under its assumptions f is surjective. Moreover, in the case when $s = n$, Theorem 6.1 essentially implies Theorem 2.3 saying that f is a homeomorphism. By "essentially" we mean that some simple (not involving any extra high-level tools) additional argument is needed in order to demonstrate that in this case, a mapping $x(\cdot)$ defined according to Theorem 6.1 is surjective. The latter evidently implies that f is a homeomorphism with $f^{-1}(\cdot) = x(\cdot)$. We next provide such an argument, assuming for concreteness that $x(\cdot)$ defined for $\bar{x} = 0$.

In order to demonstrate surjectivity of $x(\cdot)$, it is sufficient to show that $x(\mathbb{R}^n)$ is both open and closed, as the only nonempty set in $\mathbb{R}^n$ possessing both these properties simultaneously is $\mathbb{R}^n$ itself.

We first demonstrate openness of $x(\mathbb{R}^n)$. According to Theorem 2.1, f is a local homeomorphism, implying that for every $\bar{y} \in \mathbb{R}^n$ and for $\bar{x} = x(\bar{y})$, there exists a neighborhood O of $\bar{x}$ such that $x(\cdot)$ coincide with the local inverse $f^{-1} : f(O) \to O$ of f. Therefore, $x(\mathbb{R}^n) \supset O$, and hence, $x(\mathbb{R}^n)$ is open.

In order to prove that $x(\mathbb{R}^n)$ is closed, consider any sequence $\{y_j\} \subset \mathbb{R}^n$ such that $\{x(y_j)\}$ converges to some $\bar{x} \in \mathbb{R}^n$. Since f is continuous, it holds that $\{f(x(y_j))\} \to f(\bar{x})$. Furthermore, since $f(x(y_j)) = y_j$ for all j, we have that $\{y_j\} \to f(\bar{x})$. Continuity of $x(\cdot)$ then implies

$$x(f(\bar{x})) = x\left(\lim_{j\to\infty} \{y_j\}\right) = \lim_{j\to\infty} x(y_j) = \bar{x}.$$

Therefore, $\bar{x}$ belongs to $x(\mathbb{R}^n)$, and thus $x(\mathbb{R}^n)$ is closed.

7 Conclusions and Open Questions

Under natural assumptions, we have established the existence of local, semilocal, and global continuous single-valued selections of the right inverse to a locally Lipchitzian mapping. In addition, these selections were shown to satisfy the estimates providing the property known as calmness at points in question.

These results answer in full and go much beyond some long-standing open questions of nonsmooth analysis.

However, this is certainly not the end of the story as this development gives rise to new intriguing questions. One of them is whether such selections (even local!) can be chosen not just continuous and calm, but (locally) Lipschitz-continuous. This question was also mentioned in [3], and it appears highly nontrivial, and a positive answer to it (if possible) will likely require the development of some different tools.

Another long-standing interesting question arising in this context is the following: assuming that (8) holds, does there exist a linear subspace L in $\mathbb{R}^n$, of dimension s, and such that the restriction of every $A \in \partial f(\bar{x})$ to L is nonsingular? Answering this question in affirmative would immediately imply the existence of Lipschitz-continuous local selections by applying Theorem 2.1 to the restriction of f to L. This approach to establishing existence of continuous single-valued selections has been a subject of considerable interest. In particular, the stated question was answered in affirmative in [20], but only for min-type functions, and only for $s \leq 3$. To the best of our knowledge, for more general cases, this question remains open so far.

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