PROXIMAL METHOD FOR ℓ_0-NORM BASED SPARSE ENHANCED CONTROL PROBLEMS IN LARGE-SCALE INTERCONNECTED SYSTEMS

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Abstract. This paper considers linear quadratic optimal control problem of large-scale interconnected systems. An algorithmic framework is constructed to design controllers that provide a desired tradeoff between the system performance and the sparsity of the static feedback matrix. This is accomplished by introducing a minimization problem involving ℓ_0-norm of the feedback matrix subject to a maximum allowable compromise in performance. To address the computational difficulty caused by the use of ℓ_0-norm, we propose to approximate the ℓ_0-norm by its Moreau envelope and the proximal algorithm with extrapolation is constructed to solve the approximated optimization problem. Convergence analysis based on the Kurdyka-Lojasiewicz (KL) properties is presented. Our numerical examples show that the proposed framework can obtain feedback matrices with higher sparsity when compared with the model based on the ℓ_1-norm relaxation.

Key words. Interconnected systems; sparse feedback matrix; ℓ_0-norm minimization; proximal method; Kurdyka-Lojasiewicz properties.

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1. Introduction.

Consider the following control system

$$\dot{x} = Ax + B_1 d + B_2 u \tag{1.1}$$

$$z = Ex + Du, \tag{1.2}$$

where $x \in R^n$, $d \in R^q$, $u \in R^m$ and $z \in R^s$ is the state variable, disturbance, (input) control, and performance output, respectively, $E = [Q^{1/2} \;\; 0]^T \in R^{s\times n}$ and $D = [0 \;\; R^{1/2}]^T \in R^{s\times m}$, where $Q = Q^T \succeq 0$ and $R = R^T \succ 0$ is the state and performance weights, respecively, and it is assumed $A \in R^{n\times n}$ be given such that (A, B_2) is stabilizable and $(A, Q^{1/2})$ is detectable.

By letting $u = -Kx$, the corresponding closed-loop system is then given by

$$\dot{x} = (A - B_2K)x + B_1 d \tag{1.3}$$

$$z = \left[Q^{1/2} \;\; -R^{1/2}K\right]^T x, \tag{1.4}$$

where $K \in R^{m\times n}$ denotes the state-feedback matrix. This kind of control systems arises in the analysis of distributed controllers for interconnected systems [7, 10, 13, 22] where the interconnection structure of the system is often described by matrix A. The embedded sturcture is then transferred to the feedback matrix that governs the controllers via the Lyapunov-type relation [1]. The design of optimal control strategies based on the closed-loop system (1.3)-(1.4) was well-studied, see for example [23]. If there is no structural constraints imposed on K, the corresponsinding optimal control

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problem would be a standard linear-quadratic regulator (LQR) problem where the stabilizing K minimizes the $\mathcal{H}_2$ norm of the communication cost function from d to z (for example, see [23])

$$\min_{K \in R^{m \times n}} \text{trace}(P(K) B_1 B_1^T) \tag{1.5}$$

where $P(K) \in R^{n \times n}$ is the observability Gramian of (1.3)-(1.4) given by

$$P(K) = \int_0^\infty \exp((A - B_2 K)^T t)(Q + K^T R K) \exp((A - B_2 K)t) dt. \tag{1.6}$$

For simplicity, throughout the paper, we shall denote $P(K)$ by P. Using the standard Lyapunov approach (see for example [1, 23]), P can be computed by solving the following equation

$$L(K, P) = (A - B_2 K)^T P + P(A - B_2 K) + (Q + K^T R K) = 0. \tag{1.7}$$

The solution of (1.5) subjected to (1.7) is then given by

$$\bar{K} = R^{-1} B_2^T \bar{P}, \tag{1.8}$$

where $\bar{P}$ is the unique positive definite solution of the algebraic Riccati equation

$$A^T P + P A + Q - P B_2 R^{-1} B_2^T P = 0. \tag{1.9}$$

The classicial optimal control design often results a dense feedback matrix, implying that the optimal controllers are formed using information from all subsystems carried in K. However, in large networks of dynamical systems controllers that based on dense feedback matrix may impose prohibitively expensive setup cost and computation burden [5, 22]. Moreover in many applications, the communication graph does not have to be fully connected since the subsystems are dynamically coupled to each other and allowed to control their own states. Hence, it is clearly desirable to obtain a higher sparsity feedback matrix so as to reduce the setup cost for controllers that control the interconnected systems, especially when the size of the system is large. On the other hand, any alternation on the optimal feedback matrix will result in the inflation of communication cost. Therefore the main aim of this paper is to formulate an optimization problem that can obtain some meaningful combination between these two conflicting objectives, namely maximizing the sparsity of K while keeping the communication cost of the system near the one obtained via a traditional optimal control solution.

2. Model formulation and its approximation. The idea of sparsity has long been used in signal and image processing and many areas of system science. Technically, it is the problem of minimizing the number of nonzero components of a vector, which is referred to as the ℓ_0- norm. Hence, this paper proposes a new algorithmic framework to maximize the sparsity of the state-feedback matrix, i.e. to minimize the ℓ_0-norm of the feedback matrix subject to a maximum allowable increase from the optimal cost obtained via a traditional optimal control solution. The aim is to allow users to identify control configurations that strike a balance between the cost and the sparsity of the controller and to examine the influence of different control configurations on the performance of distributed systems. For this purpose, we consider the

following optimization problem

$$\min_{K\in R^{m\times n}} \|K\|_0 \tag{2.1}$$

$$\text{s.t.} \quad J(P) \leq (1+\gamma)J(\bar{P}), \tag{2.2}$$

$$L(K\,, P) = 0, \tag{2.3}$$

where $\|K\|_0$ denotes the number of nonzero entries or the cardinality of the matrix K, $J(P) = \text{trace}(PB_1B_1^T)$ is the cost function at P and $J(\bar{P})$ is the benchmarking cost index, obtained via the traditional optimal control problem (1.8). The positive scalar γ characterizes the trade-off between the relative performance and the level of sparsity; a larger γ encourages higher sparsity for K, while renders a higher cost when compared to $J(\bar{P})$.

The non-convexity and discontinuity of the ℓ_0-norm inevitably impose algorithmic difficulties on solving the optimization problem (2.1). In practice, for instance in compressive sensing and sparse representation [19, 22], the ℓ_0-norm is often apprximated by the convex ℓ_1-norm. However, it is found that such an approximation is not always satisfactory for many real practical problems [14, 17]. In contrast to this convex approximation, we shall employ another approximation for the ℓ_0- norm via the Moreau envelope. The notion of the Moreau envelope function is fundamental in variational analysis [21]; for a proper, lower semi-continuous function $\phi : R^n \to R\cup\{+\infty\}$, and a parameter value $\lambda > 0$, the Moreau envelope (or Moreau-Yosida regularization) function $\text{env}_{\lambda\phi}(x)$ and its proximal mapping $\text{prox}_{\lambda\phi}(x)$ are defined by

$$\text{env}_{\lambda\phi}(x) := \inf_{z\in R^n}\left\{\frac{1}{2\lambda}\|x-z\|_2^2 + \phi(z)\right\} \leq \phi(x) \tag{2.4}$$

$$\text{prox}_{\lambda\phi}(x) := \arg\min_{z\in R^n}\left\{\frac{1}{2\lambda}\|x-z\|_2^2 + \phi(z)\right\}. \tag{2.5}$$

Several properties can be drawn on $\text{env}_{\lambda\phi}(x)$ for $\lambda > 0$: it is continuous and finite-valued even if ϕ is only lower semi-continuous and may take on the extended value $+\infty$. Moreover, it can be shown [21] that $\text{env}_{\lambda\phi}(x) \to \phi(x)$ whenever $\lambda \to 0^+$. These proporties motivate us to approximate the ℓ_0-norm $\|\cdot\|_0$ in (2.1) by its Moreau envelope $\text{env}_{\lambda\|\cdot\|_0}$ for some $\lambda > 0$.

Denote the set C associated with the constraint (2.2) by

$$C = \{K \in R^{n\times m} | J(P) \leq \sigma\}, \tag{2.6}$$

where $\sigma = (1+\gamma)J(\bar{P})$, and the indicator functon of C, by

$$I_C(K) = \begin{cases} 0 & , \quad \text{if } K \in C, \\ +\infty & , \quad \text{otherwise.} \end{cases} \tag{2.7}$$

Without the equality constraint (2.3) and by using the above notations, problem (2.1)-(2.2) can be written as the following unconstrained problem

$$\min_{K\in R^{m\times n}} \|K\|_0 + I_C(K). \tag{2.8}$$

Hence, an approximate model using the Moreau envelope of ℓ_0-norm for some positive λ is given by

$$\min_{K\in R^{n\times m}} \text{env}_{\lambda\|\cdot\|_0}(K) + I_C(K). \tag{2.9}$$

Based on the definition of the Moreau envelope, (2.9) can be written as the following two variables set unconstrained optimization problem

$$\min_{K,Z\in R^{n\times m}} F(K,Z) := \frac{1}{2\lambda}\|K-Z\|_F^2 + \|Z\|_0 + I_C(K), \tag{2.10}$$

where $\|\cdot\|_F$ denotes the Frobenius norm. The following lemma gives the connection between the solution of (2.9) and the solution pair of (2.10)

LEMMA 2.1. *Let C be the set given by (2.6). For any $\lambda > 0$, if the pair (K^*, Z^*) is a solution of (2.10), then*

$$Z^* \in prox_{\lambda\|\cdot\|_0}(K^*). \tag{2.11}$$

Moreover, K^* is a solution of problem (2.9) with $Z^* \in prox_{\lambda\|\cdot\|_0}(K^*)$ if and only if (K^*, Z^*) is a solution of problem (2.10).

Proof. Suppose that (K^*, Z^*) is a solution of problem (2.10). Then $K^* \in C$ and

$$F(K^*, Z^*) \leq F(K^*, Z), \tag{2.12}$$

for all $Z \in \text{prox}_{\lambda\|\cdot\|_0}(K^*)$. Suppose for the sake of contradiction that $Z^* \notin \text{prox}_{\lambda\|\cdot\|_0}(K^*)$. Then, there exist some $Z \in \text{prox}_{\lambda\|\cdot\|_0}(K^*)$ such that

$$\frac{1}{2\lambda}\|K^*-Z^*\|_F^2 + \|Z^*\|_0 > \frac{1}{2\lambda}\|K^*-Z\|_F^2 + \|Z\|_0. \tag{2.13}$$

Hence, the definition of the proximity operator of $\|\cdot\|_0$ implies that $F(K^*, Z^*) > F(K^*, Z)$, which contradicts (2.12).

We shall now proceed to show that a pair (K^*, Z^*) is a solution of problem (2.10) if and only if K^* is a solution of problem (2.9) with $Z^* \in \text{prox}_{\lambda\|\cdot\|_0}(K^*)$. Let $G(K) := \text{env}_{\lambda\|\cdot\|_0}(K) + I_C(K)$. Using the definition of the Moreau envelope we have

$$G(K) = F(K, Z), \tag{2.14}$$

for all $Z \in \text{prox}_{\lambda\|\cdot\|_0}(K)$. Suppose that (K^*, Z^*) is a solution of problem (2.10), but K^* is not a solution of problem (2.9). Then there exists a matrix $\tilde{K} \in R^{n\times m}$ such that $G(\tilde{K}) < G(K^*)$. By (2.14) one can get

$$F(\tilde{K}, \tilde{Z}) = G(\tilde{K}) < G(K^*), \tag{2.15}$$

for all $\tilde{Z} \in \text{prox}_{\lambda\|\cdot\|_0}(\tilde{K})$. On the other hand, since (K^*, Z^*) is a solution of problem (2.10) the first part of this lemma ensures that $Z^* \in \text{prox}_{\lambda\|\cdot\|_0}(K^*)$. Again by (2.14) we then have

$$F(\tilde{K}, \tilde{Z}) = G(\tilde{K}) < G(K^*) = F(K^*, Z^*), \tag{2.16}$$

which violates the assumption of (K^*, Z^*) being a solution of problem (2.10).

Conversely, suppose that K^* is a solution of problem (2.9) with $Z \in \text{prox}_{\lambda\|\cdot\|_0}(K^*)$, then (K^*, Z^*) is a solution of problem (2.10). Indeed, if it is not the case, then there exists a pair $(\tilde{K}, \tilde{Z})$ such that $F(\tilde{K}, \tilde{Z}) < F(K^*, Z^*)$ for some $\tilde{Z} \in \text{prox}_{\lambda\|\cdot\|_0}(\tilde{K})$. This implies that $G(\tilde{K}) < G(K^*)$, which contradicts the assumption on K^*. □

The above lemma clearly suggested that one can solve (2.8) by solving the equivalent approximate problem (2.10). Thus, our main focus now is on the development of algorithm to solve (2.10).

3. Proximity algorithm and convergence analysis. This section motivates an iterative algorithm for problem (2.1)-(2.3) using the fixed-point proximity that characterize the critical points of the problem. We begin by considering the following unconstrained nonsmooth minimization problem:

$$\min_{x,y} \mathcal{H}(x,y) := \mathcal{F}(x) + \mathcal{G}(y) + \mathcal{Q}(x,y) \tag{3.1}$$

where $x \in R^n$ and $y \in R^m$, $\mathcal{F}(x), \mathcal{G}(y)$ are proper lower semi-continuous functions, and $\mathcal{Q}(x,y)$ is a smooth function with Lipschitz gradient on any bounded set. The proximal method proposed in [6] updates the estimate of (x,y) via solving the following proximal problems:

$$x^{k+1} \in \arg\min_x \mathcal{F}(x) + (x-x^k)^T \nabla_x \mathcal{Q}(x^k, y^k) + \frac{1}{2}\alpha_1^k \|x - x^k\|_2^2; \tag{3.2}$$

$$y^{k+1} \in \arg\min_y \mathcal{G}(y) + (y-y^k)^T \nabla_y \mathcal{Q}(x^{k+1}, y^k) + \frac{1}{2}\alpha_2^k \|y - y^k\|_2^2; \tag{3.3}$$

where α_1^k and α_2^k are two appropriately chosen steplengths. In fact if the Lipschitz continuity moduli for $\nabla\mathcal{Q}$ are known, the steplengths can be taken as $\alpha_1^k = \alpha L_1(y_k)$ and $\alpha_2^k = \alpha L_2(x_{k+1})$, respectively where $\alpha > 1$, and $L_1(y)$ and $L_2(x)$ are the corresponding Lipschitz moduli of $\nabla\mathcal{Q}$ such that

$$\|\nabla_x \mathcal{Q}(x_1, y) - \nabla_x \mathcal{Q}(x_2, y)\| \le L_1(y)\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^n, \tag{3.4}$$

$$\|\nabla_y \mathcal{Q}(x, y_1) - \nabla_y \mathcal{Q}(x, y_2)\| \le L_2(x)\|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathbb{R}^m. \tag{3.5}$$

Using the proximal operator defined as (2.5), the minimization of (3.1) is equivalent to the following proximal problem:

$$x^{k+1} \in \mathrm{prox}_{\alpha_1 \mathcal{F}}\left(x^k - \frac{1}{\alpha_1^k}\nabla_x \mathcal{Q}(x^k, y^k)\right); \tag{3.6}$$

$$y^{k+1} \in \mathrm{prox}_{\alpha_2 \mathcal{G}}\left(y^k - \frac{1}{\alpha_2^k}\nabla_y \mathcal{Q}(x^{k+1}, y^k)\right). \tag{3.7}$$

Obviously, the ℓ_0-norm minimization problem (2.1)-(2.3) can be expressed in the form of (3.1) by setting

$$\mathcal{F}(K) = \|K\|_0; \tag{3.8}$$

$$\mathcal{G}(P) = I_{\mathcal{X}}(P); \tag{3.9}$$

$$\mathcal{Q}(K,P) = \mu\|L(K,P)\|_F^2, \tag{3.10}$$

where $I_{\mathcal{X}}(P)$ denotes the indicator function of P that satisfies $I_{\mathcal{X}}(P) = 0$ if $P \in \mathcal{X} = \{P \in R^{n\times n} | J(P) \le \sigma\}$ and $+\infty$ otherwise, $\mathcal{Q}$ is the penalty function for the equality constraint (2.3) with an appropriate penalty parameter μ.

REMARK 1. *One can view the equality constraint $L(K,P) = 0$, as a system of quadratic polynomials, and thus $\mathcal{Q}(K,P)$ is nothing else but a forth degree polynomial of the components of K and P. Therefore, we can define the partial defivative operators with respect to both K and P as follows:*

$$\nabla_K = \left[\frac{\partial}{\partial K_{ij}} : i = 1, \ldots, m;\ j = 1, \ldots, n\right]; \tag{3.11}$$

$$\nabla_P = \left[\frac{\partial}{\partial P_{ij}} : i,j = 1,\ldots,n\right]. \tag{3.12}$$

To simplify the proximal problem (3.6)-(3.7), it is possible to segregate $\mathcal{Q}$ from the proximal equations. Suppose that a feasible candidate solution (K^k, P^k) is given such that $L(K^k, P^k) = 0$. Hence, (K^k, P^k) is the (global) minimum point of $\mathcal{Q}(K, P)$ and since $\mathcal{Q}$ is at least twice differentiable, then the first order optimality condition implies that $\nabla_K \mathcal{Q}(K^k, P^k) = \nabla_P \mathcal{Q}(K^k, P^k) = 0$. Using these observation, the proximal problem (3.6)-(3.7) can be relaxed as follows:

Given (K^k, P^k) such that $L(K^k, P^k) = 0$.

$$\begin{cases} K^{k+1} \in \text{prox}_{\alpha_1 \mathcal{F}}\left(K^k\right); \\ \text{Compute } \tilde{P}^k \text{ s.t. } L(K^{k+1}, \tilde{P}^k) = 0; \end{cases} \tag{3.13}$$

$$\begin{cases} P^{k+1} \in \text{prox}_{\alpha_2 \mathcal{G}}\left(\tilde{P}^k, K^{k+1}\right); \\ \text{Compute } \tilde{K}^{k+1} \text{ s.t. } L(\tilde{K}^{k+1}, P^{k+1}) = 0; \end{cases} \tag{3.14}$$

$$(K^{k+1}, P^{k+1}) \leftarrow (\tilde{K}^{k+1}, P^{k+1}). \tag{3.15}$$

To establish the convergence properties of the proposed framework, we give a definition of subdifferential for nonconvex and nonsmooth mappings. For a proper closed function $h : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, the (limiting) subdifferential (see, e.g. [9]) of h at $x \in \text{dom } h$ is defined as

$$\partial h(x) = \left\{ v : \exists x^k \to x, v^k \to v, \liminf_{y \to x^k} \frac{h(y) - h(x^k) - v^k \cdot (y - x^k)}{\|y - x^k\|} \geq 0 \right\}, \tag{3.16}$$

where it is assumed that for all k, $x^k \to x$ implies that $h(x^k) \to h(x)$ and dom $\partial h := \{x \in \mathcal{R}^n : \partial h(x) \neq \emptyset\}$. It is known that if h is continuously differentiable, the subdifferential (3.16) reduces to the gradient of h, denoted by ∇h; see, for example, [40, Exercise 8.8(b)]. When h is convex, the above subdifferential reduces to the classical Fréchet subdifferential, see, for example, [40, Proposition 8.12]. Moreover, the point x is called a critical point of h if $0 \in \partial h(x)$. It follows that if x is a local minimizer of h then $0 \in \partial h(x)$.

Before proceeding to the convergence study, we recall the Kurdyka-Łojasiewicz (KL) property, which plays a central role in our analysis. We begin by introducing some notation. For any subset $\mathcal{S} \subset \mathbb{R}^n$ and any point $x \in \mathbb{R}^n$, the distance from x to $\mathcal{S}$ is given by

$$\text{dist}(x, \mathcal{S}) : \inf\{\|y - x\| : y \in \mathcal{S}\}. \tag{3.17}$$

Note that when $\mathcal{S} = \emptyset$, it is defined that $\text{dist}(x, \mathcal{S}) = \infty$ for all x. Let $a \in (0, \infty]$ and we denote Φ_a the class of all concave and continuous functions $\varphi : [0, a) \to [0, \infty)$ that are continuously differentiable on $(0, a)$ with $\varphi(0) = 0$ and $\varphi\prime(s) > 0$ for all $s \in [0, a)$. We shall state the KL property.

Definition 3.1. *[6] A proper and lower semi-continuous function $h : \mathbb{R}^n \to (-\infty, \infty]$ is said to have the KL property at $\bar{s} \in$ dom $\partial h := \{s \in \mathbb{R}^n : \partial h(s) \neq 0\}$ if there exist $a \in (0, \infty]$, a neighborhood $\mathcal{S}$ of $\bar{s}$ and a function $\varphi \in \Phi_a$, such that for all $s \in S$ and $h(\bar{s}) < h(s) < h(s) + a$, the following inequality holds*

$$\varphi\prime(h(s) - h(\bar{s}) dist(0, \partial h(u)) \geq 1. \tag{3.18}$$

Moreover, if h satisfy the KL property at each point of dom ∂h then h is called a KL function.

The KL function is closely related to a so-called semi-algebraic function. Hence we shall also state the definition of the semi-algebraic sets and functions as below.

DEFINITION 3.2. *[6] A subset $\mathcal{S}$ of $\mathbb{R}^n$ is called the semi-algebraic set if there exists a finite number of real polynomial functions p_{ij} and q_{ij} such that*

$$\mathcal{S} = \bigcup_j \bigcap_i \{s \in \mathbb{R}^n : p_{ij}(s) = 0, q_{ij}(s) < 0\}. \tag{3.19}$$

Moreover, a function $h(s)$ is called the semi-algebraic function if its graph $(s,t) \in \mathbb{R}^n \times \mathbb{R} : t = h(s)\}$ is a semi-algebraic set.

THEOREM 3.3 ([6]). *The sequence generated by the iteration (7) converges to the critical point of (6), if the following conditions hold:*

i. $\mathcal{H}(x,y)$ is a KL function;

ii. $\{(x^k, y^k)\}$, $k = 0, 1, 2, \ldots$ is a bounded sequence and there exists some positive constant c_1 and c_2 such that $\alpha_1^k, \alpha_2^k \in (c_1, c_2)$, $k = 0, 1, 2, \ldots$;

iii. $\nabla \mathcal{Q}$ has Lipschitz constant on any bounded set.

The first condition requires that the objective function satisfies the so-called Kurdyka-Lojasiewicz (KL) properties in its effective domain; see Definition 3 in [14] for more details on KL properties. It is shown in Remark 5 and Theorem 11 in [5] that any so-called semi-algebraic function satisfy the Kurdyka-Lojasiewicz property. In the next, we first give the definition of the semi-algebraic sets and functions, followed by the proof that the objective function (6) defined via (10) is a semi-algebraic function.

THEOREM 3.4 (Global convergence). *The sequence generated by the Algorithm 1, $\{K^k, P^k\}$ is a Cauchy sequence and converges to a critical point of (5).*

Proof.

i. We shall show that $\mathcal{H}(K,P)$ is a KL function term by term. Firstly, since $\mathcal{Q}(K,P)$ is a real polynomial function, it is a semi-algebraic function [14]. For the indicator function $I_{\mathcal{X}}(P)$, it is shown that (see [3]) the indicator function of semi-algebraic sets is semi-algebraic function. Since $J(P) = \mathrm{tr}(BB^TP)$ is a linear polynomial in term of P, it is quite obvious that $\mathcal{X} = \{P \in R^{n\times n} | J(P) \leq \sigma\}$ is semi-algebraic set and thus $I_{\mathcal{X}}(P)$ is a semi-algebraic function. Lastly, for $\mathcal{F}(K) = \|K\|_0$, the graph of $\mathcal{F}$ is given by

$$\mathcal{S} = \cup_{i=0}^{mn} \{(K,i) : \|K\|_0 = i\}. \tag{3.20}$$

For each $i = 0, \ldots, mn$, let $\mathcal{S}_i = \{J : J \subset \{1, \ldots, mn\}, |J| = i\}$, then $\{(K,i) : \|K\|_0 = i\} = \cup_{J\in\mathcal{S}_k} \{(K,i) : K_{J'} = 0, K_J \neq 0\}$, where $J' = \{1, \ldots, mn\} \backslash J$. It is easy to see that the set $\{(K,i) : K_{J'} = 0, K_{J_1} < 0, -K_{J_2} < 0\}$; $J = J_1 \cup J_2$ is a semi-algebraic set and hence, $\mathcal{F}(K)$ is a semi-algebraic function since the finite union of the semi-algebraic set is also semi-algebraic.

ii. If the initial approximation $(K^0, P^)$ is chosen as $(\bar{K}, \bar{P})$ then one can construct a level set $\mathcal{S}_P = \{P : J(\bar{P}) \leq J(P) \leq \gamma\}$, which is is closed and compact. Thus, any $P^k \in \{P : J(\bar{P}) \leq J(P) \leq \gamma\}$, $\forall k$ are bounded since $J(P)$ is a linear function with non-negative coefficients. Moreover, for every bounded P^k, the equality constraint $L(K, P^k) = 0$ is reduced to an algebraic matrix Riccati equation in term of K. Since the system (A, B_2) is stabilizable and $(A, Q^{1/2})$ is detectable, then these imply that the corresponding

matrix Riccati equation will have bounded solution [16]. On the other hand, since $\mathcal{Q}(K,P)$ is a real polynomial in term of K and P, then there exist $c_1^-, c_1^+, c_2^-, c_2^+ > 0$ such that

$$\inf\{L_1(P^k) : k \in \mathbb{N}\} \geq c_1^- \text{ and } \sup\{L_1(K^k) : k \in \mathbb{N}\} \leq c_1^+; \tag{3.21}$$

$$\inf\{L_2(P^k) : k \in \mathbb{N}\} \geq c_2^- \text{ and } \sup\{L_2(K^k) : k \in \mathbb{N}\} \leq c_2^+. \tag{3.22}$$

Then, for $c_1 = \min\{c_1^-, c_2^-\}$ and $c_2 = \max\{c_1^+, c_2^+\}$, we have $\alpha_1^k, \alpha_2^k \in (c_1, c_2)$, $k = 0, 1, 2, \ldots$;

iii. The satisfaction of condition (iii) is obvious due to the fact that $\mathcal{Q}$ in our context is a real polynomial.

□

Based on the proximal iteration (3.13)-(3.15), we shall propose some closed form of the proximity operators so that the execution of the above iterative scheme is possible. Firstly, let us consider the proximity operator of the ℓ_0-norm. It is easy to see that for every $\lambda > 0$ and $Y \in R^{n\times m}$, the proximity operator of the ℓ_0-norm can be expressed in the form of component-wise

$$\text{prox}_{\lambda\|\cdot\|}(Y) = \Big[\text{prox}_{\lambda|\cdot|_0}(Y_{ij})\Big]_{n\times m}, \quad i = 1, \ldots, n, \;\; j = 1, \ldots, m. \tag{3.23}$$

Note that $\text{prox}_{\lambda|\cdot|_0}$ can be seen as the hard thresholding operator (see, e.g. [21]) that forces every variables except the large one to be zero. It can then be defined by

$$\text{prox}_{\lambda|\cdot|_0}(Y_{ij}) = \begin{cases} \{0\} & , \; |Y_{ij}| < \sqrt{2\lambda}, \\ \{0, Y_{ij}\} & , \; |Y_{ij}| = \sqrt{2\lambda}, \\ \{Y_{ij}\} & , \; \text{otherwise.} \end{cases} \tag{3.24}$$

Now we turn our attention to the proximity operator of the indicator function. Note that the role of K and P is interchangable. Once K is obtained, P can be computed by solving (1.7) and vice-versa. From the definition of the proximity operator of the indicator function,

$$\text{prox}_{I_C}(Y) := \arg\min\left\{\frac{1}{2\lambda}\|Y - P\|_F^2 + I_C(Y) : P \in R^{n\times m}\right\} \tag{3.25}$$

Since C is a closed non-empty set, the proximity operator of I_C reduces to Euclidean projection of a $n \times m$ matrix Y onto C, in which

$$\text{prox}_{I_C}(Y) := \min\{J(Y), \sigma\}\frac{Y}{J(Y)}. \tag{3.26}$$

To motivate a practical algorithm based on these proximity operators, let us investigate the strategy on the selection of the parameters and some accrelarating technique that is useful to enhance the effectiveness of the algorithm.

3.1. Selection on the parameters. An iterative scheme may be suggested as below. Given K^0 and Z^0,

$$Z^{k+1} \in \text{prox}_{\alpha\lambda\|\cdot\|_0}(\alpha K^k + (1-\alpha)Z^k), \tag{3.27}$$

where $\alpha \in (0,1)$ is a parameter to balance the two iterates K^k and Z^k. This is inspired by the so-called proximity gradient method discussed as follows: Recall the definition of $\text{prox}_{\lambda\|\cdot\|_0}$ for $\lambda > 0$:

$$\text{prox}_{\lambda\|\cdot\|_0}(Z) := \arg\min_{Z\in R^{n\times m}} \left\{ \frac{1}{2\lambda}\|K - Z\|_F^2 + \|Z\|_0 \right\}, \tag{3.28}$$

where the first term of the objective, $h(Z) = \frac{1}{2\lambda}\|K - Z\|_F^2$ is differentiable. Thus, the corresponding proximity gadient method is given by

$$Z^{k+1} \in \text{prox}_{\alpha^k\lambda\|\cdot\|_0}(Z^k - \alpha^k\nabla h(Z^k)) := \text{prox}_{\alpha^k\lambda\|\cdot\|_0}((\alpha^k/\lambda)K^k + (1 - (\alpha^k/\lambda)Z^k), \tag{3.29}$$

for which $\alpha^k > 0$ is a step size. Equation (3.29) coincides with (3.27) when a fixed step size $\alpha^k = \alpha\lambda$ is used. Since ∇h is Lipschitz continuous with constant L, it can be shown that the proximity gradient method converges, i. e. $\nabla h(Z^k) \to 0$ (see, for e.g. [19]) when a fixed step $\alpha^k = \alpha\lambda \in (0, 1/L]$ is selected. In this context, choosing a fixed step size is much desirable since the Lipschitz constant is known, i.e. $1/\lambda$. Note that having $\alpha\lambda \in (0, \lambda]$ implies that any $\alpha \in (0, 1]$ is acceptable. Because line search methods often prefer largest possible step size, $\alpha = 1$ should be used. However, this choice would result in the neglection of K^k in (3.29). Hence we make a small trade off on the step size by taking $\alpha = \frac{99}{100}$.

We now turn our discussion on the selection of λ. It is known that λ serves as the threshold parameter of the hard thresholding operator in (3.29). Hence, a large λ encourages sparser solution in a faster pace. On the other hand, the discrepancy between the approximate problem (2.10) and the original model (2.9) becomes smaller as λ decreasing. So a reasonable strategy that can combine both of the outcomes of large and small λ is to gradually decrease the value of λ during the iterations. The smallest $\lambda_{\min}$ is then set to determine on the quality of the solution. Under this setting, we take the following steps: Choose an initial value $\lambda_{\max}$ and a reducing factor $\rho < 1$, update $\max\{\rho\lambda, \lambda_{\min}\} \leftarrow \lambda$. To encourage more variety of solutions set, we use both fixed and random reducing factors. The one that gives the better solution is adopted.

3.2. Accerelation for the proximity algorithm. Given P^k, the feedback matrix K^k can be computed by solving the corresponding algebraic Riccati equation. However, this is not preferred because the solution process of the algebraic Riccati equation can be time-consuming and most importantly, the sparsity of K obtained in the previous iteration may be jeopardized through the solution process. Hence, a simple remedy to accerelate the algorithm is to compute a new candidate by including a gradient descent step in the algorithm:

$$K^{k+1} := Z^{k+1} - \alpha^k\nabla J(P(Z^{k+1})), \tag{3.30}$$

where α^k is the step size. Furthermore to avoid the compute of ∇J, one can employ extrapolation step in the form of

$$K^{k+1} := Z^{k+1} + \omega^k(Z^{k+1} - K^k), \tag{3.31}$$

where $\omega^k \in [0, 1)$ is an extrapolation parameter. The parameter ω^k must be chosen in specific ways to achieve the convergence acceralation consistent to that of a gradient

descent step. A scheme for ω suggested in [19] has the form: Given $\tau^0 = 1$,

$$\tau^{k+1} = \frac{\sqrt{1 + (2\tau^k)^2} + 1}{2},$$
$$\omega^k = \frac{\tau^k - 1}{\tau^{k+1}}. \tag{3.32}$$

Finally, based on the proximity operators, selection of the parameters and accerelation scheme, we can now propose the following accerelated method for problem (2.10).

Accerelated Proximal Algorithm (APA)

Input: $K^0 = Z^0 = \bar{K}$, $\alpha = \frac{99}{100}$, $\lambda_{\max} = 8$, $\lambda_{\min} = 5 \times 10^{-4}$, $\rho = \frac{9}{10}, \frac{3}{5}, \frac{1}{2}$ (only the best result is reported), $\epsilon = 10^{-4}$.

Step 0. Given K^0, Z^0, α, λ, $\lambda_{\min}$, ρ, and τ^0. Set $k := 0$.

Step 1. Compute $Z^{k+1} \in \text{prox}_{\alpha\lambda\|\cdot\|_0}(\alpha K^k + (1-\alpha)Z^k)$.

Step 2. Compute $P^k = P(Z^{k+1})$ by solving $L(P, Z^{k+1}) = 0$ where L is defined by (1.7).

Step 3. Compute $P^{k+1} = \text{prox}_{I_C}(P^k)$.

Step 4. Compute ω^k according to (3.32) and K^{k+1} by (3.31).

Step 5. If the criterion

$$\frac{\|K^{k+1} - K^k\|_F}{\|K^k\|_F} \leq \epsilon, \tag{3.33}$$

is met, set $K^0 = \tilde{K}^0 \leftarrow K^{k+1}$, $Z^0 \leftarrow Z^{k+1}$, $\lambda \leftarrow \rho\lambda > \lambda_{\min}$, and go to Step 0. Else, set $k := k + 1$ and go to Step 1.

Remark. In Step 3, given Z^{k+1}, one can solve the following Lyapunov equation for P:

$$(A - B_2 Z^{k+1})^T P + P(A - B_2 Z^{k+1}) + (Q + (Z^{k+1})^T R Z^{k+1}) = 0. \tag{3.34}$$

The Bartels-Stewart method [4] has been the method of choice for solving small-to-medium scale Sylvester and Lyapunov equations. The main idea of the Bartels-Stewart algorithm is to apply the Schur decomposition [15] to transform (3.34) into a triangular system which can be solved efficiently by forward or backward substitutions. This method is implemented as `lyap` in Matlab toolbox. On the other hand given P, (1.7) in Step 4 reduces to an algebraic Riccati equation in term of K, and can be solved effectively using the algorithm by Arnold and Laub [2]. The Matlab tool `care` implements this algorithm.

4. Numerical illustrations. In this section we evaluate the performance of the proposed algorithm on the task of finding sparse controllers for three problems inspired by a bio-chemical reaction, a network control system with N^2 unstable nodes [14] and the power distribution of IEEE 39 New England power grid problem. Computationally, the ℓ_0-norm is often approximated by the ℓ_1-norm. However, we shall show that our method can produce better quality solution than the model that based on the minimization of the relaxed norm $\min \|K\|_1$ subject to the cost constraint (2.2). Smoothing methods (see for example, [8] and the references herewithin) are the common technique for nonsmooth optimization problems. For the purpose of illustration, the relaxed nonsmooth constrained optimization problem is solved by the Matlab optimization solver `fmincon`. The following numerical examples are constructed to validate the performance of the methods:

Example 1. Consider a network of N systems coupled through the dynamics (1.3)-(1.4) where

$$[A]_{ii} = \begin{bmatrix} -1 & 0 & -3 \\ 3 & -1 & 0 \\ 0 & 3 & -1 \end{bmatrix}, \quad [B_1]_{ii} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad [B_2]_{ii} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with $[\cdot]_{ij}$ denotes the $ij-$th block of a matrix. Each system models a cyclic interconnection that arises in bio-chemical reactions. Here A is a block diagonal matrix while B_1 is a diagonal matrix and B_2 is the Korenecker product of $[B_2]_{ii}$ and the identity matrix. The performance weights Q and R are set to identity matrices.

Example 2. The goal of this example is to investigate the structural properties of optimal control of N^2 nodes, which are randomly and uniformly distributed in a square region of $N \times N$. Each node is an unstable second order linear system coupled with other nodes through the dynamics (1.3)-(1.4) where:

$$[A]_{ii} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad [B_1]_{ii} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The diagonal of A is defined by $[A]_{ii}$ and the off-diagonal $[A]_{ij}$ $(i \neq j)$ is the identity matrix while $B_1 = B_2$ are the Kronecker product of $[B_1]_{ii}$ and the identity matrix. Again the performance weights Q and R are set to identity matrices.

Example 3 This example considers the IEEE 39 New England Power Grid Model with 39 buses and 10 generators. Generator 10 is an equivalent aggregated model for the part of the network that it is assumed to be uncontrollable; generators 1 to 9 are equipped with Power System Stabilizers, which provide good damping of local modes and stabilize otherwise unstable open-loop system. Detailed description of the model can be found in [11, 12]. We use a sparsity-promoting optimal control strategy to design a supplementary wide area control signal aimed at achieving a desirable trade-off between damping of the inter-area oscillations and communication requirements in the distributed controller. Inter-area oscillations are associated with the dynamics of power transfers and they are characterized by groups of coherent machines that swing relative to each other. These oscillations are caused by weakly damped modes of the linearized swing equations and they physically correspond to active power transfer between different generator groups.

In the absence of higher-order generator dynamics, for purely inductive lines and constant-current loads, the dynamics of each generator can be represented by the electromechanical swing equations

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \sum_{j=1}^{N} |Y_{i,j}|E_iE_j \sin(\theta_i - \theta_j), \tag{4.1}$$

where N denotes the number of generators, P_i is the generator power injection in the network-reduced model, and $Y_{i,j}$ is the Kron-reduced admittance matrix describing the interactions among generators. After the linearization at an operating point the swing equations reduce to

$$M\ddot{\theta} + D\dot{\theta} + L\theta = 0, \tag{4.2}$$

where M and D are diagonal matrices of inertia and damping coefficients, and the coupling among generators is entirely described by the weighted graph induced by the Laplacian matrix L.

Let the state of the power network be partitioned as $x = [\theta^T \quad \dot{\theta}^T \quad x_r^T]^T$ where θ and $\dot{\theta}$ are the rotor angles and frequencies of 10 synchronous generators and x_r are the state variables which correspond to fast electrical dynamics. Hence, the linearized dynamics is given by

$$\begin{aligned} \dot{x} &= Ax + B_1 d + B_2 u \\ u &= -Kx. \end{aligned}$$

An equivalent but more versatile formulation of the optimal control problem is given by Dörfler et al. [11] via the closed-loop observability Gramian P as

$$\begin{aligned} &\min J(P) = \text{trace}(PB_1B_1^T) \\ &\text{s. t.}\,(A - B_2K)^T P + P(A - B_2K) = -(Q + K^T RK). \end{aligned} \tag{4.3}$$

Therefore the problem considered is reduced to that similar to our main problem. Unlike Example 1-2 where the corresponding feedback gain matrix is square, here in this example, it is a 9×75 non-square matrix. For the purpose of illustration $R = I$, and $Q = \epsilon I - \left(I - \frac{1}{N} ee^T\right)$ are chosen, where the choice of Q is inspired by slow coherency theory with $\epsilon > 0$ denoting a small regularization parameter and e is the vector of all ones. The data set for A, B_1 and B_2 are available in [18].

To illustrate the performance of the methods, we define the following indicators:

$$r_0 = \frac{\text{Number of non-zero element in } K}{\text{Number of non-zero element in } \bar{K}}; \;\; r_C = \frac{J(P)}{J(\bar{P})}, \tag{4.4}$$

where $\bar{K}$ and $\bar{P}$ are given in (1.8). In all runs for Example 1 and Example 2, the cost tolerence γ considered is up to 0.2, i.e. the maximum allowable increased in cost is 20% of the optimal cost. As for Example 3, note that the optimal cost without imposing any sparsity requirement is 344.7694. We are particularly interested to obtain the efficiency curve of sparsity and cost against γ. Hence γ is allowed to grow up to 1.0, i.e. up to twice of the optimal cost. The corresponding efficiency curve is plotted in Figure 1 while the performances of the methods for Example 1 and Example 2 are given in Table 1 and Table 2, respectively. An observation from the efficiency curve of APA in Figure 1 reveals that the number of nonzero component in K decreases rapidly initially with only a small grow in cost. However, the decreasing rate is slowing down significantly when γ is increased beyond 0.05. Therefore a desirable structure for K would be the solution that associated with $\gamma = 0.05$, for which the cost is 356.7155 with 72 nonzero components.

From the tables, we observe that although `fmincon` solver can generate lower cost, APA can always produce a better quality solutions i.e. a much sparser K. Moreover, we demonstrate that for large problems with hundreds of states, our method is able to reach a sufficiently sparse solution while `fmincon` solver fails.

5. Conclusion. When the ℓ_0-norm function in (2.1) is replaced by its Moreau envelope, problem (2.8) can be viewed as a relaxation of the combinatorial problem (2.1)-(2.3). As the parameter varies over, the solution of (2.8) traces the trade-off path between the performance and the feedback gain sparsity. When the solution is the centralized feedback gain, we then allow a slightly increase the cost and employ an iterative algorithm the alternating direction method of multipliers (ADMM) initialized by the optimal feedback matrix at the previous. After a desired level of sparsity is achieved, we fix the sparsity structure and find the approximated optimal

TABLE 4.1
Example 1: `fmincon` vs APA

N	γ	`fmincon`		APA	
		r_0	r_C	r_0	r_C
15	1.02	0.9688	1.0199	0.9022	1.0197
	1.10	0.8266	1.1000	0.3688	1.0931
	1.20	0.8444	1.2000	0.3688	1.0931
30	1.02	0.9422	1.0055	1.0000	1.0000
	1.10	0.9422	1.0999	0.4200	1.0952
	1.20	0.9422	1.2000	0.4200	1.1557
45	1.02	0.9200	1.0001	0.8034	1.0001
	1.10	0.8962	1.0144	0.4352	1.0912
	1.20	0.8843	1.0447	0.4352	1.1211

TABLE 4.2
Example 2: `fmincon` vs APA

N	γ	`fmincon`		APA	
		r_0	r_C	r_0	r_C
50	1.02	1.0000	1.0000	0.5312	1.0019
	1.10	0.9046	1.0999	0.5296	1.0021
	1.20	0.9412	1.1821	0.5440	1.0019
100	1.02	out of memory	out of memory	0.5166	1.0062
	1.10	out of memory	out of memory	0.5136	1.0047
	1.20	out of memory	out of memory	0.5146	1.0050
200	1.02	out of memory	out of memory	0.4965	1.0005
	1.10	out of memory	out of memory	0.4944	1.0003
	1.20	out of memory	out of memory	0.4941	1.0011

structured feedback gain by an extrapolation scheme. Since the set of stabilizing feedback gains is in general not convex, it makes it difficult to establish convergence to the global minimum of (2.8). Even in problems for which we cannot establish the convergence, our computational experiments suggest that the algorithm developed provides an effective means for attaining a desired trade-off between the performance and the sparsity of the controller.

REFERENCES

[1] B. D. O. Anderson and J. B. Moore, Optimal Control: Linear Quadratic Methods, Englewood Cliffs New Jersey USA: Prentice-Hall, 1990.

[2] W. F. Arnold III and A.J. Laub, Generalized eigenproblem algorithms and software for algebraic Riccati equations, Proceedings of the IEEE 72: pp. 1746-1754, 1984

[3] H. Attouch and J. Bolte. On the convergence of the proximal algorithm for nonsmooth functions involving analytic features, Mathematical Programming 116: pp. 5–16, 2009.

[4] R. H. Bartels and G. W. Stewart, Solution of the matrix equation AX+XB = C, Communications of the ACM 15: pp. 820–826, 1972.

[5] B. Bamieh, F. Paganini, and M. A. Dahleh, Distributed control of spatially invariant systems. IEEE Trans. Automat. Control 47: 109–1107, 2002.

[6] J. Bolte, S. Sabach and M. Teboulle, Proximal alternating linearized minimization for nonconvex and nonsmooth problems, Mathematical Programming 146: pp. 459-494, 2014.

[7] G. A. de Castro and F. Paganini, Convex synthesis of localized controllers for spatially invariant

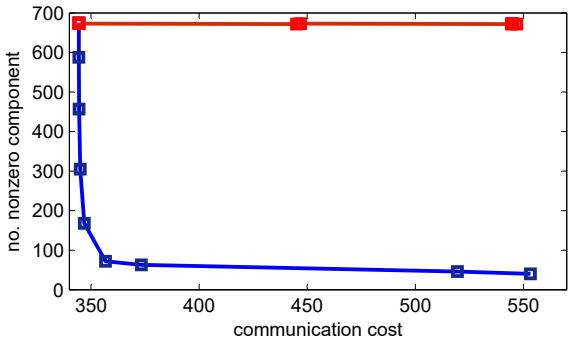

FIG. 4.1. *Example 3: Efficient curve of sparsity and cost (blue=APA; red=fmincon)*

system, Automatica 38: pp.445–456, 2002.
[8] X. Chen, Smoothing methods for nonsmooth, nonconvex minimization, Mathematical Programming 134: pp. 71-99, 2012.
[9] F. H. Clarke, Optimization and Nonsmooth Analysis, Wiley: New York, 1983.
[10] R. D'Andrea and G. E. Dullerud, Distributed control design for spatially interconnected systems, IEEE Transactions on Automatic Control 48: pp. 1478–1495, 2003.
[11] F. Dörfler, M. R. Jovanović, M. Chertkov, and F. Bullo, Sparsity-promoting optimal wide-area control of power networks, IEEE Transactions on Power Systems 29: pp. 2281-2291, 2014.
[12] F. Dorfler, M. R. Jovanovic, M. Chertkov, and F. Bullo, Sparse and optimal wide-area damping control in power networks, in Proceedings of the 2013 American Control Conference, pp. 4295-4300, 2013.
[13] G. Dullerud and R. D'Andrea, Distributed control of heterogeneous systems, IEEE Transactions on Automatic Control 49: pp. 2113–2128, 2004.
[14] M. Fardad, F. Lin, and M. R. Jovanovic, On the optimal design of structured feedback gains for interconnected systems, in Proceedings of the 48th IEEE Conference on Decision and Control, pp. 978-983, 2009.
[15] G. H. Golub and C. F. van Loan, Matrix Computations, 3rd ed. Baltimore, MD: Johns Hopkins Univ. Press, 1996.
[16] V. Kučera, A review of the matrix Riccati equation, Kybernetika 9: pp. 42–61, 1973.
[17] F. Lin, M. Fardad, and M. R. Jovanovic. Design of optimal sparse feedback gains via the alternating direction method of multipliers. IEEE Trans. Automat. Control, 58: pp. 2426–2431, 2013.
[18] LQRSP Sparsity-Promoting Linear Quadratic Regulator. Retrieved June 30, 2017, from http://people.ece.umn.edu/users/mihailo/software/lqrsp/.
[19] N. Parikh, and S. Boyd, Proximal algorithms, Foundations and Trends in Optimization 1: pp. 123–231, 2003
[20] R. T. Rockafellar, Convex analysis, Princeton University Press: Princeton, 1970.
[21] R. T. Rockafellar, and R. J. B. Wets, Variational Analysis, Springer-Verlag: New York, 1998 .
[22] M. Rotkowitz and S. Lall, A characterization of convex problems in decentralized control, IEEE Transactions on Automatic Control 51: pp. 274–286, 2006.
[23] K. L. Teo, K. H. Wong, and W. Y. Yan, A gradient flow approach to computing nonlinear quadratic optimal output feedback gain matrix, Journal of Optimization Thoery and Applications 85: pp. 75-96, 1995.