Dynamic Portfolio Selection with Linear Control Policies for Coherent Risk Minimization

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Abstract

This paper is concerned with a linear control policy for dynamic portfolio selection. We develop this policy by incorporating time-series behaviors of asset returns on the basis of coherent risk minimization. Analyzing the dual form of our optimization model, we demonstrate that the investment performance of linear control policies is directly connected to the intertemporal covariance of asset returns. To mitigate overfitting to training data (*i.e.*, historical asset returns), we apply robust optimization. For this optimization, we prove that the worst-case coherent risk measure can be decomposed into the empirical risk measure and the penalty terms. Numerical results demonstrate that when the number of assets is small, linear control policies deliver good out-of-sample investment performance. When the number of assets is large, the penalty terms improve the out-of-sample investment performance.

Keywords: portfolio selection, control policy, coherent risk measure, robust optimization

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1. Introduction

Portfolio selection problems involve determining an optimal investment allocation (*i.e.*, portfolio weights) of financial assets to produce low-risk high-return investments [13, 34]. The investment risk is reduced by exploiting correlations between asset returns in the single-period portfolio selection;

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this was pioneered by Markowitz [40]. In this standard approach, however, serial dependence in asset returns is disregarded, even though historical asset returns are collected as time-series data. A number of empirical studies have discovered that past asset returns contain information about expected returns and volatility [12, 21, 22, 30, 31, 38]. These results suggest that investment performance can be upgraded by fully considering serial dependence in asset returns.

Accordingly, we focus on a stochastic control approach [14, 29, 42], which involves designing control policies (*i.e.,* decision rules) for dynamically rebalancing a portfolio. This approach enables investors to make decisions conditioned on past outcomes. In general, however, selecting the best control policy from among all nonlinear functions is computationally intractable. A computational framework based on the kernel method [50] was proposed to deal with strong nonlinearity of control policies [4, 11, 53, 54]. However, the kernel method can be applied to portfolio selection problems only when there is a small number of assets, and nonlinear control policies are likely to overfit noisy financial data.

An effective remedy for these drawbacks is to use linear control policies, that is, decision rules restricted to the class of affine functions of past outcomes. Although this restriction may cause a substantial loss of optimality [9, 25], linear control policies have lower computational complexity for dynamic decision-making problems under uncertainty [7, 51]. Applications of linear control policies can be found in dynamic asset allocation [16, 17, 18, 43, 49], arbitrage trading [2, 3, 56], international portfolio management [23], and electricity portfolio management [44]. Additionally, linear control policies have been used to exploit various useful information, such as stock characteristics [15], industry momentum [5], and return predictability [24, 41].

Coherent risk measures are defined as functionals that satisfy desirable properties for quantifying financial risks [1, 46]. Conditional value-at-risk (CVaR) [47, 48] is the most popular of various coherent risk measures to be minimized; however, several coherent risk measures are often subject to estimation errors because the risk estimates depend on only a small portion of the sampled scenarios [35, 37]. To mitigate the fragility of these coherent risk measures, Gotoh *et al.* [26] used some techniques for robust optimization [6, 8, 33] of single-period portfolio selection.

The motivation behind our research is to extend the robust optimization techniques developed by Gotoh *et al.* [26] to dynamic portfolio selection with linear control policies. Fonseca and Rustem [23] also used a robust optimization technique to minimize the worst-case value-at-risk for portfolio selection with linear control policies. However, since that optimization model was posed as a semidefinite optimization problem, applying it to large-scale portfolio selection problems is difficult. Additionally, value-at-risk lacks subadditivity, which is a property required of coherent risk measures [1, 46].

In this study, we address the dynamic portfolio selection problem of minimizing a coherent risk measure to construct an effective linear control policy. First, we formulate this problem as a time-series-based optimization model. Next, we analyze the optimization model in the dual form to understand that the intertemporal covariance in asset returns is crucial for better investment performance of linear control policies. Finally, we adopt robust optimization techniques [10, 26] for our optimization model in order to cope with uncertainty about asset returns. Our main result is that the worst-case coherent risk measure can be decomposed into the empirical risk measure and the penalty terms.

The efficacy of our method is assessed through numerical experiments using select historical data on stock returns from the United States. For small numbers of assets, our control policies achieved better out-of-sample investment performance than both an equally weighted portfolio and application of the standard single-period portfolio selection method. In particular, our method clearly outperformed the single-period portfolio selection when high returns were overwhelmingly preferred. In addition, when the number of assets was large, the penalty terms based on robust optimization techniques substantially improved on the out-of-sample investment performance of the linear control policies.

The remainder of this paper is organized as follows. Section 2 gives the formulation of our optimization model for dynamic portfolio selection, and Section 3 shows approaching the optimization model via its dual. Section 4 presents the robust optimization techniques applied to our optimization model. Section 5 reports our numerical results to evaluate the effectiveness of our method for dynamic portfolio selection. Section 6 concludes with a brief summary of our work and a discussion of future research directions.

2. Problem formulation

This section introduces linear control policies and coherent risk measures. We then formulate our optimization model for dynamic portfolio selection.

2.1. Linear control policy

We write the historical data on asset returns as $\boldsymbol{R} := (r_{jt})_{(j,t)\in J\times T} \in \mathbb{R}^{|J\times T|}$, where $J, T \subseteq \mathbb{N}$ are the index sets of financial assets and time periods, respectively. The mean return of asset j is $\bar{r}_j := (\sum_{t\in T} r_{jt})/|T|$. We introduce $\boldsymbol{y} := (y_{jt})_{(j,t)\in J\times T} \in \mathbb{R}^{|J\times T|}$, the vector composed of decision variables representing *portfolio weights*, which satisfy the constraints

$$\sum_{j\in J} y_{jt} = 1, \quad t \in T,$$
$$y_{jt} \geq 0, \quad (j,t) \in J \times T.$$

To adjust the portfolio weights in every period, we use the following *linear control policy*:

$$y_{jt} = b_j + \sum_{k\in K}\sum_{i\in I} a_{ijk}(r_{i,t-k} - \bar{r}_i), \quad (j,t) \in J \times (T \setminus K), \tag{1}$$

where $I, K \subseteq \mathbb{N}$ are the index sets of input assets and input time periods, respectively. Additionally, $\boldsymbol{b} := (b_j)_{j\in J} \in \mathbb{R}^{|J|}$ and $\boldsymbol{a} := (a_{ijk})_{(i,j,k)\in I\times J\times K} \in \mathbb{R}^{|I\times J\times K|}$ are the vectors composed of decision variables; b_j corresponds to a nominal portfolio weight on asset j, and a_{ijk} represents linear feedback to the weight of asset j from the past returns of asset i. For example, when $K = \{1,2,3\}$, the portfolio weights are dynamically adjusted in response to excess asset returns (*i.e.*, $r_{i,t-k} - \bar{r}_i$) during the most recent three periods. For simplicity, we assume for the remainder of the paper that $I = J$.

2.2. Coherent risk measure

The *portfolio return* generated by the linear control policy (1) in period t is expressed as

$$r_t(\boldsymbol{y}) := \sum_{j\in J} r_{jt} y_{jt} = \sum_{j\in J} r_{jt}\Big(b_j + \sum_{k\in K}\sum_{i\in I} a_{ijk}(r_{i,t-k} - \bar{r}_i)\Big). \tag{2}$$

The vector of the portfolio returns is then given by

$$\boldsymbol{r}(\boldsymbol{y}) := (r_t(\boldsymbol{y}))_{t\in T\setminus K} = \Big(\sum_{j\in J} r_{jt} y_{jt}\Big)_{t\in T\setminus K} \in \mathbb{R}^{|T\setminus K|}. \tag{3}$$

We define the *portfolio loss* as the additive inverse of the portfolio return (*i.e.*, $-\boldsymbol{r}(\boldsymbol{y})$). We then denote by $\rho(-\boldsymbol{r}(\boldsymbol{y}))$ a *coherent risk measure* of the portfolio loss. Namely, for vectors $\boldsymbol{\ell}, \boldsymbol{\ell}' \in \mathbb{R}^{|T\setminus K|}$ corresponding to portfolio losses, the function $\rho(\,\cdot\,)$ satisfies the following properties [1]:

1. $\rho(\boldsymbol{\ell}) \geq \rho(\boldsymbol{\ell}')$ for all $\boldsymbol{\ell}$ and $\boldsymbol{\ell}'$ satisfying $\boldsymbol{\ell} \geq \boldsymbol{\ell}'$ (*monotonicity*),
2. $\rho(\boldsymbol{\ell} + c\mathbf{1}) = \rho(\boldsymbol{\ell}) + c$ for all $\boldsymbol{\ell}$ and $c \in \mathbb{R}$ (*translation invariance*),
3. $\rho(\alpha\boldsymbol{\ell}) = \alpha\rho(\boldsymbol{\ell})$ for all $\boldsymbol{\ell}$ and $\alpha \in \mathbb{R}_+$ (*positive homogeneity*),
4. $\rho(\boldsymbol{\ell} + \boldsymbol{\ell}') \leq \rho(\boldsymbol{\ell}) + \rho(\boldsymbol{\ell}')$ for all $\boldsymbol{\ell}$ and $\boldsymbol{\ell}'$ (*subadditivity*),

where $\mathbf{1} := (1, 1, \ldots, 1)^\top \in \mathbb{R}^{|T \setminus K|}$.

The set of probability distributions $\boldsymbol{p} := (p_t)_{t \in T \setminus K} \in \mathbb{R}^{|T \setminus K|}$ is given by

$$P := \left\{ \boldsymbol{p} \in \mathbb{R}^{|T \setminus K|} \;\middle|\; \sum_{t \in T \setminus K} p_t = 1; \quad p_t \geq 0, \quad t \in T \setminus K \right\}.$$

It is known that any coherent risk measure can be expressed by the following *dual representation* [1, 46]:

$$\rho(-\boldsymbol{r}(\boldsymbol{y})) = \max_{\boldsymbol{q} \in Q} \left\{ -\sum_{t \in T \setminus K} q_t r_t(\boldsymbol{y}) \right\}, \tag{4}$$

where $\boldsymbol{q} := (q_t)_{t \in T \setminus K} \in \mathbb{R}^{|T \setminus K|}$ is the vector composed of the decision variables representing a probability distribution, and $Q \subseteq P$ is a nonempty convex set that uniquely defines the coherent risk measure. The dual representation (4) characterizes a coherent risk measure as the expected portfolio loss in the worst-case probability distribution $\boldsymbol{q} \in Q$.

A typical example of a coherent risk measure is the *β-conditional value-at-risk* (*β-CVaR*) [47, 48], which is defined as

$$\phi(-\boldsymbol{r}(\boldsymbol{y})) := \min_{v \in \mathbb{R}} \left\{ v + \frac{1}{(1-\beta)|T \setminus K|} \sum_{t \in T \setminus K} \max\left\{ -\sum_{j \in J} r_{jt} y_{jt} - v, 0 \right\} \right\},$$

where $v \in \mathbb{R}$ is a decision variable corresponding to the value-at-risk, and $\beta \in [0, 1)$ is a user-defined parameter of probability level (*e.g.,* $\beta = 0.9$). The computation of β-CVaR can be reduced to solving the following linear optimization problem:

$$\underset{v, \boldsymbol{z}}{\text{minimize}} \quad v + \frac{1}{(1-\beta)|T \setminus K|} \sum_{t \in T \setminus K} z_t \tag{5}$$

$$\text{subject to} \quad z_t \geq -\sum_{j \in J} r_{jt} y_{jt} - v, \quad z_t \geq 0, \quad t \in T \setminus K, \tag{6}$$

where $\boldsymbol{z} := (z_t)_{t \in T \setminus K} \in \mathbb{R}^{|T \setminus K|}$ is the vector composed of auxiliary decision variables. The dual representation of problem (5)–(6) is posed as

$$\underset{\boldsymbol{q}}{\text{maximize}} \quad -\sum_{t \in T \setminus K} q_t \sum_{j \in J} r_{jt} y_{jt} \tag{7}$$

$$\text{subject to} \quad \sum_{t \in T \setminus K} q_t = 1, \tag{8}$$

$$0 \le q_t \le \frac{1}{(1-\beta)|T \setminus K|}, \quad t \in T \setminus K. \tag{9}$$

This representation makes it clear that β-CVaR is the expected value of large losses [47, 48]. In particular, when $\beta = 0$, the β-CVaR coincides with the additive inverse of the expected portfolio return

$$-\frac{1}{|T \setminus K|} \sum_{t \in T \setminus K} \sum_{j \in J} r_{jt} y_{jt}, \tag{10}$$

which results from a uniform distribution (*i.e.*, $q_t = 1/|T \setminus K|$ for $t \in T \setminus K$).

2.3. Time-series-based optimization model

We are now in a position to formulate our time-series-based optimization model for dynamic portfolio selection. Specifically, we determine an optimal linear control policy (1) such that the coherent risk measure (4) will be minimized, as follows:

$$\underset{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{y}}{\text{minimize}} \quad \rho(-\boldsymbol{r}(\boldsymbol{y})) \tag{11}$$

$$\text{subject to} \quad y_{jt} = b_j + \sum_{k \in K} \sum_{i \in I} a_{ijk} (r_{i,t-k} - \bar{r}_i), \quad (j,t) \in J \times (T \setminus K), \tag{12}$$

$$\sum_{j \in J} b_j = 1, \tag{13}$$

$$\sum_{j \in J} a_{ijk} = 0, \quad (i,k) \in I \times K, \tag{14}$$

$$b_j \ge 0, \quad j \in J, \tag{15}$$

$$y_{jt} \ge 0, \quad (j,t) \in J \times (T \setminus K). \tag{16}$$

Here, constraints (13)–(14) ensure that the sum of portfolio weights will be 1; indeed, it follows from constraints (12)–(14) that

$$\sum_{j \in J} y_{jt} = \sum_{j \in J} b_j + \sum_{k \in K} \sum_{i \in I} (r_{i,t-k} - \bar{r}_i) \sum_{j \in J} a_{ijk} = \sum_{j \in J} b_j = 1. \tag{17}$$

Constraints (15)–(16) are non-negativity constraints on portfolio weights to prevent short sales.

Note that the additive inverse of the expected portfolio return (10) is a coherent risk measure, and that the non-negative weighted-sum operation preserves coherency [46]. Therefore, the mean-risk objective is also applicable in our optimization model. For example, the mean-CVaR optimization can be reduced to the following linear optimization problem:

$$\underset{\boldsymbol{a},\boldsymbol{b},v,\boldsymbol{y},\boldsymbol{z}}{\text{minimize}} \quad \frac{\alpha-1}{|T\setminus K|}\sum_{t\in T\setminus K}\sum_{j\in J} r_{jt}y_{jt} + \alpha\Big(v + \frac{1}{(1-\beta)|T\setminus K|}\sum_{t\in T\setminus K} z_t\Big) \tag{18}$$

$$\text{subject to} \quad \text{Eqs. (6) and (12)–(16)}, \tag{19}$$

where $\alpha \in [0,1]$ is a user-defined parameter of risk aversion. The expected portfolio return is maximized when $\alpha = 0$, and the β-CVaR is minimized when $\alpha = 1$.

3. Analysis of the dual problem

In this section, we consider the dual form of the optimization model (11)–(16).

Theorem 1. *The dual formulation of problem* (11)–(16) *is given by*

$$\underset{\boldsymbol{q},\gamma,\boldsymbol{\eta},\boldsymbol{\xi}}{\text{maximize}} \quad \gamma \tag{20}$$

$$\text{subject to} \quad -\sum_{t\in T\setminus K}(q_t r_{jt} + \xi_{jt}) \ge \gamma, \quad j\in J, \tag{21}$$

$$\sum_{t\in T\setminus K}(q_t r_{jt} + \xi_{jt})(r_{i,t-k} - \bar{r}_i) = \eta_{ik}, \quad (i,j,k)\in I\times J\times K, \tag{22}$$

$$\xi_{jt} \ge 0, \quad (j,t)\in J\times(T\setminus K), \tag{23}$$

$$\boldsymbol{q}\in Q, \tag{24}$$

where $\boldsymbol{q} := (q_t)_{t\in T\setminus K} \in \mathbb{R}^{|T\setminus K|}$, $\gamma\in\mathbb{R}$, $\boldsymbol{\eta} := (\eta_{ik})_{(i,k)\in I\times K} \in \mathbb{R}^{|I\times K|}$, *and* $\boldsymbol{\xi} := (\xi_{jt})_{(j,t)\in J\times(T\setminus K)} \in \mathbb{R}^{|J\times(T\setminus K)|}$ *are dual decision variables.*

Proof. We first eliminate the decision variable $\boldsymbol{y}$ by substitution with constraint (12). It follows from Eqs. (2) and (4) that

$$\rho(-\boldsymbol{r}(\boldsymbol{y})) = \max_{\boldsymbol{q}\in Q} f(\boldsymbol{a},\boldsymbol{b},\boldsymbol{q}),$$

where

$$f(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{q}) := -\sum_{t \in T \setminus K} q_t \sum_{j \in J} r_{jt} \underbrace{\left(b_j + \sum_{k \in K} \sum_{i \in I} a_{ijk}(r_{i,t-k} - \bar{r}_i) \right)}_{y_{jt}}.$$

We can also rewrite constraint (16) as

$$b_j + \sum_{k \in K} \sum_{i \in I} a_{ijk}(r_{i,t-k} - \bar{r}_i) \geq 0, \quad (j,t) \in J \times (T \setminus K). \tag{25}$$

After doing so, problem (11)–(16) reduces to

$$\min_{(\boldsymbol{a},\boldsymbol{b}) \in F} \max_{\boldsymbol{q} \in Q} f(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{q}),$$

where

$$F := \{(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^{|I \times J \times K|} \times \mathbb{R}^{|J|} \mid \text{Eqs. (13)–(15) and (25)}\}.$$

Note that F and Q are non-empty closed convex sets, and Q is bounded. In addition, $f(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{q})$ is a continuous finite convex-concave function on $F \times Q$ because of Eqs. (16) and (17). From the minimax theorem (see, *e.g.,* Corollary 37.3.2 of Rockafellar [45]), we have

$$\min_{(\boldsymbol{a},\boldsymbol{b}) \in F} \max_{\boldsymbol{q} \in Q} f(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{q}) = \max_{\boldsymbol{q} \in Q} \min_{(\boldsymbol{a},\boldsymbol{b}) \in F} f(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{q}).$$

The dual formulation (20)–(24) is then derived by applying the duality theorem of linear optimization to the inner minimization problem (*i.e.,* $\min_{(\boldsymbol{a},\boldsymbol{b}) \in F} f(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{q})$). □

In what follows, we analyze the dual problem (20)–(24). The left-hand side of constraint (21) contains the expected return (*i.e.,* $\sum_{t \in T \setminus K} q_t r_{jt}$) of asset j in the probability distribution $\boldsymbol{q} \in Q$, and the sum of non-negative increments (*i.e.,* $\sum_{t \in T \setminus K} \xi_{jt}$). These increments, which are dual decision variables corresponding to constraint (12), represent the effect of control policies. In addition, these increments push down the objective value (20) through constraint (21), so they improve the objective value (11) in the primal form through the duality relationship.

Now we define

$$s_{ijkt} := r_{jt}(r_{i,t-k} - \bar{r}_i), \tag{26}$$

$$c_{ijk}(\boldsymbol{q}) := \sum_{t \in T \setminus K} q_t s_{ijkt} = \sum_{t \in T \setminus K} q_t r_{jt}(r_{i,t-k} - \bar{r}_i), \tag{27}$$

which are regarded as, respectively, the *intertemporal product* and *intertemporal covariance* of asset returns in the probability distribution $\boldsymbol{q} \in Q$. Constraint (22) can then be rewritten as

$$c_{ijk}(\boldsymbol{q}) + \sum_{t \in T \setminus K} \xi_{jt}(r_{i,t-k} - \bar{r}_i) = \eta_{ik}.$$

This proves that if $c_{ijk}(\boldsymbol{q}) = \eta_{ik}$ holds for all $j \in J$, then all ξ_{jt} can be set to zero, and this is optimal. Consequently, the increments ξ_{jt} become large only when the intertemporal covariance $c_{ijk}(\boldsymbol{q})$ of asset returns has different values for different $j \in J$. This highlights the importance of the intertemporal covariance of asset returns to enhance the investment performance of linear control policies.

In summary, the coherent risk (11) can be reduced by linear control policies when the returns of assets $j \in J$ respond differently to the past return of a particular asset i. The main reason for this is that control policies can exploit these distinct response patterns to improve the investment performance.

4. Robust optimization techniques

This section presents the robust optimization techniques applied to our optimization model. We first derive a robust counterpart of the non-negativity constraint on portfolio weights. We then formulate the worst-case coherent risk measures.

4.1. Robust counterpart of non-negativity constraint

The non-negativity constraint (16) ensures that the portfolio weights are non-negative only when the asset returns used for optimization are entered into the control policies. Thus, this constraint can be violated in out-of-sample investments. To resolve this, we simulate out-of-sample asset returns by introducing uncertainty into the historical asset returns.

We introduce a vector $\boldsymbol{\varepsilon} := (\varepsilon_{jt})_{(j,t)\in J\times T} \in \mathbb{R}^{|J\times T|}$ of perturbations belonging to the uncertainty set of the form

$$\mathcal{E} := \{\boldsymbol{\varepsilon} \in \mathbb{R}^{|J\times T|} \mid \|\boldsymbol{\varepsilon}_t\| \le \lambda, \quad t \in T\},$$

where $\boldsymbol{\varepsilon}_t := (\varepsilon_{jt})_{j\in J}$ for $t \in T$. Note that $\|\cdot\|$ is a norm in a real coordinate space of appropriate dimensions, and $\lambda \in \mathbb{R}_+$ is a user-defined parameter that characterizes the degree of possible uncertainty. The portfolio weights under the perturbed asset returns are then given by

$$y_{jt}(\boldsymbol{\varepsilon}) := b_j + \sum_{k\in K}\sum_{i\in I} a_{ijk}(r_{i,t-k} - \varepsilon_{i,t-k} - \bar{r}_i).$$

The robust counterpart of the non-negativity constraint (16) is then expressed as

$$y_{jt}(\boldsymbol{\varepsilon}) \ge 0, \quad \boldsymbol{\varepsilon} \in \mathcal{E}, \tag{28}$$

which means that the portfolio weight must be non-negative for all realizations $\boldsymbol{\varepsilon} \in \mathcal{E}$.

Let us define the dual norm of $\|\cdot\|$ as

$$\|\boldsymbol{a}\|^\circ := \max\{\boldsymbol{a}^\top\boldsymbol{\varepsilon} \mid \|\boldsymbol{\varepsilon}\| \le 1\}.$$

Following the proof of Theorem 2 of Bertsimas *et al.* [10], we can reduce an infinite number of constraints (28) to a single convex constraint with the penalty terms based on the dual norm.

Theorem 2. *Suppose that constraint* (12) *holds. Then, constraint* (28) *can be rewritten in the equivalent form*

$$y_{jt} - \lambda \sum_{k\in K} \|\boldsymbol{a}_{jk}\|^\circ \ge 0, \tag{29}$$

where $\boldsymbol{a}_{jk} := (a_{ijk})_{i\in I}$ *for* $(j,k) \in J \times K$*.*

Proof. Constraint (28) can be transformed as follows.

$$
\begin{aligned}
& y_{jt}(\boldsymbol{\varepsilon}) \geq 0, \quad \boldsymbol{\varepsilon} \in \mathcal{E} \\
\iff\ & b_j + \sum_{k \in K} \sum_{i \in I} a_{ijk}(r_{i,t-k} - \bar{r}_i) - \sum_{k \in K} \sum_{i \in I} a_{ijk} \varepsilon_{i,t-k} \geq 0, \quad \boldsymbol{\varepsilon} \in \mathcal{E} \\
\iff\ & y_{jt} - \sum_{k \in K} \max\{(\boldsymbol{a}_{jk})^\top \boldsymbol{\varepsilon}_{t-k} \mid \|\boldsymbol{\varepsilon}_{t-k}\| \leq \lambda\} \geq 0 \\
\iff\ & y_{jt} - \lambda \sum_{k \in K} \max\{(\boldsymbol{a}_{jk})^\top \boldsymbol{\varepsilon}_{t-k} \mid \|\boldsymbol{\varepsilon}_{t-k}\| \leq 1\} \geq 0 \\
\iff\ & y_{jt} - \lambda \sum_{k \in K} \|\boldsymbol{a}_{jk}\|^\circ \geq 0.
\end{aligned}
$$

Here, note that $\boldsymbol{\varepsilon}_{t-k}$ is replaced by $\lambda \boldsymbol{\varepsilon}_{t-k}$ to derive the fourth line. □

4.2. Worst-case coherent risk measure

We next consider coherent risk measures in the worst case of out-of-sample investments. To characterize uncertainty about the intertemporal products (26) of asset returns, we introduce a vector $\boldsymbol{\delta} := (\delta_{ijkt})_{(i,j,k,t) \in I \times J \times K \times T} \in \mathbb{R}^{|I \times J \times K \times T|}$ of perturbations belonging to an uncertainty set of the form

$$
\mathcal{D} := \{\boldsymbol{\delta} \in \mathbb{R}^{|I \times J \times K \times T|} \mid \|\boldsymbol{\delta}_{kt}\| \leq \lambda_k, \quad (k,t) \in K \times T\},
$$

where $\boldsymbol{\delta}_{kt} := (\delta_{ijkt})_{(i,j) \in I \times J}$ for $(k,t) \in K \times T$, and $\lambda_k \in \mathbb{R}_+$ is a user-defined parameter that characterizes the degree of possible uncertainty of $k \in K$.

The portfolio return (2) under the perturbations is then expressed as

$$
r_t(\boldsymbol{a}, \boldsymbol{b} \mid \boldsymbol{\varepsilon}, \boldsymbol{\delta}) := \sum_{j \in J} b_j (r_{jt} - \varepsilon_{jt}) + \sum_{k \in K} \sum_{i \in I} \sum_{j \in J} a_{ijk} (s_{ijkt} - \delta_{ijkt}), \tag{30}
$$

and the worst-case coherent risk is given by

$$
\max\{\rho(-\boldsymbol{r}(\boldsymbol{a}, \boldsymbol{b} \mid \boldsymbol{\varepsilon}, \boldsymbol{\delta})) \mid (\boldsymbol{\varepsilon}, \boldsymbol{\delta}) \in \mathcal{E} \times \mathcal{D}\}, \tag{31}
$$

where $\boldsymbol{r}(\boldsymbol{a}, \boldsymbol{b} \mid \boldsymbol{\varepsilon}, \boldsymbol{\delta}) := (r_t(\boldsymbol{a}, \boldsymbol{b} \mid \boldsymbol{\varepsilon}, \boldsymbol{\delta}))_{t \in T \setminus K} \in \mathbb{R}^{|T \setminus K|}$. Although s_{ijkt} depends on r_{jt} by definition (26), the associated perturbations vary independently in Eq. (31). In this sense, Eq. (31) can be regarded as an upper limit on the worst-case coherent risk, so minimizing Eq. (31) reduces the worst-case coherent risk. For simplicity, in what follows, we call Eq. (31) the *worst-case coherent risk measure*.

By extending Theorem 3.1 of Gotoh *et al.* [26] to our dynamic portfolio selection, we prove that the worst-case coherent risk measure can be decomposed into the empirical component $\rho(-\boldsymbol{r}(\boldsymbol{y}))$ and the penalty terms based on the dual norm.

Theorem 3. *Suppose that constraint* (12) *holds. Then, the worst-case coherent risk measure* (31) *can be written as*

$$\rho(-\boldsymbol{r}(\boldsymbol{y})) + \lambda\|\boldsymbol{b}\|^\circ + \sum_{k\in K} \lambda_k \|\boldsymbol{a}_k\|^\circ, \tag{32}$$

where $\boldsymbol{a}_k := (a_{ijk})_{(i,j)\in I\times J}$ *for* $k \in K$*.*

Proof. The worst-case coherent risk measure (31) can be decomposed based on Eqs. (4) and (30) as follows.

$$\begin{aligned}
&\max\{\rho(-\boldsymbol{r}(\boldsymbol{a},\boldsymbol{b} \mid \boldsymbol{\varepsilon},\boldsymbol{\delta})) \mid (\boldsymbol{\varepsilon},\boldsymbol{\delta}) \in \mathcal{E}\times\mathcal{D}\} \\
=&\underbrace{\max_{\boldsymbol{q}\in Q}\Big\{-\sum_{t\in T\setminus K} q_t\Big(\sum_{j\in J} b_j r_{jt} + \sum_{k\in K}\sum_{i\in I}\sum_{j\in J} a_{ijk}s_{ijkt}\Big)\Big\}}_{\rho(-\boldsymbol{r}(\boldsymbol{y}))} \\
&+\max\Big\{\sum_{t\in T\setminus K} q_t \sum_{j\in J} b_j\varepsilon_{jt} \;\Big|\; \boldsymbol{q}\in Q;\quad \|\boldsymbol{\varepsilon}_t\| \le \lambda,\quad t\in T\Big\} \\
&+\max\Big\{\sum_{t\in T\setminus K} q_t \sum_{k\in K}\sum_{i\in I}\sum_{j\in J} a_{ijk}\delta_{ijkt} \;\Big|\; \boldsymbol{q}\in Q;\quad \|\boldsymbol{\delta}_{kt}\| \le \lambda_k,\quad (k,t)\in K\times T\Big\}.
\end{aligned} \tag{33}$$

For all $\boldsymbol{q} \in Q$, we have

$$\begin{aligned}
&\max\Big\{\sum_{t\in T\setminus K} q_t \sum_{j\in J} b_j\varepsilon_{jt} \;\Big|\; \|\boldsymbol{\varepsilon}_t\| \le \lambda,\quad t\in T\Big\} \\
=&\sum_{t\in T\setminus K} q_t\lambda \max\Big\{\boldsymbol{b}^\top\boldsymbol{\varepsilon}_t \;\Big|\; \|\boldsymbol{\varepsilon}_t\| \le 1\Big\} \\
=&\sum_{t\in T\setminus K} q_t\lambda\|\boldsymbol{b}\|^\circ = \lambda\|\boldsymbol{b}\|^\circ \sum_{t\in T\setminus K} q_t = \lambda\|\boldsymbol{b}\|^\circ,
\end{aligned} \tag{34}$$

where $\boldsymbol{\varepsilon}_t$ was replaced by $\lambda\boldsymbol{\varepsilon}_t$ to derive the second line.

Similarly, for all $\boldsymbol{q} \in Q$ we have

$$\begin{aligned}
&\max\Big\{\sum_{t\in T\setminus K} q_t \sum_{k\in K}\sum_{i\in I}\sum_{j\in J} a_{ijk}\delta_{ijkt} \;\Big|\; \|\boldsymbol{\delta}_{kt}\| \le \lambda_k, \quad (k,t)\in K\times T)\Big\} \\
&= \sum_{t\in T\setminus K} q_t \sum_{k\in K} \lambda_k \max\Big\{\sum_{i\in I}\sum_{j\in J} (\boldsymbol{a}_k)^\top \boldsymbol{\delta}_{kt} \;\Big|\; \|\boldsymbol{\delta}_{kt}\| \le 1\Big\} \\
&= \sum_{t\in T\setminus K} q_t \sum_{k\in K} \lambda_k \|\boldsymbol{a}_k\|^\circ = \sum_{k\in K} \lambda_k \|\boldsymbol{a}_k\|^\circ \sum_{t\in T\setminus K} q_t = \sum_{k\in K} \lambda_k \|\boldsymbol{a}_k\|^\circ, \qquad (35)
\end{aligned}$$

where $\boldsymbol{\delta}_{kt}$ was replaced by $\lambda_k \boldsymbol{\delta}_{kt}$ to derive the second line. The proof is completed by Eqs. (33)–(35). $\square$

Theorem 3 states that when a dual norm $\|\cdot\|^\circ$ and a convex set Q are selected appropriately, the problem of minimizing the worst-case coherent risk can be reduced to a convex optimization problem, which is computationally tractable. It is also notable that when $\lambda = 0$ and $\lambda_k = \infty$ for $k \in K$, the corresponding optimization model reduces to the standard single-period portfolio selection. In addition, the penalty term $\lambda\|\boldsymbol{b}\|^\circ$ is consistent with the norm constraint [19, 27, 28] used for single-period portfolio selection.

Suppose that the L_1-norm is used as the dual norm $\|\cdot\|^\circ$. We then introduce $\boldsymbol{a}^{(+)} := (a^{(+)}_{ijk})_{(i,j,k)\in I\times J\times K}$ and $\boldsymbol{a}^{(-)} := (a^{(-)}_{ijk})_{(i,j,k)\in I\times J\times K}$ as the vectors of auxiliary decision variables. When the mean-CVaR objective (18) is used as a coherent risk measure, the problem of minimizing the worst-case coherent risk is formulated as the following linear optimization problem:

$$\begin{aligned}
\underset{\substack{\boldsymbol{a},\boldsymbol{a}^{(+)},\boldsymbol{a}^{(-)},\\ \boldsymbol{b},v,\boldsymbol{y},\boldsymbol{z}}}{\text{minimize}} \quad & \frac{\alpha-1}{|T\setminus K|}\sum_{t\in T\setminus K}\sum_{j\in J} r_{jt}y_{jt} + \alpha\Big(v + \frac{1}{(1-\beta)|T\setminus K|}\sum_{t\in T\setminus K} z_t\Big) \\
& + \sum_{k\in K}\lambda_k \sum_{i\in I}\sum_{j\in J}(a^{(+)}_{ijk} + a^{(-)}_{ijk}) && (36)\\
\text{subject to} \quad & a_{ijk} = a^{(+)}_{ijk} - a^{(-)}_{ijk}, \quad (i,j,k)\in I\times J\times K, && (37)\\
& a^{(+)}_{ijk} \ge 0,\ a^{(-)}_{ijk} \ge 0, \quad (i,j,k)\in I\times J\times K, && (38)\\
& \text{Eqs. (6) and (12)–(16).} && (39)
\end{aligned}$$

Note here that the penalty term $\lambda\|\boldsymbol{b}\|^\circ$ has been omitted because $\|\boldsymbol{b}\|^\circ = \sum_{j\in J} b_j = 1$ from Eqs. (13) and (15).

5. Numerical experiments

This section reports results from numerical experiments designed to evaluate the effectiveness of our method of dynamic portfolio selection.

5.1. Experimental design

To see how our method behaves with real data, we used the following four historical datasets of US stock returns, which were downloaded from the data library of K.R. French's website (`https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/`).

5Ind: 5 industry portfolios ($|J| = 5$)

6SiMa: 6 portfolios formed on size and book-to-market ($|J| = 6$)

10Ind: 10 industry portfolios ($|J| = 10$)

25SiMa: 25 portfolios formed on size and book-to-market ($|J| = 25$)

We used the monthly asset returns from 2001 to 2010 (120 months) for training our portfolio selection strategies, and those from 2011 to 2018 (96 months) for testing the out-of-sample investment performance of the obtained strategies.

Tables 1 and 2 give summary statistics of the monthly asset returns for the 5Ind and 6SiMa datasets, respectively. Here, $\bar{r}_j$ and σ_j are the mean and standard deviation, respectively, of the return from asset j, and c_{ijk} is the empirical intertemporal covariance (27) between the returns of assets i and j. The column labeled "stdev" shows the standard deviation of $(c_{ijk})_{j \in J}$ for each $i \in I$. As explained in Section 3, when c_{ijk} has different values for different $j \in J$, linear control policies can improve the investment performance. The intertemporal covariance varies more widely in Table 2 than in Table 1 (see the stdev column), which implies that the linear control policies are expected to perform better for the 6SiMa dataset than for the 5Ind dataset.

We compared the performance of the following portfolio selection strategies.

EWP: equally weighted portfolio (*i.e.*, $y_{jt} = 1/|J|$ for $(j, t) \in J \times T$)

SPP: single-period portfolio selection (*i.e.*, problem (18), (6), (13), and (15) with $y_{jt} = b_j$ for $(j, t) \in J \times T$)

Table 1: Summary statistics of monthly asset returns for the period 2001–2018 (5Ind dataset)

		$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	stdev
$\bar{r}_j$ (%)		0.75	0.70	0.61	0.60	0.51	
σ_j (%)		3.70	4.40	5.85	3.93	5.05	
c_{ijk} (%^{2})	$i=1$	1.15	1.81	0.31	0.01	1.99	0.79
$(k=1)$	$i=2$	1.05	1.51	1.05	0.63	2.49	0.64
	$i=3$	3.13	3.92	1.80	2.02	4.53	1.06
	$i=4$	1.42	0.82	3.27	−0.07	1.31	1.09
	$i=5$	2.24	3.91	2.41	0.66	4.14	1.26

Table 2: Summary statistics of monthly asset returns for the period 2001–2018 (6SiMa dataset)

		$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	stdev
$\bar{r}_j$ (%)		0.71	0.94	0.98	0.62	0.62	0.57	
σ_j (%)		6.18	5.32	5.76	4.08	4.17	5.45	
c_{ijk} (%^{2})	$i=1$	3.08	2.90	4.57	1.03	2.80	5.35	1.38
$(k=1)$	$i=2$	1.51	1.72	3.56	0.14	1.99	3.87	1.27
	$i=3$	1.23	2.16	4.39	−0.26	2.25	4.29	1.63
	$i=4$	4.83	3.85	4.99	1.62	2.76	4.77	1.24
	$i=5$	3.95	3.65	5.05	1.08	2.70	4.45	1.29
	$i=6$	4.25	3.99	5.83	0.74	2.92	5.05	1.64

LC($|K|$): linear control policies developed by our optimization model (18)–(19)

LC-W($|K|$): linear control policies developed by our optimization model (36)–(39) that minimizes the worst-case coherent risk

For these, $|K|$ is the number of input time periods for the linear control policies. The equally weighted portfolio often shows good out-of-sample investment performance [20]. The probability level for β-CVaR was set as $\beta = 0.9$. The penalty parameters λ_k for $k \in K$ in LC-W($|K|$) were tuned by hold-out validation, where the monthly asset returns from 2001 to 2006 were used for training, those from 2007 to 2010 were used for validation, and $\lambda_k \in \{10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}\}$ were set to the same value for all $k \in K$. FICO Xpress Optimization (`https://www.fico.com/en/products/fico-xpress-optimization`) was used to solve the optimization problems.

Let $\hat{T}$ be the index set of the time periods being tested. We denote by $\hat{\boldsymbol{y}} := (\hat{y}_{jt})_{(j,t)\in J\times\hat{T}}$ and $\hat{\boldsymbol{r}}(\hat{\boldsymbol{y}}) := (\hat{r}_t(\hat{\boldsymbol{y}}))_{t\in\hat{T}}$ the vectors of portfolio weights and portfolio returns, respectively, in the testing time periods. We investigated the out-of-sample investment performance on the basis of the following metrics.

Cumulative total return: $\prod_{t\in\hat{T}}(1+\hat{r}_t(\hat{\boldsymbol{y}}))$

Average return: $\frac{1}{|\hat{T}|}\sum_{t\in\hat{T}}\hat{r}_t(\hat{\boldsymbol{y}})$

Standard deviation of return: $\sqrt{\frac{1}{|\hat{T}|}\sum_{t\in\hat{T}}\Big(\hat{r}_t(\hat{\boldsymbol{y}})-\frac{1}{|\hat{T}|}\sum_{u\in\hat{T}}\hat{r}_u(\hat{\boldsymbol{y}})\Big)^2}$

Number of short sales: $|\{\hat{y}_{jt} \mid \hat{y}_{jt}<0, \quad (j,t)\in J\times\hat{T}\}|$

Recall that short sales can be made by linear control policies in out-of-sample investments. In our experiments, these short sales were replaced by borrowing money at a monthly interest rate of 1%, and the associated interest payment was subtracted from the portfolio returns.

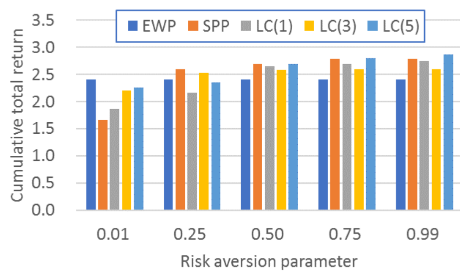
5.2. Out-of-sample investment performance

Figures 1–4 show the out-of-sample investment performance of the portfolio selection strategies, where the risk aversion parameter is $\alpha\in\{0.01,0.25,0.50,0.75,0.99\}$. Note that the performance of EWP is independent of the risk aversion parameter. We first examine the performance of linear control policies when the number of assets is small (Figures 1 and 2). We next focus on the advantages of minimizing the worst-case coherent risk when the number of assets is large (Figures 3 and 4).

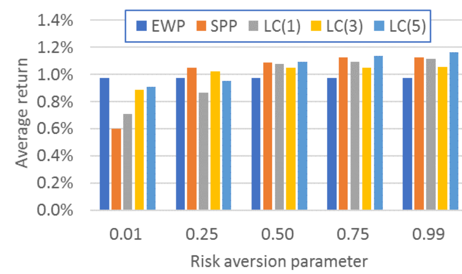
Figure 1 shows the out-of-sample investment performance for the 5Ind dataset. When the value of the risk aversion parameter was 0.75 or 0.99, LC(5) achieved the highest returns among methods (Figures 1a and 1b). Notably, SPP provided very low returns when the value of the risk aversion parameter was 0.01. Additionally, the standard deviation of return provided by SPP was slightly larger than the standard deviations for the other methods when the value of the risk aversion parameter was 0.01 (Figure 1c). If high returns are strongly preferred (*i.e.,* α is nearly 0), all the money will be invested in one asset by SPP; this investment strategy did not work well. The number of short sales made by LC($|K|$) increased with the number $|K|$ of input time periods (Figure 1d).

Figure 2 shows the out-of-sample investment performance for the 6SiMa dataset. In this case, LC(2) achieved the highest returns among methods when the value of the risk aversion parameter was 0.75 and 0.99 (Figures 2a and 2b). As in the case shown in Figure 1, SPP provided very low returns when the value of the risk aversion parameter was 0.01. The standard deviation of return when using EWP was relatively high, and more importantly, that provided by SPP was the largest when the value of the risk aversion parameter was 0.01 (Figure 2c). Short sales were made less frequently by the linear control policies in Figure 2d than in Figure 1d.

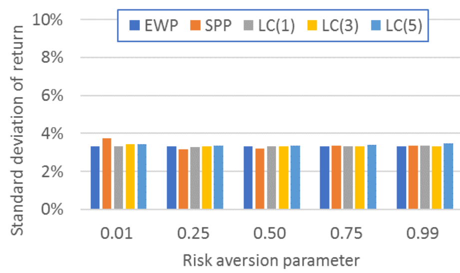
Figure 3 shows the out-of-sample investment performance for the 10Ind dataset. Note that if the number of assets is large, linear control policies are likely to be overfit to the training datasets. Thus, the penalty terms (32), which can prevent overfitting, are expected to enhance the out-of sample investment performance. Indeed, LC-W(5) produced higher returns than LC(5) did except when the value of the risk aversion parameter was 0.01, whereas LC(5) always generated lower returns than EWP did (Figures 3a and 3b). Moreover, LC-W(5) always generated higher returns than SPP did, and LC-W(5) achieved the highest returns among methods when the value of the risk aversion parameter was 0.99. The standard deviation of return provided LC-W(5) was also smaller than that provided by LC(5) except when the value of the risk aversion parameter was 0.01 (Figure 3c). A large number of short sales were made by LC(5) (Figure 3d), which greatly increased the associated interest payment. In contrast, LC-W(5) made no short sales, and this is one of the main reasons that LC-W(5) substantially outperformed LC(5). Recall that similar penalty terms are contained in Eqs. (29) and (32), so minimizing the worst-case coherent risk is also effective in preventing short sales.



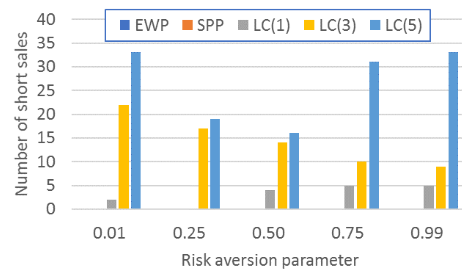
(a) Cumulative total return



(b) Average return

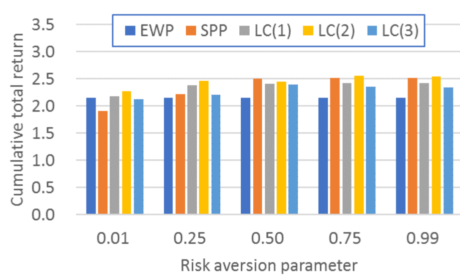


(c) Standard deviation of return

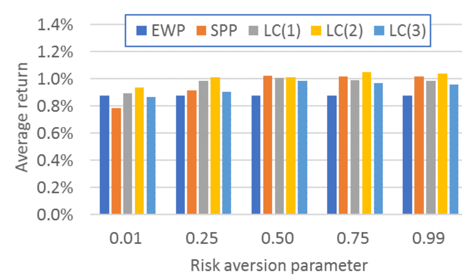


(d) Number of short sales

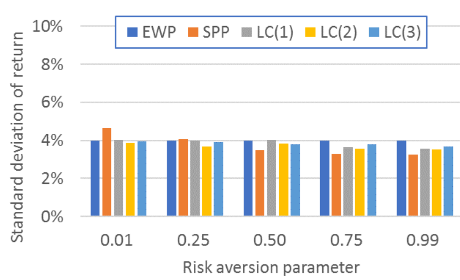
Figure 1: Out-of-sample investment performance for the 5Ind dataset



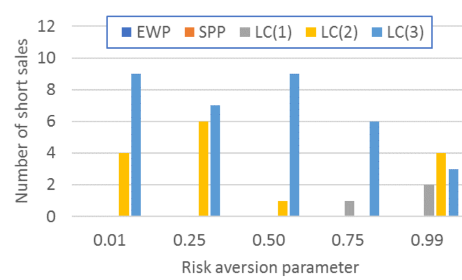
(a) Cumulative total return



(b) Average return

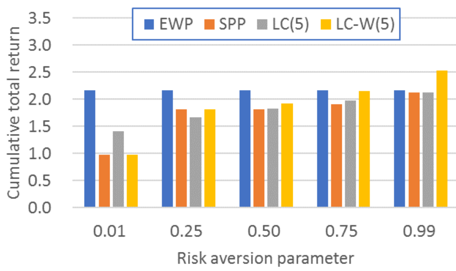


(c) Standard deviation of return

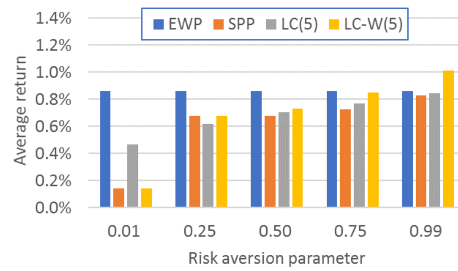


(d) Number of short sales

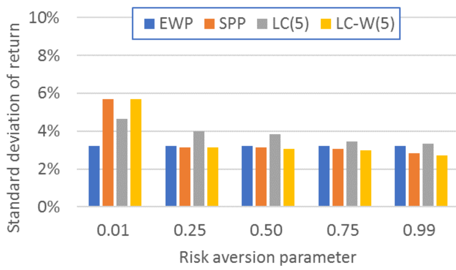
Figure 2: Out-of-sample investment performance for the 6SiMa dataset



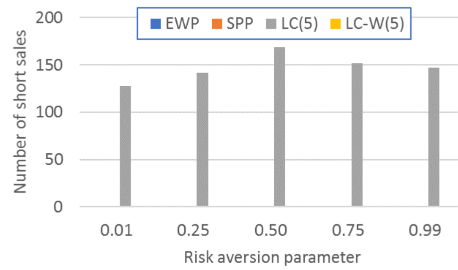
(a) Cumulative total return



(b) Average return

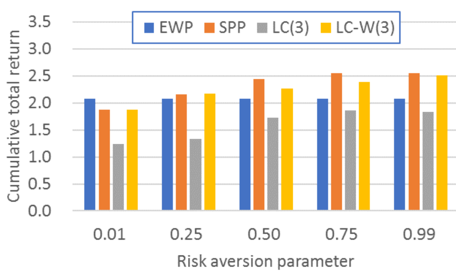


(c) Standard deviation of return

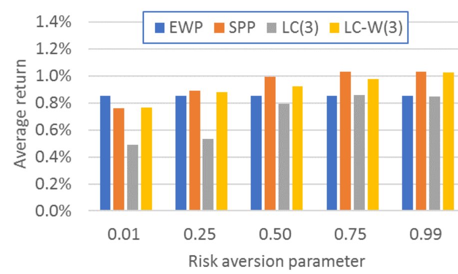


(d) Number of short sales

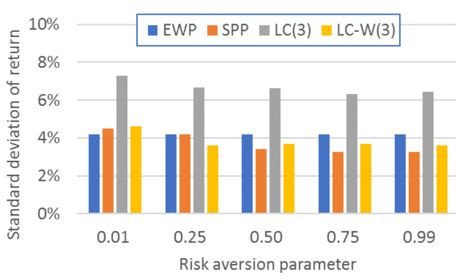
Figure 3: Out-of-sample investment performance for the 10Ind dataset



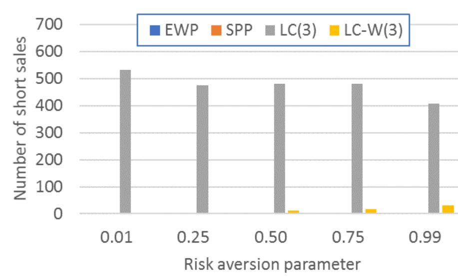
(a) Cumulative total return



(b) Average return



(c) Standard deviation of return



(d) Number of short sales

Figure 4: Out-of-sample investment performance for the 25SiMa dataset

Figure 4 shows the out-of-sample investment performance for the 25SiMa dataset. In this case, LC-W(3) always produced much higher returns than LC(3) did. LC-W(3) also generated higher returns than EWP, except when the value of the risk aversion parameter was 0.01 (Figures 4a and 4b). The performance of LC-W(3) was comparable to that of SPP, which showed the best performance in Figure 4. The standard deviation of return provided by LC-W(3) was always much smaller than that provided by LC(3) (Figure 4c). As in the case shown in Figure 3, LC(3) made a large number of short sales, whereas LC-W(3) did not (Figure 4d).

5.3. Optimal control policies

Table 3 lists the optimal control policies of LC(1) for the 5Ind dataset. Recall that assets 1 and 2 provided high average returns, and that assets 1 and 4 had low standard deviations of return (Table 1).

When the value of the risk aversion parameter was 0.01, the nominal portfolio weights were $b_1 = 0.39$ and $b_2 = 0.61$. This means that the optimal policy was to invest in high-return assets 1 and 2. In addition, the coefficients $a_{111} = -896.7$ and $a_{221} = -416.0$ imply that this policy is closely related to a contrarian investment strategy, in which one purchases assets that produced low returns in the preceding period and sells assets that produced high returns in the preceding period. This strategy is known to be effective in long-term investments.

When the value of the risk aversion parameter was 0.99, the optimal policy is to invest mostly in low-risk assets 1 and 4; this is reflected in the nominal portfolio weights $b_1 = 0.43$ and $b_4 = 0.52$. Note also that asset $j = 1$ had three positive coefficients ($a_{111} = 439.7$, $a_{211} = 361.9$, and $a_{411} = 332.8$) and two negative coefficients ($a_{311} = -99.2$ and $a_{511} = -597.0$). This implies that when many industries generate high returns (*e.g.,* during a boom economy), the weight on the high-return asset 1 should be increased.

Table 4 gives optimal control policies of LC(1) for the 6SiMa dataset. Recall that assets 2 and 3 provided high average returns, and that assets 4 and 5 had low standard deviations of return (Table 2).

Table 3: Optimal control policies for the 5Ind dataset

(a) $\alpha = 0.01$		$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
b_j		0.39	0.61	0	0	0
a_{ijk}	$i=1$	-896.7	896.7	0	0	0
$(k=1)$	$i=2$	416.0	-416.0	0	0	0
	$i=3$	148.8	-148.8	0	0	0
	$i=4$	183.0	-183.0	0	0	0
	$i=5$	-104.8	104.8	0	0	0
(b) $\alpha = 0.99$		**$j=1$**	**$j=2$**	**$j=3$**	**$j=4$**	**$j=5$**
b_j		0.43	0.05	0	0.52	0
a_{ijk}	$i=1$	439.7	-104.8	0	-334.9	0
$(k=1)$	$i=2$	361.9	48.4	0	-410.3	0
	$i=3$	-99.2	-8.5	0	107.7	0
	$i=4$	332.8	-25.6	0	-307.3	0
	$i=5$	-597.0	52.1	0	544.9	0

Table 4: Optimal control policies for the 6SiMa dataset

(a) $\alpha = 0.01$		$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
b_j		0	0.10	0.59	0.31	0	0
a_{ijk}	$i=1$	0	53.0	74.1	-127.1	0	0
$(k=1)$	$i=2$	0	236.4	-911.1	674.7	0	0
	$i=3$	0	-285.7	599.6	-313.9	0	0
	$i=4$	0	-63.2	136.2	-73.0	0	0
	$i=5$	0	-45.9	549.0	-503.1	0	0
	$i=6$	0	104.7	-96.7	-8.0	0	0
(b) $\alpha = 0.99$		**$j=1$**	**$j=2$**	**$j=3$**	**$j=4$**	**$j=5$**	**$j=6$**
b_j		0	0.24	0	0.49	0.26	0
a_{ijk}	$i=1$	0	263.8	0	-415.5	151.7	0
$(k=1)$	$i=2$	0	-834.6	0	1548.7	-714.0	0
	$i=3$	0	253.0	0	-350.5	97.5	0
	$i=4$	0	-22.7	0	232.1	-209.5	0
	$i=5$	0	426.6	0	-931.4	504.9	0
	$i=6$	0	27.9	0	-222.4	194.5	0

When the value of the risk aversion parameter was 0.01, we obtained the nominal portfolio weights $b_2 = 0.10$, $b_3 = 0.59$, and $b_4 = 0.31$. In other words, the optimal policy is to invest in a combination of high-return assets 2 and 3, and low-risk asset 4. In addition, the absolute values of the coefficients of asset $i = 2$ were relatively large ($a_{221} = 236.4$, $a_{231} = -911.1$, and $a_{241} = 674.7$), so asset 2 contributes strongly to the portfolio construction. Specifically, if asset 2 produces a high return, the portfolio weights on assets 2 and 4 will be increased, and that on asset 3 will be decreased.

When the value of the risk aversion parameter was 0.99, the nominal portfolio weights were $b_2 = 0.24$, $b_4 = 0.49$, and $b_5 = 0.26$, so the optimal policy is to invest in a combination of the high-return asset 2 and the low-risk assets 4 and 5. As in the case of $\alpha = 0.01$, asset 2 can have a strong effect on portfolio construction because the absolute values of its coefficients were relatively large ($a_{221} = -834.6$, $a_{241} = 1548.7$, and $a_{251} = -714.0$). Accordingly, when asset 2 generates a high return, the portfolio weights on assets 2 and 5 will be decreased, and that on asset 3 will be increased.

6. Conclusion

We considered the problem of developing a linear control policy for dynamic portfolio selection. We first formulated the time-series-based optimization model on the basis of coherent risk measures. We next analyzed the optimization model in its dual form and highlighted the importance of the intertemporal covariance of asset returns. We finally derived the penalty terms via robust optimization techniques so that our optimization model can perform better in out-of-sample investments.

The numerical results confirmed that our linear control policies can deliver good out-of-sample investment performance when the number of assets is small. In particular, our policies clearly outperformed the single-period portfolio selection when high returns were strongly preferred. Moreover, when the number of assets was large, the penalty terms based on robust optimization techniques substantially improved the out-of-sample investment performance of the linear control policies.

Our research opens up possibilities for dynamic portfolio selection using coherent risk measures and robust optimization techniques. Indeed, our analysis of the dual problem offers a new insight into dynamic portfolio selection. In addition, the penalty terms obtained by applying robust optimization techniques have the advantage of enhancing the out-of-sample investment performance of dynamic portfolio selection.

A future direction of study will be to combine our technique with dimensionality reduction [54], which can be helpful in removing noise in the historical data. Another direction of future research is to incorporate transaction costs and lots into our optimization model [32, 36, 39, 52, 55]. We are now working on testing the investment performance of our optimization model using additional input variables (other than past asset returns) for control policies.

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