Dynamic string-averaging CQ-methods for the split feasibility problem with percentage violation constraints arising in radiation therapy treatment planning

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Abstract

In this paper we study a feasibility-seeking problem with percentage violation constraints. These are additional constraints, that are appended to an existing family of constraints, which single out certain subsets of the existing constraints and declare that up to a specified fraction of the number of constraints in each subset is allowed to be violated by up to a specified percentage of the existing bounds. Our motivation to investigate problems with percentage violation constraints comes from the field of radiation therapy treatment planning wherein the fully-discretized inverse planning problem is formulated as a split feasibility problem and the percentage violation constraints give rise to non-convex constraints. We develop a string-averaging CQ method that uses only projections onto the individual sets which are half-spaces represented by linear inequalities. The question of extending our theoretical results to the non-convex sets case is still open. We describe how our results apply to radiation therapy treatment planning.

Keywords: String-averaging, CQ-algorithm, split feasibility, percentage violation constraints, radiation therapy treatment planning, dose-volume constraints, common fixed points, inverse problem, Landweber operator, cutter operator.

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1 Introduction

1.1 Motivation

In this work we are motivated by a linear split feasibility problem with percentage violation constraints arising in radiation therapy treatment planning. We first provide the background in general terms.

Inverse radiation therapy treatment planning (RTTP). This problem, in its fully-discretized modeling approach, leads to a linear feasibility problem. This is a system of linear interval inequalities

$$c \leq Ax \leq b, \tag{1.1}$$

wherein the "dose matrix" A is pre-calculated by techniques called in RTTP "forward calculation" or "forward planning" and the vector x is the unknown vector of "intensities" that, when used in setting up the treatment machine, will realize this specific "treatment plan". The vectors b and c contain upper and lower bounds on the total dose Ax permitted and required in volume elements (voxels) of sensitive organs/tissues and target areas, respectively, inside the irradiated body. The components of b and c are prescribed by the attending oncologist and given to the treatment planner.

Percentage violation constraints (PVCs). In general terms, these are additional constraints that are appended to an existing family of constraints. They single out certain subsets of the existing constraints and declare that up to a specified fraction of the number of constraints in each subset is allowed to be violated by up to a specified percentage of the existing bounds. Such PVCs are useful in the inverse problem of RTTP, mentioned above, where they are called "dose volume constraints" (DVCs). When the system of linear interval inequalities is inconsistent, that is, there is no solution vector that satisfies all inequalities, the DVCs allow the oncologist and the planner to relax the original constraints in a controlled manner to achieve consistency and find a solution.

Split feasibility. PVCs are, by their very nature, integer constraints which change the feasibility problem to which they are attached from being a continuous feasibility problem into becoming a mixed integer feasibility problem. An alternative to the latter is to translate the PVCs into constraints sets that are appended to the original system of linear interval inequalities but are formulated on the vectors Ax, rather than directly on x. This gives rise to a "split feasibility problem" which is split between two spaces: the space of "intensity vectors" x and the space of "dose vectors" $d := Ax$ in which A is the operator mapping one space onto the other.

Non-convexity. The constraints sets, that arise from the PVCs, in the space of "dose vectors" are non-convex sets but, due to their special form enable the calculation of orthogonal projections of points onto them. This opens the door for applying our proposed dynamic string-averaging CQ-method to the RTTP inverse problem with PVCs. Mathematical analysis for the case of non-convex sets remains an open question. Looking at it from the practical point of view one may consider also alternatives such as reformulating PVCs as ℓ_1-norm constraints. See, for example, [6, 22].

Group-structure of constraints. Each row in the system (1.1) represents a constraint on a single voxel. Lumping together constraints of voxels, according to the organ/tissue to which they belong, divides the matrix A and the whole system into "groups" of constraints, referred to below as "blocks of constraints" in a natural manner. These groups affect the formulation of the split feasibility problem at hand by demanding that the space of intensity vectors x be mapped separately by each group of rows of the matrix A into another space of dose vectors d.

1.2 Contribution

Motivated by the above we deal in this paper with the "multiple-operator split common fixed point problem" (MOSCFPP) defined next.

Problem 1 ***The multiple-operator split common fixed point problem (MOSCFPP).*** *Let $\mathcal{H}$ and $\mathcal{K}$ be two real Hilbert spaces, and let r and p be two natural numbers. Let $U_i : \mathcal{H} \to \mathcal{H}$, $1 \leq i \leq p$, and $T_j : \mathcal{K} \to \mathcal{K}$, $1 \leq j \leq r$, be given operators with nonempty fixed point sets $\operatorname{Fix}(U_i)$ and $\operatorname{Fix}(T_j)$, respectively. Further, let $A_j : \mathcal{H} \to \mathcal{K}$, for all $1 \leq j \leq r$, be given bounded linear operators. In addition let Ω be another closed and convex subset of $\mathcal{H}$. The MOSCFPP is:*

$$\text{Find an } x^* \in \Omega \text{ such that } x^* \in \mathcal{H} \text{ such that } x^* \in \cap_{i=1}^{p} \operatorname{Fix}(U_i) \text{ and,} \tag{1.2}$$

$$\text{for all } 1 \leq j \leq r,\ A_j x^* \in \operatorname{Fix}(T_j). \tag{1.3}$$

This problem formulation unifies several existing "split problems" formulations and, to the best of our knowledge, has not been formulated before. We analyze it and propose a "dynamic string-averaging CQ-method" to solve it, based on techniques used in some of those earlier formulations. We show in detail how this problem covers and applies to the linear split feasibility problem with DVCs in RTTP. Our convergence results about the dynamic string-averaging CQ-algorithm presented here rely on convexity assumptions. Therefore, there remains an open question whether our work can be expanded to cover the case of the non-convex constraints in the space of dose vectors d used in RTTP. Recent work in the field report on strides made in the field of projection methods when the underlying sets are non-convex. This encourages us to expand the results presented here in the same way.

1.3 Structure of the paper

We begin by briefly reviewing relevant "split problem" formulations which have led to our proposed MOSCFPP and a "dynamic string-averaging CQ-method" to solve it. Starting from a general formulation of two concurrent inverse problems in different vector spaces connected by a bounded linear operator, we explore the inclusion of multiple convex constraint sets within each vector space. Defining operators that act on each of these sets allows us to formulate equivalent fixed point problems, which naturally leads to our MOSCFPP. We then provide some insight into how one may solve such a problem, using constrained minimization, or successive metric projections as part of a CQ-type method [3]. These projection methods form the basis of our "dynamic string-averaging CQ-method", which is introduced in Section 4. Important mathematical foundations for this method are provided in Section 3, which serve to describe the conditions under which the method converges to a solution in Section 5. Finally, we bring percentage violation constraints (PVCs) into our problem formulation (Section 6) and consolidate our work by providing examples of how the MOSCFPP and "dynamic string-averaging CQ-method" may be applied in inverse radiation therapy treatment planning (Section 7).

2 A brief review of "split problems" formulations and solution methods

The following brief review of "split problems" formulations and solution methods will help put our work in context. The review is non-exhaustive and focuses only on split problems that led to our new formulation that appears in Problem 1. Other split problems such as "split variational inequality problems", see, e.g., [23] or "split Nash equilibrium problems for non-cooperative strategic games", see, e.g., [21] and many others are not included here. The "split inverse problem" (SIP), which was introduced in [14] (see also [5]), is formulated as follows.

Problem 2 ***The split inverse problem (SIP)***. *Given are two vector spaces X and Y and a bounded linear operator $A : X \to Y$. In addition, two inverse problems are involved. The first one, denoted by IP_1, is formulated in the space X and the second one, denoted by IP_2, is formulated in the space Y. The SIP is:*

$$\text{Find an } x^* \in X \text{ that solves } IP_1 \text{ such that } y^* := Ax^* \in Y \text{ solves } IP_2. \tag{2.1}$$

The first published instance of a SIP is the "split convex feasibility problem" (SCFP) of Censor and Elfving [11], formulated as follows.

Problem 3 ***The split convex feasibility problem (SCFP)****. Let $\mathcal{H}$ and $\mathcal{K}$ be two real Hilbert spaces. Given are nonempty, closed and convex sets $C \subseteq \mathcal{H}$ and $Q \subseteq \mathcal{K}$ and a bounded linear operator $A : \mathcal{H} \to \mathcal{K}$. The SCFP is:*

$$\textit{Find an } \ x^* \in C \textit{ such that } Ax^* \in Q. \tag{2.2}$$

This problem was employed, among others, for solving an inverse problem in intensity-modulated radiation therapy (IMRT) treatment planning, see [10]. More results regarding the SCFP theory and algorithms, can be found, for example, in [32, 24, 17], and the references therein. The SCFP was extended in many directions to Hilbert and Banach spaces formulations. It was extended also to the following "multiple sets split convex feasibility problem" (MSSCFP).

Problem 4 ***The multiple sets split convex feasibility problem (MSSCFP)****. Let $\mathcal{H}$ and $\mathcal{K}$ be two real Hilbert spaces and r and p be two natural numbers. Given are sets C_i, $1 \leq i \leq p$ and Q_j, $1 \leq j \leq r$, that are closed and convex subsets of $\mathcal{H}$ and $\mathcal{K}$, respectively, and a bounded linear operator $A : \mathcal{H} \to \mathcal{K}$. The MSSCFP is:*

$$\textit{Find an } x^* \in \cap_{i=1}^{p} C_i \textit{ such that } Ax^* \in \cap_{j=1}^{r} Q_j. \tag{2.3}$$

Masad and Reich [25] proposed the "(constrained) multiple set split convex feasibility problem" (MSCFP) which is phrased as follows (see also [20]).

Problem 5 ***The (constrained) multiple set split convex feasibility problem (MSCFP)****. Let $\mathcal{H}$ and $\mathcal{K}$ be two real Hilbert spaces and r and p be two natural numbers. Given are sets C_i, $1 \leq i \leq p$ and Q_j, $1 \leq j \leq r$, that are closed and convex subsets of $\mathcal{H}$ and $\mathcal{K}$, respectively, and for $1 \leq j \leq r$, given bounded linear operators $A_j : \mathcal{H} \to \mathcal{K}$. In addition let Ω be another closed and convex subset of $\mathcal{H}$. The MSCFP is:*

$$\textit{Find an } x^* \in \Omega \textit{ such that } x^* \in \cap_{i=1}^{p} C_i \textit{ and } A_j x^* \in Q_j, \textit{ for } 1 \leq j \leq r. \tag{2.4}$$

Another extension, due to Censor and Segal [15], is the following "split common fixed points problem" (SCFPP).

Problem 6 ***The split common fixed points problem (SCFPP)****. Let $\mathcal{H}$ and $\mathcal{K}$ be two real Hilbert spaces and r and p be two natural numbers. Given are operators $U_i : \mathcal{H} \to \mathcal{H}$, $1 \leq i \leq p$, and $T_j : \mathcal{K} \to \mathcal{K}$, $1 \leq j \leq r$, with nonempty fixed point sets $\operatorname{Fix}(U_i)$, $1 \leq i \leq p$ and $\operatorname{Fix}(T_j)$, $1 \leq j \leq r$, respectively, and a bounded linear operator $A : \mathcal{H} \to \mathcal{K}$. The SCFPP is:*

$$\textit{Find an } x^* \in \cap_{i=1}^{p} \operatorname{Fix}(U_i) \textit{ such that } Ax^* \in \cap_{j=1}^{r} \operatorname{Fix}(T_j). \tag{2.5}$$

Problems 3–6 are SIPs but, more importantly, they are special cases of our MOSCFPP of Problem 1.

Focusing in a telegraphic manner on algorithms for solving some of the above SIPs, we observe that the SCFP of Problem 3 can be reformulated as the constrained minimization problem:

$$\min_{x\in C}\frac{1}{2}\|P_Q(Ax)-Ax\|^2, \tag{2.6}$$

where P_Q is the orthogonal (metric) projection onto Q. Since the objective function is convex and continuously differentiable with Lipschitz continuous gradients, one can apply the projected gradient method, see, e.g., Goldstein [19], and obtain Byrne's well-known *CQ-algorithm [3]*. The iterative step of the CQ-algorithm has the following structure:

$$x^{k+1}=P_C(x^k-\gamma A^\star(I-P_Q)Ax^k), \tag{2.7}$$

where $A^\star$ stands for the adjoint ($A^\star$=A^T transpose in Euclidean spaces) of A, γ is some positive number, I is the identity operator, and P_C and P_Q are the orthogonal projections onto C and Q, respectively. For the MSSCFP of Problem 4, the minimization model considered in [12], is

$$\min_{x\in\mathbb{R}^M}\left(\sum_{i=1}^{p}\text{dist}^2(x,C)+\sum_{j=1}^{r}\text{dist}^2(Ax,Q)\right), \tag{2.8}$$

leading, for example, to a gradient descent method which has an iterative simultaneous projections nature:

$$x^{k+1}=x^k-\gamma\sum_{i=1}^{p}\alpha_i\left(I-P_{C_i}\right)x^k+\sum_{j=1}^{r}\beta_jA^\star\left(I-P_{Q_j}\right)Ax^k, \tag{2.9}$$

where $\gamma\in\left(0,\frac{2}{L}\right)$ with

$$L:=\sum_{i=1}^{p}\alpha_i+\sum_{j=1}^{r}\beta_j\|A\|_F^2 \tag{2.10}$$

where $\|A\|_F^2$ is the squared Frobenius norm of A.

Inspired by the above and the work presented in [29], we propose in the sequel a "dynamic string-averaging CQ-method" for solving the MOSCFPP of Problem 1.

3 Preliminaries

Through this paper $\mathcal{H}$ and $\mathcal{K}$ are two real Hilbert spaces and let $D\subset\mathcal{H}$. For every point $x\in\mathcal{H}$, there exists a unique nearest point in D, denoted by $P_D(x)$ such that

$$\|x-P_D(x)\|\leq\|x-y\|,\ \text{for all}\ y\in D. \tag{3.1}$$

The operator $P_D : \mathcal{H} \to \mathcal{H}$ is called the *metric projection* onto D.

Definition 7 *Let $T : \mathcal{H} \to \mathcal{H}$ be an operator and $D \subset \mathcal{H}$.*

(i) The operator T is called ***Lipschitz continuous*** *on D with constant $L > 0$ if*

$$\|T(x) - T(y)\| \leq L\|x - y\|, \text{ for all } x, y \in D. \tag{3.2}$$

(ii) The operator T is called ***nonexpansive*** *on D if it is 1-Lipschitz continuous.*

(iii) The ***Fixed Point set*** *of T is*

$$\mathrm{Fix}(T) := \{x \in \mathcal{H} \mid T(x) = x\}. \tag{3.3}$$

*(iv) The operator T is called c-**averaged** (c-av) [1] if there exists a nonexpansive operator $N : D \to \mathcal{H}$ and a number $c \in (0, 1)$ such that*

$$T = (1 - c)I + cN. \tag{3.4}$$

In this case we also say that T is c-av [4]. If two operators T_1 and T_2 are c_1-av and c_2-av, respectively, then their composition $S = T_1T_2$ is $(c_1 + c_2 - c_1c_2)$-av. See [4, Lemma 2.2].

*(v) The operator T is called ν-**inverse strongly monotone** (ν-ism) on D if there exists a number $\nu > 0$ such that*

$$\langle T(x) - T(y), x - y\rangle \geq \nu\|T(x) - T(y)\|^2, \text{ for all } x, y \in D. \tag{3.5}$$

(vi) The operator T is called ***firmly nonexpansive*** *(FNE) on D if*

$$\langle T(x) - T(y), x - y\rangle \geq \|T(x) - T(y)\|^2, \text{ for all } x, y \in D, \tag{3.6}$$

A useful fact is that T is firmly nonexpansive if and only if its complement $I - T$ is firmly nonexpansive. Moreover, T is firmly nonexpansive if and only if T is $(1/2)$-av (see [18, Proposition 11.2] and [4, Lemma 2.3]). In addition, T is averaged if and only if its complement $I - T$ is ν-ism for some $\nu > 1/2$; see, e.g., [4, Lemma 2.1].

(vii) The operator T is called ***quasi-nonexpansive*** *(QNE)*

$$\|T(x) - w\| \leq \|x - w\| \text{ for all } (x, w) \in \mathcal{H} \times \mathrm{Fix}(T) \tag{3.7}$$

(viii) The operator T is called is called a ***cutter*** *(also* ***firmly quasi- nonexpansive****) ($T \in \mathfrak{T}$) if $\mathrm{Fix}(T) \neq \emptyset$ and*

$$\langle T(x) - x, T(x) - w\rangle \leq 0 \text{ for all } (x, w) \in \mathcal{H} \times \mathrm{Fix}(T). \tag{3.8}$$

(ix) Let $\lambda \in [0,2]$, the operator $T_\lambda := (1-\lambda)I + \lambda T$ is called λ-`relaxation` *of the operator T. With respect to cutters above it is known that for $\lambda \in [0,1]$, the λ-relaxation of a cutter is also a cutter, see, e.g., [7, Remark 2.1.32].*

(x) The operator T is called is called ρ-`strongly quasi-nonexpansive` *(ρ-SQNE), where $\rho \geq 0$, if $\operatorname{Fix}(T) \neq \emptyset$ and*

$$\|T(x) - w\| \leq \|x - w\| - \rho\|T(x) - x\|, \text{ for all } (x, w) \in \mathcal{H} \times \operatorname{Fix}(T). \tag{3.9}$$

A useful fact is that a family of SQNE operators with non-empty intersection of fixed point sets is closed under composition and convex combination, see, e.g., [7, Corollary 2.1.47].

(xi) The operator T is called is called `demiclosed` *at $y \in \mathcal{H}$ if for any sequence $\{x^k\}_{k=0}^\infty$ in D such that $x^k \to \overline{x} \in D$ and $T(x^k) \to y$, we have $T(\overline{x}) = y$.*

Next we recall the well-known *Demiclosedness Principle* [2].

Lemma 8 *Let $\mathcal{H}$ be a Hilbert space, D a closed and convex subset of $\mathcal{H}$, and $N : D \to \mathcal{H}$ a nonexpansive operator. Then $I - N$ (I is the identity operator on $\mathcal{H}$) is* `demiclosed` *at $y \in \mathcal{H}$.*

Let $A : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator with $\|A\| > 0$, and $C \subseteq \mathcal{H}$ and $Q \subseteq \mathcal{K}$ are nonempty, closed and convex sets.

The operator $V : \mathcal{H} \to \mathcal{H}$ which is defined by

$$V := I - \frac{1}{\|A\|^2} A^\star (I - P_Q) A \tag{3.10}$$

is called a *Landweber operator* and $U : \mathcal{H} \to \mathcal{H}$ defined by

$$U := P_C \left(I - \frac{1}{\|A\|^2} A^\star (I - P_Q) A \right) \tag{3.11}$$

is called a *projected Landweber operator*. See, e.g., [7, 8, 9].

In the general case where $T : \mathcal{H} \to \mathcal{H}$ is quasi-nonexpansive and $A : \mathcal{H} \to \mathcal{K}$ is a bounded and linear operator with $\|A\| > 0$, an operator

$$V := I - \frac{1}{\|A\|^2} A^\star (I - T) A \tag{3.12}$$

is called *Landweber-type operator*, see e.g., [8].

4 The dynamic string-averaging CQ-method

In this section we present our "dynamic string-averaging CQ-method" for solving the MOSCFPP of Problem 1. It is actually an algorithmic scheme which encompasses many specific algorithms that are obtained from it by different choices of strings and weights. First, for all $j = 1, 2, \dots, r$, construct from the given data of Problem 1, the operators $V_j : \mathcal{H} \to \mathcal{H}$ defined by

$$V_j := I - \gamma_j A_j^\star (I - T_j) A_j, \tag{4.1}$$

where $\gamma_j \in \left(0, \frac{1}{L_j}\right)$, $L_j = \|A_j\|^2$. For quasi-nonexpansive T_j this definition coincides with that of "Landweber-type operators related to T_j" of [8, Definition 2] with a relaxation of γ_j.

For simplicity, and without loss of generality, we assume that $r = p$ in Problem 1. This is not restrictive since if $r < p$ we will define $T_j := I$ for $r + 1 \le j \le p$, and if $p < r$ we will define $U_i := I$ for $p + 1 \le i \le r$, which, in both cases, will not make any difference to the formulation of Problem 1.

Define $L := \{1, 2, \dots, p\}$ and for each $i \in L$ define the operator $R_i : \mathcal{H} \to \mathcal{H}$ by $R_i := U_i V_i$. An *index vector* is a vector $t = (t_1, t_2, \dots, t_q)$ such that $t_i \in L$ for all $i = 1, 2, \dots, q$. For a given index vector $t = (t_1, t_2, \dots, t_q)$ we denote its *length* by $\ell(t) := q$, and define the operator $Z[t]$ as the product of the individual operators R_i whose indices appear in the index vector t, namely,

$$Z[t] := R_{t_{\ell(t)}} R_{t_{\ell(t)-1}} \cdots R_{t_1}, \tag{4.2}$$

and call it a *string operator*. A finite set Θ of index vectors is called *fit* if for each $i \in L$, there exists a vector $t = (t_1, t_2, \dots, t_q) \in \Theta$ such that $t_s = i$ for some $s \in L$.

Denote by $\mathcal{M}$ the collection of all pairs (Θ, w), where Θ is a fit finite set of index vectors and

$$w : \Theta \to (0, \infty) \text{ is such that } \sum_{t \in \Theta} w(t) = 1. \tag{4.3}$$

For any $(\Theta, w) \in \mathcal{M}$ define the convex combination of the end-points of all strings defined by members of Θ by

$$\Gamma_{\Theta,w}(x) := \sum_{t \in \Theta} w(t) Z[t](x), \; x \in \mathcal{H}. \tag{4.4}$$

We fix a number $\Delta \in (0, 1/p)$ and an integer $\bar{q} \ge p$ and denote by $\mathcal{M}_* \equiv \mathcal{M}_*(\Delta, \bar{q})$ the set of all $(\Theta, w) \in \mathcal{M}$ such that the lengths of the strings are bounded and the weights are all bounded away from zero, namely,

$$\mathcal{M}_* := \{(\Theta, w) \in \mathcal{M} \mid \ell(t) \le \bar{q} \text{ and } w(t) \ge \Delta \text{ for all } t \in \Theta\}. \tag{4.5}$$

The dynamic string-averaging CQ-method with variable strings and variable weights is described by the following iterative process.

Algorithm 9 ***The dynamic string-averaging CQ-method with variable strings and variable weights***

Initialization: *Select an arbitrary* $x^0 \in \mathcal{H}$*,*

Iterative step: *Given a current iteration vector* x^k *pick a pair* $(\Theta_k, w_k) \in \mathcal{M}_*$ *and calculate the next iteration vector by*

$$x^{k+1} = \Gamma_{\Theta_k, w_k}(x^k). \tag{4.6}$$

The iterative step of (4.6) amounts to calculating, for all $t \in \Theta_k$, the strings' end-points

$$Z[t](x^k) = R_{i^t_{\ell(t)}} \cdots R_{i^t_2} R_{i^t_1}(x^k), \tag{4.7}$$

and then calculating

$$x^{k+1} = \sum_{t \in \Theta_k} w_k(t) Z[t](x^k). \tag{4.8}$$

This algorithmic scheme applies to x^k successively the operators $R_i := U_i V_i$ whose indices belong to the string t. This can be done in parallel for all strings and then the end-points of all strings are convexly combined, with weights that may vary from iteration to iteration, to form the next iterate x^{k+1}. This is indeed an algorithm provided that the operators $\{R_i\}_{i=1}^p$ all have algorithmic implementations. In this framework we get a *sequential algorithm* by allowing a single string created by the index vector $t = L$ and a *simultaneous algorithm* by the choice of p different strings of length one each containing one element of L. Intermediate structures are possible by judicious choices of strings and weights.

5 Convergence

Next we prove the equivalence between Problem 1 and a common fixed point problem which is not split, give a description of $\mathrm{Fix}(V_j)$, and state a property of V_j.

Lemma 10 *Denote the solution set of Problem 1 by* Ω *and assume that it is nonempty. Then, for* V_j *as in (4.1),*

(i) $x^* \in \Omega$ *if and only if* x^* *solves the common fixed point problem:*

$$\textit{Find } x^* \in \left(\cap_{i=1}^p Fix(U_i)\right) \cap \left(\cap_{j=1}^r Fix(V_j)\right), \tag{5.1}$$

(ii) for all $j = 1, 2, \ldots, r$:

$$Fix\ (V_j) = \{x \in \mathcal{H} \mid A_j x \in Fix\ (T_j)\} = A_j^{-1}(Fix\ (T_j)), \tag{5.2}$$

(iii) if, in addition, all operators T_j are cutters then all V_j are cutters (i.e., are 1-SQNE),

(iv) if T_j is ρ-SQNE, $A_j \cap FixT_j \neq \emptyset$ and satisfies the demi-closedness principle then V_j also satisfies the demi-closedness principle.

Proof. (i) We need to show only that

$$x^* \in \cap_{j=1}^r \mathrm{Fix}(V_j) \Leftrightarrow A_j x^* \in \mathrm{Fix}\ (T_j) \text{ for all } j = 1, 2, \ldots, r. \tag{5.3}$$

Indeed, for any $j = 1, 2, \ldots, r$,

$$\begin{aligned} A_j x^* \in \mathrm{Fix}\ (T_j) &\Leftrightarrow A_j x^* - T_j A_j x^* = 0 \\ \Leftrightarrow A_j^T(I - T_j)A_j x^* = A_j^T 0 &\Leftrightarrow -\gamma_j A_j^T(I - T_j)A_j x^* = 0 \\ \Leftrightarrow x^* - \gamma_j A_j^T(I - T_j)A_j x^* = x^* &\Leftrightarrow x^* \in \mathrm{Fix}(V_j). \end{aligned} \tag{5.4}$$

(ii) Follows from (5.4).

(iii) To show that V_j is a cutter take $w \in \mathrm{Fix}(V_j)$, $\gamma_j \in \left(0, \frac{1}{L_j}\right)$ and $\xi \in \mathcal{H}$.

$$\begin{aligned} &\frac{1}{\gamma_j}\langle w - V_j(\xi), \xi - V_j(\xi)\rangle \\ &= \left\langle w - \xi - \gamma_j A_j^T(T_j - I)A_j\xi, A_j^T(I - T_j)A_j\xi\right\rangle \\ &= \left\langle w - \xi, A_j^T(I - T_j)A_j\xi\right\rangle + \gamma_j\|A_j^T(I - T_j)A_j\xi\|^2 \\ &= \langle A_j w - A_j\xi, (I - T_j)A_j\xi\rangle + \gamma_j\|A_j^T(I - T_j)A_j\xi\|^2 \\ &= \langle A_j w - T_j(A_j\xi), (I - T_j)A_j\xi\rangle + \gamma_j\|A_j^T(I - T_j)A_j\xi\|^2 \\ &\quad - \|(I - T_j)A_j\xi\|^2. \end{aligned} \tag{5.5}$$

Since T_j is a cutter and $A_j w \in \mathrm{Fix}\ (T_j)$, we have

$$\langle A_j w - T_j(A_j\xi), (I - T_j)A_j\xi\rangle \leq 0. \tag{5.6}$$

Also,

$$\gamma_j\|A_j^T(I - T_j)A_j\xi\|^2 \leq \gamma_j\|A_j\|^2\|(I - T_j)A_j\xi\|^2 \leq \|(I - T_j)A_j\xi\|^2, \tag{5.7}$$

for all $\gamma_j \in (0, 1/L_j)$. Using the above we get that

$$\langle w - V_j(\xi), \xi - V_j(\xi)\rangle \leq 0. \tag{5.8}$$

which proves that V_j is a cutter.

(iv) Proved in [8, Theorem 8(iv)]. ■

The special case where in Problem 1 there is only one operator $A : \mathcal{H} \to \mathcal{K}$ and (1.3) in Problem 1 is replaced by

$$\text{for all } 1 \le j \le r, \; Ax^* \in \text{Fix } (T_j) \tag{5.9}$$

which amounts to $Ax^* \in \cap_{j=1}^r \text{Fix } (T_j)$ was treated in the literature, see, e.g., Cegielski's papers [8, 9] and Wang and Xu [31]. The extensions to our more general case, necessitated by the application to RTTP at hand, follow the patterns in those earlier papers. Our convergence theorem for the dynamic string-averaging CQ-method now follows. We rely on the convergence result [30, Theorem 4.1] who, motivated by [13, Algorithm 3.3], invented and investigated the "modular string averaging (MSA) method" [30, Procedure 1.1].

Theorem 11 *Let $p \ge 1$ be an integer and suppose that Problem 1 with $r = p$ has a nonempty solution set Ω. Let $\{U_i\}_{i=1}^p$ and $\{T_i\}_{i=1}^p$ be cutters on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Further assume that $U_i - I$ and $T_i - I$ are demiclosed at zero for all i. Then any sequence $\{x^k\}_{k=0}^\infty$, generated by Algorithm 9 with $R_i := U_i V_i$ for all i, converges weakly to a point $x^* \in \Omega$.*

Proof. To apply the convergence result [30, Theorem 4.1] we show that our dynamic string-averaging CQ-method fits into the MSA [30, Procedure 1.1] and that the assumptions of [30, Theorem 4.1] hold. Indeed, we identify our Γ_{Θ_k, w_k} from (4.6) with the right-hand side of Equation (1.12) of [30] (being careful with regard to the duplicity of symbols that represent different things in [30] and here). This means that Algorithm 9 can be represented by the iterative process of Equation (1.2) (or Equation (4.2)) of [30].

Next we inspect the validity of the assumptions needed by [30, Theorem 4.1].

Assumption (i): The operators $\{U_i\}_{i=1}^M$ of [30, Theorem 4.1] are our $R_i := U_i V_i$. Although our R_i are not necessarily cutters, the arguments in the proof of [30, Theorem 4.1] are based on the strongly quasi-nonexpansiveness of the operators T_k there (our Γ_{Θ_k, w_k}) and by (iii) of Lemma 10 above, our operators $\{V_i\}_{i=1}^p$ (defined in (4.1)) are cutters and this together with the assumption on our $\{U_i\}_{i=1}^p$ and $\{T_i\}_{i=1}^p$, yields that the composition operators $R_i := U_i V_i$ are ρ-SQNE for all i and, thus, so are also our Γ_{Θ_k, w_k}.

Assumption (ii)+(iii): Since the construction of the operators Γ_{Θ_k, w_k} is based on $\mathcal{M}_*$ ((4.5)) which includes a fit Θ, it guarantees that every index $i \in L$ appears in the construction of Γ_{Θ_k, w_k} for all $k > 0$, thus Assumption (ii) in [30, Theorem 4.1] holds. Following the same reasoning, it is clear that the number of steps N_k, defined in the MSA [30, Procedure 1.1]), is bounded.

The weak convergence part of the proof of [30, Theorem 4.1] requires that all (their) $\{U_i\}_{i=1}^M$ satisfy Opial's demiclosedness principle (i.e., that $U_i - I$ are demiclosed at zero). In our case, we assume that $U_i - I$ and $T_i - I$ are demiclosed at zero for all i. By (iv) of Lemma 10 above $V_i - I$ are also demiclosed at zero. So, we identify $\{U_i\}_{i=1}^M$ of [30] with our U_is and V_is and construct first the operators $R_i = U_i V_i$, and then use them as the building bricks of the operators Γ_{Θ_k, w_k}.

Thus, the desired result is obtained. ■

Remark 12 *(i) If one assumes that the T_j operators are firmly nonexpansive, then similar arguments as in the proof of [28, Theorem 3.1] show that the V_j operators are also averaged and then [30, Theorem 4.1] can be adjusted to hold for averaged operators.*

(ii) It is possible to propose inexact versions of Algorithm 9 following [30, Theorem 4.5] and Combettes' "almost cyclic sequential algorithm (ACA)" [16, Algorithm 6.1].

(iii) Our work can be extended to cover also underrelaxed operators, i.e., by defining $R_i := (U_i)_\lambda (V_i)_\delta$ for $\lambda, \delta \in [0, 1]$. This is allowed due the fact that if an operator is firmly quasi-nonexpansive, then so is its relaxation.

(iv) [30, Theorem 4.1] also includes a strong convergence part under some additional assumptions on their operators $\{U_i\}_{i=1}^M$. It is possible to adjust this theorem for our case as well.

6 Percentage violation constraints (PVCs) arising in radiation therapy treatment planning

6.1 Transforming problems with a PVC

Given p closed convex subsets $\Omega_1, \Omega_2, \cdots, \Omega_p \subseteq \mathbb{R}^n$ of the n-dimensional Euclidean space $\mathbb{R}^n$, expressed as level sets

$$\Omega_j = \{x \in \mathbb{R}^n \mid f_j(x) \leq v_j\}, \text{ for all } j \in J := \{1, 2, \ldots, p\}, \tag{6.1}$$

where $f_j : \mathbb{R}^n \to \mathbb{R}$ are convex functions and v_j are some given real numbers, the convex feasibility problem (CFP) is to find a point $x^* \in \Omega := \cap_{j \in J} \Omega_j$. If $\Omega = \emptyset$ where $\emptyset$ is the empty set then the CFP is said to be inconsistent.

Problem 13 ***Convex feasibility problem (CFP) with a percentage-violation constraint (PVC) (CFP+PVC).*** *Consider p closed convex subsets $\Omega_1, \Omega_2, \cdots, \Omega_p \subseteq \mathbb{R}^n$ of the n-dimensional Euclidean space $\mathbb{R}^n$, expressed as level sets*

$$\Omega_j = \{x \in \mathbb{R}^n \mid f_j(x) \leq v_j\}, \textit{ for all } j \in J, \tag{6.2}$$

where $f_j : \mathbb{R}^n \to \mathbb{R}$ are convex functions and v_j are some given real numbers. Let $0 \leq \alpha \leq 1$ and $0 < \beta < 1$ be two given real numbers. The CFP+PVC is:

Find an $x^* \in \mathbb{R}^n$ such that $x^* \in \cap_{j=1}^{p} \Omega_j$ and in up to a fraction α (i.e., $100\alpha\%$) of the total number of inequalities in (6.2) the bounds v_j may be potentially violated by up to a fraction β (i.e., $100\beta\%$) of their values.

A PVC is an integer constraint by its nature. It changes the CFP (6.2) to which it is attached from being a continuous feasibility problem into becoming a mixed integer feasibility problem. Denoting the inner product of two vectors in $\mathbb{R}^n$ by $\langle a, b\rangle := \sum_{i=1}^{n} a_i b_i$, the linear feasibility problem (LFP) with PVC (LFP+PVC) is the following special case of Problem 13.

Problem 14 ***Linear feasibility problem (LFP) with a percentage-violation constraint (PVC) (LFP+PVC).*** *Consider p closed convex subsets $\Omega_1, \Omega_2, \cdots, \Omega_p \subseteq \mathbb{R}^n$ of the n-dimensional Euclidean space $\mathbb{R}^n$, expressed as level sets*

$$\Omega_j = \left\{ x \in \mathbb{R}^n \mid \left\langle a^j, x \right\rangle \le b_j \right\}, \text{ for all } j \in J, \tag{6.3}$$

for a set of given vectors $a^j \in \mathbb{R}^n$ and b_j some given real numbers. Let $0 \le \alpha \le 1$ and $0 < \beta < 1$ be two given real numbers. The LFP+PVC is:

Find an $x^* \in \mathbb{R}^n$ such that $\left\langle a^j, x^* \right\rangle \le b_j$, for all $j \in J$, and in up to a fraction α of the total number of inequalities in (6.3) the bounds b_j may be potentially violated by up to a fraction β of their values.

Our tool to "translate" the mixed integer LFP+PVC into a "continuous" one is the notion of sparsity norm, called elsewhere the zero-norm, of a vector $x \in \mathbb{R}^n$ which counts the number of nonzero entries of x, that is,

$$\|x\|_0 := |\{x_i \mid x_i \neq 0\}|, \tag{6.4}$$

where $|\cdot|$ denotes the cardinality, i.e., the number of elements, of a set. This notion has been recently used for various purposes in compressed sensing, machine learning and more. The rectifier (or "positive ramp operation") on a vector $x \in \mathbb{R}^n$ means that, for all $i = 1, 2, \ldots, n$,

$$(x_+)_i := \max(0, x_i) = \begin{cases} x_i, & \text{if } x_i > 0, \\ 0, & \text{if } x_i \le 0. \end{cases} \tag{6.5}$$

Obviously, x_+ is always a component-wise nonnegative vector. Hence, $\|x_+\|_0$ counts the number of positive entries of x and is defined by

$$\|x_+\|_0 := |\{x_i \mid x_i > 0\}|. \tag{6.6}$$

We translate the LFP+PVC to the following.

Problem 15 ***Translated problem of LFP+PVC (for LFP with upper bounds).*** *For the data of Problem 14 the translated problem of LFP+PVC (for LFP with upper* *bounds) is:*

$$\textit{Find an } x^* \in \mathbb{R}^n \textit{ such that } \left\langle a^j, x^* \right\rangle \leq (1+\beta)b_j, \tag{6.7}$$
$$\textit{for all } j \in J, \textit{ and } \|(Ax^*-b)_+\|_0 \leq \alpha p. \tag{6.8}$$

The number of the violations in (6.7) is $\|(Ax^*-b)_+\|_0$ and $\|(Ax^*-b)_+\|_0 \leq \alpha p$ guarantees that the number of violations of up to β in the original row inequalities remains at bay as demanded. This is a split feasibility problem between the space $\mathbb{R}^n$ and the space $\mathbb{R}^m$ with the matrix A mapping the first to the latter. The constraints in $\mathbb{R}^n$ are linear (thus convex) but the constraint

$$x^* \in S := \{y \in \mathbb{R}^m \mid \|(y-b)_+\|_0 \leq \alpha p\} \tag{6.9}$$

is not convex. This makes Problem 15 similar in structure, but not identical with, Problem 3.

Similarly, if the linear inequalities in Problem 6.3 are in an opposite direction, i.e., of the form $c_j \leq \left\langle a^j, x \right\rangle$, for all $j \in J$, then the translated problem of LFP+PVC will be as follows.

Problem 16 ***Translated problem of LFP+PVC (for LFP with lower bounds).*** *For the data of Problem 14 the translated problem of LFP+PVC (for LFP with lower bounds) is:*

$$\textit{Find an } x^* \in \mathbb{R}^n \textit{ such that } (1-\beta)c_j \leq \left\langle a^j, x^* \right\rangle, \tag{6.10}$$
$$\textit{for all } j \in J, \textit{ and } \|(c-Ax^*)_+\|_0 \leq \alpha p. \tag{6.11}$$

This is also a split feasibility problem between the space $\mathbb{R}^n$ and the space $\mathbb{R}^m$ with the matrix A mapping the first to the latter. The constraints in $\mathbb{R}^n$ are linear (thus convex) but the constraint

$$x^* \in T := \{y \in \mathbb{R}^m \mid \|(c-y)_+\|_0 \leq \alpha m\} \tag{6.12}$$

is again not convex.

6.2 Translated block LFP+PVC

Consider an $m \times n$ matrix A divided into blocks A_ℓ, for $\ell = 1, 2, \ldots, L$, with each block forming an $m_\ell \times n$ matrix and $\sum_{\ell=1}^{L} m_\ell = m$. Further, the blocks are assumed to give rise to block-wise LFPs of the two kinds; those with upper bounds, say for $\ell = 1, 2, \ldots, p$, and those with lower bounds, say for $\ell = p+1, p+2, \ldots, p+r$. PVCs are imposed on each block separately with parameters α_ℓ and β_ℓ, respectively, for all $\ell = 1, 2, \ldots, L$. The original block-LFP prior to imposing the PVCs is:

$$\begin{array}{ll} A_\ell x \leq b^\ell, & \text{for all } \ell = 1, 2, \ldots, p, \\ c^\ell \leq A_\ell x, & \text{for all } \ell = p+1, p+2, \ldots, p+r. \end{array} \tag{6.13}$$

After imposing the PVCs and translating the systems according to the principles of Problems 15 and 16 we obtain the translated problem of LFP+PVC for blocks.

Problem 17 ***Translated problem of LFP+PVC for blocks.*** *Find an $x^* \in \mathbb{R}^n$ such that*

$$\begin{array}{rl} A_\ell x^* \leq (1+\beta_\ell) b^\ell, & \textit{for all } \ell = 1, 2, \ldots, p, \\ (1-\beta_\ell) c^\ell \leq A_\ell x^*, & \textit{for all } \ell = p+1, p+2, \ldots, p+r, \\ \|(A_\ell x^* - b^\ell)_+\|_0 \leq \alpha_\ell m_\ell, & \textit{for all } \ell = 1, 2, \ldots, p, \\ \|(c^\ell - A_\ell x^*)_+\|_0 \leq \alpha_\ell m_\ell, & \textit{for all } \ell = p+1, p+2, \ldots, p+r. \end{array} \tag{6.14}$$

This is a split feasibility problem between the space $\mathbb{R}^n$ and the space $\mathbb{R}^m$ but with a structure similar to Problem 5 where, for $\ell = 1, 2, \ldots, L$, each A_ℓ maps $\mathbb{R}^n$ to $\mathbb{R}^{m_\ell}$. Again, it is not identical with Problem 5 because here the constraints in $\mathbb{R}^{m_\ell}$, for $\ell = 1, 2, \ldots, L$, are not convex. . Although Problem 17 defines an upper PVC on exactly p blocks and a lower PVC on exactly r blocks, we can, without loss of generality, choose to define PVCs only on a subset of these blocks. For blocks without a PVC, the problem reverts to a standard LFP.

7 Application to radiation therapy treatment planning

The process of planning a radiotherapy treatment plan involves a physician providing dose prescriptions which geometrically constrain the distribution of dose deposited in the patient. Choosing the appropriate nonnegative weights of many individual beamlet dose kernels to achieve these prescriptions as best as possible is posed as a split inverse problem. We focus, for our purposes, on constraining the problem with upper and lower dose bounds, and dose volume constraints (DVCs), which we more generally refer to as PVCs in this work. DVCs allow dose levels in a specified proportion of a structure to fall short of, or exceed, their prescriptions by a specified amount. They largely serve to allow more flexibility in the solution space.

Problem 17 describes the split feasibility probem as it applies in the context of radiation therapy treatment planning. Each block represents a defined geometrical structure in the patient, which is classified either as an *avoidance structure* or a *target volume*. An example of an avoidance strucrue is an organ at risk (OAR), in which one wishes to deposit minimal dose. An example of a target strucrure is the planning target volume (PTV), to which a sufficient dose is prescribed to destroy the tumoural tissue. If there are p avoidance structures, any number of blocks in $\{1, 2, \ldots, p\}$ can have lower PVCs applied. Similarly, if there are r target volumes then any number of blocks in $\{p+1, p+2, \ldots, p+r\}$ can have an upper PVC applied.

This problem can be formulated as the MOSCFPP described in Problem 1 as follows. For the data of Problem 17, define $\mathit{\Gamma} \subseteq \{1, 2, \ldots, p+r\}$ and for all $i = 1, 2, \ldots, m_\ell$, let

$$C_\ell^i := \{x \in \mathbb{R}^n \mid \langle a_\ell^i, x\rangle \leq (1+\beta_\ell)b_i^\ell\} \cap \mathbb{R}_+^n, \tag{7.1}$$

for all $\ell \in \{1, 2, \ldots, p\}$ and

$$C_\ell^i := \{x \in \mathbb{R}^n \mid (1-\beta_\ell)c_i^\ell \leq \langle a_\ell^i, x\rangle\} \cap \mathbb{R}_+^n, \tag{7.2}$$

for all $\ell \in \{p+1, p+2, \ldots, p+r\}$.

Additionally, let

$$Q_\ell := \{A_\ell x \in \mathbb{R}^{m_\ell} \mid \|(A_\ell x^* - b^\ell)_+\|_0 \leq \alpha_\ell m_\ell\}, \tag{7.3}$$

for all $\ell \in \{1, 2, \ldots, p\} \cap \mathit{\Gamma}$ and

$$Q_\ell := \{A_\ell x \in \mathbb{R}^{m_\ell} \mid \|(c^\ell - A_\ell x^*)_+\|_0 \leq \alpha_\ell m_\ell\} \tag{7.4}$$

for all $\ell \in \{p+1, p+2, \ldots, p+r\} \cap \mathit{\Gamma}$.

Problem 18 ***Translated problem of MOSCFPP for RTTP.***

Let the operators $P_{C_\ell^i} : \mathbb{R}^n \to \mathbb{R}^n$ be orthogonal projections onto C_ℓ^i for all $\ell \in \{1, 2, \ldots, p+r\}$ and $i \in \{1, 2, \ldots, m_\ell\}$, and let $P_{Q_\ell} : \mathbb{R}^{m_\ell} \to \mathbb{R}^{m_\ell}$ be orthogonal projections onto Q_ℓ, for all $\ell \in \mathit{\Gamma}$. The translated MOSCFPP for RTTP is:

$$\begin{aligned} &\textit{Find an } x^* \in \mathbb{R}_+^n \textit{ such that } x^* \in \bigcap_{\ell=1}^{p+r} \bigcap_{i=1}^{m_\ell} \operatorname{Fix}(P_{C_\ell^i}) \textit{ and,} \\ &\textit{for all } \ell \in \mathit{\Gamma},\ A_\ell x^* \in \operatorname{Fix}(P_{Q_\ell}). \end{aligned} \tag{7.5}$$

We seek a solution to Problem 18 using our dynamic string-averaging CQ-method, described in Algorithm 9. We define, for all $\ell \in \Gamma$,

$$V_\ell := I - \gamma_\ell A_\ell^T (I - P_{Q_\ell}) A_\ell, \tag{7.6}$$

where $\gamma_\ell \in \left(0, \frac{1}{L_\ell}\right)$, $L_\ell = \|A_\ell\|^2$.

Remark 19 *In practical use relaxation parameters play an important role:*

(i) Each projection operator $P_{C_\ell^i} : \mathbb{R}^n \to \mathbb{R}^n$ *may be relaxed with a parameter* $\lambda_\ell \in (0, 2)$ *defined on the block* $\ell \in \{1, 2, \ldots, p + r\}$*.*

(ii) The relaxation parameters λ_ℓ*, as defined in (i), and* γ_ℓ*, as given in (7.6), are permitted to take any value within their bounds on any iterative step of Algorithm 9. That is, they may depend on (vary with) the iteration index* k *and, therefore, be labelled* λ_ℓ^k *and* γ_ℓ^k*.*

(iii) Instead of setting the operator T_ℓ *(replacing the subscript* j *for* ℓ*) in (4.1) to* P_{Q_ℓ} *in (7.6), one may, alternatively, set* T_ℓ *as a stacked projection operator, i.e., a product of projections,* $T_\ell := \prod_{q=1}^{N_q} P_{Q_\ell}$*, for some integer* N_q*, which is also a quasi-nonexpansive operator, as we did in [29].*

Tracking the percentage of elements in the current iteration of dose vectors $A_\ell x^k$ that are violating their constraints enables one to impose an adaptive version of Algorithm 9 using the comments in Remark 19. If, for example, one block has more PVC violations than LFP (dose limit constraints) violations then one could choose to alter the relaxation paramaters at the next iteration, λ_ℓ^{k+1} and γ_ℓ^{k+1}, in order to place less emphasis on the projections onto C_ℓ^i (Remark 19(i,ii)), or one could perform multiple projections onto Q_ℓ before proceeding to calculate the projections onto C_ℓ^i (Remark 19(iii)).

8 Conclusions

We introduced a new split feasibility problem called "the multiple-operator split common fixed point problem" (MOSCFPP). This problem generalizes some well-known split feasibility problems such as the split convex feasibility problem, the split common fixed point problem and more. Following the recent work of Penfold et al. [29], and motivated from the field of radiation therapy treatment planning, the MOSCFPP involves additional so-called Percentage Violation Constraints (PVCs) that give rise to non-convex constraints sets. A new string-averaging CQ method for solving the problem is introduced, which provides the user great flexibility in the weighting and order in which the projections onto the individual sets are executed.

List of abbreviations

ACA Almost cyclic sequential algorithm

CFP Convex feasibility problem

DVC	Dose volume constraint
FNE	Firmly nonexpansive
IMRT	Intensity modulated radiation therapy
LFP	Linear feasibility problem
MOSCFPP	Multiple-operator split common fixed point problem
MSA	Modular string averaging
MSCFP	(Constrained) multiple set split convex feasibility problem
MSSCFP	Multiple sets split convex feasibility problem
OAR	Organ at risk
PTV	Planning target volume
PVC	Percentage violation constraint
RTTP	Radiation therapy treatment planning
SCFP	Split convex feasibility problem
SCFPP	Split common fixed points problem
SIP	Split inverse problem
SQNE	Strongly quasi-nonexpansive

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Competing interests

The authors declare that they have no competing interests.

Authors contributions

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