manuscript No.
(will be inserted by the editor)

Multi-stage robust optimization problems: A sampled scenario tree based approach

Francesca Maggioni · Fabrizio Dabbene · Georg Ch. Pflug

Received: date / Accepted: date

Abstract In this paper, we consider multi-stage robust convex optimization problems of the minimax type. We assume that the total uncertainty set is the cartesian product of stagewise compact uncertainty sets and approximate the given problem by a sampled subproblem. Instead of looking for the worst case among the infinite and typically uncountable set of uncertain parameters, we consider only the worst case among a randomly selected subset of parameters. By adopting such a strategy, two main questions arise: (1) Can we quantify the error committed by the random approximation, especially as a function of the sample size? (2) If the sample size tends to infinity, does the optimal value converge to the "true" optimal value? Both questions will be answered in this paper. An explicit bound on the probability of violation is given and chain of lower bounds on the original multi-stage robust optimization problem provided. Numerical results dealing with a multi-stage inventory management problem show that the proposed approach works well, given that the sample size is large enough. Due to the fact that we solve a very complex problem and not a surrogate with no quality guarantee, large sample sizes cannot be avoided.

Keywords convex multi-stage robust optimization · constraint sampling · scenario approach in optimization · randomized algorithms

1 Introduction

We consider a multi-stage decision problem, i.e. a problem, where the decisions x_t have to be taken at discrete time instants $t = 1, \ldots, H+1$, where $H+1$ is the horizon length. In our setup, some relevant parameters are not known at time t of decision and become revealed only at time

F. Maggioni
Department of Management, Information and Production Engineering University of Bergamo, Italy
E-mail: francesca.maggioni@unibg.it

Fabrizio Dabbene
CNR-IEIIT, Politecnico di Torino, C.so Duca degli Abruzzi n. 24 10129 Torino, Italy
E-mail: fabrizio.dabbene@polito.it

Georg Ch. Pflug
Department of Statistics and Operations Research, University of Vienna and International Institute for Applied Systems Analysis (IIASA), Laxenburg, Austria
E-mail: georg.pflug@univie.ac.at

$t+1$. We look for the optimal decision strategy under the objective to minimize the costs for the worst case among all possible parameter values. Thus we look for a multi-stage robust solution.

The usual robust optimization models deal with *static* problems, where all the decision variables have to be determined before the uncertain parameters are revealed. A vast literature focused on uncertainty structure to obtain computationally tractable problems is available, see for instance [22] and [43] for polyhedral uncertainty sets and [6] for ellipsoidal uncertainty sets, respectively. However, this approach cannot directly handle problems that are multiperiod in nature, where a decision at any period should take into account data realizations in previous periods, and the decision maker needs to adjust his/her strategy to the information revealed over time. This means that some of the variables (non-adjustable variables) must be determined before the realization of the uncertain parameters, while the other variables (adjustable variables) have to be chosen after the uncertainty realization. For a recent overview of multiperiod robust optimization, we refer to [8,21,26]. In order to describe such a situation, and extend robust optimization to a dynamic framework, the concept of *Adjustable Robust Counterpart* (ARC) has been first introduced and analyzed in [5]. This approach opened up the research in several new application areas, such as portfolio optimization [39,46], inventory management [4,12], scheduling [48], facility location [3], revenue management [37] and energy generation [49]. ARC is clearly less conservative than the static robust approach, but in most cases it turns out to be computationally intractable. One of the most recent methods to cope with this difficulty is obtained by approximating the adjustable decisions by *decision rules*, i.e. combinations of given basis functions of the uncertainty. A particular case is the *Affinely Adjustable Robust Counterpart* (AARC) [5], where the adjustable variables are affine functions of the uncertainty. The decision rule approximation often allows to obtain a formulation which is equivalent to a tractable optimization problem (such as linear, quadratic and second-order conic [7], or semidefinite [28]), transforming the original dynamic problem into a static robust optimization problem whose decision variables are the coefficients of the linear combination. In [40] a methodology for constructing decision rules for integer and continuous decision variables has been provided. The authors show by iteratively splitting the uncertainty set into subsets, how one can determine the later-period decisions based on the revealed uncertain parameters.

However, in many practical cases, also the static robust optimization problem ensuing from the decision rule approximation is still numerically intractable. In these situations, one can recur to approximate solutions based on constraint sampling, which consists in taking into account only a finite set of constraints, chosen at random among the possible continuum of constraint instances of the uncertainty. The attractive feature of this method is to provide explicit random bounds on the measure of the original constraints of the static problem that are possibly violated by the randomized solution. The properties of the solutions provided by this approach, called scenario approach have been studied in [15,18,20], where it has been shown that most of the constraints of the original static problem are satisfied provided the number of scenarios sufficiently large. The constraint sampling method has been also extensively studied within the chance constraint approach through different directions by [24,30,36].

In [9,16,47] multi-stage convex robust optimization problems are solved by combining general nonlinear decision rules and constraint sampling techniques. This means that the dynamic robust optimization problem is transformed into a static one through decision rules approximation and then solved via a scenario counterpart. In practice, the novelty of [47] is to introduce, besides polynomial decision rules, also trigonometric monomials and basis functions based on sigmoidal and Gaussian radial functions, thus allowing more flexibility. A rigorous convergence proof for the optimal value, based on the decision rule approximation and of the constraint randomization approach is also given. Convergence is proved when both the complexity parameter (the number of basis functions in the decision rule approximation) and the number of scenarios tend to infinity.

The work [9] proposes a technique based on structured adaptability that results in sample complexity, i.e. the minimum number of samples required to achieve the desired probabilistic guarantees, that is polynomial in the number of stages. This allows to provide a hierarchy of adaptability schemes, not only for continuous problems, but also for discrete problems.

In the context of randomized methods for uncertain optimization control problems, the *scenario with certificates* approach has been proposed in [25], based on an original idea of [35]. This approach has been then extended and exploited for anti-windup augmentation problems [25]. The main idea of this approach is to distinguish between design variables (corresponding to non-adjustable variables) and certificates (corresponding to adjustable variables).

Linear decision rules have a long history also in stochastic programming (see, e.g., [27]), and have been adapted to Multi-stage Linear Stochastic Programming (MSLP) in [42], and recently in Kuhn et al. [29] who analyzed their application in the dual of the MSLP. Under certain assumption such as stagewise independence, compact and polyhedral support, if uncertainty is limited to the right-hand side of the constraints, [29,42] have shown that the static approximations obtained restricting the primal and dual policies to be linear decision rules are both tractable linear programs. Better policies have been obtained in [2,19] by considering polynomial decision rules and piecewise linear decision rules respectively, while binary decision rules have been considered in [10]. Recently [13] present a new use of linear decision rules for MLSP named *two-stage linear decision rule approach* based on the idea of partitioning the decision variables into *state* and *recourse* decisions and applying linear decision rules only to the state variables. This approach allows to reduce the problem to a two-stage stochastic linear program with a potentially improved policy and bounds. The approach is also applied to the dual of an MSLP, imposing the restriction only on the dual variables associated with the state equations and they show to obtain better bounds and policies than the the ones provided by the standard static approach.

In this paper, we consider randomized methods for robust convex multi-stage optimization problems. We approximate the given robust problem by a sampled subproblem via a scenario-tree approximation, where instead of looking for the worst case among the infinite and typically uncountable set of uncertain parameters, we consider only the worst case among a randomly selected subset of parameters. In this way, we establish a link between multi-stage robust optimization and multi-stage stochastic optimization. By adopting such a strategy, two main questions arise: (1) Can we quantify the error committed by the random approximation, especially as a function of the sample size and provide a bound on the violation probability of the ignored constraints? (2) If the sample size tends to infinity, does the optimal value converge to the "true" optimal value? Both questions will be answered in this paper.

To simplify the understanding, we first consider the two-stage case, and we show that the theoretical sample complexity depends only on the number of first-stage variables. For the multi-stage case, the contributions can be summarized as follows: (i) define the probability of violation at each decision stage; (ii) provide a bound on the probability of violation by a function of the number of nodes of the tree up to that stage, the number of decision variables at that stage and the pre-specified violation tolerance; (iii) come up with an iterative scheme to define a sufficiently large number of nodes of the tree at each stage; and lastly, (iv) define the total violation probability as the probability of violation at any stage. Moreover, lower bounds on the true optimal value by extending two commonly used relaxations from the stochastic programming literature such as the wait-and-see problem, and the two-stage relaxation are provided. The proposed ideas are illustrated on a simple inventory model for which the true optimal value can be computed exactly. The way how the proposed algorithm works is shown by analyzing the optimality gaps and empirical violation probabilities of the scenario-problem solutions, for many levels of the violation threshold, for the two- and three-stage cases. While the main application considered deals with a linear objective and convex constraints, our bounds are valid also in the more general setup of a convex objective and convex constraint sets.

The main difference between the approach proposed in this paper and the one in [47], is that we do not change the decision model to a simpler one restricting the decision functions spaces via decision-rule approximation. The asymptotic result they provide holds only, if the chosen function space is such large that any continuous function can be uniformly approximated with a sup-distance less than some chosen ϵ. By more sampling alone, the optimization gap cannot be brought to zero. In our setup, we keep the model as it is and approximate it by sampling. Moreover, the authors in [47] consider only sampled paths from the uncertain parameters, while we consider complete sampled scenario trees, leading to a much stricter notion of the so called *violation probability*, as it will be explained in detail in section 2.5. Furthermore, if the uncertainty set is finite and we have sampled all points, then our solution is exact, while the decision-rule approximation approach is typically not.

The rest of the paper is organized as follows. Section 2 discusses the formulations of two-stage, multi-stage robust linear and convex programs, and provides a result on the probability of violation. Bounds on the number of scenarios needed to obtain a user-prescribed guarantee of violation is given. Section 3 provides a chain of inequalities among lower bounds on the optimal value of the multi-stage robust optimization problem. Section 4 presents numerical results dealing with a multi-stage inventory management problem. The conclusions follow.

2 Main results

2.1 Basic facts

We consider a multi-stage discrete-time decision problem, where the decisions at times $t = 1, \ldots, H+1$, denoted by $x_t \in \mathbb{R}^{n_t}$ have to be made under the presence of parameters $\xi_t, t = 1, \ldots, H$. At time t, the values $\xi_1, \ldots, \xi_{t-1}$ are known, but for ξ_t it is known only that it lies in some uncertainty set Ξ_t. The problem is to find optimal decisions under a worst-case objective, making the problem of nested minimax type.

Typically, the uncertainty sets are uncountable and the only way of treating this problem is by approximating the large uncertainty set by well chosen finite one. Notice that, in this paper, we do not restrict the class of possible decisions.

The uncountable sets Ξ_t are replaced by finite subsets $\tilde{\Xi}_t$. These subsets may be chosen by minimizing the Pompeiu-Hausdorff distance between the large sets Ξ_t and the finite sets $\tilde{\Xi}_t$, i.e. by an optimal selection of points. However, in this paper we use the simplest way of extracting points from larger sets: we do random sampling.

Notice that for a sampling method we need to define a probability measure $\mathbb{P}$ on $\Xi = \bigtimes_{t=1}^{H} \Xi_t$. While the proposed methods work for any probability measure on Ξ_t which has a Lebesgue density which is bounded away from zero, we recommend to use, if possible, a uniform distribution on Ξ_t for a "fair" treatment of all points in Ξ_t and the product measure on Ξ. Notice that we have to use the same probability measure for sampling and for calculation of the violation probability. Theoretically, one could also try to construct a probability measure which makes the "extremal" points more likely, but by treating some points less likely, we may have difficulties in interpretating the violation probability.

By considering N_t independent random samples from $\mathbb{P}_t$, for each t, finite subsets $\hat{\Xi}_t^{N_t}$ of sizes N_t are extracted from Ξ_t, and the multi-stage worst-case problem is solved with

$$\Xi_1 \times \cdots \times \Xi_H \ ,$$

replaced by the finite sets

$$\hat{\Xi}_1^{N_1} \times \cdots \times \hat{\Xi}_H^{N_H} \ .$$

The technique to replace a possible infinite set of convex constraints by a random finite selection of these constraints was originally introduced by Calafiore and Campi [15] and later improved independently by Calafiore [14] as well as by Campi and Garatti [18]. The sampled sets $\hat{\Xi}_t^{N_t}$ are referred to as *sampled scenarios*. We remark that in the stochastic optimization literature the use of a finite number of scenarios to represent the infinite possible realization of the uncertain quantities ξ_t is rather popular, see e.g. [23,38,41].

We report here the Calafiore, Campi and Garatti main result, which is crucial for this paper.

Proposition 2.1 (CCG Theorem, Calafiore [14] and Campi and Garatti [18]) *Consider the robust optimization problem:*

$$\begin{aligned} RO \quad : \quad & \min_{x \in \mathbb{X}} c^\top x \\ & s.t.\ f(x,\xi) \le 0, \qquad \forall \xi \in \Xi \ , \end{aligned} \tag{1}$$

or equivalently

$$RO \quad : \quad \min_{x \in \mathbb{X}} \left\{ c^\top x : \sup_{\xi \in \Xi} f(x,\xi) \le 0 \right\} \ , \tag{2}$$

where $x \in \mathbb{X} \subseteq \mathbb{R}^n$ is the optimization variable, $\mathbb{X}$ is convex and closed and $f(x,\xi) : \mathbb{X} \times \Xi \to \mathbb{R}$ is a convex function in x for all $\xi \in \Xi$. The optimal objective value $v(\cdot)$ of problem (1) is denoted by $v(RO)$. Suppose that Ξ is a compact set and $\mathbb{P}$ is a probability measure on it with nonvanishing density. Let $\xi^{(1)}, \dots, \xi^{(N)}$ be independent scenarios from Ξ, sampled according to $\mathbb{P}^N = \mathbb{P} \times \cdots \times \mathbb{P}$, N times. The "scenario" approximation of problem (2) is defined as follows:

$$\widehat{RO}^N \quad : \quad \min_{x \in \mathbb{X}} \left\{ c^\top x : \max_{i=1,\dots,N} f(x,\xi^{(i)}) \le 0 \right\} \ , \tag{3}$$

with the understanding that, in (3), we let the optimal solution $v(\widehat{RO}^N) = \infty$ whenever the random extraction $\xi^{(1)}, \dots, \xi^{(N)}$ leads to an infeasible problem. [1] *The "violation probability" $V(\cdot)$ of the sample $\hat{\Xi}^N := \left\{\xi^{(1)}, \dots, \xi^{(N)}\right\}$ is defined as:*

$$V(\hat{\Xi}^N) := \mathbb{P}\left\{ \xi^{(N+1)} : \min_{x \in \mathbb{X}} \left\{ c^\top x : \max_{i=1,\dots,N+1} f(x,\xi^{(i)}) \le 0 \right\} > v(\widehat{RO}^N) \right\} \ , \tag{4}$$

where also $\xi^{(N+1)}$ is sampled from $\mathbb{P}$. Notice that V is a random variable taking its values in $[0,1]$. Then the tail probability of V under $\mathbb{P}$ can be bounded by

$$\mathbb{P}\{V(\hat{\Xi}^N) > \epsilon\} \le B(N,\epsilon,n) = \sum_{j=0}^{n} \binom{N}{j} \epsilon^j (1-\epsilon)^{N-j} \ . \tag{5}$$

For any probability level $\epsilon \in (0,1)$ and confidence level $\beta \in (0,1)$, let:

$$N(\epsilon,\beta,n) := \min \left\{ N \in \mathbb{N} : \sum_{j=0}^{n} \binom{N}{j} \epsilon^j (1-\epsilon)^{N-j} \le \beta \right\} \ . \tag{6}$$

Then $N(\epsilon,\beta,n)$ is a sample size which guarantees that the ϵ-violation probability given in (5) lies below β.

[1] Notice that $v(\widehat{\mathrm{RO}}^N) = \infty$ in (3) implies that also $v(\mathrm{RO}) = \infty$ in (2) and detecting this, one may stop sampling.

Remark 2.1 (On the CCG Theorem) Notice that, in the CCG Theorem, we defined the violation probability in terms of the *cost function*, following the approach in [14]. This allows to define the violation also for possible situations in which the ensuing scenario problem (3) turns out to be infeasible. It should also be remarked that the previous result holds under some assumptions on the scenario problem (3). Namely, the CCG Theorem requires that it is guaranteed that when problem (3) admits an optimal solution, this solution is unique (uniqueness), and that it is nondegenerate with probability one (nondegeneracy). These assumptions are of technical nature, and, as observed in [14], can be usually relaxed. For instance, uniqueness of the solution can essentially be always obtained by imposing some suitable tie-breaking rule. Regarding the definition degeneracy, we refer the reader to [14, 3.4] for a detailed discussion.

We remark that, in the literature, the minimum number of samples for which $B(N, \epsilon, n) \leq \beta$ holds for given $\epsilon \in (0, 1)$ and $\beta \in (0, 1)$ is referred to as *sample complexity*, see for instance [45]. There exist several results in the literature about bounding the sample complexity. In particular, in Lemma 1 and 2 in [1], it is proved that given $\epsilon \in (0, 1)$ and $\beta \in (0, 1)$:

$$N(\epsilon, \beta, n) \leq N^*(\epsilon, \beta, n) := \frac{1}{\epsilon} \frac{e}{e-1} \left(\ln \frac{1}{\beta} + n \right), \tag{7}$$

where e is the Euler constant and n is the dimension $dim(\cdot)$ of vector x, i.e. $n = dim(x)$. This bound gave a (numerically) significant improvement upon other bounds available in the literature [14, 17]. Notice that while bound (7) is certainty useful for estimating N, the problem in (6) can be solved by using bisection or ready-made tools such as Matlab betainv, in order to get the exact (tight) value of N.

The CCG Theorem can also be applied to the problem:

$$\min_x \sup_{\xi \in \Xi} \left\{ g(x, \xi) : x \in \mathbb{X}(\xi) \right\}, \tag{8}$$

where $g : x \mapsto g(x, \xi)$ is convex in x and $\mathbb{X}(\xi)$ are convex sets for all $\xi \in \Xi$. To see this, set:

$$f(x, \xi) := g(x, \xi) + \psi_{\mathbb{X}(\xi)}(x), \tag{9}$$

where

$$\psi_{\mathbb{B}}(x) := \begin{cases} 0 & \text{if } x \in \mathbb{B} \\ \infty & \text{otherwise.} \end{cases} \tag{10}$$

Then f is convex in x and (8) can be written as:

$$\min_x \sup_{\xi \in \Xi} f(x, \xi).$$

Finally, observe that this problem is equivalent to:

$$\min_{x,\gamma} \left\{ \gamma : \sup_{\xi \in \Xi} f(x, \xi) - \gamma \leq 0 \right\}.$$

This problem is of the standard form (2). Note that, in this case, the dimension of the decision variable vector is $dim(x) + dim(\gamma) = n + 1$.

2.2 Two-stage robust linear case

To simplify our exposition, we first analyze a *two-stage robust linear program*, formally defined as follows:[2]

$$\begin{aligned}\mathrm{RO}_2 \quad : \quad & \min_{x_1} c_1^\top x_1 + \sup_{\xi_1 \in \Xi_1} \left[\min_{x_2(\xi_1)} c_2^\top(\xi_1) x_2(\xi_1) \right] \\ & \text{s.t. } Ax_1 = h_1, \quad x_1 \geq 0 \\ & \qquad T_1(\xi_1)x_1 + W_2(\xi_1)x_2(\xi_1) = h_2(\xi_1), \qquad x_2(\xi_1) \geq 0, \qquad \forall \xi_1 \in \Xi_1 \ ,\end{aligned} \tag{11}$$

where $c_1 \in \mathbb{R}^{n_1}$ and $h_1 \in \mathbb{R}^{m_1}$ are known vectors and $A \in \mathbb{R}^{m_1 \times n_1}$ is a given (known) matrix. The uncertain parameter vectors and matrices as functions of the uncertain factor ξ_1 are given by $h_2(\xi_1) \in \mathbb{R}^{m_2}$, $c_2(\xi_1) \in \mathbb{R}^{n_2}$, $T_1(\xi_1) \in \mathbb{R}^{m_2 \times n_1}$, and $W_2(\xi_1) \in \mathbb{R}^{m_2 \times n_2}$. Ξ_1 is a compact set in $\mathbb{R}^{k_1}$. The goal is to find a first-stage decision x_1 and a second-stage decision function $\xi_1 \mapsto x_2(\xi_1)$, such that the cost function in the worst-case realization of $\xi_1 \in \Xi_1$ is minimized. To this end, we first remark that problem (11) can equivalently be rewritten as follows:

$$\mathrm{RO}_2 \quad : \quad \min_{x_1 \in \mathbb{X}_1} \left\{ c_1^\top x_1 + \mathscr{R}(x_1) \right\}, \tag{12}$$

where

$$\mathbb{X}_1 := \{ x_1 \geq 0 : Ax_1 = h_1 \} \ , \tag{13}$$

and $\mathscr{R}(x_1)$ is the *worst-case recourse function*

$$\mathscr{R}(x_1) := \sup_{\xi_1 \in \Xi_1} \mathcal{Q}\left(x_1, \xi_1\right) \ ,$$

with $\mathcal{Q}\left(x_1, \xi_1\right)$ being the *recourse function*

$$\begin{aligned}\mathcal{Q}(x_1, \xi_1) := & \min_{x_2(\xi_1)} c_2^\top(\xi_1) x_2(\xi_1) \\ & \text{s.t. } T_1(\xi_1)x_1 + W_2(\xi_1)x_2(\xi_1) = h_2(\xi_1) \\ & \qquad x_2(\xi_1) \geq 0 \ .\end{aligned} \tag{14}$$

Since RO_2 in (11) is of minimax type, we have to make the used notion of feasibility more precise. This is discussed in the next remark.

Remark 2.2 (On the feasibility of RO_2) Define the feasible set at stage 2 for given x_1 and ξ_1 as follows:

$$\mathbb{X}_2(x_1, \xi_1) := \{ x_2 \geq 0 : T_1(\xi_1)x_1 + W_2(\xi_1)x_2 = h_2(\xi_1) \} \ . \tag{15}$$

Notice that the worst-case recourse function can be expressed in terms of (15):

$$\mathscr{R}(x_1) = \sup_{\xi_1 \in \Xi_1} \min_{x_2 \in \mathbb{X}_2(x_1, \xi_1)} c_2^\top(\xi_1) x_2(\xi_1) \ .$$

Define $\mathbb{X}_2(x_1) := \bigcap_{\xi_1 \in \Xi_1} \mathbb{X}_2(x_1, \xi_1)$. For all x_1 such that $\mathbb{X}_2(x_1) = \emptyset$, we set $\mathscr{R}(x_1) = \infty$. Then, we set $\mathrm{Feas} = \{x_1 \geq 0 : Ax_1 = h_1, \mathbb{X}_2(x_1) \neq \emptyset\}$. Notice that the problem has *relatively complete recourse* iff $\mathrm{Feas} = \mathbb{X}_1$. If $\mathrm{Feas} = \emptyset$, we set the optimal objective value $v(\mathrm{RO}_2)$ to ∞.

It may happen that all second-stage problems are unbounded, i.e. $\mathscr{R}(x_1) = -\infty$ for some $x_1 \in \mathrm{Feas} \neq \emptyset$. In this case we assign the value $v(\mathrm{RO}_2) = -\infty$. However an infeasible first-stage

[2] We adopt the convention of putting as lower indices the number of stages of the problem, e.g. RO_2 denotes a two-stage robust linear problem ($H = 1$).

(i.e. Feas $= \emptyset$ with $v(\mathrm{RO}_2) = \infty$) is not compensated by an unbounded second stage and gives the value $v(\mathrm{RO}_2) = \infty$. This resolves the problem about $\infty - \infty$. An infeasible second stage for some ξ_1 makes the problem infeasible, even if the first-stage would be unbounded.

If RO_2 is feasible and bounded, its optimal value is neither ∞ nor $-\infty$. In this case, the optimum $v(\mathrm{RO}_2)$ may be attained or not in general (but by our assumptions the optimum is always attained). Suppose that the optimal value $v(\mathrm{RO}_2)$ with $-\infty < v(\mathrm{RO}_2) < \infty$ is attained. Then a solution set consists of all pairs $(x_1, \xi_1 \mapsto x_2(\xi_1))$ such that $x_1 \in$ Feas and $x_2(\xi_1) \in \mathbb{X}_2(x_1, \xi_1)$ for all $\xi_1 \in \Xi_1$ such that

$$c_1^\top x_1 + \sup_{\xi_1 \in \Xi_1} c_2^\top(\xi_1) x_2(\xi_1) = v(\mathrm{RO}_2) \ .$$

Notice that we do not require that $x_2(\xi_1)$ are in the argmins of $\min_{x_2(\xi_1)}\{c_2^\top(\xi_1)x_2(\xi_1) : x_2(\xi_1) \in \mathbb{X}_2(x_1, \xi_1)\}$ for all $\xi_1 \in \Xi_1$.

It is immediate to observe that problem RO_2 rewrites as follows:

$$\mathrm{RO}_2 \quad : \quad \min_{x_1 \in \mathbb{X}_1} \left\{ c_1^\top x_1 + \gamma : \sup_{\xi_1 \in \Xi_1} \mathcal{Q}\left(x_1, \xi_1\right) - \gamma \le 0 \right\}. \tag{16}$$

A key observation of this section is the fact that the above problem is exactly in the form of the CCG Theorem. Indeed, we remark that the function $\mathcal{Q}\left(x_1, \xi_1\right)$ as defined in (14) is a convex function in x_1. This follows from the structure of (14) and the fact that if $(x, y) \mapsto f(x, y)$ is jointly convex, then $x \mapsto \min_y f(x, y)$ is also convex.

The above observation justifies the adoption of a sampling approach, based on the random extraction of N_1 independent identically distributed (iid) scenarios:

$$\hat{\Xi}_1^{N_1} := \left\{\xi_1^{(1)}, \ldots, \xi_1^{(N_1)}\right\},$$

of the random variable ξ_1, similarly to what is proposed in [47]. Recall that $\mathbb{P}$ has a nonvanishing density on the compact set Ξ. Let $T_1(\xi_1^{(i)})$, $h_2(\xi_1^{(i)})$, $c_2(\xi_1^{(i)})$ be the realization of $T_1(\xi_1)$, $h_2(\xi_1)$ and $c_2(\xi_1)$ under scenario $\xi_1^{(i)}$, $i = 1, \ldots, N_1$, and let $x_2^{(i)}$ be the second-stage (adjustable) design variables created for the scenarios $\xi_1^{(i)}$, $i = 1, \ldots, N_1$.

These scenarios are used to construct the following *sample-based approximation* based on N_1 instances of the uncertain constraints:

$$\widehat{\mathrm{RO}}_2^{N_1} \quad : \quad \min_{x_1 \in \mathbb{X}_1, \gamma} \left\{ c_1^\top x_1 + \gamma : \max_{i=1,\ldots,N_1} \mathcal{Q}\left(x_1, \xi_1^{(i)}\right) - \gamma \le 0 \right\} .$$

We note that the above problem explicitly rewrites as follows:

$$\begin{aligned} \widehat{\mathrm{RO}}_2^{N_1} : \min_{x_1 \in \mathbb{X}_1, \gamma, x_2^{(1)}, \ldots, x_2^{(N_1)}} \quad & c_1^\top x_1 + \gamma \\ \text{s.t. } & c_2^\top(\xi_1^{(i)}) x_2^{(i)} \le \gamma, \quad i = 1, \ldots, N_1 \\ & T_1(\xi_1^{(i)}) x_1 + W_2(\xi_1^{(i)}) x_2^{(i)} = h_2(\xi_1^{(i)}), \quad i = 1, \ldots, N_1 \\ & x_2^{(i)} \ge 0, \ i = 1, \ldots, N_1 \ . \end{aligned} \tag{17}$$

We define now the violation probability V_1 for the two-stage case as:

$$V_1(\hat{\Xi}_1^{N_1}) := \mathbb{P}\left\{\xi^{(N_1+1)} : v(\widehat{\mathrm{RO}}_2^{N_1+1}) > v(\widehat{\mathrm{RO}}_2^{N_1})\right\} \ . \tag{18}$$

The interpretation of the violation probability is as follows: if we consider the sampled problem $\widehat{\mathrm{RO}_2}^{N_1}$, then $V_1(\hat{\Xi}_1^{N_1})$ is the probability that we encounter an (yet unseen) uncertainty realization $\xi_1^{(N_1+1)}$ leading to a cost $v(\widehat{\mathrm{RO}_2}^{N_1+1})$ larger than $v(\widehat{\mathrm{RO}_2}^{N_1})$. Notice that, in the light of Remark 2, a larger cost could also mean that the problem becomes infeasible at stage two (we have in this case that the cost is infinite). Hence, the smaller is V_1, the higher is the probability that the solution at stage one will lead to a feasible stage two problem, and that no cost increase is observed.

We are hence in the position of providing a rigorous result connecting the violation probability to the number of scenarios N_1 adopted in the construction of the $\widehat{\mathrm{RO}_2}^{N_1}$ problem.

An easy consequence of the basic Proposition 2.1 is the following result (see [25]).

Theorem 2.1 (two-stage robust linear case) *Given an accuracy level $\epsilon \in (0,1)$, the violation probability of the sample-based problem $\widehat{RO_2}^{N_1}$, based on the random extraction of N_1 iid scenarios of ξ_1, is bounded as:*

$$\mathbb{P}\left\{ V_1(\hat{\Xi}_1^{N_1}) > \epsilon \right\} \leq B(N_1, \epsilon, n_1 + 1) , \tag{19}$$

where $B(N_1, \epsilon, n_1 + 1)$ is as in (5) with $n_1 = dim(x_1)$ and $1 = dim(\gamma)$.

Note that equation (7) can be used to obtain a priori the number of scenarios N_1 (i.e. the sample complexity) necessary to guarantee the desired level of confidence β that the violation probability $V_1(\hat{\Xi}_1^{N_1})$ is less than a pre-determined desired level ϵ. It is important to highlight that the number of scenarios N_1 in formula (7) depends only on the dimension of first-stage variables (non-adjustable variables); thus it reduces the number of scenarios needed to satisfy a prescribed level of violation with respect to that proposed in [47].

2.3 Connections with scenario with certificates approach

It is interesting to observe that problem RO_2 can be restated as the following *robust with certificates* RwC_2 problem:

$$\begin{aligned}
\mathrm{RwC}_2 \quad : \quad & \min_{x_1 \in \mathbb{X}_1, \gamma} c_1^\top x_1 + \gamma \\
& \text{s.t. } \forall \xi_1 \in \Xi_1, \; \exists \; x_2(\xi_1) \text{ satisfying} \\
& \qquad c_2^\top (\xi_1) \, x_2(\xi_1) \leq \gamma \\
& \qquad x_2(\xi_1) \geq 0, \; T_1(\xi_1) x_1 + W_2(\xi_1) x_2(\xi_1) = h_2(\xi_1) ,
\end{aligned}$$

where we distinguish between *design variables* (x_1, γ) and certificates $x_2(\xi_1)$. This problem does not contain a nested optimization as RO_2. It is just a standard optimization problem with possibly infinitely many variables and infinitely many constraints. RwC_2 is feasible, if its constraint set is non-empty (otherwise we set its optimal value to ∞). It is bounded, if it is feasible and its optimal value is not $-\infty$. We set the optimal value $v(\mathrm{RwC}_2)$ to ∞ if RwC_2 is not feasible and to $-\infty$ if it is unbounded.

It should be noted that the two formulations are equivalent, as formally proved in the next Theorem. We remark that a similar result can be found in [44]. We provide the proof in Appendix A for completeness.

Theorem 2.2 *RO_2 and RwC_2 are equivalent formulations, i.e. RO_2 is feasible and bounded if and only if RwC_2 is feasible and bounded. In the case of feasibility and boundedness, the optimal value is either attained by both or by none. If the optimal value is attained, the optimal solution values coincide.*

We note that in problem $\widehat{\mathrm{RO}}_2^{N_1}$, a *certificate* $x_2^{(i)}$ is constructed for every scenario $\xi_1^{(i)}$. The rationale behind this approach is the following: We are not interested in the explicit knowledge of the function $x_2(\xi_1)$, we are content with the fact that for every possible value of the uncertainty *there exists* a possible choice of x_2 compatible with the ensuing realization of the constraints. Note that this represents a key difference with respect to other sampling based approaches. In particular, in [47] different explicit parametrizations of the decision function $x_2(\xi_1)$ forming an M-dimensional subspace are introduced. It is easy to infer how this latter approach is bound to being more conservative, since the an extra constraint on the solution space is introduced.

Indeed, in the proof of Corollary 1 in [47], the number of required scenarios of the decision-rule approximation, which has extra first-stage decision variables corresponding to the basis functions, depends on the size of the basis and on the number of decision variables at each stage i.e. in (7) they need to set the dimension n of the decision vector to $n = 1 + n_1 \cdot M$ instead of $n = 1 + n_1$ of our approach. On the other hand, the number of decision variables used in our approach is larger than those used in [47], due to the introduction of sample-dependent certificates (or second-stage decision variables). In Section 2.5 we will analyze in details the main differences between our approach and the decision-rule approximation one proposed in [47].

It is clear that the approximate solution returned by problem $\widehat{\mathrm{RO}}_2^{N_1}$ is optimistic, since it considers only a subset of possible scenarios. That is, the following bound holds for all N_1:

$$v(\widehat{\mathrm{RO}}_2^{N_1}) \leq v(\mathrm{RO}_2) \ . \tag{20}$$

Hence, we have derived a lower bound, which by construction is better than bounds derived using wait-and-see approaches, as discussed in Section 3. Moreover, it is easy to show that the formulation is consistent, that is:

$$\lim_{N_1 \to \infty} v(\widehat{\mathrm{RO}}_2^{N_1}) = v(\mathrm{RO}_2) \qquad a.s.$$

A proof of the convergence of $\widehat{\mathrm{RO}}_2^{N_1}$ to RO_2 is given in the Appendix B in a more general setting.

2.4 Multi-stage robust linear case

We are now ready to introduce the multi-stage generalization of RO_2, see (11). We denote by $\underline{\xi}_t := (\xi_1, \ldots, \xi_t)$ the history of the uncertainty up to time t. We consider the following robust linear program over $H+1$ stages:

$$\begin{aligned}
\mathrm{RO}_{H+1} \quad : \quad & \min_{x_1} c_1^\top x_1 + \qquad\qquad (21)\\
& + \sup_{\xi_1\in\Xi_1}\left[\min_{x_2(\xi_1)} c_2^\top(\xi_1)\, x_2(\xi_1) + \sup_{\xi_2\in\Xi_2}\left[\cdots + \sup_{\xi_H\in\Xi_H}\left[\min_{x_{H+1}(\underline{\xi}_H)} c_{H+1}^\top(\xi_H)\, x_{H+1}\left(\underline{\xi}_H\right)\right]\right]\right]\\
& \text{s.t. } Ax_1 = h_1,\ x_1 \geq 0\\
& \qquad T_1(\xi_1)x_1 + W_2(\xi_1)x_2(\xi_1) = h_2(\xi_1),\ \forall \xi_1 \in \Xi_1\\
& \qquad\qquad \vdots\\
& \qquad T_H(\xi_H)x_H(\underline{\xi}_{H-1}) + W_{H+1}(\xi_H)x_{H+1}(\underline{\xi}_H) = h_{H+1}(\xi_H),\ \forall \xi_H \in \Xi_H\\
& \qquad x_t(\underline{\xi}_{t-1}) \geq 0 \quad \forall \xi_{t-1} \in \Xi_{t-1}; \quad t = 2,\ldots,H+1\ ,
\end{aligned}$$

where $c_1 \in \mathbb{R}^{n_1}$ and $h_1 \in \mathbb{R}^{m_1}$ are known vectors and $A \in \mathbb{R}^{m_1 \times n_1}$ is a known matrix. The uncertain parameter vectors and matrices depending on the parameters $\xi_t \in \Xi_t$ are then given by $h_t \in \mathbb{R}^{m_t}$, $c_t \in \mathbb{R}^{n_t}$, $T_{t-1} \in \mathbb{R}^{m_t \times n_{t-1}}$, and $W_t \in \mathbb{R}^{m_t \times n_t}$, $t = 2,\ldots,H+1$. Ξ_t are compact sets in $\mathbb{R}^{k_t}$.

There is an important difference between the two-stage case and the three- (or more-) stage case, due to the dynamic character of the multi-stage model: Since the optimization problems in (21) over stages are nested, they cannot be written as one big optimization problem unless new additional constraints are formulated, that is decisions at stage t are not allowed to depend on ξ-values from later stages. This property of *non-anticipativity* requires to reconsider the notion of random sampling and constraint violation. Indeed, the correct data structure for a multi-stage (non-anticipative) robust optimization problem is a tree of ξ-values and not just a collection of ξ-vectors. This tree has height H. A relevant random draw from the uncertainty set $\Xi = \times_{t=1}^{H} \Xi_t$ is the collection of independently sampled values:

$$\begin{aligned}
\hat{\Xi}_1^{N_1} &= \{\xi_1^{(1)},\ldots,\xi_1^{(N_1)}\}\ ,\\
\hat{\Xi}_2^{N_2} &= \{\xi_2^{(1)},\ldots,\xi_2^{(N_2)}\}\ ,\\
&\ \vdots\\
\hat{\Xi}_H^{N_H} &= \{\xi_H^{(1)},\ldots,\xi_H^{(N_H)}\}\ ,
\end{aligned}$$

which can be organized as a tree $\hat{\mathcal{T}}^{N_1,\ldots,N_H}$, where $\{\xi_1^{(1)},\ldots,\xi_1^{(N_1)}\}$ are the successors of the root, and recursively all nodes at stage $t+1$ get all values from $\Xi_t^{N_t}$ as successors. Notice that this is a random tree (as it depends on the sample) and that the number of nodes at stage $t+1$ of the tree is $\bar{N}_t := \prod_{s=1}^{t} N_s$. The total number of nodes of the tree is hence:

$$N_{\text{tot}} := 1 + \sum_{i=1}^{H} \bar{N}_i = 1 + N_1 + N_1N_2 + \cdots + N_1N_2\cdots N_H\ .$$

For each node of the finite tree $\hat{\mathcal{T}}^{N_1,\ldots,N_H}$ one has to consider a decision variable x. Let $\xi_1^{(i_1)}, \xi_2^{(i_2)}, \ldots, \xi_H^{(i_H)}$ with $i_1 = 1,\ldots,N_1,\ i_2 = 1,\ldots,N_2,\ldots,i_H = 1,\ldots,N_H$ be a path of the tree and let $x_1^{(i_1)}, x_2^{(i_1 i_2)}, \ldots, x_H^{(i_1\ldots i_H)}$ the corresponding decision variables[3]. The finite problem

[3] Because of the general setup of the model, the decisions may be path dependent.

on the sampled tree can be written as:

$$\widehat{\mathrm{RO}}_{H+1}^{N_1\cdots N_H} \quad : \quad \begin{aligned}[t] & \min_{x_1} c_1^\top x_1 + \\ & + \max_{i_1}\left[\min_{x_2^{(i_1)}} c_2^\top(\xi_1^{(i_1)})x_2^{(i_1)} + \max_{i_2}\left[\cdots + \max_{i_H}\left[\min_{x_{H+1}^{(i_1\dots i_H)}} c_{H+1}^\top(\xi_H^{(i_H)})x_{H+1}^{(i_1\dots i_H)}\right]\right]\right] \\ & \text{s.t. } Ax_1 = h_1,\ x_1 \geq 0 \\ & \quad T_1(\xi_1^{(i_1)})x_1 + W_2(\xi_1^{(i_1)})x_2^{(i_1)} = h_2^{(i_1)},\ \forall i_1 = 1,\dots,N_1 \\ & \qquad \vdots \\ & \quad T_H(\xi_H^{(i_H)})x_H^{(i_1\dots i_{H-1})} + W_{H+1}(\xi_H^{(i_H)})x_{H+1}^{(i_1\dots i_H)} = h_{H+1}(\xi_H^{(i_H)}),\ \forall i_H = 1,\dots,N_H \\ & \quad x_t^{(i_1\dots i_t)} \geq 0\ ,\ t = 1,\dots,H+1;\quad \forall i_1,\dots,i_H\ . \end{aligned} \tag{22}$$

The purpose of this paper is to compute a bound for the violation probability of the optimal value obtained by the sampled version $\widehat{\mathrm{RO}}_{H+1}^{N_1\cdots N_H}$ given by (22), and to show that the optimal values of the sampled version converge to those of the basic problem RO_{H+1} (see (21)), when the sampling rates tend to infinity.

2.5 The violation probability at stage t

First note that there is a one-to-one correspondence between a multisample $\hat{\Xi}_1^{N_1} \times \cdots \times \hat{\Xi}_H^{N_H}$ and the sample scenario tree $\hat{\mathcal{T}}^{N_1,\dots,N_H}$. For a fixed tree $\hat{\mathcal{T}}^{N_1,\dots,N_H}$, the robust optimization problem (22) may be solved, leading to an optimal value of:

$$v(\hat{\mathcal{T}}^{N_1,\dots,N_H}) := v(\widehat{\mathrm{RO}}_{H+1}^{N_1\cdots N_H})\ .$$

Notice that this value is by construction a lower bound to the optimal value $v(RO_{H+1})$ of the original infinite problem (21).

As a first step, we define the probability of violation at stage t. To this end, we add a new scenario $\xi_t^{(N_t+1)}$ from Ξ_t to the original data set, and form the associated tree $\hat{\mathcal{T}}^{N_1,\dots,N_t+1,\dots,N_H}$. Then, a violation at stage t occurs, if solving the finite problem on this new extended tree leads to a higher value than for the smaller tree $\hat{\mathcal{T}}^{N_1,\dots,N_H}$. Given the previously sampled tree $\hat{\mathcal{T}}^{N_1,\dots,N_H}$, the probability of stage t violation is therefore given by:

$$V_t(\hat{\mathcal{T}}^{N_1,\dots,N_H}) := V_t(\hat{\Xi}_1^{N_1},\dots,\hat{\Xi}_H^{N_H}) := \mathbb{P}\left\{\xi_t^{(N_t+1)} : v(\hat{\mathcal{T}}^{N_1,\dots,N_t+1,\dots,N_H}) > v(\hat{\mathcal{T}}^{N_1,\dots,N_H})\right\}. \tag{23}$$

Before discussing how to derive bounds on the probability distribution of V_t, we illustrate with a simple example the meaning of the concepts introduced so far.

Illustration. For a simple illustration, assume that $\Xi_1 = \Xi_2 = [0,1]$. We sampled from Ξ_1 the three values $\xi_1^{(1)} = 0.2$, $\xi_1^{(2)} = 0.6$, $\xi_1^{(3)} = 0.8$ and from Ξ_2 the two values $\xi_2^{(1)} = 0.3$, $\xi_2^{(2)} = 0.8$. The corresponding sampled tree $\hat{\mathcal{T}}^{3,2}$ is shown in Figure 1. As before, we denote the optimal value based on the (random) data of tree $\hat{\mathcal{T}}^{3,2}$ by $v(\hat{\mathcal{T}}^{3,2})$. In order to define the stage 1 violation probability, we sample a new point $\xi_1^{(4)} = 0.4 \in \Xi_1$ and form the new, extended tree $\hat{\mathcal{T}}^{4,2}$ (see Figure 2). A violation V_1 occurs, if $v(\hat{\mathcal{T}}^{4,2}) > v(\hat{\mathcal{T}}^{3,2})$ and the stage 1 violation probability is the probability of the random draw ξ_1^4 which results in a violation. Similarly, we may define the stage 2 violation probability. Sample a new point $\xi_2^{(3)} = 0.5$ and form the tree given in Figure 3. A stage 2 violation occurs, if the optimal value on tree $\hat{\mathcal{T}}^{3,3}$ is larger than the optimal value on tree $\hat{\mathcal{T}}^{3,2}$, i.e. whenever $v(\hat{\mathcal{T}}^{3,3}) > v(\hat{\mathcal{T}}^{3,2})$.

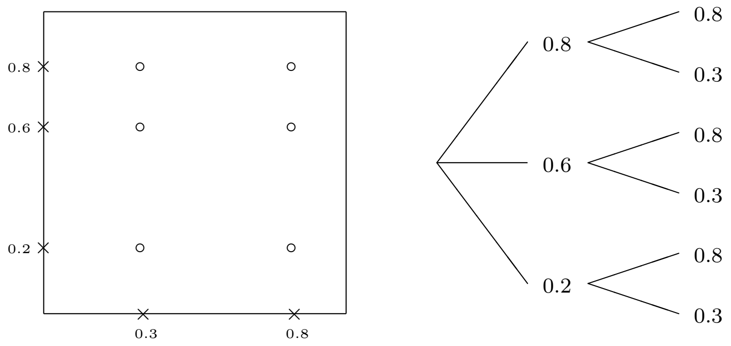

Fig. 1 The original sampled tree $\hat{\mathcal{T}}^{3,2}$.

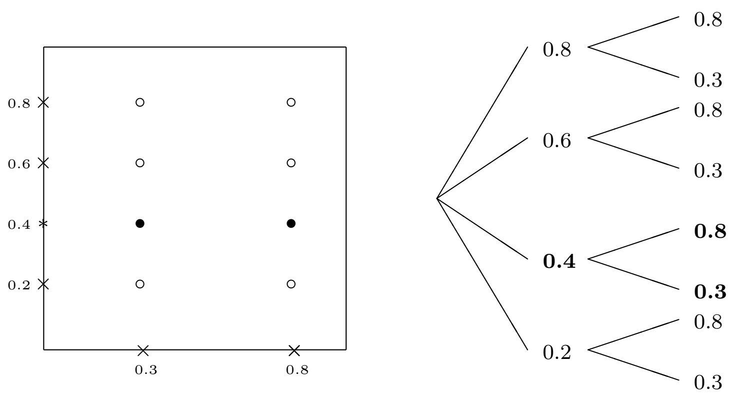

Fig. 2 The randomly extended tree $\hat{\mathcal{T}}^{4,2}$. The new nodes are in bold.

There is an important difference between the work of Vayanos et al. [47] and our approach. We assume that the total uncertainty set is the cartesian product set $\Xi = \times_{t=1}^{H} \Xi_t$ and consider therefore trees, while [47] consider always paths. To illustrate it, suppose that we sample N paths from the product set $\Xi = \Xi_1, \times \cdots \times \Xi_H$ denoted by $(\xi_1^{(i)}, \ldots, \xi_H^{(i)})$ for $i = 1, \ldots, N$. Since Ξ has product structure, any combination of points from Ξ_t, $t = 1, \ldots, H$ is a valid point in Ξ. Thus

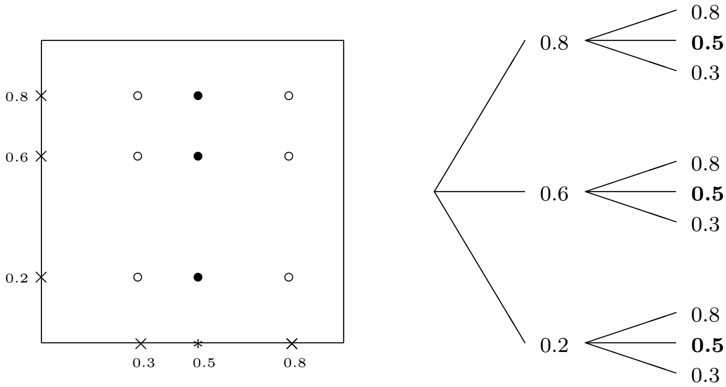

Fig. 3 The randomly extended tree $\hat{\mathcal{T}}^{3,3}$. The new nodes are in bold.

by sampling N path, we get in fact N^H points

$$(\xi_1^{(i_1)}, \ldots, \xi_t^{(i_t)}, \ldots, \xi_H^{(i_H)}), \qquad i_t \in \{1, \ldots, N\} ,$$

i.e. all possible selections of N points from $\hat{\Xi}_t, t = 1, \ldots, H$. These selections can be organised in a tree with N^H leaves. Since our tree contains N^H paths and not just N, a violation is much more likely to occur and be detected earlier than in the path-oriented approach.

The calculation ignoring the combined samples leads to an underestimation of the true value (the true costs) of the underlying problem and also the violation probability. Compared to [47], our notion of violation is different and stronger. The tree structure as we consider it in this paper guarantees the best (lower) bound of the true value, which is obtainable from all the samples obtained so far.

Moreover, we allow different sample sizes N_t for different stages, which may be important, since often the sizes of the uncertainty sets vary and increase by stages (see Remark 2.3). In the case that Ξ is not a product set, not all combinations of selections are valid points in Ξ and the tree may smaller, i.e. contain less leaves than N^H. However, the product form is a standard assumption in robust multi-stage optimization.

In the following, we further illustrate the difference between the path-oriented approach as in [47] and the tree structured model.

Example. Consider a three-stage problem ($H = 2$) with two sampled points $\xi_1^{(1)}, \xi_1^{(2)} \in \Xi_1$ and two sampled points $\xi_2^{(1)}, \xi_2^{(2)} \in \Xi_2$. In Figure 4, the path-oriented problem as in [47] and our tree-structured problem are depicted for illustration.

W.l.o.g. we set $A = I; h_1 = 0$ and therefore $x_1 = 0$ and $T_1(\xi_1) = 0$. Set

$$\begin{aligned}
c_{ji} &:= c_j(\xi_{j-1}^{(i)}), \qquad i = 1, 2; \quad j = 2, 3 \\
T_{ji} &:= T_j(\xi_j^{(i)}), \qquad i = 1, 2; \quad j = 2 \\
W_{ji} &:= W_j(\xi_{j-1}^{(i)}), \qquad i = 1, 2; \quad j = 2, 3 \\
h_{ji} &:= h_j(\xi_{j-1}^{(i)}), \qquad i = 1, 2; \quad j = 2, 3 \\
x_{ji} &:= x_j(\xi_j^{(i)}), \qquad i = 1, 2; \quad j = 2, 3 .
\end{aligned}$$

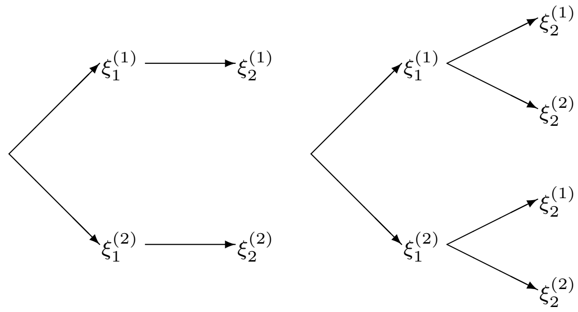

Fig. 4 Left: A fan with 2 paths. Right: The pertaining tree using the same data has $2^2 = 4$ paths.

We consider a problem including equality and inequality constraints, which can be brought to the form (21) by introducing slack variables:

$$\begin{aligned}
\min\ & (\max(c_{21}x_{21}, c_{22}x_{22}) + \max(c_{31}x_{31}, c_{32}x_{32})) \\
\text{s.t. } & W_{21}x_{21} = h_{21} \\
& W_{22}x_{22} = h_{22} \\
& T_{21}x_{21} + W_{31}x_{31} \leq h_{31} \\
& T_{22}x_{22} + W_{32}x_{32} = h_{32} \\
& T_{21}x_{21} + W_{32}x_{32} = h_{32} && (24) \\
& T_{22}x_{22} + W_{31}x_{31} \leq h_{31} && (25) \\
& x_{ji} \geq 0, \qquad i = 1, 2; j = 2, 3 \ .
\end{aligned}$$

If one argues pathwise then there are only two paths: $(\xi_1^{(1)}, \xi_2^{(1)})$ and $(\xi_1^{(2)}, \xi_2^{(2)})$ and the constraints (24) and (25) disappear.

For the concrete choice consider:

$$\begin{aligned}
(W_{21}, W_{22}, W_{31}, W_{32}) &= (1, 1, 1, 1) \\
(T_{21}, T_{22}) &= (1, 1) \\
(h_{21}, h_{22}, h_{31}, h_{32}) &= (1, 2, 2, 2) \ .
\end{aligned}$$

One can see that the path-oriented problem without constraints (24) and (25) is feasible, while the tree-structured problem including these two constraints is infeasible. Therefore the tree-structured problem may detect violation earlier than the path-oriented problem since we allow violation to occur at every stage by adding a new point from $\xi_t^{N_t+1} \in \Xi_t$ and not only sampling a complete new path $\left(\xi_1^{(N+1)}, \dots, \xi_H^{(N+1)}\right)$ as is done in [47].

In order to prove the main result of this section, we use a variable-split notation for the tree $\hat{\mathcal{T}}^{N_1,\dots,N_H}$. The tree has $\bar{N}_H = \prod_{s=1}^H N_s$ leaves, indexed by $\ell = 1, \dots, \bar{N}_H$. Notice that the leaves represent the random scenarios, i.e. there is a one-to-one correspondence between the scenarios and the tree leaves. Hence, to select a sample path $\underline{\xi}_H$, we may equivalently select a leaf index ℓ.

For every leaf ℓ the index of the predecessor at stage t is denoted by $p_t(\ell)$. Moreover, we introduce the relation $\ell_1 \sim_t \ell_2$, to denote the fact that the leaves ℓ_1, ℓ_2 share the same predecessors at stage t, i.e. $p_t(\ell_1) = p_t(\ell_2)$. We denote by $x_{t,\ell}$ the decision variable at stage t for the scenario ℓ. Note that, in principle, a different decision $x_{t,\ell}$ has to be made at each stage $t \in 1, \dots, H+1$

and for each scenario $\ell \in 1,\dots,\bar{N}_H$. However, the non-anticipativity condition requires that $x_{t,\ell_1} = x_{t,\ell_2}$, if $\ell_1 \sim_t \ell_2$, that is if the leaves ℓ_1, ℓ_2 share the same predecessor at stage t.

Formally, at each stage t we have an $\bar{N}_H$-vector $x_{t,\cdot}$, containing the different decisions at stage t corresponding to the different scenarios/sample-paths. However, as observed, the relation $\sim_t$ – and the related non-anticipativity constraints – dissects the set $\{1,2,\dots,\bar{N}_H\}$ into equivalence classes. The constraint that decisions in the same equivalence class must share the same value can be expressed by the condition $(x_{t,\cdot}) \in I_t$ where I_t is a linear subspace. For instance, all vectors $(x_{1,\cdot}) \in I_1$ have all identical components, and for $t > 1$ some subgroups of components must share the same value.

These considerations allow us to reformulate our basic multi-stage robust problem. To this end, let, $\mathbb{X}_1 = \{x_1 \geq 0 : Ax_1 = h_1\}$ as in (13), and define:

$$\mathbb{X}_t(x_{t-1},\xi_{t-1}) := \{x_t \geq 0 : T_{t-1}(\xi_{t-1})x_{t-1} + W_t(\xi_{t-1})x_t = h_t(\xi_{t-1})\} \ .$$

The basic problem $\widehat{\mathrm{RO}}_{H+1}^{N_1\cdots N_H}$ formulated on the sampled tree can be written in a compact form as:

$$\widehat{\mathrm{RO}}_{H+1}^{N_1\cdots N_H} \quad : \quad \min_{(x_{1,\cdot})\in I_1} \max_{\ell} \min_{(x_{2,\cdot})\in I_2} \max_{\ell} \min_{(x_{3,\cdot})\in I_3} \dots \max_{\ell} f_\ell(x_{1,\ell},\dots,x_{H,\ell},x_{H+1,\ell}) \ , \tag{26}$$

where the functions f_ℓ are defined as follows, for $\ell = 1,\dots,\bar{N}_H$

$$f_\ell(x_{1,\ell},\dots,x_{H+1,\ell}) := c_1^\top x_{1,\ell} + \psi_{\mathbb{X}_1}(x_{1,\ell}) + \sum_{t=2}^{H+1} (c_{t,\ell}^\top x_{t,\ell} + \psi_{\mathbb{X}_t(x_{t-1,\ell},\xi_{p_t(\ell)})}(x_{t,\ell})) \ ,$$

where $\psi_{(\cdot)}$ is defined in (10) and $c_{t,\ell}$ is the cost at stage t in scenario ℓ. Notice that, in (26), the minima can be taken over all paths ℓ, since if $\ell_1 \sim_t \ell_2$, then the function values for ℓ_1 and ℓ_2 are identical.

Let us now introduce the first-stage objective function:

$$\bar{f}(x_1,\ell) = \min_{(x_{2,\cdot})\in I_2} \max_{\ell} \min_{(x_{3,\cdot})\in I_3} \dots \max_{\ell} f_\ell(x_{1,\ell},\dots,x_{H+1,\ell}) \ .$$

Two crucial observations can be made about the function $\bar{f}(x_1,\ell)$:

1. The function $\bar{f}(x_1,\ell)$ is constant on the equivalence classes given by I_2. In particular, one may write it as $\bar{f}(x_1,\xi_1)$.
2. The function is convex in the variable x_1 for given ℓ (and hence, for given ξ_1).
   In particular, the convexity of $\bar{f}$ in x_1 can be seen from the following two facts:
   i) If $(x,y) \mapsto f(x,y)$ is jointly convex, then $x \mapsto \min_y f(x,y)$ is also convex.
   ii) The maximum of convex functions is convex.
   These properties hold for our case at hand (and it is the basic underlying reason in the scenario with certificates results in [25]).

Now, we keep the samples $\hat{\Xi}_2,\dots,\hat{\Xi}_H$ fixed and look only at the dependency on ξ_1. In particular, to analyze the first-stage violation, we introduce a previously unobserved random value $\xi_1^{(N_1+1)}$ at stage one, keeping all the other stages fixed. It is immediate to observe that we are again in the standard setup of Proposition 1. Indeed, we find that the violation probability at stage one is the violation probability of the problem:

$$\min_{x_1} \max_{\xi_1} \bar{f}(x_1,\xi_1) \ .$$

Therefore, we get the estimate:

$$\mathbb{P}\left\{V_1(\hat{\Xi}_1^{N_1},\ldots,\hat{\Xi}_H^{N_H})>\epsilon\right\} \le B(N_1,\epsilon,n_1+1)\ , \tag{27}$$

where $n_1 = dim(x_1)$ and $1 = dim(\gamma_1)$ with

$$\gamma_1 := \max_{\ell} \min_{(x_{2,\cdot})\in I_2} \max_{\ell} \min_{(x_{3,\cdot})\in I_3} \ldots \max_{\ell} f_\ell(x_{1,\ell},\ldots,x_{H,\ell},x_{H+1,\ell})\ .$$

Similarly, at stage t, there are $\bar{N}_{t-1} = \prod_{s=1}^{t-1} N_s$ nodes of the tree. Denoting with $\hat{\mathcal{T}}_j^{N_t,\ldots,N_H}$ the sub-tree born from node j, the violation probability at stage t and a fixed node j defined as

$$V_{t,j}(\hat{\Xi}_1^{N_1},\ldots,\hat{\Xi}_H^{N_H}) := \mathbb{P}\left\{\xi_t^{(N_t+1)} : v(\hat{\mathcal{T}}_j^{N_t+1,\ldots,N_H}) > v(\hat{\mathcal{T}}_j^{N_t,\ldots,N_H})\right\}\ , \tag{28}$$

and it follows that

$$\mathbb{P}\left\{V_{t,j}(\hat{\Xi}_1^{N_1},\ldots,\hat{\Xi}_H^{N_H})>\epsilon\right\} \le B(N_t,\epsilon,n_t+1),$$

where as before $n_t = dim(x_t)$ and $1 = dim(\gamma_{t-1})$ with

$$\gamma_{t-1} := \min_{(x_{t,\cdot})\in I_t} \ldots \max_{\ell} f_\ell(x_{1,\ell},\ldots,x_{H,\ell},x_{H+1,\ell})\ .$$

Notice that this bound does not depend on j. Now:

$$\begin{aligned} V_t(\hat{\Xi}_1^{N_1},\ldots,\hat{\Xi}_H^{N_H}) &= \mathbb{P}\left\{\text{ Violation at any node at stage } t | \hat{\Xi}_1^{N_1},\ldots,\hat{\Xi}_H^{N_H}\right\} \\ &\le \sum_{j=1}^{\bar{N}_{t-1}} \mathbb{P}\left\{\text{ Violation at node } j \text{ at stage } t\ | \hat{\Xi}_1^{N_1},\ldots,\hat{\Xi}_H^{N_H}\right\} \\ &= \sum_{j=1}^{\bar{N}_{t-1}} V_{t,j}(\hat{\Xi}_1^{N_1},\ldots,\hat{\Xi}_H^{N_H})\ , \end{aligned}$$

where the inequality follows by the fact that $V_{t,j}$, $j = 1,\ldots,\bar{N}_{t-1}$ are possibly dependent random variables. Now, we use the following result (whose proof is reported in Appendix C):

Lemma 2.1 *Let $Z_1,\ldots,Z_K$ be a sequence of identically distributed, but possibly dependent random variables. Then*

$$\mathbb{P}\left\{\sum_{i=1}^{K} Z_i \ge z\right\} \le K\mathbb{P}\left\{Z_i \ge z/K\right\}. \tag{29}$$

This gives:

$$\mathbb{P}\left\{\sum_{j=1}^{\bar{N}_{t-1}} V_{t,j}(\hat{\Xi}_1^{N_1},\ldots,\hat{\Xi}_H^{N_H})>\epsilon\right\} \le \bar{N}_{t-1}B(N_t,\epsilon/\bar{N}_{t-1},n_t+1)\ ,$$

assuming that the probability distribution $\mathbb{P}$ has nonvanishing Lebesgue density.
The above line of reasoning proves the following theorem, which constitutes the main result of this paper.

Theorem 2.3 (Violation probability at stage t of sampled scenario tree) *Given an accuracy level $\epsilon \in (0,1)$, let $\bar{N}_{t-1} = \prod_{s=1}^{t-1} N_s$ and $\epsilon_t := \epsilon/\bar{N}_{t-1}$. Then, the probability of violation at stage t, $V_t(\hat{\Xi}^{N_1}, \dots, \hat{\Xi}^{N_H})$ defined in (23), is bounded as:*

$$\mathbb{P}\left\{V_t(\hat{\Xi}^{N_1}, \dots, \hat{\Xi}^{N_H}) > \epsilon\right\} \leq \bar{N}_{t-1} B(N_t, \epsilon_t, n_t + 1) \ , \tag{30}$$

where $n_t = dim(x_t)$.

Remark 2.3 In the light of the above result, we can derive the required sample size to guarantee an ϵ-exceedance of the stagewise violations V_t, $t = 1, \dots, H$ being smaller than β:

- N_1 has to be chosen larger than $N_1^* = \frac{1}{\epsilon}\frac{e}{e-1}(\ln\frac{1}{\beta} + n_1 + 1)$.
- Given N_1^*, the number N_2 has to be at least $N_2^* = \frac{N_1^{*2}}{\epsilon}\frac{e}{e-1}(\ln\frac{1}{\beta} + n_2 + 1)$.
- Given the values $N_1^*, \dots, N_{t-1}^*$, form $\bar{N}_{t-1}^* = \prod_{s=1}^{t-1} N_s^*$ and choose N_t at least $N_t^* = \frac{\bar{N}_{t-1}^{*2}}{\epsilon}\frac{e}{e-1}(\ln\frac{1}{\beta} + n_t + 1)$.

Example. Notice that these sample sizes are calculated under a worst-case setup. In practical cases one needs much fewer samples. Here is a table for the required sample size resulting from our above calculations, assuming that $n_t = 2$.

ϵ	β	N_1^*	N_2^*
0.2	0.1	42	74088
0.1	0.1	84	592704
0.2	0.05	48	110592
0.1	0.05	95	857375

Remark 2.4 Notice that the sample complexity result given in Theorem 2.3 can be easily extended with similar considerations to multi-stage convex robust programs of the following type:

$$\begin{aligned} \text{CRO}_{H+1} \quad : \quad & \min_{x_1, x_2(\xi_1), \dots, x_{H+1}(\underline{\xi}_H)} \sup_{\underline{\xi}_H \in \Xi} g(x_1, x_2(\xi_1), \dots, x_{H+1}(\underline{\xi}_H), \underline{\xi}_H) \\ & \text{s.t. } h(x_1, x_2(\xi_1), \dots, x_{H+1}(\underline{\xi}_H), \underline{\xi}_H) \leq 0, \ \forall \underline{\xi}_H \in \Xi \\ & x_1 \geq 0 \ , \quad x_t(\underline{\xi}_{t-1}) \geq 0 \ , \ t = 2, \dots, H+1 \ , \end{aligned}$$

where $g : \mathbb{R}^{\sum_{t=1}^{H+1} n_t} \times \Xi \to \mathbb{R}$ and $h : \mathbb{R}^{\sum_{t=1}^{H+1} n_t} \times \Xi \to \mathbb{R}$ are convex in $x_t \in \mathbb{R}_+^{n_t}$, $t = 1, \dots, H+1$ and continuous in $(x_t, \underline{\xi}_H)$.

2.6 The "total" violation probability

Theorem 2.3 provides a way to bound the probability of violation at stage t. This result can be used to bound the probability of "total" violation, which we define as the probability of violating *at any* stage t. Formally we write:

$$V_{\text{TOT}}(\hat{\Xi}_1^{N_1}, \dots, \hat{\Xi}_H^{N_H}) := \mathbb{P}\left\{\xi_1^{(N_1+1)}, \dots, \xi_H^{(N_H+1)} : \exists t \text{ s.t. } v(\hat{\mathcal{T}}^{N_1, \dots, N_t+1, \dots, N_H}) > v(\hat{\mathcal{T}}^{N_1, \dots, N_H})\right\} \ . \tag{31}$$

We note that that the above quantity may be immediately bounded as follows:

$$V_{\text{TOT}}(\hat{\Xi}_1^{N_1}, \dots, \hat{\Xi}_H^{N_H}) \leq V_1(\hat{\Xi}_1^{N_1}, \dots, \hat{\Xi}_H^{N_H}) + \dots + V_H(\hat{\Xi}_1^{N_1}, \dots, \hat{\Xi}_H^{N_H}) \ .$$

3 Lower Bounds for Multi-stage Linear Robust Optimization Problems

Due to the large number of required samples, problem $\widehat{RO}_{H+1}^{N_1\ldots N_H}$ is typically difficult to solve. Consequently, it is advisable to solve simpler problems allowing to obtain at least guaranteed bounds for it. Notice that $\widehat{RO}_{H+1}^{N_1\ldots N_H}$ gives a lower bound for the original problem RO_{H+1} for any size of the random extractions. As to upper bounds, any feasible decision of the original problem gives an upper bound. Thus, by extending the solution of the sampled subproblem to a solution of the original problem, guaranteed upper bounds are obtained. An extension would assign the decision x_t to a history $(\xi_1, \ldots, \xi_t)$, by taking the same value as assigned to the nearest history $(\xi_1^{(i_1)}, \xi_2^{(i_1 1_2)}, \ldots, \xi_t^{(i_1,\ldots,i_t)})$ in the sample.

Several construction priciples for lower bounds are known in the context of stochastic programming, see for instance [31,32,33,34]. Here we adapt them for the sampled scenario approach and compare them in terms of optimal objective function values for the case of robust multi-stage linear programs. We remark that a general principle for obtaining lower bounds is to relax some of the constraints. Relaxing non-anticipativity constraints leads typically to a computationally much simpler problem, especially for the sampled approximations (see later).

First, we introduce the *robust multi-stage wait-and-see* problem RWS_{H+1}, where the realizations of all the history of the random parameters $\underline{\xi}_H = (\xi_1, \ldots, \xi_H)$ are assumed to be known at the first-stage. This problem takes the following form:

$$\begin{aligned}
\mathrm{RWS}_{H+1} \quad : \quad & \sup_{\underline{\xi}_H} \min_{x_1(\underline{\xi}_H),\ldots,x_{H+1}(\underline{\xi}_H)} c_1^\top x_1(\underline{\xi}_H) + \ldots + c_{H+1}^\top(\xi_H) x_{H+1}(\underline{\xi}_H) \\
& \text{s.t. } Ax_1(\underline{\xi}_H) = h_1,\ x_1(\underline{\xi}_H) \geq 0 \\
& \qquad T_1(\xi_1)x_1(\underline{\xi}_H) + W_2(\xi_1)x_2(\underline{\xi}_H) = h_2(\xi_1) \\
& \qquad \vdots \\
& \qquad T_H(\xi_H)x_H(\underline{\xi}_H) + W_{H+1}(\xi_H)x_{H+1}(\underline{\xi}_H) = h_{H+1}(\xi_H) \\
& \qquad x_t(\underline{\xi}_H) \geq 0\ ,\ t = 2, \ldots, H+1\ .
\end{aligned} \tag{32}$$

Notice that, in the above setup, the minimum and supremum have been exchanged. Hence, the decision process has become *anticipative*, since the decisions $x_1, x_2, \ldots, x_{H+1}$ depend on a given realization of $\underline{\xi}_H$.

We introduce the following definition, which is an immediate extension of the concept of *Expected Value of Perfect Information* for stochastic programs:

Definition 3.1 The difference

$$\mathrm{RVPI}_{H+1} := v(\mathrm{RO}_{H+1}) - v(\mathrm{RWS}_{H+1})\ , \tag{33}$$

denotes the *Robust Value of Perfect Information* and compares robust multi-stage wait-and-see RWS_{H+1} with robust multi-stage RO_{H+1}.

Note that the RVPI_{H+1} can be interpreted as a measure of the advantage of reaching perfect information in advance: A small RVPI_{H+1} indicates a small advantage for reaching the perfect information since all possible realizations of uncertainty have similar costs. In particular, the following inequality can be proven.

Proposition 3.1 (lower bound for $\mathbf{RO}_{H+1}$) *Given the robust multi-stage linear optimization problem RO_{H+1} defined in (21), and the robust multi-stage wait-and-see problem RWS_{H+1} defined in (32), the following inequality holds true:*

$$v(RWS_{H+1}) \leq v(RO_{H+1})\ . \tag{34}$$

The proof is given in Appendix D.

A second lower bound for problem RO_{H+1} can be obtained by relaxing the non-anticipativity constraints only at stages $2,\ldots,H$ and replacing the future from stage 2 with a single sample path (see [31]). The ensuing program is the so-called *robust two-stage relaxation* RT_{H+1}. Formally, consider the discrete random process as follows:

$$\underline{\tilde{\xi}}_t := (\xi_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_t),\ t = 2, \ldots, H\ ,$$

where $\tilde{\xi}_t$, is a deterministic realization of the random process ξ_t. We denote the robust two-stage relaxation problem RT_{H+1}, as follows:

$$\begin{aligned}
\mathrm{RT}_{H+1} : \min_{x_1} c_1^\top x_1 + \sup_{\xi_1} \Big[& \min_{x_2,\ldots,x_{H+1}} c_2^\top(\xi_1) x_2(\underline{\tilde{\xi}}_H) + c_3^\top(\tilde{\xi}_2) x_3(\underline{\tilde{\xi}}_H) + \ldots + c_{H+1}^\top(\tilde{\xi}_H) x_{H+1}(\underline{\tilde{\xi}}_H) \Big] \quad (35)\\
\text{s.t. } & Ax_1 = h_1,\ x_1 \geq 0\\
& T_1(\xi_1)x_1 + W_2(\xi_1)x_2(\underline{\tilde{\xi}}_H) = h_2(\xi_1),\ \forall \xi_1 \in \varXi_1\\
& \vdots\\
& T_H(\tilde{\xi}_H)x_H(\underline{\tilde{\xi}}_H) + W_{H+1}(\tilde{\xi}_H)x_{H+1}(\underline{\tilde{\xi}}_H) = h_{H+1}(\tilde{\xi}_H), \forall \xi_1 \in \varXi_1\\
& x_t(\underline{\tilde{\xi}}_H) \geq 0,\ t = 2, \ldots, H+1,\ \forall \xi_1 \in \varXi_1\ .
\end{aligned}$$

There are no non-anticipativity conditions here (except for the first-stage decisions).

Finally we remark that one may introduce intermediate relaxation steps by just relaxing some of the later non-anticipativities (or moving the max-operators to left only for stages later than a given stage P). Relaxing the non-anticipativity constraints in stages $P,\ldots,H$ with $P = 3,\ldots,H-1$ and replacing the future from stage P with a single sample path, hence considering a discrete random process:

$$\underline{\tilde{\xi}}_{P,H} := (\xi_1, \ldots, \xi_{P-1}, \tilde{\xi}_P, \ldots, \tilde{\xi}_H)\ ,$$

we can get a sequence of lower bounds by stepwise relaxation from the end to the beginning. Denoting by $v(\mathrm{RO}_{P,H+1})$ the value of this robust P-stage relaxation, and following reasons similar to those in the proof of Proposition 3.1, the following bounds can be proven. In particular, it is clear that $\mathrm{RO}_{1,H+1} = \mathrm{RWS}_{H+1}$ and $\mathrm{RO}_{2,H+1} = \mathrm{RT}_{H+1}$.

Proposition 3.2 (Chain of lower bounds for RO_{H+1}) *Given the robust multi-stage linear optimization problem RO_{H+1} (21), the robust multi-stage wait-and-see problem RWS_{H+1} (32), the robust two-stage relaxation problem RT_{H+1} and the robust P-stage relaxation problem $RO_{P,H+1}$, $P = 3,\ldots,H-1$, the following inequalities hold true*

$$v(RWS_{H+1}) = v(RO_{1,H+1}) \leq v(RT_{H+1}) = v(RO_{2,H+1}) \leq \ldots \leq v(RO_{P,H+1}) \leq \ldots \leq v(RO_{H+1}). \quad (36)$$

The above results have a clear theoretical meaning. However, it should be remarked that, in the general case, problems $\mathrm{RO}_{P,H+1}$ may be hard to solve in practice. In such case, it becomes of great interest to introduce and study the sampled versions of them. In particular, given $\widehat{RO}_{H+1}^{N_1\ldots N_H}$ and a collection of independently sampled values $\hat{\varXi}_1^{N_1},\ldots,\hat{\varXi}_H^{N_H}$, we can introduce the *sampled* *robust wait-and-see* problem $\widehat{\mathrm{RWS}}^{\bar{N}_H}_{H+1}$, based on the extraction of $\bar{N}_H = N_1 \cdot N_2 \cdot \ldots \cdot N_H$ iid samples $\underline{\xi}_H^{(1)}, \ldots, \underline{\xi}_H^{(\bar{N}_H)}$ from $\hat{\Xi}_1^{N_1}, \hat{\Xi}_2^{N_2}, \ldots, \hat{\Xi}_H^{N_H}$:

$$\begin{aligned}
\widehat{\mathrm{RWS}}^{\bar{N}_H}_{H+1} : \max\nolimits_{i=1,\ldots,\bar{N}_H} \quad & \min_{x_1(\underline{\xi}_H^{(i)}),\ldots,x_H(\underline{\xi}_H^{(i)})} c_1^\top x_1(\underline{\xi}_H^{(i)}) + \ldots + c_{H+1}^\top(\xi_H^{(i)}) x_{H+1}(\underline{\xi}_H^{(i)}) \\
& \text{s.t. } A x_1(\underline{\xi}_H^{(i)}) = h_1,\ x_1(\underline{\xi}_H^{(i)}) \geq 0 \\
& \quad T_1(\xi_1^{(i)}) x_1(\underline{\xi}_H^{(i)}) + W_2(\xi_1^{(i)}) x_2(\underline{\xi}_H^{(i)}) = h_2(\xi_1^{(i)}) \\
& \quad \vdots \\
& \quad T_H(\xi_H^{(i)}) x_H(\underline{\xi}_H^{(i)}) + W_{H+1}(\xi_H^{(i)}) x_{H+1}(\underline{\xi}_H^{(i)}) = h_{H+1}(\xi_H^{(i)}) \\
& \quad x_t(\underline{\xi}_H^{(i)}) \geq 0\ ,\ t = 2, \ldots, H+1,\ i = 1, \ldots, \bar{N}_H\ .
\end{aligned} \tag{37}$$

Similarly, one can extract N_1 iid scenarios $\xi_1^{(i)}$, $i = 1, \ldots, N_1$ and keep the rest $\tilde{\xi}_2, \ldots, \tilde{\xi}_H$ deterministic such that $\underline{\tilde{\xi}}_H^{(i)} := (\xi_1^{(i)}, \tilde{\xi}_2, \ldots, \tilde{\xi}_H)$ and construct the *sampled robust two-stage relaxation* problem $\widehat{\mathrm{RT}}^{N_1}_{H+1}$ given by:

$$\begin{aligned}
\widehat{\mathrm{RT}}^{N_1}_{H+1} \quad : \quad & \min_{x_1,\gamma_1} c_1^\top x_1 + \gamma_1 \\
& \text{s.t. } A x_1 = h_1,\ x_1 \geq 0 \\
& \qquad \mathcal{Q}_1(x_1, \underline{\tilde{\xi}}_H^{(i)}) \leq \gamma_1, \quad i = 1, \ldots, N_1\ ,
\end{aligned} \tag{38}$$

where

$$\begin{aligned}
\mathcal{Q}_1(x_1, \underline{\tilde{\xi}}_H^{(i)}) := & \min_{x_2,\ldots,x_{H+1}} c_2^\top(\xi_1^{(i)}) x_2(\xi_1^{(i)}) + c_3^\top(\tilde{\xi}_2) x_3(\underline{\tilde{\xi}}_2^{(i)}) + \ldots + c_{H+1}^\top(\tilde{\xi}_H) x_{H+1}(\underline{\tilde{\xi}}_H^{(i)}) \\
& \text{s.t. } T_1(\xi_1^{(i)}) x_1 + W_2(\xi_1^{(i)}) x_2(\xi_1^{(i)}) = h_2(\xi_1^{(i)}) \\
& \quad \vdots \\
& \quad T_H(\tilde{\xi}_H) x_H(\underline{\tilde{\xi}}_H^{(i)}) + W_{H+1}(\tilde{\xi}_H) x_{H+1}(\underline{\tilde{\xi}}_H^{(i)}) = h_{H+1}(\tilde{\xi}_H) \\
& \quad x_t(\underline{\tilde{\xi}}_{t-1}^{(i)}) \geq 0, \qquad t = 2, \ldots, H+1\ .
\end{aligned} \tag{39}$$

The violation probability at stage one, $V_1(\hat{\Xi}_1^{N_1}, \tilde{\xi}_2, \ldots, \tilde{\xi}_H)$, of the objective function value returned by $\widehat{\mathrm{RT}}^{N_1}_{H+1}$ depends only on $dim(x_1) + dim(\gamma_1) = n_1 + 1$, i.e.:

$$\mathbb{P}\left\{ V_1(\hat{\Xi}_1^{N_1}, \tilde{\xi}_2, \ldots, \tilde{\xi}_H) > \epsilon \right\} \leq B(N_1, \epsilon, n_1 + 1)\ .$$

The dimension of the variables at stages greater than 1 is irrelevant, which is a quite remarkable fact.

Similarly a *sampled robust P-stage relaxation* $\widehat{\mathrm{RO}}^{N_1 \ldots N_{P-1}}_{P,H+1}$ of problem $\mathrm{RO}_{P,H+1}$ can be defined. Again, probabilistic guarantees of the solution of problem $\widehat{\mathrm{RO}}^{N_1 \ldots N_{P-1}}_{P,H+1}$ can be obtained on the same lines of Theorem 2.3.

We conclude this section by providing the following proposition, which shows the relationship between the various lower bounds based on sampling presented in this paper.

Proposition 3.3 (Chain of sampling-based lower bounds for RO$_{H+1}$) *Given the robust multi-stage linear optimization problem RO_{H+1} (21), the sampled robust optimization problem $\widehat{RO}_{H+1}^{N_1\ldots N_H}$ (22), the sampled robust multi-stage wait-and-see problem $\widehat{RWS}_{H+1}^{\bar{N}_H}$ (37), the sampled robust two-stage relaxation $\widehat{RT}_{H+1}^{N_1}$ (38) and the sampled robust P-stage relaxation $\widehat{RO}_{P,H+1}^{N_1\ldots N_{P-1}}$ for a fixed collection of independently sampled values $\hat{\Xi}_1^{N_1},\ldots,\hat{\Xi}_N^{N_H}$, the following chain of inequalities holds true:*

$$v(\widehat{RWS}_{H+1}^{\bar{N}_H}) \le v(\widehat{RT}_{H+1}^{N_1}) \le \ldots \le v(\widehat{RO}_{P,H+1}^{N_1\ldots N_{P-1}}) \le \ldots \le v(\widehat{RO}_{H+1}^{N_1\ldots N_H}) \le v(RO_{H+1}) \ . \quad (40)$$

4 Numerical Results: Inventory Management with Cumulative Orders Constraints

In this section, to show the effectiveness of the proposed approach, we consider a problem from inventory management which was originally considered in [4], describing the negotiation of flexible contracts between a retailer and a supplier in the presence of uncertain orders from customers. In particular, we analyze the performance of the approach proposed in this paper on a simplified version discussed in [11] and in [47]. We remark that the considered numerical problem is such that the optimal solution of the original multi-stage robust optimization problem can be assessed: This allows to evaluate the performance of the scenario tree based approach.

The problem setting can be summarized as follows: A retailer received orders ξ_t at the beginning of each time period $t \in \{1,\ldots,H\}$, $\underline{\xi}_t$ represents the demand history up to time t. The demand needs to be satisfied from an inventory with filling level s_t^{inv} by means of orders x_t^o at a cost d_t per unit of product. Unsatisfied demand may be backlogged at cost p_t and inventory may be held in the warehouse with a unitary holding cost h_t. Lower and upper bounds on the orders x_t^o ($\underline{x}_t^o$ and $\bar{x}_t^o$) at each period as well as on the cumulative orders s_t^{co} ($\underline{s}_t^{co}$ and $\bar{s}_t^{co}$) up to period t are imposed. We assume that there is no demand at time $t = 1$ and that the demand at time t lies within an interval centered around a nominal value $\bar{\xi}_t$ and uncertainty level $\rho \in [0,1]$ resulting in a box uncertainty set as follows: $\Xi = \times_{t=1}^{H} \left\{\xi_t \in \mathbb{R} : \left|\xi_t - \bar{\xi}_t\right| \le \rho\bar{\xi}_t\right\}$. Denoting with x_t^c the retailer's cost at stage t, the problem with *Cumulative Order Constraints* ($\mathcal{COC}$) can be modeled as a convex poblem of the following form

$$\text{RO}_{H+1}(\mathcal{COC}) : \quad \min_{x_t^o, x_t^c, s_t^{co}, s_t^{inv}} \left[x_1^c + \max_{\underline{\xi}\in\Xi} \sum_{t\in\mathbb{T}} x_{t+1}^c(\underline{\xi}_t) \right] \quad (41a)$$

$$\text{s.t. } x_1^c \ge d_1 x_1^o + \max\left\{h_1 s_1^{inv}, -p_1 s_1^{inv}\right\} \quad (41b)$$

$$x_{t+1}^c(\underline{\xi}_t) \ge d_{t+1} x_{t+1}^o(\underline{\xi}_t) + + \max\left\{h_{t+1} s_{t+1}^{inv}(\underline{\xi}_t), -p_{t+1} s_{t+1}^{inv}(\underline{\xi}_t)\right\}, \ t = 1,\ldots,H-1 \quad (41c)$$

$$x_{H+1}^c(\underline{\xi}_H) \ge \max\left\{h_{H+1} s_{H+1}^{inv}(\underline{\xi}_H), -p_{H+1} s_{H+1}^{inv}(\underline{\xi}_H)\right\} \quad (41d)$$

$$s_2^{inv}(\underline{\xi}_1) = s_1^{inv} + x_1^o - \xi_1 \quad (41e)$$

$$s_{t+1}^{inv}(\underline{\xi}_t) = s_t^{inv}(\underline{\xi}_{t-1}) + x_t^o(\underline{\xi}_{t-1}) - \xi_t \ , \quad t = 2,\ldots,H \quad (41f)$$

$$s_2^{co}(\underline{\xi}_1) = s_1^{co} + x_1^o \quad (41g)$$

$$s_{t+1}^{co}(\underline{\xi}_t) = s_t^{co}(\underline{\xi}_{t-1}) + x_t^o(\underline{\xi}_{t-1}) \ , \quad t = 2,\ldots,H \quad (41h)$$

$$\underline{x}_1^o \le x_1^o \le \bar{x}_1^o, \quad \underline{s}_1^{co} \le s_1^{co} \le \bar{s}_1^{co} \quad (41i)$$

$$\underline{x}_t^o \le x_t^o(\underline{\xi}_{t-1}) \le \bar{x}_t^o, \quad \underline{s}_t^{co} \le s_t^{co}(\underline{\xi}_{t-1}) \le \bar{s}_t^{co}, \quad t = 2,\ldots,H+1 \ . \quad (41j)$$

The objective function (41a) corresponds to minimizing the worst-case cumulative cost. Constraints (41b)-(41c)-(41d) define the stagewise costs $x^c_{t+1}(\underline{\xi}_t)$, $t = 1, \dots, H$ while constraints (41e)-(41f) and (41g)-(41h) respectively define the dynamics of the inventory level and cumulative orders. Finally, constraints (41i)-(41j) denote the lower and upper bounds on the instantaneous and cumulative orders. Notice that the decision process is non-anticipative.

We consider specific instances of problem $\mathrm{RO}_{H+1}(\mathcal{COC})$ as summarized in Table 1 under the assumption of two-stage ($H = 1$) and three-stage ($H = 2$) and uncertainty level $\rho = 30\%$ meaning that for a given value v, the uncertainty set is $[v(1-\rho), v(1+\rho)]$. The data presents some slight modifications of the data presented in [47].

Parameters of the Problem $\mathcal{COC}$	$t = 1$	$t = 2$
(p_t, d_t, h_t)	(11,1,10)	(11,1,10)
s_1^{inv}	0	
$(\underline{x}_t^o, \bar{x}_t^o)$	$(0, \infty)$	$(0, \infty)$
$\underline{s}_t^{co}$	47	134
$\bar{s}_t^{co}$	94	248
$\bar{\xi}_t = 100\left(1 + \frac{1}{2}\sin\left(\frac{\pi(t-2)}{6}\right)\right)$	75	100

Table 1 Input data for the inventory management problem.

We define optimality gaps of the scenario problem $\widehat{\mathrm{RO}}_{H+1}^{N_1 \dots N_H}(\mathcal{COC})$ as:

$$\textit{optimality gap} := \frac{v(\widehat{\mathrm{RO}}_{H+1}^{N_1 \dots N_H}(\mathcal{COC})) - v(\mathrm{RO}_{H+1}(\mathcal{COC}))}{v(\mathrm{RO}_{H+1}(\mathcal{COC}))} \, . \tag{42}$$

We note that the optimality gap in (42) can be computed, since problem $\mathrm{RO}_{H+1}(\mathcal{COC})$ can be solved exactly by using a scenario tree that consists of the vertices of the polytopic uncertainty set Ξ reported in Table 2 (see [11]).

Vertex	Ξ_1
1	52.5
2	97.5

Vertex	Ξ_1	Ξ_2
1	52.5	70
2	52.5	130
3	97.5	70
4	97.5	130

Table 2 Vertices of Ξ for the inventory management problem in the two-stage ($H = 1$) and three-stage ($H = 2$) cases.

To assess the performance of our approach, we compute the *empirical violation probability* $\hat{V}_t(\hat{\mathcal{T}}^{N_1,\dots,N_H})$ at stage $t = 1, \dots, H$ of the solution of a given scenario tree $\hat{\mathcal{T}}^{N_1,\dots,N_H}$ associated with the scenario problem $\widehat{\mathrm{RO}}_{H+1}^{N_1 \dots N_H}(\mathcal{COC})$, defined as:

$$\hat{V}_t(\hat{\mathcal{T}}^{N_1,\dots,N_H}) := \sum_{i=1}^{1000} \frac{\Big(v(\hat{\mathcal{T}}_i^{N_1,\dots,N_t+1,\dots,N_H}) - v(\hat{\mathcal{T}}^{N_1,\dots,N_H})\Big)}{1000} \quad , \quad t = 1, \dots, H \; , \tag{43}$$

where $\hat{\mathcal{T}}_i^{N_1,\dots,N_t+1,\dots,N_H}$, $i = 1, \dots, 1000$ is a new scenario tree with one new independent scenario $\xi_t^{(N_t+1)}$ from Ξ_t with respect to the tree $\hat{\mathcal{T}}^{N_1,\dots,N_H}$ and

$$(\alpha) := \begin{cases} 1 \text{ if } \alpha > 0 \\ 0 \text{ otherwise} \, ; \end{cases}$$

notice that the extended tree $\hat{\mathcal{T}}_i^{N_1,\dots,N_t+1,\dots,N_H}$ contains $\prod_{j=t+1}^{H} N_j$ new independent scenarios belonging to the sub-tree generated by the new scenario $\xi_t^{N_t+1}$ at stage t. In the two-stage case the extended tree $\hat{\mathcal{T}}_i^{N_1+1}$ contains only one new independent scenario extracted from Ξ_1.
The numerical results are obtained as follows:

- we fix a confidence level of $\beta = 0.001$ for the two-stage case and $\beta = 0.1$ for the three-stage case;
- we select the target violation probability $\epsilon = 0.0005, 0.001, 0.005, 0.01, 0.05, 0.1, 0.2, 0.3$;
- we compute the corresponding sample size $N_1^* = \frac{1}{\epsilon}\frac{e}{e-1}(\ln\frac{1}{\beta}+n_1+1)$ and $N_2^* = \frac{N_1^{*2}}{\epsilon}\frac{e}{e-1}(\ln\frac{1}{\beta}+n_2+1)$.
- we solve 100 instances of problem $\widehat{\text{RO}}_{H+1}^{N_1^*\dots N_H^*}$ each based on a different scenario tree $\hat{\mathcal{T}}^{N_1^*,\dots,N_H^*}$;
- for each instance, we compute the optimality gap given in formula (42) and empirical violation probability given in formula (43);
- we compute statistics over 100 instances.

The problems derived from the case study have been formulated and solved under *AMPL* environment along with *CPLEX* 20.1.0.0 solver. All computations have been performed on a 64-bit machine with 32 GB of RAM and an Intel Core i7-1065G7 CPU 1.30 GHz processor.

First, we evaluate the performance of the sample-based approximation $\widehat{\text{RO}}_2^{N_1^*}(\mathcal{COC})$ in the two-stage case ($H = 1$). Figure 5 displays the optimality gaps of problem $\widehat{\text{RO}}_2^{N_1^*}(\mathcal{COC})$ with respect to $\text{RO}_2(\mathcal{COC})$ for different values of violation probability ϵ (%) ranging from 30% down to 0.05%. The number of scenarios N_1^*, constraints and variables of the corresponding optimization models are reported in Table 3.

ϵ (%)	N_1^*	# of const.	# of var.
30	35	420	246
20	53	636	372
10	105	1260	736
5	209	2508	1464
1	1045	12540	7316
0.5	2090	25080	14631
0.1	10450	125400	73151
0.05	20899	250788	146294

Table 3 Number of scenarios N_1^*, constraints and variables for decreasing values of ϵ (%) in the two-stage case ($H = 1$) for the inventory management problem.

From the results shown in Figure 5 we can observe that the variance of $\widehat{\text{RO}}_2^{N_1^*}(\mathcal{COC})$ decreases substantially as ϵ decreases as well as the optimality gaps passing from -4.4% (in average) to $-10^{-5}\%$. The distribution of the empirical violation probability as function of ϵ is plotted in Figure 6, for the two-stage case. As expected, as ϵ decreases, the violation converges to 0. We also note that the empirical violation probability is smaller than ϵ in all the considered cases.

Finally, Figure 7 shows the average solver time (solid lines) and the number of scenarios (dashed lines) for problem $\widehat{\text{RO}}_2^{N_1^*}(\mathcal{COC})$ as a function of $Log(1/\epsilon)$. In particular, they are considerably lower than those used in [47], where the number of scenarios depends on the size of the basis and on the number of decision variables at each stage. On the other hand, we should remark that the number of variables used in our approach is larger, due to the introduction of sample-dependent certificates (or second-stage decision variables).

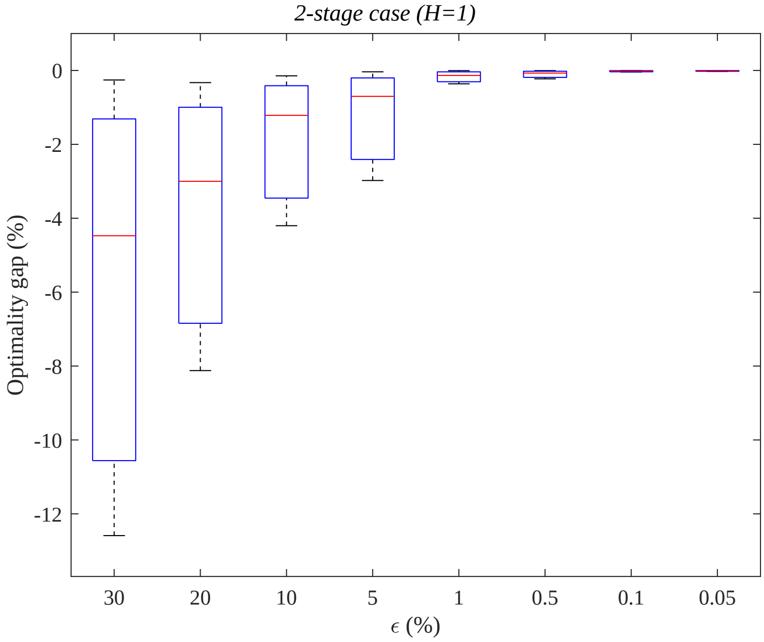

Fig. 5 Optimality gaps for $\widehat{\mathrm{RO}}_2^{N_1^*}(\mathcal{COC})$ (boxes and whiskers) for decreasing values of ϵ for the two-stage ($H=1$) case.

ϵ (%)	N_1^*	N_2^*	$\bar{N}_2$	# of const.	# of var.	average CPU time
30	23	12003	276069	1380488	828304	1586
20	35	41691	1459185	7296140	4377700	42149.7

Table 4 Number of scenarios at first period N_1^*, at second period N_2^* and in total $\bar{N}_2$, constraints, variables and average CPU time (in seconds) for $\epsilon = 30, 20$ (%) in the three-stage case ($H = 2$) for the inventory management problem.

Secondly, we evaluate the performance of the sample-based approximation $\widehat{\mathrm{RO}}_3^{N_1^* N_2^*}(\mathcal{COC})$ in the three-stage case ($H = 2$). The number of scenarios N_1^*, N_2^* and $\bar{N}_2$, constraints and variables of the corresponding optimization models with average CPU time over 100 instances are reported in Table 4 for $\epsilon = 20\%$ and 30%.

Results shows that the average solver time to solve problem $\widehat{\mathrm{RO}}_3^{N_1^* N_2^*}(\mathcal{COC})$ pass from 1586 CPU seconds (with $\epsilon = 30\%$) with a scenario tree with $N_1^* = 23$ and $N_2^* = 12003$ to 42149.7 CPU seconds (with $\epsilon = 20\%$), for a tree with $N_1^* = 35$, $N_2^* = 41691$ and $\overline{N}_2 = N_1^* N_2^* = 1459185$ scenarios.

From the results shown in Figure 8 we can observe that the optimality gaps of $\widehat{\mathrm{RO}}_3^{N_1^* N_2^*}(\mathcal{COC})$ decrease as ϵ decreases passing from -0.03% (in average) when $\epsilon = 30\%$ to -0.02% when $\epsilon = 20\%$.

The distribution of the empirical violation probabilities $\hat{V}_1(\hat{\mathcal{T}}^{N_1^* N_2^*})$ and $\hat{V}_2(\hat{\mathcal{T}}^{N_1^* N_2^*})$ as function of ϵ are plotted in Figures 9, for the three-stage case. We note that both the empirical violation probabilities are always smaller than ϵ. Results on $\hat{V}_1(\hat{\mathcal{T}}^{N_1^* N_2^*})$ and $\hat{V}_2(\hat{\mathcal{T}}^{N_1^* N_2^*})$ show that as ϵ decreases from 30% to 20%, both the empirical violation probabilities decrease passing from an average value of 6.8% to 5% and of 0.04% to 0%, respectively. Results also show that $\hat{V}_1(\hat{\mathcal{T}}^{N_1^* N_2^*})$ is always larger than $\hat{V}_2(\hat{\mathcal{T}}^{N_1^* N_2^*})$.

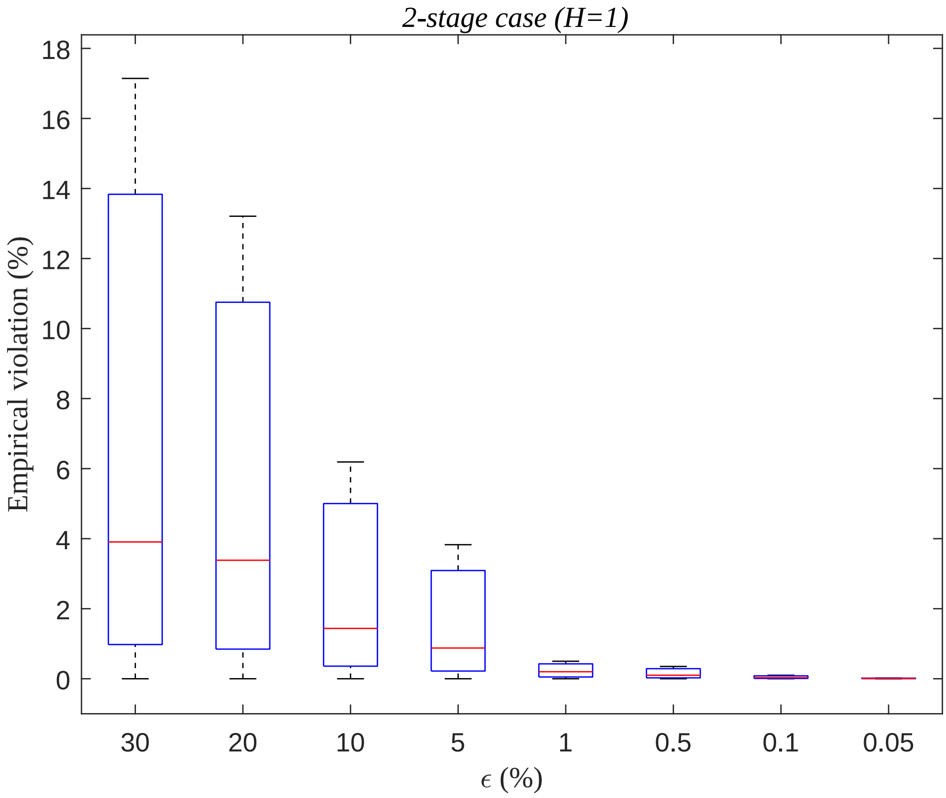

Fig. 6 Empirical violation probability for $\widehat{\mathrm{RO}}_2^{N_1^*}(\mathcal{COC})$ (boxes and whiskers) for increasing values of ϵ for the two-stage ($H = 1$) case.

4.1 Bounds for the Inventory Management with Cumulative Orders Constraints

In this section, we evaluate possible relaxations to problem $\mathrm{RO}_{H+1}(\mathcal{COC})$ as described in Section 3. In particular we consider the multi-stage wait-and-see problem $\mathrm{RWS}_{H+1}(\mathcal{COC})$ for problem $\mathrm{RO}_{H+1}(\mathcal{COC})$, and the robust two-stage relaxation problem $\mathrm{RT}_H(\mathcal{COC})$ where the non-anticipativity constraints are relaxed in stages $2, \ldots, H$. Again, we remark that for the case at hand these two problems can be computed exactly by considering only the vertices of Ξ. Similarly to formula (42), we define optimality gaps of the problem $\mathrm{RWS}_{H+1}(\mathcal{COC})$ as:

$$(\textit{optimality gap})_{\mathrm{RWS}_{H+1}(\mathcal{COC})} := \frac{v(\mathrm{RWS}_{H+1}(\mathcal{COC})) - v(\mathrm{RO}_{H+1}(\mathcal{COC}))}{v(\mathrm{RO}_{H+1}(\mathcal{COC}))} \,, \tag{44}$$

and in the same way for $\mathrm{RT}_{H+1}(\mathcal{COC})$.

The optimality gap of $\mathrm{RWS}_3(\mathcal{COC})$ turned out to be equal to -68%, passing from an objective function value of 725.35 for $\mathrm{RO}_3(\mathcal{COC})$ to 227.5; consequently the Robust Value of Perfect Information $\mathrm{RVPI}_3((\mathcal{COC}))$ is 497.85.

The optimality gap of $\mathrm{TP}_3(\mathcal{COC})$ turned out to be equal to -39%, passing from an objective function value of 725.35 for $\mathrm{RO}_3(\mathcal{COC})$ to 439.64.

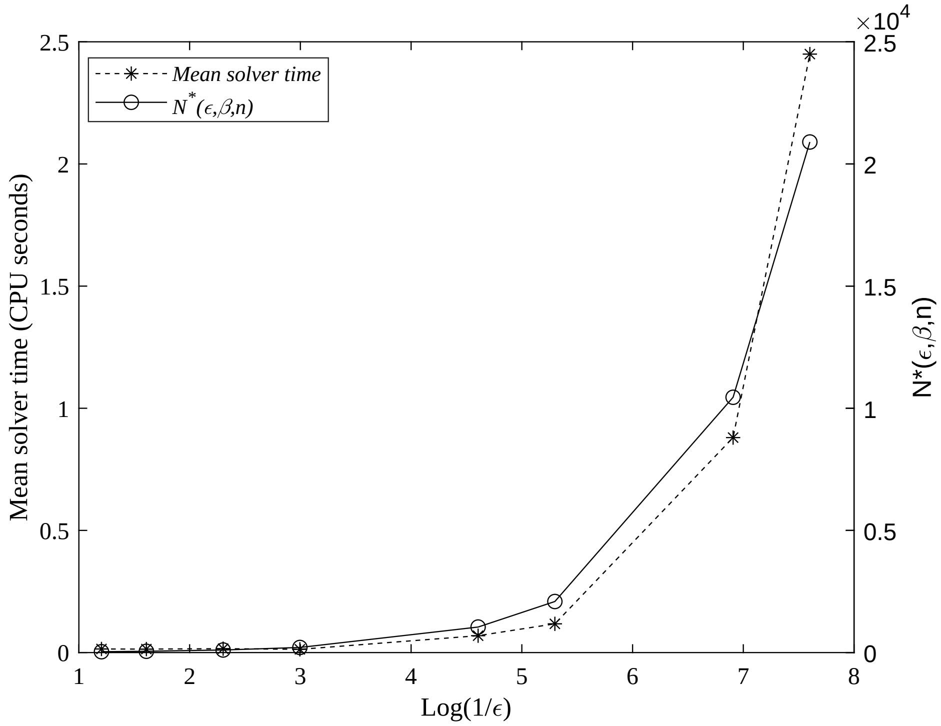

Fig. 7 Mean solver times (black line) and number of scenarios (dashed line) as a function of $\text{Log}(1/\epsilon)$ for problem $\widehat{\text{RO}}_2^{N_1^*}(\mathcal{COC})$ in the two-stage ($H = 1$) case.

We now compute the optimality gaps by using the scenario approach. Figures 10 shows that the optimality gaps of $\widehat{\text{RT}}_3^{N_1^*}(\mathcal{COC})$ with respect to $\text{RO}_3(\mathcal{COC})$ slightly decrease as ϵ decreases passing from -44% (in average) to -43%. Notice that the best optimality gap which can be attained by the sampled two-stage relaxation $\widehat{\text{RT}}_3^{N_1^*}(\mathcal{COC})$ is given by the two-stage relaxation itself $\text{RT}_3(\mathcal{COC})$ i.e., -39%.

The distribution of the empirical violation probability $\hat{V}_1(\widehat{RT}_3^{N_1^*})$ as function of ϵ is plotted in Figure 11, for the three-stage case. We note that the empirical violation probability $\hat{V}_1(\widehat{RT}_3^{N_1^*})$ is always smaller than ϵ and it decreases as ϵ decreases from 30% to 0.1%, passing from an average value of 6.6% to 0.06%. On the other hand, the empirical violation probability at stage 2, $\hat{V}_2(\widehat{RT}_3^{N_1^*})$, is equal to 1, independently on the value of ϵ showing the inappropriateness of the two-stage relaxation consisting in just one scenario per sub-tree at stage two.

Finally Figure 12 the average solver time (solid line) and the number of scenarios (dashed line). We again note that the number of required scenarios is considerably smaller than the one corresponding to the sampled robust problem $\widehat{\text{RO}}_3^{N_1^* N_2^*}$ allowing us to solve the approximated problem $\widehat{\text{RT}}_3^{N_1^*}$ in a reasonable amount of time (4 CPU seconds in the case of $\beta = 0.1$, $\epsilon = 0.1\%$ and $N_1^* = 6807$) at expenses of larger optimality gaps.

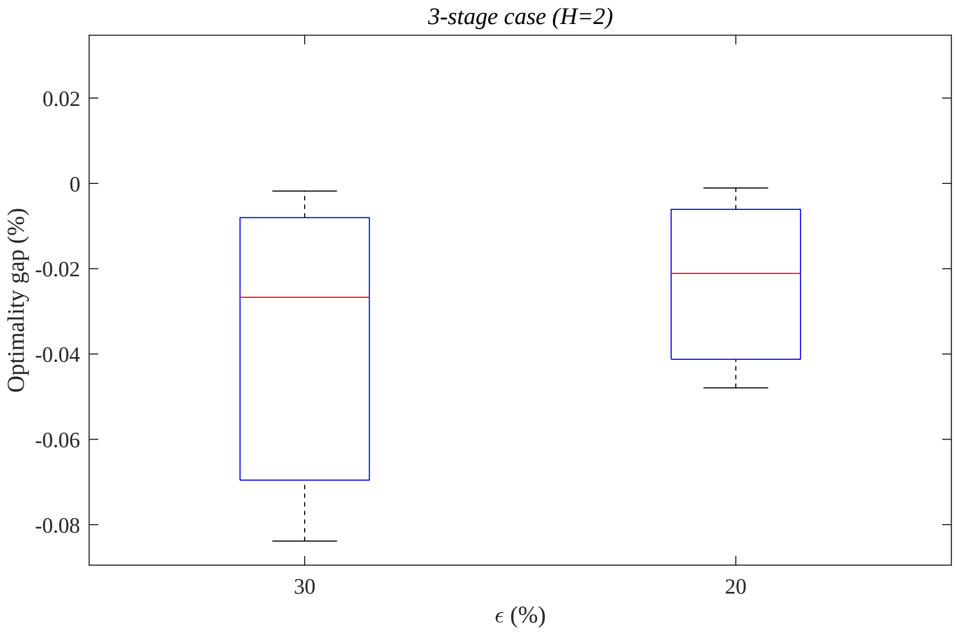

Fig. 8 Optimality gaps for $\widehat{\mathrm{RO}}_3^{N_1^* N_2^*}(\mathcal{COC})$ (boxes and whiskers) for decreasing values of ϵ for the three-stage $(H = 2)$ case.

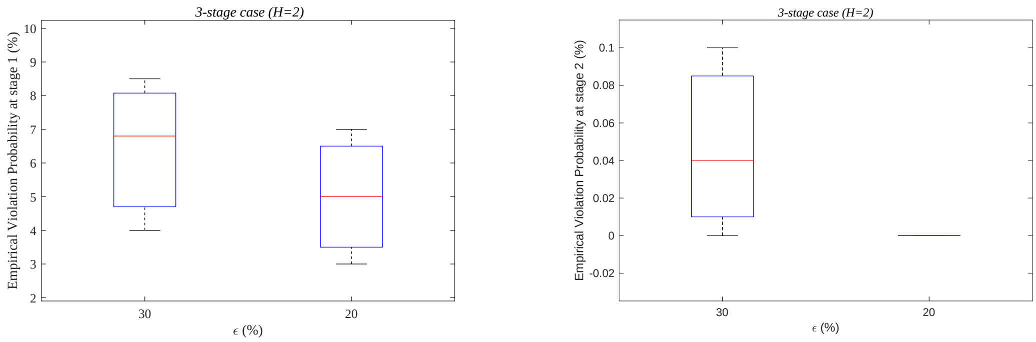

Fig. 9 Empirical violation probabilities $\hat{V}_1(\hat{\mathcal{T}}^{N_1^* N_2^*})$ (on the left) and $\hat{V}_2(\hat{\mathcal{T}}^{N_1^* N_2^*})$ (on the right) for $\widehat{\mathrm{RO}}_3^{N_1^* N_2^*}(\mathcal{COC})$ for decreasing values of ϵ for the three-stage $(H = 2)$ case.

5 Conclusions

In this paper probabilistic guarantees for constraint sampling in multi-stage convex robust optimization problems have been proposed. A sampled-based problem taking into account the non-anticipativity of the decision process has been considered. For this approach, which avoids the conservative use of parametrization through decision rules proposed before in literature, a bound on the probability of violation of the randomized solution and a proof of convergence have been provided. Chains of lower bounds by relaxing the non-anticipativity constraints and sampling are also discussed. Because we use a worst-case approach, the numbers for needed sample

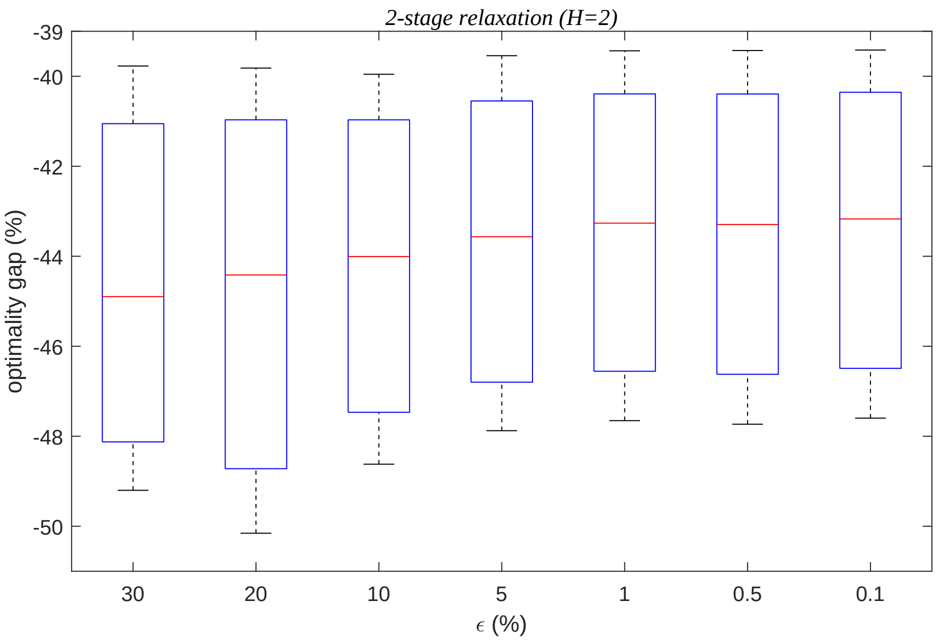

Fig. 10 Optimality gaps for $\widehat{\mathrm{RT}}_3^{N_1^*}(\mathcal{COC})$ (boxes and whiskers) for decreasing values of ϵ for the three-stage $(H = 2)$ case.

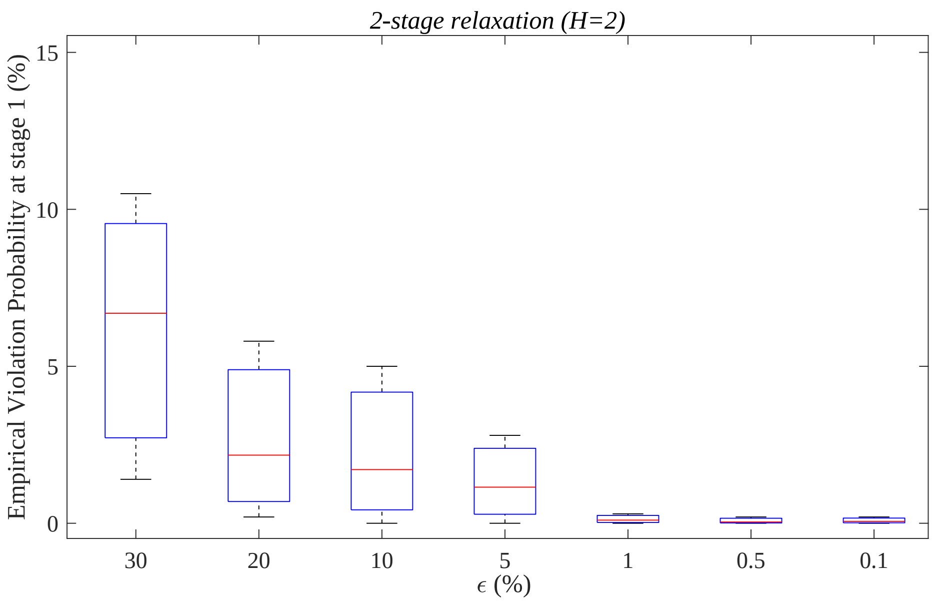

Fig. 11 Empirical violation probabilities $\hat{V}_1(\widehat{RT}_3^{N_1^*})$ for the two-stage relaxation $\widehat{RT}_3^{N_1^*}(\mathcal{COC})$ for decreasing values of ϵ for the three-stage $(H = 2)$ case.

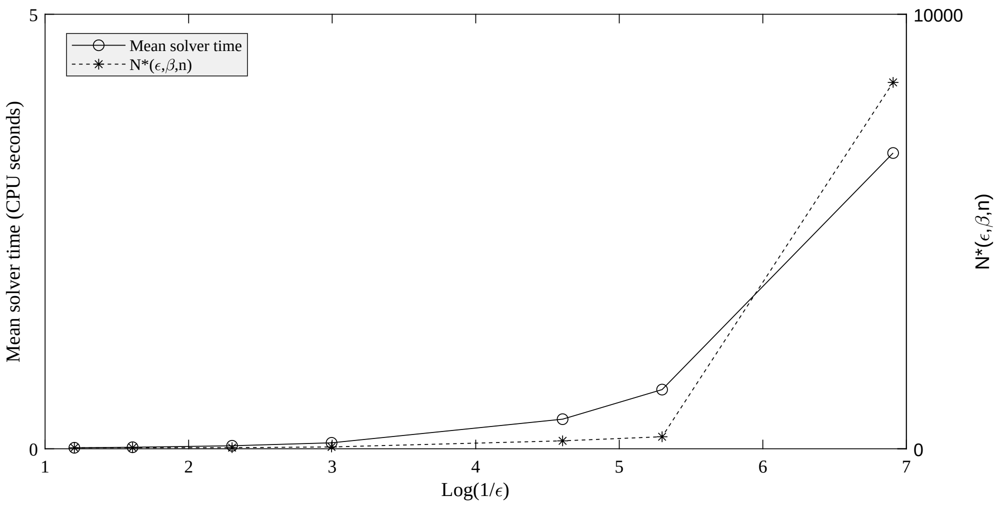

Fig. 12 Mean solver times (solid lines) and number of scenarios (dashed lines) as a function of $\mathrm{Log}(1/\epsilon)$ for problem $\widehat{\mathrm{RT}}_3^{N_1^*}(\mathcal{COC})$ for the three-stage ($H = 2$) case.

sizes are very high. However, our numerical results show that the empirical violation probabilities are much smaller than their predetermined values used in the calculation of the sample sizes. Morevoer, since we distinguish between violations at different stages, it was observed that violation in earlier stages are more probable than in later ones. It was also observed that for a three stage problem with 2-stage relaxation, the approximation provides good decisions at stage 1, but at stage 2 the violation probability is always 1, independently of the chosen ϵ. This shows that such a relaxation may be inappropriate for later stage decisions.

Acknowledgements

This work has been supported by "ULTRA OPTYMAL - Urban Logistics and sustainable TRAnsportation: OPtimization under uncertainTY and MAchine Learning", a PRIN2020 project funded by the Italian University and Research Ministry (grant number 20207C8T9M, official website: https://ultraoptymal.unibg.it), by Gruppo Nazionale per il Calcolo Scientifico (GNCS-INdAM) and by funds of the CNR-JST Joint International Lab COOPS. The authors FM and FD would like to express their gratitude to Roberto Tempo and Marida Bertocchi, both sadly passed away, for helpful discussions in an early stage of this work.

A Proof of Theorem 2

Proof Let $v(\mathrm{RO}_2) = -\infty$. Fix any $\gamma \in \mathbb{R}$. Then there is a $x_1 \in \mathrm{Feas}$ such that $\mathscr{R}(x_1) = -\infty$, meaning that there are functions $\xi_1 \mapsto x_2(\xi_1)$ such that:

$$\sup_{\xi_1 \in \Xi_1} \{c_2^\top(\xi_1) x_2(\xi_1) : x_2(\xi_1) \in \mathbb{X}_2(x_1, \xi_1)\} \leq \gamma - \nu \ ,$$

where $\nu = c_1^\top x_1 < \infty$. Notice that:

$$\mathrm{Feas} = \{x_1 \geq 0 : Ax_1 = h_1; \forall \xi_1 \in \Xi_1 \text{ there exists a } x_2(\xi_1) \in \mathbb{X}_2(x_1, \xi_1)\} \ .$$

Thus γ together with $(x_1, \xi_1 \mapsto x_2(\xi_1))$ is feasible for RwC_2 and since γ is arbitrary, $v(\mathrm{RwC}_2) = -\infty$. The same argumentation shows that $v(\mathrm{RwC}_2) = -\infty$ implies that $v(\mathrm{RO}_2) = -\infty$.

Suppose now that $v(\mathrm{RO}_2) = \infty$. This means that either the first-stage problem or at least one second-stage problem is infeasible and this implies and is implied by the fact that RwC_2 is infeasible.

It remains to show what happens in the case $-\infty < v(\mathrm{RO}_2) < \infty$. In case the the optimal value is attained, let $(x_1, \xi_1 \mapsto x_2(\xi_1))$ be in the solution set of RO_2, then $(x_1, \gamma, \xi_1 \mapsto x_2(\xi_1))$ is feasible for RwC_2, iff $\gamma \geq v(\mathrm{RO}_2)$ and it is in the solution set of RwC_2, if $\gamma = v(\mathrm{RO}_2)$. Conversely, if a $(\gamma, x_1, \xi_1 \mapsto x_2(\xi_1))$ is feasible for RwC_2, then $(x_1, \xi_1 \mapsto x_2(\xi_1))$ is feasible for RO_2 and $v(\mathrm{RO}_2) \leq \gamma$. The optimal γ equals $v(\mathrm{RwC}_2)$.

B Proof of convergence

In this section we show that by letting the sample sizes $N_1, N_2, \ldots, N_H$ tend to infinity, the optimal value of the sampled problem (22) converges almost surely to the optimal value of the basic problem (21). To this end, introduce the following assumptions.

Assumption A.

(i) The sets Ξ_t are compact with nonempty interior.
(ii) The probability $\mathbb{P}$ defined on $\Xi = \Xi_1 \times \cdots \times \Xi_H$ has a nonvanishing density, such that the probability of all relative open sets in Ξ is positive.
(iii) The functions $\xi_t \mapsto c_{t+1}(\xi_t)$, $\xi_t \mapsto T_t(\xi_t)$, $\xi_t \mapsto W_{t+1}(\xi_t)$, $\xi_t \mapsto h_{t+1}(\xi_t)$ defined on Ξ are continuous and therefore uniformly continuous for $t = 1, \ldots, H$.
(iv) There is a constant K such that the optimal values of (21) are uniformly bounded, i.e. $\|x_t\| \leq K$, $t = 1, \ldots, H+1$.
(v) The rank of the matrices $W_{t+1}(\xi_t)$ is m_{t+1} for all $\xi_t \in \Xi_t$.

Recall that we sample N_t independent replicates from distribution $\mathbb{P}_t$ in Ξ_t. Notice the following Lemma.

Lemma B.1 *Under Assumption A,*

$$\lim_{N_t \to \infty} \max_{\xi_t \in \Xi_t} \min_{\xi_t^{(i)} \in \widehat{\Xi}_t^{N_t}} \|\xi_t^{(i)} - \xi_t\| = 0$$

almost surely.

Proof Notice that $\max_{\xi_t \in \Xi_t} \min_{\xi_t^{(i)} \in \widehat{\Xi}_t^{N_t}} \|\xi_t^{(i)} - \xi_t\|$ is decreasing in N_t. Suppose that there is a $\xi_t \in \Xi_t$ such that $\min_{\xi_t \in \widehat{\Xi}_t^{N_t}} \|\xi_t^{(i)} - \xi_t\| \geq \eta$ for all N_t. Let B_η be the closed ball with radius η and center ξ_t. By assumption $\mathbb{P}(B_\eta) = \delta$ (say) with $\delta > 0$. Now

$$\mathbb{P}\{\xi_t^{(1)} \notin B_\eta, \ldots, \xi_t^{(N_t)} \notin B_\eta\} = (1-\delta)^{N_t} \ .$$

By choosing N_t such large that $(1-\delta)^{N_t} < \eta$ leads to a contradiction.

Corollary B.1 *If g is a continuous functions on Ξ_t, then $\min_{\xi_t^{(i)} \in \widehat{\Xi}_t^{N_t}} g(\xi_t^{(i)})$ converges to $\min_{\xi_t \in \Xi_t} g(\xi_t)$ a.s. for $N_t \to \infty$. The same is true for the maximum.*

Proposition B.1 *Under Assumption A, if $\min(N_1, \ldots, N_H) \to \infty$, then the optimal value of the sampled problem (22) converges to the optimal value of the basic problem (21).*

Proof For the sake of simplicity we give the proof for the three-stage problem, i.e. we assume that $H = 2$. The proof of the general case is analogous. The optimization problem (21) can be written as follows:

$$\begin{aligned} \mathrm{RO}_3 := \min_{x_1, \gamma_1} \ & c_1^\top x_1 + \gamma_1 \\ \text{s.t. } & Ax_1 = h_1, \ x_1 \geq 0 \\ & \mathcal{Q}_1(x_1, \xi_1) \leq \gamma_1, \quad \forall \xi_1 \in \Xi_1 \ , \end{aligned} \tag{45}$$

where the function $\mathcal{Q}_1(x_1, \xi_1)$ can be written as

$$\begin{aligned} \mathcal{Q}_1(x_1, \xi_1) := \min_{x_2, \gamma_2} \ & c_2^\top(\xi_1) x_2 + \gamma_2 \\ \text{s.t. } & T_1(\xi_1) x_1 + W_2(\xi_1) x_2(\xi_1) = h_2(\xi_1) \\ & \mathcal{Q}_2(x_2, \xi_2) \leq \gamma_2, \quad \forall \xi_2 \in \Xi_2 \\ & x_2 \geq 0 \ , \end{aligned} \tag{46}$$

with

$$\begin{aligned}\mathcal{Q}_2(x_2,\xi_2) := \min_{x_3} & \, c_3^\top(\xi_2)x_3 \\ \text{s.t. } & T_2(\xi_2)x_2 + W_3(\xi_2)x_3 = h_3(\xi_2) \\ & x_3 \geq 0 \ .\end{aligned} \tag{47}$$

Now let $\widehat{\Xi}_1^{N_1} = \{\xi_1^{(i_1)} : i_1 = 1,\dots,N_1\}$ resp. $\widehat{\Xi}_2^{N_2} = \{\xi_2^{(i_2)} : i_2 = 1,\dots,N_2\}$, be independent random scenarios from of Ξ_1 resp. Ξ_2. We set:

$$\begin{aligned}\widehat{\text{RO}}_3 := \min_{x_1,\gamma_1} & \, c_1^\top x_1 + \gamma_1 \\ \text{s.t. } & Ax_1 = h_1, \ x_1 \geq 0 \\ & \widehat{\mathcal{Q}}_1(x_1,\xi_1) \leq \gamma_1, \quad \forall \xi_1 \in \widehat{\Xi}_1^{N_1} \ ,\end{aligned} \tag{48}$$

where the function $\widehat{\mathcal{Q}}_1(x_1,\xi_1)$ can be written as

$$\begin{aligned}\widehat{\mathcal{Q}}_1(x_1,\xi_1) := \min_{x_2,\gamma_2} & \, c_2^\top(\xi_1)x_2 + \gamma_2 \\ \text{s.t. } & T_1(\xi_1)x_1 + W_2(\xi_1)x_2 = h_2(\xi_1) \\ & \mathcal{Q}_2(x_2,\xi_2) \leq \gamma_2, \quad \forall \xi_2 \in \widehat{\Xi}_2^{N_2} \\ & x_2 \geq 0 \ ,\end{aligned} \tag{49}$$

with

$$\begin{aligned}\mathcal{Q}_2(x_2,\xi_2) := \min_{x_3} & \, c_3^\top(\xi_2)x_3 \\ \text{s.t. } & T_2(\xi_2)x_2 + W_3(\xi_2)x_3 = h_3(\xi_2) \\ & x_3 \geq 0 \ .\end{aligned} \tag{50}$$

as before.

Notice that the functions $\mathcal{Q}_2$ are identical for the original problem and the sampled problem. We show that the functions $\widehat{\mathcal{Q}}_t(x_t,\xi_t)$, $t = 1,2$ are continuous in x_t and ξ_t. For the function $\mathcal{Q}_2 = \widehat{\mathcal{Q}}_2$ this follows from the fact that $W_3(\xi_2)$ has rank $m_3 < n_3$ for all $\xi_2 \in \Xi_2$. Suppose that the polyhedron $\mathbb{X}_3(x_2,\xi_2) = \{x_3 \geq 0 : W_3(\xi_2)x_3 = h_3(\xi_2) - T_2(\xi_2)x_2\}$ has extremals $x_{3,1}(x_2,\xi_2),\dots,x_{3,\ell}(x_2,\xi_2)$. The fact that $W_3(\xi_2)$ has maximal rank for all ξ_2 implies that the extremals are continuous in W_3 as well as in the r.h.s. $h_3(\xi_2) - T_2(\xi_2)x_2$. Therefore, by assumption, the extremals are continuous in ξ and in x. By continuity of c_3, one sees that $\mathcal{Q}_2(x_2,\xi_2)$ is continuous in both arguments and since both x_2 and ξ_2 lie in a compact set, it is uniformly continuous, that is:

$$\sup_{\|x_2\| \leq K} \Big| \sup_{\xi_2 \in \Xi_2} \mathcal{Q}_2(x_2,\xi_2) - \sup_{\xi_2 \in \widehat{\Xi}_2^{N_2}} \mathcal{Q}_2(x_2,\xi_2) \Big| \to 0 \tag{51}$$

almost surely as $N_2 \to \infty$.

Now consider the function $\widehat{\mathcal{Q}}(x_1,\xi_1)$. As in the previous case, the extremals of $\mathbb{X}_2(x_1,\xi_1) = \{x_2 \geq 0 : W_2(\xi_1)x_2 = h_2(\xi_1) - T_1(\xi_1)x_1\}$ are continuous in x_1 and ξ_1. Together with (51) this implies that both $\mathcal{Q}_1$ and $\widehat{\mathcal{Q}}_1$ are continuous in x_1 and ξ_1 and therefore:

$$\sup_{\|x_1\| \leq K} \Big| \sup_{\xi_1 \in \Xi_1} \mathcal{Q}_1(x_1,\xi_1) - \sup_{\xi_1 \in \widehat{\Xi}_1^{N_1}} \widehat{\mathcal{Q}}_1^{N_2}(x_1,\xi_1) \Big| \to 0 \tag{52}$$

almost surely as $\min(N_1,N_2) \to \infty$.

The function $\mathcal{Q}_2(x_2,\xi_2)$ is uniformly continuous in ξ_2 and x_2. Therefore for $\epsilon > 0$ there is an $\eta > 0$ such that $\|\xi_2^1 - \xi_2^2\| \leq \eta$ implies that $|\mathcal{Q}_2(x_2,\xi_2^1) - \mathcal{Q}_2(x_2,\xi_2^2)| \leq \epsilon$. Thus by the previous Lemma:

$$\max_{\xi_2 \in \Xi_2} \mathcal{Q}_2(x_2,\xi_2) - \max_{\xi_2 \in \widehat{\Xi}_2^{N_2}} \mathcal{Q}_2(x_2,\xi_2) \to 0 \ .$$

The same argument applies also to the function $\mathcal{Q}_1(x_1,\xi_1)$.

C Proof of Lemma 2.4

Proof

$$\mathbb{P}\left\{\sum_{i=1}^{K} Z_i \geq z\right\} \leq \mathbb{P}\left(\bigcup_{i=1}^{K}\{Z_i \geq z/K\}\right) \leq K\mathbb{P}\{Z_i \geq z/K\}.$$

This inequality is sharp: To see this, consider a discrete probability space having $K > 1$ atoms $\{\omega_1, \ldots, \omega_K\}$, each with same probability $P\{\omega_i\} = 1/K$. On ω_i define the random variables $Z_1, \ldots, Z_K$ as

$$Z_i = (z + K - 1)/K; \qquad Z_j = (z - 1)/K \text{ for } j \neq i.$$

Then the Z_i have all identical distributions and $\sum_i Z_i = z$. Consequently

$$P\left\{\sum_{i=1}^{K} Z_i \geq z\right\} = 1 = K \cdot (1/K) = K \cdot P\{Z_i \geq z/K\}.$$

D Proof of Proposition 3.2

Proof Since in RWS_{H+1} the non-anticipativity constraints are relaxed, we get the inequality (34). More formally, denoting by $f\left[\left(x_1(\underline{\xi}_H), \ldots, x_{H+1}(\underline{\xi}_H)\right), \underline{\xi}_H\right]$ in a compact way the objective function and constraints of problem (32), we can write:

$$\mathrm{RWS}_{H+1} \quad : \quad \sup_{\underline{\xi}_H} \min_{(x_1(\underline{\xi}_H), \ldots, x_{H+1}(\underline{\xi}_H))} f\left[\left(x_1(\underline{\xi}_H), \ldots, x_{H+1}(\underline{\xi}_H)\right), \underline{\xi}_H\right].$$

For every realization, $\underline{\xi}_H$, we have the relation:

$$f\left[\left(\tilde{x}_1(\underline{\xi}_H), \ldots, \tilde{x}_{H+1}(\underline{\xi}_H)\right), \underline{\xi}_H\right] \leq f\left[\left(x_1^*, \ldots, x_{H+1}^*(\underline{\xi}_H)\right), \underline{\xi}_H\right],$$

where $\left(x_1^*, \ldots, x_{H+1}^*(\underline{\xi}_H)\right)$ denotes an optimal solution to the RO_{H+1} problem (21), and $\left(\tilde{x}_1(\underline{\xi}_H), \ldots, \tilde{x}_{H+1}(\underline{\xi}_H)\right)$ denotes the optimal solution for each realization of $\underline{\xi}_H$. Taking the supremum of both sides yields the required inequality.

References

1. T. Alamo, R. Tempo, A. Luque, and D. R. Ramirez. Randomized methods for design of uncertain systems: Sample complexity and sequential algorithms. *Automatica*, 52:160 – 172, 2015.
2. D. Bampou and D. Kuhn. Scenario-free stochastic programming with polynomial decision rules. In *2011 50th IEEE Conference on Decision and Control and European Control Conference*, pages 7806–7812. IEEE, 2011.
3. O. Baron, J. Milner, and H. Naseraldin. Facility location: A robust optimization approach. *Production and Operations Management*, 20(5):772–785, 2011.
4. A. Ben-Tal, B. Golany, A. Nemirovski, and J.-P. Vial. Retailer-supplier flexible commitments contracts: A robust optimization approach. *Manufacturing & Service Operations Management*, 7(3):248–271, July 2005.
5. A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2):351–376, 2003.
6. A. Ben-Tal and A. Nemirovski. Robust solutions of uncertain linear programs. *Operations Research Letters*, 25(1):1 – 13, 1999.
7. A. Ben-Tal, A. Nemirovski, and C. Roos. Robust solutions of uncertain quadratic and conic-quadratic problems. *SIAM Journal on Optimization*, 13(2):535–560, 2002.
8. D. Bertsimas, D. B. Brown, and C. Caramanis. Theory and applications of robust optimization. *SIAM Review*, 53(3):464–501, 2011.
9. D. Bertsimas and I. Dunning. Multistage robust mixed-integer optimization with adaptive partitions. *Operations Research*, 64(4):980–998, 2016.

10. D. Bertsimas and A. Georghiou. Binary decision rules for multistage adaptive mixed-integer optimization. *Mathematical Programming*, 167(2):395–433, 2018.
11. D. Bertsimas, D. Iancu, and P. Parrilo. A hierarchy of near-optimal policies for multistage adaptive optimization. *Automatic Control, IEEE Transactions on*, 56(12):2809–2824, Dec 2011.
12. D. Bertsimas and A. Thiele. A robust optimization approach to inventory theory. *Operations Research*, 54(1):150–168, 2006.
13. M. Bodur and J. R. Luedtke. Two-stage linear decision rules for multi-stage stochastic programming. *Mathematical Programming*, pages 1–34, 2018.
14. G. Calafiore. Random convex programs. *SIAM Journal on Optimization*, 20(6):3427–3464, 2010.
15. G. Calafiore and M. Campi. Uncertain convex programs: randomized solutions and confidence levels. *Mathematical Programming*, 102(1):25–46, 2004.
16. G. Calafiore and A. Nilim. Multi-stage resource allocation under uncertainty. In *Decision and Control, 2004. CDC. 43rd IEEE Conference on*, volume 1, pages 417–423 Vol.1, Dec 2004.
17. G. C. Calafiore, F. Dabbene, and R. Tempo. Research on probabilistic methods for control system design. *Automatica*, 47(7):1279 – 1293, 2011.
18. M. C. Campi and S. Garatti. The exact feasibility of randomized solutions of uncertain convex programs. *SIAM Journal on Optimization*, 19(3):1211–1230, 2008.
19. X. Chen, M. Sim, P. Sun, and J. Zhang. A linear decision-based approximation approach to stochastic programming. *Operations Research*, 56(2):344–357, 2008.
20. D. P. de Farias and B. V. Roy. On constraint sampling in the linear programming approach to approximate dynamic programming. *Math. Oper. Res.*, 29(3):462–478, Aug. 2004.
21. E. Delage and D. A. Iancu. Robust multistage decision making. In *The Operations Research Revolution*, pages 20–46. Institute for Operations Research and the Management Sciences (INFORMS), sep 2015.
22. M. S. Dimitris Bertsimas. The price of robustness. *Operations Research*, 52(1):35–53, 2004.
23. J. Dupačová, N. Gröwe-Kuska, and W. Römisch. Scenario reduction in stochastic programming. *Mathematical programming*, 95(3):493–511, 2003.
24. E. Erdogan and G. Iyengar. Ambiguous chance constrained problems and robust optimization. *Math. Program.*, 107(1):37–61, June 2006.
25. S. Formentin, F. Dabbene, R. Tempo, L. Zaccarian, and S. M. Savaresi. Robust static anti-windup augmentation with probabilistic certificates. *IEEE Transactions on Automatic Control*, accepted, doi:10.1109/TAC.2016.2586606, 2017.
26. V. Gabrel, C. Murat, and A. Thiele. Recent advances in robust optimization and robustness: An overview. *Technical Report, LAMSADE, Universite Paris-Dauphine, Paris, France 2012*, 2012.
27. S. J. Garstka and R. J.-B. Wets. On decision rules in stochastic programming. *Mathematical Programming*, 7(1):117–143, 1974.
28. L. E. Ghaoui, F. Oustry, and H. Lebret. Robust solutions to uncertain semidefinite programs. *SIAM Journal on Optimization*, 9(1):33–52, 1998.
29. D. Kuhn, W. Wiesemann, and A. Georghiou. Primal and dual linear decision rules in stochastic and robust optimization. *Mathematical Programming*, 130(1):177–209, 2011.
30. J. Luedtke and S. Ahmed. A sample approximation approach for optimization with probabilistic constraints. *SIAM Journal on Optimization*, 19(2):674–699, 2008.
31. F. Maggioni, E. Allevi, and M. Bertocchi. Bounds in multistage linear stochastic programming. *Journal of Optimization Theory and Applications*, 163(1):200–229, 2014.
32. F. Maggioni, E. Allevi, and M. Bertocchi. Monotonic bounds in multistage mixed-integer stochastic programming. *Computational Management Science*, 13(3):423–457, 2016.
33. F. Maggioni and G. Pflug. Guaranteed bounds for general nondiscrete multistage risk-averse stochastic optimization programs. *SIAM Journal on Optimization*, 29(1):454–483, 2019.
34. F. Maggioni and G. C. Pflug. Bounds and approximations for multistage stochastic programs. *SIAM Journal on Optimization*, 26(1):831–855, 2016.
35. Y. Oishi. *Probabilistic and Randomized Methods for Design under Uncertainty*, chapter Probabilistic Design of a Robust Controller Using a Parameter-Dependent Lyapunov Function, pages 303–316. Springer London, London, 2006.
36. B. K. Pagnoncelli, S. Ahmed, and A. Shapiro. Sample average approximation method for chance constrained programming: Theory and applications. *Journal of Optimization Theory and Applications*, 142(2):399–416, 2009.
37. G. Perakis and G. Roels. Robust controls for network revenue management. *Manufacturing & Service Operations Management*, 12(1):56–76, 2010.
38. G. C. Pflug and A. Pichler. *Multistage stochastic optimization*. Springer, 2014.
39. M. c. Pınar and R. H. Tütüncü. Robust profit opportunities in risky financial portfolios. *Oper. Res. Lett.*, 33(4):331–340, July 2005.
40. K. Postek and D. den Hertog. Multistage adjustable robust mixed-integer optimization via iterative splitting of the uncertainty set. *INFORMS Journal on Computing*, 28(3):553–574, 2016.

41. A. Shapiro, D. Dentcheva, and A. Ruszczyński. *Lectures on stochastic programming: modeling and theory.* MPS-SIAM series on optimization. Society for Industrial and Applied Mathematics, Philadelphia, 2009.
42. A. Shapiro and A. Nemirovski. On complexity of stochastic programming problems. In *Continuous optimization*, pages 111–146. Springer, 2005.
43. A. L. Soyster. Convex programming with set-inclusive constraints and applications to inexact linear programming. *Operations Research*, 21(5):1154–1157, 1973.
44. A. Takeda, S. Taguchi, and R. H. Tütüncü. Adjustable robust optimization models for a nonlinear two-period system. *Journal of Optimization Theory and Applications*, 136(2):275–295, Feb 2008.
45. R. Tempo, G. Calafiore, and F. Dabbene. *Randomized algorithms for analysis and control of uncertain systems. With applications. 2nd revised ed.* London: Springer, 2nd revised ed. edition, 2013.
46. R. Tütüncü and M. Koenig. Robust asset allocation. *Annals of Operations Research*, 132(1):157–187, 2004.
47. P. Vayanos, D. Kuhn, and B. Rustem. A constraint sampling approach for multi-stage robust optimization. *Automatica*, 48(3):459 – 471, 2012.
48. D. S. Yamashita, V. A. Armentano, and M. Laguna. Robust optimization models for project scheduling with resource availability cost. *Journal of Scheduling*, 10(1):67–76, 2007.
49. C. Zhao, J. Wang, J. P. Watson, and Y. Guan. Multi-stage robust unit commitment considering wind and demand response uncertainties. *IEEE Transactions on Power Systems*, 28(3):2708–2717, Aug 2013.