EXPLOITING PARTIAL CONVEXITY OF PUMP CHARACTERISTICS IN WATER NETWORK DESIGN

MARC E. PFETSCH AND ANDREAS SCHMITT

Abstract. The design of water networks consists of selecting pipe connections and pumps to ensure a given water demand to minimize investment and operating costs. Of particular importance is the modeling of variable speed pumps, which are usually represented by degree two and three polynomials approximating the characteristic diagrams. In total, this yields complex mixed-integer (non-convex) nonlinear programs.

This work investigates a reformulation of these characteristic diagrams, eliminating rotating speed variables and determining power usage in terms of volume flow and pressure increase. We characterize when this formulation is convex in the pressure variables. This structural observation is applied to design the water network of a high-rise building in which the piping is tree-shaped. For these problems, the volume flow can only attain finitely many values. We branch on these flow values, eliminating the non-convexities of the characteristic diagrams. Then we apply perspective cuts to strengthen the formulation. Numerical results demonstrate the advantage of the proposed approach.

1. Introduction

In this paper the optimal design and operation of water networks using mixed-integer nonlinear programming (MINLP) is considered, see [2] for an overview. More precisely, we investigate the optimal design of tree-shaped high-rise water supply systems in which the floors need to be connected by pipes and pumps must be placed, such that all floors are supplied by water under minimal investment and running costs in a stationary setting. A customized branch and bound approach has been developed in [1], which aims to deal with the inherent combinatorial complexity for deciding the topology. Another challenge of the problem lies in the operation of pumps. Their nonlinear and non-convex behavior is described by so called characteristic diagrams, see for an example Figure 1, which determine for a given volume flow q the corresponding possible range of pressure increase Δh and power consumption p if one varies the normalized operating speed ω. These nonlinearities are modeled in the high-rise problem using a quadratic

$$\Delta H \colon [\underline{q}, \overline{q}] \times [\underline{\omega}, 1] \to \mathbb{R}, \quad (q, \omega) \mapsto \alpha^H q^2 + \beta^H q\omega + \gamma^H \omega^2$$

and a cubic polynomial

$$P \colon [\underline{q}, \overline{q}] \times [\underline{\omega}, 1] \to \mathbb{R}, \quad (q, \omega) \mapsto \alpha^P q^3 + \beta^P q^2\omega + \gamma^P q\omega^2 + \delta^P \omega^3 + \epsilon^P.$$

Since the piping has to form a tree, the volume flow in a given floor and pump attains only finitely many distinct values $Q \coloneqq \{q_1, \ldots, q_n\} \subset \mathbb{R}_+$ with $q_i < q_{i+1}$.

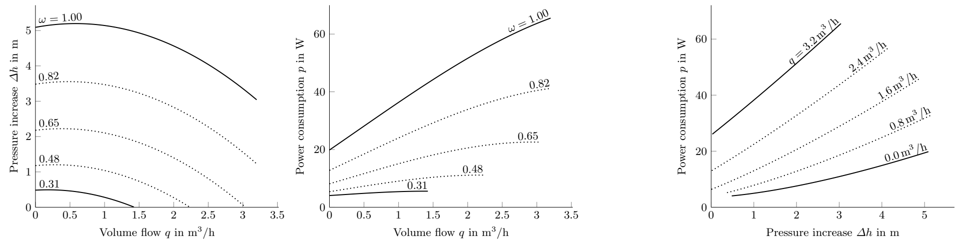

FIGURE 1. Exemplary characteristic diagram (left) and graph of $\tilde{P}(q, \Delta h)$ (right).

Therefore a pump is modeled by the non-convex set

$$D := \big\{(y, p, \Delta h, q, \omega) \in \{0,1\} \times \mathbb{R}_+^2 \times Q \times [\underline{\omega}, 1] \ \mid \\ p \geq P\,(q, \omega)\ y,\ \Delta h = \Delta H\,(q, \omega)\ y,\ \underline{q}\, y \leq q \leq q_n + (\overline{q} - q_n)\, y\big\},$$

where y is 1 iff the pump is used. Note that q also models the volume flow in a floor and thus can attain values exceeding $\underline{q}$ or $\overline{q}$ if $y = 0$.

Whereas Δh and q are linked to other variables in the model by further constraints, the operating speed ω is only constrained by D. In the ensuing paper we first introduce an alternative formulation X that eliminates ω and is convex in Δh for fixed q. Afterwards we present a simple test to check for this convexity. Subsequently we derive valid inequalities for X involving perspective cuts by lifting. The benefit of projecting out ω and using these cuts in a branch and cut framework is demonstrated on a representative testset.

2. Reformulation and convexity

To derive a formulation X of the projection of D onto the variables y, p, Δh and q, we will work with the following two assumptions: All coefficients appearing in the approximations are nonzero to avoid edgecases. More importantly, $\Delta H(q, \omega)$ is strictly increasing in ω on $[\underline{\omega}, 1]$ for all $q \in [\underline{q}, \overline{q}]$; this is supported by scaling laws. This implies the existence of the inverse function $\Omega\colon [\underline{q}, \overline{q}] \times \mathbb{R}_+ \to \mathbb{R}_+$ of $\Delta H(q, \omega)$ with respect to ω for fixed q, i.e., $\Omega(q, \Delta H(q, \omega)) = \omega$ for all $\omega \in [\underline{\omega}, 1]$. It is given by

$$\Omega(q, \Delta h) = \frac{1}{2\gamma^H}\left(-\beta^H q + \sqrt{\left({\beta^H}^2 - 4\gamma^H \alpha^H\right) q^2 + 4\gamma^H \Delta h}\right)$$

and its composition with $P(q, \omega)$ leads to the function

$$\tilde{P}\colon [\underline{q}, \overline{q}] \times \mathbb{R}_+ \to [\underline{\omega}, 1], \quad (q, \Delta h) \mapsto P(q, \Omega(q, \Delta h)),$$

which calculates the power consumption of a pump processing a volume flow q and increasing the pressure by Δh. See Figure 1 for an example. With $\Delta \underline{H}(q) := \Delta H\,(q, \underline{\omega})$ and $\Delta \overline{H}(q) := \Delta H\,(q, 1)$, the projection is then given by

$$X := \big\{(y, p, \Delta h, q) \in \{0,1\} \times \mathbb{R}_+^2 \times Q \ \mid \\ p \geq \tilde{P}(q, \Delta h)\, y,\ \Delta \underline{H}(q)\, y \leq \Delta h \leq \Delta \overline{H}(q)\, y,\ \underline{q}\, y \leq q \leq q_n + (\overline{q} - q_n)\, y\big\}.$$

Both X and D present obstacles for state-of-the-art optimization software and methods. The pumps in our instances, however, satisfy a convexity property characterized in the following lemma, making the usage of X beneficial.

Lemma 2.1. *For each fixed* $q \in [\underline{q}, \overline{q}]$, *the function* $\tilde{P}(q, \Delta h)$ *is convex in* $\Delta h \in [\Delta\underline{H}(q), \Delta\overline{H}(q)]$ *if and only if*

$$\max_{q\in[\underline{q},\overline{q}]} (\gamma^H \beta^P - \beta^H \gamma^P) q^2 - 3\,\beta^H \delta^P q\,\tilde{\omega} \le 3\,\gamma^H \delta^P \tilde{\omega}^2,$$

where $\tilde{\omega} = 1$ *if* $\delta^P < 0$ *and* $\tilde{\omega} = \underline{\omega}$ *otherwise.*

Proof. Convexity can be checked by the minimization of $\frac{\partial^2 \tilde{P}}{\partial^2 \Delta h}$ over $q \in [\underline{q}, \overline{q}]$ and $\Delta h \in [\Delta\underline{H}(q), \Delta\overline{H}(q)]$, which can (after some calculations) be written as

$$\min_{q\in[\underline{q},\overline{q}],\,\omega\in[\underline{\omega},1]} (\beta^H \gamma^P - \gamma^H \beta^P) q^2 + 3\,\beta^H \delta^P q\,\omega + 3\,\gamma^H \delta^P \omega^2 \ge 0.$$

The sign of the partial derivative in ω of the objective function over $[\underline{q}, \overline{q}] \times [\underline{\omega}, 1]$ is equal to the sign of δ^P, since $\Delta H(q, \omega)$ is increasing in ω. Since this is constant, the minimum is attained either for $\omega = 1$ or $\omega = \underline{\omega}$. □

Note that $\Delta H(q, \omega)$ being concave ($\gamma^H \le 0$) and $P(q, \omega)$ being convex as well as non decreasing in ω for fixed q is a sufficient condition for convexity. This, however, is not satisfied by our real-world testdata, compare Table 1.

3. Perspective cuts and lifted valid inequalities for X

In the following we use the convexity property and present several families of valid inequalities for X. We first consider the case when $Q = \{q_1\} \subseteq [\underline{q}, \overline{q}]$ is a singleton. Then perspective cuts introduced by Frangioni and Gentile [3] are valid for X. Defining $\tilde{P}_q(\Delta h) := \tilde{P}(q, \Delta h)$, these are given for $\Delta h^* \in [\Delta\underline{H}(q_1), \Delta\overline{H}(q_1)]$ by

$$\tilde{P}'_{q_1}(\Delta h^*)\,\Delta h + \left(\tilde{P}_{q_1}(\Delta h^*) - \tilde{P}'_{q_1}(\Delta h^*)\Delta h^*\right) y \le p.$$

Validity can be seen by case distinction on the value of y. For $y = 0$ also Δh must be zero, leading to a vanishing left-hand side. If y is one, the cut corresponds to a gradient cut, which is valid by convexity.

For more general Q, another family of valid inequalities is obtained by combination of different perspective cuts, where we denote $\tilde{N} := \{1 \le i \le n \,|\, \underline{q} \le q_i \le \overline{q}\}$.

Lemma 3.1. *For parameters* $\Delta h_i^* \in [\Delta\underline{H}(q_i), \Delta\overline{H}(q_i)]$ *with* $i \in \tilde{N}$, *the inequality*

$$\left(\min_{i\in\tilde{N}} \tilde{P}'_{q_i}(\Delta h_i^*)\right)\Delta h + \left(\min_{i\in\tilde{N}} \tilde{P}_{q_i}(\Delta h_i^*) - \tilde{P}'_{q_i}(\Delta h_i^*)\Delta h_i^*\right) y \le p$$

is valid for X.

Proof. This follows from the validity of the perspective cuts for X with fixed $q = q_i$, $i \in \tilde{N}$, and that Δh and y are zero for $q \notin [\underline{q}, \overline{q}]$. □

Another family of valid inequalities can be formed by considering a perspective cut on the set $X \cap \{q = \underline{q}\}$ or $X \cap \{q = \overline{q}\}$ and lifting the variable q into it:

Lemma 3.2. *For parameters* $(\tilde{q}, q^*) \in \{(q_1, \underline{q}), (q_n, \overline{q})\}$, $\Delta h^* \in [\Delta\underline{H}(q^*), \Delta\overline{H}(q^*)]$ *and* $\gamma \in \mathbb{R}$ *such that* $\gamma(q - \tilde{q}) \le 0$ *holds for each* $q \in Q$ *and*

$$\min_{\Delta h^*\in[\Delta\underline{H}(q_i),\Delta\overline{H}(q_i)]} \tilde{P}_{q_i}(\Delta h) - \tilde{P}'_{q^*}(\Delta h^*)\,\Delta h \ge \tilde{P}_{q^*}(\Delta h^*) - \tilde{P}'_{q^*}(\Delta h^*)\Delta h^* + \gamma(q_i - \tilde{q})$$

holds for each $i \in \tilde{N}$, *the following inequality is valid for* X

$$\tilde{P}'_{q^*}(\Delta h^*)\,\Delta h + \left(\tilde{P}_{q^*}(\Delta h^*) - \tilde{P}'_{q^*}(\Delta h^*)\Delta h^*\right) y + \gamma(q - \tilde{q}) \le p.$$

TABLE 1. Pump parameters.

α^P	β^P	γ^P	δ^P	ϵ^P	α^H	β^H	γ^H	$\underline{q}$	$\overline{q}$	$\underline{\omega}$
-0.095382	0.25552	13.6444	18.0057	6.3763	-0.068048	0.26853	4.1294	0.0	8.0	0.517
-0.13	0.79325	18.2727	33.8072	6.2362	-0.066243	0.30559	6.1273	0.0	10.0	0.429
-0.14637	1.1882	23.0823	53.0306	6.0431	-0.065158	0.34196	8.1602	0.0	11.0	0.350
-0.32719	0.36765	16.4571	16.2571	3.5722	-0.31462	0.36629	5.0907	0.0	3.2	0.308
0.35512	-4.4285	17.4687	30.5853	0.013785	-0.083327	-0.10738	4.2983	0.0	6.0	0.498

Proof. The inequality is valid for $X \cap \{y = 0\}$, since its left-hand side simplifies to $\gamma(q - \tilde{q}) \leq 0$ and p is non-negative. Moreover, the minimum condition on γ makes sure that the inequality is valid for $X \cap \{y = 1, q = q_i\}$ for all $i \in \tilde{N}$. $\square$

This lifting idea leads to a further family of valid inequalities by first lifting q and then y into a gradient cut.

Lemma 3.3. *Let $q^* \in \{\underline{q}, \overline{q}\}$, $\Delta h^* \in [\Delta\underline{H}(q^*), \Delta\overline{H}(q^*)]$ and $\beta, \gamma \in \mathbb{R}$ such that for $i \in \tilde{N}$ with $q_i \neq q^*$*

$$\min_{\Delta h \in [\Delta\underline{H}(q_i), \Delta\overline{H}(q_i)]} \tilde{P}_{q_i}(\Delta h) - \tilde{P}'_{q^*}(\Delta h^*)\, \Delta h \geq \tilde{P}_{q^*}(\Delta h^*) - \tilde{P}'_{q^*}(\Delta h^*)\, \Delta h^* + \gamma(q_i - q^*)$$

and for $1 \leq i \leq n$ with $q_i \neq q^*$

$$\beta \geq \tilde{P}_{q^*}(\Delta h^*) - \tilde{P}'_{q^*}(q^*, \Delta h^*)\, \Delta h^* + \gamma(q_i - q^*)$$

holds. Then the following inequality is valid for X

$$\tilde{P}_{q^*}(\Delta h^*) - \tilde{P}'_{q^*}(\Delta h^*)\, \Delta h^* + \tilde{P}'_{q^*}(\Delta h^*)\, \Delta h + \beta(y - 1) + \gamma(q - q^*) \leq p.$$

Proof. We again show the validity for subsets of X. First of all, the inequality corresponds to a gradient cut on $X \cap \{y = 1, q = q^*\}$. By the minimum condition on γ, the inequality is valid for $X \cap \{y = 1\}$. The last condition states that the left-hand side of the inequality must be bounded by zero for $X \cap \{y = 0\}$. $\square$

The inequalities derived in Lemma 3.1–3.3 are able to strengthen the relaxations used by MINLP solvers. Since there are infinitely many and to obtain small relaxations, usually only inequalities violated by solution candidates are added. Given a relaxation solution with pressure increase value $\Delta\tilde{h}$, the following heuristic for separating the above families of inequalities works well. The parameter Δh_i^* in Lemma 3.1 is chosen as $\Delta\tilde{h}$ if it belongs to $[\Delta\underline{H}(q_i), \Delta\overline{H}(q_i)]$, otherwise as the midpoint of the interval. To separate the inequalities given by Lemma 3.2 or 3.3, we try both choices for q^* and/or $\tilde{q}$ and use $\Delta\tilde{h}$ for Δh^* if it belongs to the interval $[\Delta\underline{H}(q^*), \Delta\overline{H}(q^*)]$; otherwise we set it to the lower or the upper bound according on which side of the interval $\Delta\tilde{h}$ lies. We then maximize or minimize γ depending on q^* and minimize β by solving $\min\{\tilde{P}_{q^*}(\Delta h) - \alpha\, \Delta h \mid \Delta h \in [\Delta\underline{H}(q^*), \Delta\overline{H}(q^*)]\}$ with given values of $q^* \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. This is, for appropriate speed bounds $\underline{\omega}$ and $\overline{\omega}$, equivalent to $\min\{P(q^*, \omega) - \alpha\, \Delta H(q^*, \omega) \mid \omega \in [\underline{\omega}, \overline{\omega}]\}$, i.e, the trivial problem to minimize a one-dimensional cubic function over an interval.

4. Computational experiments

We conducted experiments on a testset containing instances of a high-rise problem resembling a real life downscaled testrig. We use five different basic pump types, c.f. Table 1, and also pump types derived by placing each basic type up to three

TABLE 2. Overview of test results. "Time" and "Nodes" give the shifted geometric means (see [4]) of solving time and number of branch and bound nodes, respectively. "Gap" gives the arithmetic mean over the gap between primal and dual bound after one hour. "# solved" gives the number of solved instances in this time.

Formulation/Setting	Time	Nodes	Gap	# solved
D	295.5	22428.6	53.41	61
X	120.7	4548.7	88.11	89
`CH`	44.4	19210.6	29.03	90
`CH+SEP`	13.4	1464.4	0.27	117

times in parallel. Usage of Lemma 2.1 verifies that each type possesses the convexity property. We investigated 5, 7 and 10 floors, with a one meter height difference between consecutive floors. The pressure increase demanded in a floor is independent of volume flow and lies between 1.2 and 1.44 times its height. Furthermore, we include an energy cost weight of 10 and 100, which determines the importance of the nonlinearities in the objective value. The volume flow demand in each floor was sampled according to $\max\{0, \mathcal{N}(\mu, \sigma^2)\}$ for $(\mu, \sigma) \in \{(1, 0.5), (0.5, 0.25)\}$. For each of these different settings, ten instances were created, leading to 120 instances. To perform the test, we used SCIP 6.0.1, see [4], compiled with IPOPT 3.12, see [5], and CPLEX 12.8 running on a Linux cluster with Intel Xeon E5 CPUs with 3.50GHz, 10MB cache, and 32GB memory using a one hour time limit. We implemented a constraint handler, which enforces X by branching on the volume flow variables and using perspective cuts for fixed volume flows. Furthermore, it heuristically separates the three families presented in Lemma 3.1 to 3.3 and propagates flow and pressure increase bounds.

We compared the formulation involving speed variables, i.e., the set D and a formulation involving X without the constraint handler. Furthermore, we tested the constraint handler without (`CH`) and with the heuristic cut separation (`CH+SEP`). In Table 2, we show the performance of the different approaches. The formulation X replaces polynomials by composite functions involving square-roots; nonetheless, the elimination of ω is able to solve 28 more instances in one hour than formulation D. The worse average of the final gaps between primal and dual bounds is due to the lack of good primal solutions. Using `CH` one can solve only slightly more instances and more branch and bound nodes need to be inspected, but the solving time decreases substantially. The best performance albeit is given by also separating the lifted cuts, which also results in the least amount of visited branch and bounds nodes on average. Further results, not included for the ease of presentation, show: Separating the cuts of Lemma 3.2 has the biggest impact. Their sole usage already solves 116 instances, whereas the exclusive usage of cuts from either Lemma 3.1 or 3.3 leads to only 102 and 104 solved instances, respectively.

Acknowledgments We thank Tim Müller (TU Darmstadt) for the pump approximations. This research was funded by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project Number 57157498 – SFB 805.

References

[1] L. C. Altherr, P. Leise, M. E. Pfetsch, and A. Schmitt. Resilient layout, design and operation of energy-efficient water distribution networks for high-rise buildings using MINLP. *Optim Eng*, 20(2):605–645, 2019.

[2] C. D'Ambrosio, A. Lodi, S. Wiese, and C. Bragalli. Mathematical programming techniques in water network optimization. *Eur J Oper Res*, 243(3):774–788, 2015.

[3] A. Frangioni and C. Gentile. Perspective cuts for a class of convex 0–1 mixed integer programs. *Math Program*, 106(2):225–236, 2006.

[4] A. Gleixner, M. Bastubbe, L. Eifler, T. Gally, G. Gamrath, R. L. Gottwald, G. Hendel, C. Hojny, T. Koch, M. E. Lübbecke, S. J. Maher, M. Miltenberger, B. Müller, M. E. Pfetsch, C. Puchert, D. Rehfeldt, F. Schlösser, C. Schubert, F. Serrano, Y. Shinano, J. M. Viernickel, M. Walter, F. Wegscheider, J. T. Witt, and J. Witzig. The SCIP Optimization Suite 6.0. Technical report, Optimization Online, 2018.

[5] A. Wächter and L. T. Biegler. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Math Program*, 106(1):25–57, 2006.

Email address: {`pfetsch, aschmitt`}`@mathematik.tu-darmstadt.de`

DEPARTMENT OF MATHEMATICS, TU DARMSTADT, DOLIVOSTR. 15, 64293 DARMSTADT, GERMANY