

# THE FERMAT RULE FOR SET OPTIMIZATION PROBLEMS WITH LIPSCHITZIAN SET-VALUED MAPPINGS

GEMAYQZEL BOUZA, ERNEST QUINTANA, CHRISTIANE TAMMER,  
AND VU ANH TUAN

ABSTRACT. In this paper, we consider set optimization problems where the solution concept is given by the set approach. Specifically, we deal with the lower less and the upper less set relations. First, we derive the convexity and Lipschitzianity of suitable scalarizing functionals under the assumption that the set-valued objective mapping has certain convexity and Lipschitzianity properties. Then, we obtain upper estimates of the limiting subdifferential of these functionals. These results, together with the properties of the scalarization functionals, allow us to obtain a Fermat rule for set optimization problems with Lipschitzian data.

*Dedicated to Franco Giannessi on the occasion of his 85th birthday*

## 1. INTRODUCTION

Set optimization is a class of mathematical problems that consist in minimizing a given set-valued objective mapping. This type of problems generalizes vector optimization models and has received a lot of attention during the last decade due to their applications in finance [16, 23], socio-economics [7, 48], robotics [29] and robust multiobjective decision making [15, 27].

There are two main approaches for defining optimal solutions of a set optimization problem, namely the vector approach and the set approach. In the vector approach, we look for efficient points of the image set of the set-valued mapping [29]. Hence, in this case, one element completely determines the quality of a given set, while ignoring the rest of its elements. This is an important drawback for modeling practical problems, and the set approach is an attempt at fixing this problem. The idea in the set approach is to introduce a preorder relation on the power set of the image space, and to define minimal solutions accordingly. The first set relations were introduced independently by Young [62] and Nishnianidze [49], and later by Kuroiwa [42, 44]. More recently, new ones were derived by Jahn and Ha [28], and Karaman et al. [33]. Since then, there have been a lot of research related

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to the existence of solutions, duality statements, optimality conditions and algorithms for solving these set optimization problems. We refer the reader to [35] and the references therein for a comprehensive overview of the field.

In this paper, we are concerned with necessary optimality conditions for solutions to set optimization problems given by the set approach. The literature on the topic is rich and different results have been obtained using generalized differentiation objects lying in the primal as well as in the dual spaces (see Giannessi [18]).

The techniques employed in the primal space are mainly based on some type of directional derivatives and can be roughly separated into the following classes:

- Directional derivatives based on set differences [11, 30, 34, 54].  
The main idea is to consider a suitable operation that resembles subtraction in the power of the image space. These operations are based on the well known differences of sets of Minkowski and Demjanov [22, 56], but usually slight modifications are introduced in order to make it useful in set optimization. Then, with the help of the set difference, a directional derivative is defined as a limit of an associated incremental quotient. Furthermore, the optimality conditions obtained in this setting establish the nonnegativity of the directional derivative, according to the treated set relation.
- Directional derivatives based on a distance type functional [20, 21].  
In contrast to the previous technique, a directional derivative is introduced in [20] with the help of the standard algebraic difference of sets and a distance type functional. The distance functional is a modification of the well known Hausdorff distance for sets and is based on the classical Hiriart-Urruty functional [25]. The directional derivative is in this case defined as the minimal set of some compact set to which the incremental quotient converges (in the sense of the modified distance). A similar idea is used in [21] to introduce a concept of slope for a set-valued map at a given point, together with necessary conditions for minimal solutions of the set optimization problem in the convex case.
- Directional derivative based on embedding [45].  
The idea in [45] is to embed the class of convex and bounded sets (with respect to the ordering cone) into a suitable normed space. With this construction, the original set optimization problem is equivalent to a standard vector optimization problem having as a target function the composition of the embedding map and the set-valued objective mapping. Hence, a directional derivative of set-valued mapping is defined in a standard way as the directional derivative of this composition.
- Directional derivatives of selections of the set-valued objective mapping [1, 2].

In this approach, there is no explicit definition of directional derivatives for a set-valued mapping, but rather the authors use those of its continuous selections. Roughly speaking, the optimality conditions establish the nonnegativity, in the sense of the ordering cone, of these directional derivatives.

- Contingent derivatives and variations [40, 41, 51, 55].

Contingent derivatives and epiderivatives have been successfully employed in obtaining optimality conditions for set optimization problems with the vector approach [29]. Consequently, it was a natural idea to apply them also in the set approach setting. In this direction, other modifications of the derivatives were also studied, like those of Shi [58] and Studniarski [59].

On the other hand, optimality conditions using generalized differentiation objects lying in the dual space have been considered in the literature, see [32, 36, 40]. In particular:

- In [32], the case in which the set-valued mapping is given by functional constraints was analyzed. Using a vectorization result by Jahn [31], the set-valued problem was transformed into a vector-valued one (with an infinite dimensional image space), and hence classical optimality conditions for vector optimization problems were applied.
- In [36, 40] the idea is that, under different assumptions, set approach solutions of the set-valued problem are also solutions in the vector approach. Hence, under these conditions, well studied optimality conditions for vector approach solutions can be applied in the context of the set approach.

However, we want to mention that some of these optimality conditions are derived under somewhat strong assumptions on the set-valued objective mapping. For example, in [1, 2, 40, 41, 51, 55], it is required that the optimal set has a strongly minimal element in order to verify optimality. In addition, either the convexity or compactness (mostly both) of the images of the set-valued objective mapping are needed in [11, 20, 21, 30, 32, 54].

Recently, it also caught our attention that, independently, Amahroq and Oussarhan in [3, 4, 50] and Huerga, Jiménez and Novo in [26] were working with similar ideas to ours for deriving optimality conditions in set optimization. The main differences between the results derived in these papers and our optimality conditions are the following:

- In [3, 4], [26], the authors studied only solution concepts based on the lower less relation. In our paper, we also examine solution concepts based on the upper less relation.
- In [4], [26], the case in which the set-valued mapping is convex was analyzed under different assumptions to ours. In addition, the optimality conditions in [4] require that the optimal set has a strongly minimal element.

- In [3], the case in which the set-valued objective mapping is locally Lipschitzian is analyzed. However, the authors assume the compactness of the images of the set valued objective mapping in [3]. The optimality conditions are not established using the initial data, but rather they are expressed in a limiting form. Also in [26], certain compactness assumptions concerning the involved set-valued mappings are supposed. In our paper, we derive our results without compactness assumptions concerning the set-valued objective mapping.

In this paper, as mentioned above, we deal with the lower less relation and the upper less relation, and obtain optimality conditions using generalized differentiation objects lying in the dual spaces. By means of a suitable scalarizing functional, we construct a scalar problem that characterizes the solutions of the set-valued problem. Then, based on the initial data, the necessary conditions for the scalar problem are obtained by using well known results from variational analysis. Our results extend those in [14] for vector optimization problems.

The paper is organized as follows: In Section 2, we introduce fundamental notations, definitions and auxiliary results that will be used through the text. In Section 3, we derive the convexity and Lipschitzianity of suitable scalarizing functionals under certain convexity and Lipschitzianity assumptions on the set-valued objective mapping. Sections 4 and 5 are devoted to obtaining upper estimates of the limiting subdifferential of these scalarizing functionals. The previous results are employed in Section 6 to derive the optimality conditions for solutions to set optimization problems. Finally, we close the paper by summarizing our contributions and establishing some further remarks in Section 7.

## 2. PRELIMINARIES

We start this section by establishing the main notations used in the paper. Given a normed space  $(X, \|\cdot\|_X)$ , we will denote by  $(X^*, \|\cdot\|_X^*)$  its topological dual. In addition, the closed unit balls in  $X$  and  $X^*$  will be denoted as  $\mathbb{B}_X$  and  $\mathbb{B}_X^*$ , respectively. We omit the subscript  $X$  if there is no risk of confusion. For a nonempty set  $A \subseteq X$ ,  $\text{int } A$ ,  $\text{cl } A$ ,  $\text{bd } A$ ,  $\text{conv } A$  stand for the interior, closure, boundary and convex hull of  $A$ . If  $B \subseteq X^*$ , we denote by  $\overline{\text{conv}}^* B$  the closure of the convex hull of  $B$  in the weak\* topology of  $X^*$ . We always use lowercase letters to denote a vector or scalar-valued function, and capital letters for a set-valued mapping.

**Definition 2.1.** *Let  $X$  be a normed space. Then:*

- (i) *A nonempty set  $C \subseteq X$  is said to be a cone if  $\lambda c \in C$  for every  $c \in C$  and every  $\lambda \geq 0$ . The cone  $C$  is called:*
- *convex if  $C + C \subseteq C$ ,*
  - *proper if  $C \neq \{0\}$  and  $C \neq X$ ,*
  - *solid if  $\text{int } C \neq \emptyset$ ,*

- pointed if  $C \cap (-C) = \{0\}$ .

(ii) For a cone  $C \subseteq X$ , the continuous dual cone of  $C$  is given by

$$C^* := \{x^* \in X^* \mid \forall c \in C : \langle x^*, c \rangle \geq 0\}.$$

The support function  $\sigma_A : X \rightarrow \overline{\mathbb{R}}$  of a set  $A \subseteq X^*$  is defined by

$$\sigma_A(x) := \sup_{x^* \in A} \langle x^*, x \rangle, \quad (x \in X).$$

From now on, we work with the following assumption:

**Assumption 1.** Let  $X$  and  $Y$  be Banach spaces,  $K \subseteq Y$  be a closed, convex, pointed and solid cone and  $e \in \text{int } K$ . Let  $\Omega \subseteq X$  be nonempty and closed, and fix  $\bar{x} \in \Omega$ . Furthermore, let  $F : X \rightrightarrows Y$  be a set-valued mapping such that  $\Omega \subseteq \text{int dom } F$ .

Of course, in Assumption 1 above,

$$\text{dom } F := \{x \in X \mid F(x) \neq \emptyset\}.$$

Furthermore, the graph and epigraph of  $F$  are defined respectively as

$$\begin{aligned} \text{gph } F &:= \{(x, y) \in X \times Y \mid y \in F(x)\}, \\ \text{epi } F &:= \{(x, y) \in X \times Y \mid y \in F(x) + K\}. \end{aligned}$$

We define the epigraphical and hypergraphical multifunctions associated to  $F$  respectively as the set-valued mappings  $\mathcal{E}_F, \mathcal{H}_F : X \rightrightarrows Y$  given by

$$(2.1) \quad \mathcal{E}_F(x) := F(x) + K,$$

$$(2.2) \quad \mathcal{H}_F(x) := F(x) - K.$$

The cone  $K$  generates a partial order  $\preceq_K$  on  $Y$  as follows:  $y_1 \preceq_K y_2$  if and only if  $y_2 - y_1 \in K$ . Associated to  $\preceq_K$  is the strict inequality  $\prec_K$  which is defined as:  $y_1 \prec_K y_2$  if and only if  $y_2 - y_1 \in \text{int } K$ . We also recall that if  $y_1, y_2 \in Y$  and  $y_1 \preceq_K y_2$ , the interval  $[y_1, y_2]$  is defined by

$$[y_1, y_2] := (y_1 + K) \cap (y_2 - K).$$

**Definition 2.2.** Let Assumption 1 be fulfilled and let  $A \subseteq Y$ .

(i) The set of weakly minimal elements of  $A$  with respect to  $K$  is defined as

$$\text{WMin}(A, K) := \{y \in A \mid (y - \text{int } K) \cap A = \emptyset\}.$$

(ii) The set of minimal elements of  $A$  with respect to  $K$  is defined as

$$\text{Min}(A, K) := \{y \in A \mid (y - K) \cap A = \{y\}\}.$$

(iii) The set of weakly maximal elements of  $A$  with respect to  $K$  is defined as

$$\text{WMax}(A, K) := \{y \in A \mid (y + \text{int } K) \cap A = \emptyset\}.$$

(iv) The set of maximal elements of  $A$  with respect to  $K$  is defined as

$$\text{Max}(A, K) := \{y \in A \mid (y + K) \cap A = \{y\}\}.$$

(v) The set of strongly minimal elements of  $A$  with respect to  $K$  is defined as

$$\text{SMin}(A, K) := \{y \in A \mid A \subseteq y + K\}.$$

As mentioned in the introduction, set optimization problems in our context are based on some preorder relations between subsets of  $Y$ . These preorders are defined below.

**Definition 2.3** ([42], [43]). *Let Assumption 1 be fulfilled and suppose  $A, B, D \subseteq Y$ .*

(i) The lower- less relation  $\preceq_D^{(l)}$  is defined as

$$A \preceq_D^{(l)} B \iff B \subseteq A + D.$$

(ii) The upper- less relation  $\preceq_D^{(u)}$  is defined as

$$A \preceq_D^{(u)} B \iff A \subseteq B - D.$$

When we take the order with respect to  $\text{int } D$ , we write  $\prec_D^{(r)}$  instead of  $\preceq_{\text{int } D}^{(r)}$ , for  $r \in \{l, u\}$ . Furthermore, for  $\preceq_D^{(l)}$  and  $\preceq_D^{(u)}$ , we recall the equivalence relations on  $2^Y$  with respect to a set  $D \subseteq Y$  as follows:

$$A \sim_D^{(l)} B \iff A \preceq_D^{(l)} B \text{ and } B \preceq_D^{(l)} A \iff A + D = B + D,$$

$$A \sim_D^{(u)} B \iff A \preceq_D^{(u)} B \text{ and } B \preceq_D^{(u)} A \iff A - D = B - D.$$

These family of equivalence classes were first introduced by Hernández and Rodríguez-Marín [24]. For a set  $A \subseteq Y$ , we will denote the corresponding equivalence classes by  $[A]^{(l)}$  and  $[A]^{(u)}$  respectively, depending on the set relation. Under our assumption that  $K$  is a closed, convex, pointed and solid cone we get

$$A \sim_K^{(l)} B \iff \text{Min}(A, K) = \text{Min}(B, K),$$

(compare [35, Remark 2.6.11]). Next we define convexity of a set-valued mapping with respect to a set relation.

**Definition 2.4.** Let Assumption 1 be fulfilled and let  $r \in \{l, u\}$ . We say that  $F$  is  $\preceq_K^{(r)}$ -convex if

$$\forall x_1, x_2 \in \text{dom } F, \lambda \in (0, 1) : F(\lambda x_1 + (1 - \lambda)x_2) \preceq_K^{(r)} \lambda F(x_1) + (1 - \lambda)F(x_2).$$

**Remark 2.5.** Recall that the classical concept of convexity for a set-valued mapping is that  $F : X \rightrightarrows Y$  is convex if its graph is a convex subset of  $X \times Y$ . It can be shown that  $F$  is  $\preceq_K^{(l)}$ -convex if and only if  $\text{epi } F$  is a convex set or, equivalently, if the epigraphical multifunction  $\mathcal{E}_F$  is convex.

In the next definition we consider different concepts of boundedness associated to a set-valued mapping.

**Definition 2.6.** Let Assumption 1 be fulfilled and let  $U \subseteq X$ ,  $A \subseteq Y$ . We say that:

(i)  $A$  is  $K$ -lower (upper) bounded if there exists  $\mu > 0$  such that

$$A \subseteq -\mu e + K \quad (\text{respectively, } A \subseteq \mu e - K).$$

(ii)  $F$  is  $l$ -upper bounded on the set  $U$  if there exists a constant  $\mu > 0$  such that:

$$\forall x \in U : F(x) \cap (\mu e - K) \in [F(x)]^{(l)}.$$

(iii)  $F$  is  $l$ -lower bounded on  $U$  if  $F[U]$  is  $K$ -lower bounded. Equivalently, there exists  $\mu > 0$  such that

$$F[U] \subseteq -\mu e + K.$$

(iv)  $F$  is  $l$ -bounded on  $U$  if  $F$  is  $l$ -upper bounded and  $l$ -lower bounded on  $U$ . Equivalently, there exists a constant  $\mu > 0$  such that:

$$\forall x \in U : F(x) \cap [-\mu e, \mu e] \in [F(x)]^{(l)}.$$

(v)  $F$  is locally  $l$ -(upper, lower) bounded at  $\bar{x}$  if it is  $l$ -(upper, lower) bounded on a neighborhood  $U$  of  $\bar{x}$ .

(vi)  $F$  is  $u$ -upper bounded on  $U$  if  $F[U]$  is  $K$ -upper bounded. Equivalently, there exists  $\mu > 0$  such that

$$F[U] \subseteq \mu e - K.$$

(vii)  $F$  is  $u$ -lower bounded on  $U$  if there exists a constant  $\mu > 0$  such that:

$$\forall x \in U : F(x) \cap (-\mu e + K) \in [F(x)]^{(u)}.$$

(viii)  $F$  is  $u$ -bounded on  $U$  if  $F$  is  $u$ -upper bounded and  $u$ -lower bounded on  $U$ , it means that there exists a constant  $\mu > 0$  such that:

$$\forall x \in U : F(x) \cap [-\mu e, \mu e] \in [F(x)]^{(u)}.$$

(ix)  $F$  is locally  $u$ -(upper, lower) bounded at  $\bar{x}$  if it is  $u$ -(upper, lower) bounded on a neighborhood  $U$  of  $\bar{x}$ .

In the following definition, we introduce different topological notions of a set-valued mapping.

**Definition 2.7.** *Let Assumption 1 be fulfilled. We say that:*

(i)  $F$  is locally bounded at  $\bar{x}$  if there exists  $L > 0$  and a neighborhood  $U$  of  $\bar{x}$  such that

$$F[U] \subseteq L\mathbb{B}.$$

(ii)  $F$  is locally Lipschitzian at  $\bar{x}$  if there is a neighborhood  $U$  of  $\bar{x}$  and a constant  $\ell \geq 0$  such that

$$(2.3) \quad \forall x, x' \in U : \quad F(x) \subseteq F(x') + \ell \|x - x'\| \mathbb{B}.$$

(iii)  $F$  is inner semicompact at  $\bar{x} \in \text{dom } F$  if for every sequence  $x_k \rightarrow \bar{x}$  there is a sequence  $y_k \in F(x_k)$  that contains a convergent subsequence as  $k \rightarrow \infty$ . In particular,  $\bar{x} \in \text{int dom } F$ .

(iv)  $F$  is closed at  $\bar{x}$  if, for any sequence  $\{(x_k, y_k)\}_{k \geq 1} \subseteq \text{gph } F$  with  $(x_k, y_k) \rightarrow (\bar{x}, \bar{y})$ , we have  $(\bar{x}, \bar{y}) \in \text{gph } F$ .

For a scalar function  $f : X \rightarrow \overline{\mathbb{R}}$ , the domain and epigraph of  $f$  are given by

$$\text{dom } f := \{x \in X \mid f(x) < +\infty\},$$

$$\text{epi } f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \leq t\}.$$

Recall that a function  $f : X \rightarrow \mathbb{R}$  is convex if  $\text{epi } f$  is a convex set. The function  $f$  is said to be Lipschitzian on  $A$  provided that  $f$  is finite on  $A$  and there exists  $\ell > 0$  such that

$$\forall x, x' \in A : |f(x) - f(x')| \leq \ell \|x - x'\|_X.$$

This is also referred to as a Lipschitzian condition of rank  $\ell$ . We say that  $f$  is locally Lipschitzian at  $\bar{x}$  if there is a neighborhood  $U$  of  $\bar{x}$  such that  $f$  is Lipschitzian on  $U$ . In addition,  $f$  is said to be locally Lipschitzian on  $A$ , if  $f$  is locally Lipschitzian at every point  $x \in A$ . Hence, in this case,  $A \subseteq \text{int dom } f$ .

We now introduce the tools from variational analysis that will be employed in the text. First, we need a notion of limits of sets. For a set-valued mapping  $F : X \rightrightarrows X^*$ , we define the Painlevé-Kuratowski outer limit of  $F$  at  $\bar{x}$  with respect to the norm topology of  $X$  and the  $w^*$ -topology of  $X^*$  by

$$\limsup_{x \rightarrow \bar{x}} F(x) := \{x^* \in X^* \mid \forall k \in \mathbb{N}, \exists (x_k, x_k^*) \in \text{gph } F : x_k \rightarrow \bar{x}, x_k^* \xrightarrow{w^*} x^*\}.$$

In the next definition, we use the notation  $x' \xrightarrow{\Omega} x$  for  $x' \rightarrow x$  with  $x' \in \Omega$ .



**Definition 2.1.** ([46, Definition 1.1]) *Let Assumption 1 be fulfilled.*

(i) *Given  $x \in X$  and  $\epsilon \geq 0$ , the set of  $\epsilon$ -normals to  $\Omega$  at  $x$  is defined by*

$$(2.4) \quad \hat{N}_\epsilon(x, \Omega) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq \epsilon \right\}.$$

*When  $\epsilon = 0$ , the set (2.4) is called the Fréchet normal cone to  $\Omega$  at  $x$ , and is denoted by  $\hat{N}(x, \Omega)$ . If  $x \notin \Omega$ , we put  $\hat{N}_\epsilon(x, \Omega) := \emptyset$  for all  $\epsilon \geq 0$ .*

(ii) *The limiting normal cone to  $\Omega$  at  $\bar{x}$  is defined by*

$$(2.5) \quad N(\bar{x}, \Omega) := \limsup_{\substack{x \rightarrow \bar{x} \\ \epsilon \downarrow 0}} \hat{N}_\epsilon(x, \Omega).$$

*We also put  $N(\bar{x}, \Omega) := \emptyset$  for  $\bar{x} \notin \Omega$ .*

**Definition 2.2.** ([46, Definition 1.77]) *Let Assumption 1 be fulfilled. The limiting subdifferential of a given functional  $f : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$  at a point  $\bar{x}$  with  $|f(\bar{x})| < +\infty$  is defined by*

$$\partial f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, f(\bar{x})), \text{epi } f)\}.$$

*We put  $\partial f(\bar{x}) := \emptyset$  if  $|f(\bar{x})| = +\infty$ .*

**Remark 2.8.** *It is well known, see [46, Theorem 1.93], that if  $f$  is convex and finite at  $\bar{x}$ , then*

$$\partial f(\bar{x}) = \{x^* \in X^* \mid \forall x \in X : f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle\},$$

*and hence  $\partial f(\bar{x})$  coincides with the subdifferential of convex analysis. In case  $\Omega$  is a convex set, we also have [46, Proposition 1.5]:*

$$N(\bar{x}, \Omega) = \{x^* \in X^* \mid \forall x \in \Omega : \langle x^*, x - \bar{x} \rangle \leq 0\},$$

*and hence  $N(\bar{x}, \Omega)$  equals the normal cone in the sense of convex analysis.*

**Remark 2.9.** *Note that, if in addition to Assumption 1 the space  $X$  is Asplund and  $f : X \rightarrow \mathbb{R}$  is locally Lipschitzian at  $\bar{x}$ , the following relation holds:*

$$\partial(-f)(\bar{x}) \subseteq -\overline{\text{conv}}^*(\partial f(\bar{x})).$$

*Indeed, taking into account [46, Theorem 3.57], [10, Proposition 2.3.1] and [46, Theorem 3.57], we obtain*

$$\begin{aligned} \partial(-f)(\bar{x}) &\subseteq \overline{\text{conv}}^*(\partial(-f)(\bar{x})) \\ &= \partial^\circ(-f)(\bar{x}) \\ &= -\partial^\circ f(\bar{x}) \\ &= -\overline{\text{conv}}^*(\partial f(\bar{x})). \end{aligned}$$

*Here,  $\partial^\circ f$  represents Clarke's subdifferential of  $f$ , see [10] for details.*

We continue by defining the basic coderivative of a set-valued mapping at a point of its graph.

**Definition 2.3.** ([46, Definition 1.32]) *Let Assumption 1 be fulfilled. The basic coderivative of  $F$  at  $(\bar{x}, \bar{y}) \in \text{gph } F$  is the multifunction  $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$  defined by*

$$(2.6) \quad D^*F(\bar{x}, \bar{y})(y^*) = \{x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}), \text{gph } F)\}.$$

We put  $D^*F(\bar{x}, \bar{y})(y^*) := \emptyset$  for all  $y^* \in Y^*$  if  $(\bar{x}, \bar{y}) \notin \text{gph } F$ .

**Remark 2.10.** *We can omit  $\bar{y}$  in the coderivative notation above if  $F = f : X \rightarrow Y$  is a vector-valued function. It can be shown [46, Theorem 1.38], that if  $f$  is continuously differentiable at  $\bar{x}$ , then*

$$D^*f(\bar{x})(y^*) = \{\nabla f(\bar{x})^* y^*\} \quad \text{for all } y^* \in Y^*.$$

In the above equation,  $\nabla f(\bar{x})^* : Y^* \rightarrow X^*$  denotes the adjoint operator of  $\nabla f(\bar{x})$ .

In the last part of the section, we quickly recall two results concerning the subdifferential of marginal functions. This problem is naturally linked to the computation of the subdifferentials of the scalarization functionals, as we will see in the rest of the sections. The setting is as follows:

**Assumption 2.** *In addition to Assumption 1, let  $f : X \times Y \rightarrow \overline{\mathbb{R}}$  be a given functional and consider the associated marginal function  $\varphi : X \rightarrow \overline{\mathbb{R}}$  defined as*

$$\varphi(x) := \inf_{y \in F(x)} f(x, y).$$

Furthermore, consider the solution map  $S : X \rightrightarrows Y$  defined as

$$S(x) = \{y \in F(x) : f(x, y) = \varphi(x)\}.$$

The first of the results is concerned with the subdifferential of  $\varphi$  in the case that  $F$  and  $f$  are assumed to be convex.

**Theorem 2.11.** ([5, Theorem 4.2]) *Let Assumption 2 be fulfilled. Suppose in addition that  $\text{gph } F$  is convex,  $f$  is a proper and convex function, and that at least one of the following regularity conditions is satisfied:*

- (i)  $\text{int } \text{gph } F \cap \text{dom } f \neq \emptyset$ ,
- (ii)  $f$  is continuous at a point  $(x^0, y^0) \in \text{gph } F$ .

Then,  $\varphi$  is convex and, for any  $\bar{x} \in \text{dom } \varphi$  with  $\varphi(\bar{x}) \neq -\infty$  and any  $\bar{y} \in S(\bar{x})$ , we have

$$\partial\varphi(\bar{x}) = \bigcup_{(x^*, y^*) \in \partial f(\bar{x}, \bar{y})} \left[ x^* + D^*F(\bar{x}, \bar{y})(y^*) \right].$$

For nonconvex  $F$ , many results already exist in the literature. We conclude by establishing a weaker version of [46, Theorem 3.38 (ii)], which will be enough for our purposes. The proof is omitted since it is easy to verify that our assumptions imply those of [46, Theorem 3.38 (ii)]. Recall that a set  $A$  is locally closed around a point  $\bar{z} \in A$  if there exists a neighborhood  $V$  of  $\bar{z}$  such that  $A \cap V$  is a closed set.

**Theorem 2.12.** *In addition to Assumption 2, suppose that  $X$  and  $Y$  are Asplund spaces. Furthermore, assume that:*

- (i)  $F$  is closed at  $\bar{x}$ ,
- (ii)  $S$  is inner semicompact at  $\bar{x}$ ,
- (iii) there exists a neighborhood  $U$  of  $\bar{x}$  such that  $f$  is Lipschitzian on  $U \times Y$ ,
- (iv)  $\text{gph } F$  is locally closed around every point of the set  $\{\bar{x}\} \times S(\bar{x})$ .

Then,

$$(2.7) \quad \partial\varphi(\bar{x}) \subseteq \bigcup_{\substack{\bar{y} \in S(\bar{x}) \\ (x^*, y^*) \in \partial f(\bar{x}, \bar{y})}} \left[ x^* + D^*F(\bar{x}, \bar{y})(y^*) \right].$$

### 3. CONVEXITY AND LIPSCHITZIANITY OF THE SCALARIZING FUNCTIONALS IN SET OPTIMIZATION

With the purpose of deriving optimality conditions for set optimization problems we introduce in this section, for a given set-valued mapping  $F$ , two associated scalarizing functionals.

We proceed to show that certain nonlinear scalarizing functionals inherit the convexity and Lipschitzianity of  $F$ . First, under Assumption 1, we consider the functional  $\Psi_e : Y \rightarrow \bar{\mathbb{R}}$  given by:

$$(3.1) \quad \Psi_e(y) := \inf\{t \in \mathbb{R} \mid y \in te - K\}.$$

Recent results concerning characterizations of set relations by scalarizing functionals and corresponding subdifferentials are given in [8, 9].

The nonlinear scalarizing functional  $\Psi_e$  has been widely applied in vector optimization [17, 19, 35] and in the next proposition we collect several well known properties of  $\Psi_e$  that will be useful later in this work. Recall that a functional  $g : Y \rightarrow \mathbb{R}$  is said to be  $K$ -monotone if  $y_1, y_2 \in Y, y_1 \preceq_K y_2 \Rightarrow g(y_1) \leq g(y_2)$ . Moreover, we say that  $g$  is strictly  $K$ -monotone if  $y_1 \prec_K y_2 \Rightarrow g(y_1) < g(y_2)$ .

**Proposition 3.1** ([19, 13]). *Let Assumption 1 be fulfilled. Then:*

- (i)  $\Psi_e$  is a finite-valued sublinear function, i.e.,  $\Psi_e(y_1 + y_2) \leq \Psi_e(y_1) + \Psi_e(y_2)$  and  $\Psi_e(\lambda y) = \lambda \Psi_e(y)$  for all  $\lambda > 0$  and  $y_1, y_2 \in Y$ ,
- (ii)  $\Psi_e$  is Lipschitzian on  $Y$ ,
- (iii)  $\Psi_e$  satisfies the translativity property, i.e.,  $\Psi_e(y + te) = \Psi_e(y) + t$  for all  $t \in \mathbb{R}$  and  $y \in Y$ ,
- (iv)  $\Psi_e$  is  $K$ -monotone and strictly  $K$ -monotone,

- (v)  $\partial\Psi_e(\bar{y}) = \{k^* \in K^* \mid \langle k^*, e \rangle = 1, \Psi_e(\bar{y}) = \langle k^*, \bar{y} \rangle\}$ ,  
 (vi)  $\Psi_e$  satisfies the representability property, i.e.,

$$-K = \{y \in Y \mid \Psi_e(y) \leq 0\}, \quad -\text{int } K = \{y \in Y \mid \Psi_e(y) < 0\}.$$

**Remark 3.2.** According to (vi), for any  $\bar{y} \in -\text{bd } K$  we have  $\Psi_e(\bar{y}) = 0$ . Then, it follows from (v) that  $\partial\Psi_e(\bar{y}) = \{k^* \in K^* \mid \langle k^*, e \rangle = 1, \langle k^*, \bar{y} \rangle = 0\}$ . This simple fact is important to keep in mind for the results that will be obtained later.

In [37, 38, 39], a complete characterization of set order relations by means of a nonlinear scalarizing functional was shown. There, the main result is the following:

**Theorem 3.3.** ([37, 38, 39]) *Let Assumption 1 be fulfilled and consider  $A, B \subseteq Y$ . Then,*

- (i)  $A \preceq_K^{(l)} B \implies \sup_{b \in B} \inf_{a \in A} \Psi_e(a - b) \leq 0$ .  
 (ii)  $A \preceq_K^{(u)} B \implies \sup_{a \in A} \inf_{b \in B} \Psi_e(a - b) \leq 0$ .

The previous theorem motivates our next definition.

**Definition 3.4.** *Let Assumption 1 be fulfilled.*

- (i) *The lower inner function  $g_l : X \times Y \rightarrow \bar{\mathbb{R}}$  is defined as*

$$(3.2) \quad g_l(x, z) := \inf_{y \in F(x)} \Psi_e(y - z).$$

- (ii) *The upper inner function  $g_{u, \bar{x}} : Y \rightarrow \bar{\mathbb{R}}$  is defined as*

$$(3.3) \quad g_{u, \bar{x}}(y) := \inf_{\bar{y} \in F(\bar{x})} \Psi_e(y - \bar{y}).$$

- (iii) *For  $r \in \{l, u\}$ , the scalarizing functional  $f_{r, \bar{x}} : X \rightarrow \bar{\mathbb{R}}$  is defined as follows:*

$$(3.4) \quad f_{r, \bar{x}}(x) := \begin{cases} \sup_{\bar{y} \in F(\bar{x})} g_l(x, \bar{y}) = \sup_{\bar{y} \in F(\bar{x})} \inf_{y \in F(x)} \Psi_e(y - \bar{y}) & \text{if } r = l, \\ \sup_{y \in F(x)} g_{u, \bar{x}}(y) = \sup_{y \in F(x)} \inf_{\bar{y} \in F(\bar{x})} \Psi_e(y - \bar{y}) & \text{if } r = u. \end{cases}$$

As mentioned at the beginning of the section, we now show that for the  $\preceq_K^{(l)}$  and  $\preceq_K^{(u)}$  relations, the corresponding scalarizing functional inherits the convexity property of the set-valued mapping. We start with a simple proposition.

**Proposition 3.5.** *Let Assumption 1 be fulfilled and consider the functionals given in Definition 3.4. Then, the following statements are true:*

- (i) For every  $x \in X$ , the functional  $g_l(x, \cdot)$  is  $-K$ -monotone. Furthermore, for  $\bar{y} \in F(\bar{x})$ , we have that  $g_l(\bar{x}, \bar{y}) = 0$  if and only if  $\bar{y} \in \text{WMin}(F(\bar{x}), K)$ .
- (ii) The functional  $g_{u, \bar{x}}$  is  $K$ -monotone. Furthermore, for  $y \in F(\bar{x})$ , we have that  $g_{u, \bar{x}}(y) = 0$  if and only if  $y \in \text{WMax}(F(\bar{x}), K)$ .
- (iii) For any  $r \in \{l, u\}$ , we have  $f_{r, \bar{x}}(\bar{x}) \leq 0$ . Equality holds if  $r = l$  and  $\text{WMin}(F(\bar{x}), K) \neq \emptyset$ , or  $r = u$  and  $\text{WMax}(F(\bar{x}), K) \neq \emptyset$ .

*Proof.* (i) The monotonicity of  $g_l(x, \cdot)$  follows directly from the monotonicity of  $\Psi_e$ . Now, fix  $\bar{y} \in F(\bar{x})$ . Then, we have  $g_l(\bar{x}, \bar{y}) \leq 0$  and hence

$$\begin{aligned}
g_l(\bar{x}, \bar{y}) = 0 &\iff \inf_{y \in F(\bar{x})} \Psi_e(y - \bar{y}) \geq 0 \\
&\iff \forall y \in F(\bar{x}) : \Psi_e(y - \bar{y}) \geq 0 \\
&\iff \forall y \in F(\bar{x}) : y - \bar{y} \notin -\text{int } K \\
&\iff \bar{y} \in \text{WMin}(F(\bar{x}), K).
\end{aligned}$$

(ii) The monotonicity of  $g_{u, \bar{x}}$  is easily deduced from the monotonicity of  $\Psi_e$ . Now, take  $y \in F(\bar{x})$ . Then, we always have  $g_{u, \bar{x}}(y) \leq 0$ . Analogous to (i) we get

$$\begin{aligned}
g_{u, \bar{x}}(y) = 0 &\iff \inf_{\bar{y} \in F(\bar{x})} \Psi_e(y - \bar{y}) \geq 0 \\
&\iff \forall \bar{y} \in F(\bar{x}) : \Psi_e(y - \bar{y}) \geq 0 \\
&\iff \forall \bar{y} \in F(\bar{x}) : y - \bar{y} \notin -\text{int } K \\
&\iff y \in \text{WMax}(F(\bar{x}), K),
\end{aligned}$$

as desired.

(iii) The fact that  $f_{r, \bar{x}}(\bar{x}) \leq 0$  is trivial. If  $r = l$  and  $\tilde{y} \in \text{WMin}(F(\bar{x}), K) \neq \emptyset$  then, by statement (i), we get  $g_l(\bar{x}, \tilde{y}) = 0$ . From this we deduce that  $f_{l, \bar{x}}(\bar{x}) \geq g_l(\bar{x}, \tilde{y}) = 0$ , and hence the equality holds. Analogously, if  $r = u$  and  $\tilde{y} \in \text{WMax}(F(\bar{x}), K) \neq \emptyset$  then, by statement (ii), we get  $g_{u, \bar{x}}(\tilde{y}) = 0$ . Again, this implies that  $f_{u, \bar{x}}(\bar{x}) \geq 0$ , and hence the equality.  $\square$

**Remark 3.6.** Proposition 3.5 (i) together with Proposition 3.1 (iv) gives us monotonicity properties of the functionals  $g_l(x, \cdot)$  and  $\Psi_e$ . From this, it easily follows that the functionals  $g_l$  and  $f_{l, \bar{x}}$  are invariant under replacement of  $F$  by any set-valued mapping of the form  $F_A := F + A$ , with  $A \subseteq K$  and  $0 \in A$ . In particular, this is true when we replace  $F$  by  $\mathcal{E}_F$ .

Similarly, from Proposition 3.5 (ii) and Proposition 3.1 (iv) we deduce that the functionals  $f_{u, \bar{x}}$  and  $g_{u, \bar{x}}$  are invariant under replacement of  $F$  by any set-valued mapping of the form  $F_A := F - A$ , with  $A \subseteq K$  and  $0 \in A$ .

The next lemma proves some useful properties of the inner functions in the convex case

**Lemma 3.7.** *Let Assumption 1 be satisfied and consider the lower and upper inner functions given in Definition 3.4. The following statements hold:*

- (i) *If  $F$  is  $\preceq_K^{(l)}$ -convex, then  $g_l(\cdot, \bar{y})$  is convex for every  $\bar{y} \in F(\bar{x})$ . Furthermore, if  $F$  is locally  $l$ -bounded at  $\bar{x}$ , then  $\bar{x} \in \text{int dom } g_l(\cdot, \bar{y})$  and  $g_l(\cdot, \bar{y})$  is continuous at  $\bar{x}$ .*
- (ii) *If  $\mathcal{H}_F(\bar{x})$  is a convex and  $K$ -upper bounded set, then  $g_{u, \bar{x}}$  is a convex  $K$ -monotone functional that is continuous on  $Y$ .*

*Proof.* (i) Take  $\bar{y} \in F(\bar{x})$ ,  $x_1, x_2 \in X$ , and  $\lambda \in (0, 1)$ . Let  $x_\lambda := \lambda x_1 + (1 - \lambda)x_2$  and  $F_\lambda := \lambda F(x_1) + (1 - \lambda)F(x_2)$ . Since  $F$  is  $\preceq_K^{(l)}$ -convex, we have

$$(3.5) \quad F_\lambda \subseteq F(x_\lambda) + K.$$

We now have

$$\begin{aligned} & g_l(\lambda x_1 + (1 - \lambda)x_2, \bar{y}) \\ &= \inf_{y \in F(x_\lambda)} \Psi_e(y - \bar{y}) \\ & \text{(by monotonicity of } \Psi_e \text{)} \\ &= \inf_{y \in F(x_\lambda) + K} \Psi_e(y - \bar{y}) \\ & \text{(by (3.5))} \\ &\leq \inf_{y \in F_\lambda} \Psi_e(y - \bar{y}) \\ &= \inf_{(y_1, y_2) \in F(x_1) \times F(x_2)} \Psi_e(\lambda y_1 + (1 - \lambda)y_2 - \bar{y}) \\ &= \inf_{(y_1, y_2) \in F(x_1) \times F(x_2)} \Psi_e(\lambda(y_1 - \bar{y}) + (1 - \lambda)(y_2 - \bar{y})) \\ & \text{(by convexity of } \Psi_e \text{)} \\ &\leq \inf_{(y_1, y_2) \in F(x_1) \times F(x_2)} \lambda \Psi_e(y_1 - \bar{y}) + (1 - \lambda) \Psi_e(y_2 - \bar{y}) \\ &= \lambda g_l(x_1, \bar{y}) + (1 - \lambda) g_l(x_2, \bar{y}). \end{aligned}$$

Now, let us assume that  $F$  is locally  $l$ -bounded at  $\bar{x}$ . Hence, we can find  $\mu > 0$  and a neighborhood  $U$  of  $\bar{x}$  such that

$$\forall x \in U : F(x) \cap [-\mu e, \mu e] + K = F(x) + K.$$

By the monotonicity of  $\Psi_e$ , we have, for every  $x \in U$  :

$$\begin{aligned}
-\infty &< \Psi_e(-\mu e - \bar{y}) \\
&= \inf_{y \in -\mu e + K} \Psi_e(y - \bar{y}) \\
&\leq \inf_{y \in F(x) + K} \Psi_e(y - \bar{y}) \\
(3.6) \quad &= g_l(x, \bar{y}) \\
&= \inf_{y \in F(x) \cap [\mu e - K]} \Psi_e(y - \bar{y}) \\
&\leq \sup_{y \in F(x) \cap [\mu e - K]} \Psi_e(y - \bar{y}) \\
&\leq \Psi_e(\mu e - \bar{y}) \\
&< +\infty.
\end{aligned}$$

This shows that  $g_l(\cdot, \bar{y})$  is finite and bounded above around  $\bar{x}$ , from which the continuity is deduced.

(ii) The monotonicity of  $g_{u, \bar{x}}$  was already established in Proposition 3.5 (ii). In order to show the convexity, we check that  $\text{epi } g_{u, \bar{x}}$  is convex. Indeed, take  $(y_1, t_1), (y_2, t_2) \in \text{epi } g_{u, \bar{x}}$  and  $\lambda \in (0, 1)$ . Hence,  $g_{u, \bar{x}}(x_1) \leq t_1$  and  $g_{u, \bar{x}}(x_2) \leq t_2$ . Then, for any  $\epsilon > 0$ , we have

$$g_{u, \bar{x}}(x_1) < t_1 + \epsilon, \quad g_{u, \bar{x}}(x_2) < t_2 + \epsilon.$$

But then, we can find  $\bar{y}_1, \bar{y}_2 \in F(\bar{x})$  such that

$$\Psi_e(y_1 - \bar{y}_1) < t_1 + \epsilon, \quad \Psi_e(y_2 - \bar{y}_2) < t_2 + \epsilon.$$

From this, we get

$$\begin{aligned}
\Psi_e((\lambda y_1 + (1 - \lambda)y_2) - (\lambda \bar{y}_1 + (1 - \lambda)\bar{y}_2)) &= \Psi_e(\lambda(y_1 - \bar{y}_1) + (1 - \lambda)(y_2 - \bar{y}_2)) \\
&\leq \lambda \Psi_e(y_1 - \bar{y}_1) + (1 - \lambda) \Psi_e(y_2 - \bar{y}_2) \\
&\leq \lambda(t_1 + \epsilon) + (1 - \lambda)(t_2 + \epsilon) \\
&= \lambda t_1 + (1 - \lambda)t_2 + \epsilon.
\end{aligned}$$

Now, because  $F(\bar{x}) - K$  is convex, we have

$$\lambda \bar{y}_1 + (1 - \lambda)\bar{y}_2 \in \text{conv}(F(\bar{x})) \subseteq \mathcal{H}_F(\bar{x}),$$

and hence we can find  $\bar{y} \in F(\bar{x})$  such that  $\lambda \bar{y}_1 + (1 - \lambda)\bar{y}_2 \in \bar{y} - K$ . Then, by monotonicity of  $\Psi_e$ , we get

$$\begin{aligned}
g_{u, \bar{x}}(\lambda y_1 + (1 - \lambda)y_2) &\leq \Psi_e(\lambda y_1 + (1 - \lambda)y_2 - \bar{y}) \\
&\leq \Psi_e((\lambda y_1 + (1 - \lambda)y_2) - (\lambda \bar{y}_1 + (1 - \lambda)\bar{y}_2)) \\
&\leq \lambda t_1 + (1 - \lambda)t_2 + \epsilon.
\end{aligned}$$

Since  $\epsilon > 0$  was chosen arbitrarily, we conclude that  $(\lambda y_1 + (1 - \lambda)y_2, \lambda t_1 + (1 - \lambda)t_2) \in \text{epi } g_{u, \bar{x}}$ . But this means that  $\text{epi } g_{u, \bar{x}}$  is a convex set, as desired.

Now, since  $\mathcal{H}_F(\bar{x})$  is  $K$ -upper bounded, we have

$$-\infty < \Psi(y - \mu e) = \inf_{\bar{y} \in \mu e - K} \Psi_e(y - \bar{y}) \leq \inf_{\bar{y} \in \mathcal{H}_F(\bar{x}) - K} \Psi_e(y - \bar{y}) \stackrel{\text{(Remark 3.6)}}{=} g_{u, \bar{x}}(y).$$

This means that  $g_{u, \bar{x}}$  is finite on  $Y$ . The continuity of  $g_{u, \bar{x}}$  is now deduced by fixing  $\bar{y} \in F(\bar{x})$  and noticing that  $g_{u, \bar{x}}(\cdot) \leq \Psi_e(\cdot - \bar{y})$ , a continuous convex functional.  $\square$

We are now ready to establish the convexity of the scalarization functions  $f_{l, \bar{x}}$  and  $f_{u, \bar{x}}$ .

**Theorem 3.8.** *Let Assumption 1 be satisfied and, for  $r \in \{l, u\}$ , consider the functional  $f_{r, \bar{x}}$  given in Definition 3.4 (iii). The following statements hold:*

- (i) *If  $F$  is  $\preceq_K^{(l)}$ -convex then  $f_{l, \bar{x}}$  is convex. Furthermore, if  $F$  is locally  $l$ -bounded at  $\bar{x}$ , then  $\bar{x} \in \text{int dom } f_{l, \bar{x}}$  and  $f_{l, \bar{x}}$  is continuous at  $\bar{x}$ .*
- (ii) *If  $F$  is  $\preceq_K^{(u)}$ -convex and  $\mathcal{H}_F(\bar{x})$  is a convex set, then  $f_{u, \bar{x}}$  is convex. Furthermore, if  $F$  is locally  $u$ -upper bounded at  $\bar{x}$ , then  $\bar{x} \in \text{int dom } f_{u, \bar{x}}$  and  $f_{u, \bar{x}}$  is continuous at  $\bar{x}$ .*

*Proof.* (i) We have

$$f_{l, \bar{x}}(x) = \sup_{\bar{y} \in F(\bar{x})} g_l(x, \bar{y}).$$

By Lemma 3.7 (i), for every  $\bar{y} \in F(\bar{x})$ , the functional  $g_l(\cdot, \bar{y})$  is convex. Hence  $f_{l, \bar{x}}$  is convex as it is the supremum of convex functionals. To prove the second part, it suffices to show that  $f_{l, \bar{x}}$  is finite and upper bounded on a neighborhood of  $\bar{x}$ . In order to show that this is true, note that the assumptions on the second part of Lemma 3.7 (i) are fulfilled. Hence, from (3.6) we get the existence of  $\mu > 0$  and neighborhood  $U$  of  $\bar{x}$  on which

$$(3.7) \quad \forall x \in U : -\infty < g_l(x, \bar{y}) \leq \Psi_e(\mu e - \bar{y}).$$

Taking the supremum over  $\bar{y} \in F(\bar{x})$  in (3.7), we get

$$(3.8) \quad \forall x \in U : -\infty < f_{l, \bar{x}}(x) \leq \sup_{\bar{y} \in F(\bar{x})} \Psi_e(\mu e - \bar{y}).$$

Now, since  $F$  is locally  $l$ -lower bounded at  $\bar{x}$ , in particular  $F(\bar{x}) \subseteq -\mu e + K$ . By the monotonicity of  $\Psi_e$ , we now obtain

$$\sup_{\bar{y} \in F(\bar{x})} \Psi_e(\mu e - \bar{y}) \leq \Psi_e(2\mu e) = 2\mu.$$

This, together with (3.8), implies that  $f_{l, \bar{x}}$  is finite and upper bounded on  $U$ . The statement follows.



(ii) Let us now prove that  $f_{u,\bar{x}}$  is convex. Indeed, take any  $x_1, x_2 \in X$  and  $\lambda \in (0, 1)$ , Again, by denoting  $x_\lambda = \lambda x_1 + (1 - \lambda)x_2$  and  $F_\lambda = \lambda F(x_1) + (1 - \lambda)F(x_2)$ , we have

$$\begin{aligned}
f_{u,\bar{x}}(x_\lambda) &= \sup_{y \in F(x_\lambda)} g_{u,\bar{x}}(y) \\
&\stackrel{\text{(by convexity of } F \text{)}}{\leq} \sup_{y \in F_\lambda} g_{u,\bar{x}}(y) \\
&= \sup_{(y_1, y_2) \in F(x_1) \times F(x_2)} g_{u,\bar{x}}(\lambda y_1 + (1 - \lambda)y_2) \\
&\stackrel{\text{(by convexity of } g_{u,\bar{x}} \text{)}}{\leq} \sup_{(y_1, y_2) \in F(x_1) \times F(x_2)} \lambda g_{u,\bar{x}}(y_1) + (1 - \lambda)g_{u,\bar{x}}(y_2) \\
&= \lambda f_{u,\bar{x}}(x_1) + (1 - \lambda)f_{u,\bar{x}}(x_2),
\end{aligned}$$

as desired.

Now, assume that  $F$  is locally  $u$ -upper bounded at  $\bar{x}$  and let  $U$  be the neighborhood on which the boundedness property holds. Again, in order to prove the second part it suffices to show that  $f_{u,\bar{x}}$  is finite and upper bounded on a neighborhood of  $\bar{x}$ . We proceed as follows: since  $\bar{x} \in \text{int dom } F$ , we can assume without loss of generality that  $U \subseteq \text{int dom } F$ . Moreover, since in particular the assumptions of Lemma 3.7 (ii) are fulfilled, we get that  $g_{u,\bar{x}}(y) > -\infty$  for every  $y \in Y$ . Taking any selection  $\theta$  of  $F$  on  $U$ , we deduce that

$$\forall x \in U : -\infty < g_{u,\bar{x}}(\theta(x)) \leq f_{u,\bar{x}}(x).$$

On the other hand, recall that from Lemma 3.7 (ii) the functional  $g_{u,\bar{x}}$  is  $K$ -monotone and finite. Taking this into account and the fact that  $F(x) - K \subseteq \mu e - K$  for every  $x \in U$ , we obtain

$$\forall x \in U : f_{u,\bar{x}}(x) \leq \sup_{y \in \mu e - K} g_{u,\bar{x}}(y) = g_{u,\bar{x}}(\mu e) < +\infty.$$

The theorem is proved.  $\square$

Next, we prove that the Lipschitzianity of the set-valued mapping is also transferred to the corresponding scalarization functionals. The following proposition is crucial.

**Proposition 3.9.** *Let Assumption 1 be fulfilled and let  $f : X \times Y \rightarrow \overline{\mathbb{R}}$  be a given functional. Consider the associated marginal functions  $\varphi, \Phi : X \rightarrow \overline{\mathbb{R}}$  defined as*

$$\varphi(x) := \inf_{y \in F(x)} f(x, y), \quad \Phi(x) := \sup_{y \in F(x)} f(x, y).$$

Suppose that  $F$  is Lipschitzian on a set  $U \subseteq X$  with constant  $\ell > 0$  and that  $f$  is Lipschitzian on the set  $(U \times Y) \cap \text{gph } F$  with constant  $\ell' > 0$ . The following statements are true:

- (i) If  $\varphi(\bar{x}) > -\infty$  for some  $\bar{x} \in U$ , then  $\varphi$  is Lipschitzian on  $U$  with constant  $\ell'(1 + \ell)$ .
- (ii) If  $\Phi(\bar{x}) < +\infty$  for some  $\bar{x} \in U$ , then  $\Phi$  is Lipschitzian on  $U$  with constant  $\ell'(1 + \ell)$ .

*Proof.* We only prove (i), since the proof of (ii) is very similar. Take  $x, x' \in U$  and let  $\ell, \ell' > 0$  be the Lipschitzian constants of  $F$  and  $f$  respectively. Then, because  $F$  is Lipschitzian on  $U$ ,

$$\forall y' \in F(x'), \exists y \in F(x) : \|y - y'\| \leq \ell \|x - x'\|.$$

Taking this into account, together with the Lipschitz continuity of  $f$  on  $(U \times Y) \cap \text{gph } F$ , we have

$$\begin{aligned} \forall y' \in F(x'), \exists y \in F(x) : f(x, y) &\leq f(x', y') + \ell'(\|x - x'\| + \|y - y'\|) \\ &\leq f(x', y') + \ell'(1 + \ell)\|x - x'\|. \end{aligned}$$

This implies

$$(3.9) \quad \varphi(x) \leq \varphi(x') + \tilde{\ell}\|x - x'\|,$$

with  $\tilde{\ell} := \ell'(1 + \ell)$ . Since  $\varphi(\bar{x}) > -\infty$ , we can substitute  $x = \bar{x}$  in (3.9) to obtain that  $\varphi(x') > -\infty$  for every  $x' \in U$ . From this, it follows that  $\varphi$  is Lipschitzian on  $U$ . □

Next lemma is an immediate consequence of Proposition 3.9, Proposition 3.1 (ii) and Proposition 3.5.

**Lemma 3.10.** *Let Assumption 1 be fulfilled. Consider the lower and upper inner functions given in Definition 3.4 and let  $\rho$  be the Lipschitz constant of  $\Psi_e$ . The following statements hold:*

- (i) *If  $F$  is Lipschitzian with constant  $\ell > 0$  on a neighborhood  $U$  of  $\bar{x}$  and there exists  $\bar{y} \in Y$  with  $g_l(\bar{x}, \bar{y}) > -\infty$ , then  $g_l$  is Lipschitzian on  $U \times Y$  with constant  $\rho(1 + \ell)$ . In particular, the condition  $g_l(\bar{x}, \bar{y}) > -\infty$  can be replaced by  $\bar{y} \in \text{WMin}(F(\bar{x}), K)$ .*
- (ii) *The functional  $g_{u, \bar{x}}$  is Lipschitzian on  $Y$  with constant  $\rho$  if and only if  $g_{u, \bar{x}}(\bar{y}) > -\infty$  for some  $\bar{y} \in Y$ . In particular, this is true if  $\text{WMax}(F(\bar{x}), K) \neq \emptyset$ .*

*Proof.* (i) Consider the set-valued mapping  $\tilde{F} : X \times Y \rightrightarrows Y$  and the functional  $\tilde{f} : X \times Y \times Y \rightarrow \mathbb{R}$  defined as

$$\tilde{F}(x, y) := F(x), \quad \tilde{f}(x, y, z) := \Psi_e(z - y).$$

Apply now Proposition 3.9 (i) with  $\varphi := g_l$ ,  $F := \tilde{F}$  and  $f := \tilde{f}$  to obtain the Lipschitzianity of  $g_l$ . If  $\bar{y} \in \text{WMin}(F(\bar{x}), K)$ , then it follows from Proposition 3.5 (i) that  $g_l(\bar{x}, \bar{y}) = 0 > -\infty$ .

(ii) Follows easily from the fact that  $g_{u,\bar{x}}$  is the finite infimum of a fixed family of Lipschitzian functionals on  $Y$ . Of course, when  $\bar{y} \in \text{WMax}(F(\bar{x}), K) \neq \emptyset$ , we get  $g_{u,\bar{x}}(\bar{y}) = 0 > -\infty$  from Proposition 3.5 (ii).  $\square$

We can now establish the Lipschitzianity of the scalarizing functionals  $f_{l,\bar{x}}$  and  $f_{u,\bar{x}}$ .

**Theorem 3.11.** *Let Assumption 1 be fulfilled. For  $r \in \{l, u\}$ , consider the functional  $f_{r,\bar{x}}$  given by (3.4) and suppose that  $F$  is locally Lipschitzian at  $\bar{x}$ . The following statements hold:*

- (i) *If  $\text{WMin}(F(\bar{x}), K) \neq \emptyset$ , then  $f_{l,\bar{x}}$  is locally Lipschitzian at  $\bar{x}$ .*
- (ii) *If  $\text{WMax}(F(\bar{x}), K) \neq \emptyset$ , then  $f_{u,\bar{x}}$  is locally Lipschitzian at  $\bar{x}$ .*

*Proof.* (i) Consider the constant set-valued mapping  $\tilde{F} : X \rightrightarrows Y$  given by  $\tilde{F}(x) := F(\bar{x})$  for every  $x \in X$ . By Lemma 3.10 (i), we know that  $g_l$  is Lipschitzian on  $U \times Y$ , where  $U$  is a neighborhood of  $\bar{x}$  on which  $F$  is Lipschitzian. Furthermore, according to Proposition 3.5 (iii), we have  $f_{l,\bar{x}}(\bar{x}) = 0 < +\infty$ . Hence, the Lipschitzianity of  $f_{l,\bar{x}}$  around  $\bar{x}$  follows from Proposition 3.9 (ii) with  $\varphi := f_{l,\bar{x}}$ ,  $F := \tilde{F}$  and  $f := g_l$ .

(ii) Similarly, consider the functional  $\tilde{f} : X \times Y \rightarrow \mathbb{R}$  given by  $\tilde{f}(x, y) := g_{u,\bar{x}}(y)$  for every  $(x, y) \in X \times Y$ . From 3.10 (ii), we get that  $\tilde{f}$  is Lipschitzian on  $X \times Y$ . In addition, Proposition 3.5 (iii) tells us that  $f_{u,\bar{x}}(\bar{x}) = 0 < +\infty$ . Hence, the Lipschitzianity of  $f_{u,\bar{x}}$  around  $\bar{x}$  follows from Proposition 3.9 (ii) with  $\varphi := f_{u,\bar{x}}$ ,  $F := F$  and  $f := \tilde{f}$ .  $\square$

#### 4. SUBDIFFERENTIAL OF THE SCALARIZING FUNCTIONAL ASSOCIATED TO THE LOWER LESS RELATION

In this part, we derive upper estimates for Mordukhovich's subdifferential of the scalarizing functional  $f_{l,\bar{x}}$  studied in Section 3. Our upper estimates are given in terms of the coderivative of the set-valued objective map  $F$  and are based in Theorem 2.11 and Theorem 2.12. These motivates the definition of the following solution maps.

**Definition 4.1.** *Let Assumption 1 be fulfilled.*

- (i) *The lower inner solution map  $S_F^{l,1} : X \times Y \rightrightarrows Y$  is defined as*

$$S_F^{l,1}(x, y) := \{z \in F(x) : \Psi_\epsilon(z - y) = g_l(x, y)\}.$$

- (ii) *The lower outer solution map  $S_F^{l,2} : X \rightrightarrows Y$  is defined as*

$$S_F^{l,2}(x) := \{y \in F(\bar{x}) : f_{l,\bar{x}}(x) = g_l(x, y)\}.$$

**Remark 4.2.** *According to Remark 3.6, the functionals  $g_l$  and  $f_{l,\bar{x}}$  are invariant under replacement of  $F$  by  $\mathcal{E}_F$ . However, although the set-valued*

mappings  $S_F^{l,i}$  and  $S_{\mathcal{E}_F}^{l,i}$  are based on the same functionals ( $i = 1, 2$ ), we always have  $S_F^{l,i}(\cdot) \subseteq S_{\mathcal{E}_F}^{l,i}(\cdot)$  and the inclusions can be strict.

We divide the analysis in two cases, corresponding to whether  $F$  is  $\preceq_K^{(l)}$ -convex or locally Lipschitzian at  $\bar{x}$ . We start the study with the convex case. The next lemma shows an exact formula for the subdifferential of the inner function given in Definition 3.4 (i). It is worth mentioning that a similar version of this result was recently obtained in [21, Lemma 2], but assuming the separability of  $X$ .

**Lemma 4.3.** *Let Assumption 1 be fulfilled and, for  $\bar{y} \in \text{WMin}(\mathcal{E}_F(\bar{x}), K)$ , consider the functional  $g_{l,\bar{y}} := g_l(\cdot, \bar{y})$ . Assume in addition that  $F$  is  $\preceq_K^{(l)}$ -convex and locally  $l$ -bounded at  $\bar{x}$ . Then,*

$$(4.1) \quad \partial g_{l,\bar{y}}(\bar{x}) = D^* \mathcal{E}_F(\bar{x}, \bar{y}) [\partial \Psi_e(0)].$$

*Proof.* The result will be a simple consequence of Theorem 2.11. Indeed, note that according to Remark 3.6 we can write

$$g_{l,\bar{y}}(x) = \inf_{y \in \mathcal{E}_F(x)} f(x, y),$$

where  $f : X \times Y \rightarrow \mathbb{R}$  is defined as  $f(x, y) = \Psi_e(y - \bar{y})$ . Since  $F$  is  $\preceq_K^{(l)}$ -convex, we have that  $\mathcal{E}_F$  is a convex set-valued mapping. It is also obvious that  $f$  is proper and convex. Moreover, by Proposition 3.5 (i), we have that  $g_{l,\bar{y}}(\bar{x}) = 0 \neq -\infty$ . According to Proposition 3.1 (ii),  $f$  is Lipschitzian on  $X \times Y$  and hence the regularity condition (ii) in Theorem 2.11 is satisfied. In this case, the solution map is just  $S_{\mathcal{E}_F}^{l,1}(\cdot, \bar{y})$ . According to Proposition 3.5 (i) and Proposition 3.1 (vi), we get

$$(4.2) \quad S_{\mathcal{E}_F}^{l,1}(\bar{x}, \bar{y}) = \{y \in \mathcal{E}_F(\bar{x}) \mid \Psi_e(y - \bar{y}) = 0\} = \mathcal{E}_F(\bar{x}) \cap (\bar{y} - \text{bd } K).$$

Since  $0 \in \text{bd } K$ , it follows that  $\bar{y} \in S_{\mathcal{E}_F}^{l,1}(\bar{x}, \bar{y})$ . Applying now Theorem 2.11, we obtain

$$\begin{aligned} \partial g_{l,\bar{y}}(\bar{x}) &= \bigcup_{(x^*, y^*) \in \partial f(\bar{x}, \bar{y})} [x^* + D^* \mathcal{E}_F(\bar{x}, \bar{y})(y^*)] \\ &= \bigcup_{(x^*, y^*) \in \{0\} \times \partial \Psi_e(0)} [x^* + D^* \mathcal{E}_F(\bar{x}, \bar{y})(y^*)] \\ &= \bigcup_{y^* \in \partial \Psi_e(0)} D^* \mathcal{E}_F(\bar{x}, \bar{y})(y^*), \end{aligned}$$

which proves the statement.  $\square$

**Lemma 4.4.** *Let Assumption 1 be fulfilled and take points  $(\bar{x}, \bar{y}_1), (\bar{x}, \bar{y}_2) \in \text{gph } \mathcal{E}_F$  such that  $\bar{y}_1 \preceq_K \bar{y}_2$ . If  $F$  is  $\preceq_K^{(l)}$ -convex, then:*

$$\forall y^* \in K^* : D^* \mathcal{E}_F(\bar{x}, \bar{y}_2)(y^*) \subseteq D^* \mathcal{E}_F(\bar{x}, \bar{y}_1)(y^*).$$

*Proof.* Fix  $y^* \in K^*$  and  $x^* \in D^* \mathcal{E}_F(\bar{x}, \bar{y}_2)(y^*)$ . Since  $\bar{y}_1 - \bar{y}_2 \in -K$ , we have that  $\langle y^*, \bar{y}_1 - \bar{y}_2 \rangle \leq 0$ . Then, for every  $(x, y) \in \text{gph } \mathcal{E}_F$ , we have

$$\begin{aligned} \langle x^*, x - \bar{x} \rangle &\leq \langle y^*, y - \bar{y}_2 \rangle \\ &= \langle y^*, y - \bar{y}_1 \rangle + \langle y^*, \bar{y}_1 - \bar{y}_2 \rangle \\ &\leq \langle y^*, y - \bar{y}_1 \rangle, \end{aligned}$$

which implies that  $(x^*, -y^*) \in N((\bar{x}, \bar{y}_1), \text{gph } \mathcal{E}_F)$ . The statement is proved.  $\square$

The following concept was introduced in [60].

**Definition 4.5.** *Let Assumption 1 be fulfilled and consider  $A \subseteq Y$ . We say that  $A$  is strongly  $K$ -compact if there exists a compact set  $B \subseteq A$  such that  $B \in [A]^{(l)}$ .*

**Theorem 4.6.** *Let Assumption 1 be satisfied. Suppose that  $F$  is  $\preceq_K^{(l)}$ -convex and locally  $l$ -bounded at  $\bar{x}$ . Furthermore, assume that  $F(\bar{x})$  is strongly  $K$ -compact. Then,*

$$\partial f_{l, \bar{x}}(\bar{x}) = \overline{\text{conv}}^* \left( \bigcup_{\bar{y} \in \text{Min}(F(\bar{x}), K)} D^* \mathcal{E}_F(\bar{x}, \bar{y}) [\partial \Psi_\epsilon(0)] \right).$$

*Proof.* Under the assumptions of the theorem we can apply Theorem 3.8 (i) to obtain that the functional  $f_{l, \bar{x}}$  is convex and continuous at  $\bar{x}$ . Hence, by [53, Proposition 1.11], we have  $\partial f_{l, \bar{x}}(\bar{x}) \neq \emptyset$ . Since  $F(\bar{x})$  is strongly  $K$ -compact, there exists a compact set  $A \subseteq F(\bar{x})$  such that  $A + K = F(\bar{x}) + K$ . Applying [29, Lemma 4.7], we get that

$$(4.3) \quad \text{Min}(F(\bar{x}), K) = \text{Min}(A, K) \neq \emptyset.$$

As in Lemma 4.3 we consider, for  $\bar{y} \in F(\bar{x})$ , the functional  $g_{l, \bar{y}} := g_l(\cdot, \bar{y})$ . Then, according to Proposition 3.5 (i), the functional  $g_l(x, \cdot)$  is  $-K$  monotone for any  $x \in X$ . This implies

$$\begin{aligned} f_{l, \bar{x}}(x) &= \sup_{\bar{y} \in F(\bar{x})} g_l(x, \bar{y}) = \sup_{\bar{y} \in F(\bar{x}) + K} g_l(x, \bar{y}) = \sup_{\bar{y} \in A + K} g_l(x, \bar{y}) \\ &= \sup_{\bar{y} \in A} g_l(x, \bar{y}) = \sup_{\bar{y} \in A} g_{l, \bar{y}}(x). \end{aligned}$$

The above equation implies that  $f_{l, \bar{x}}$  can be expressed as the pointwise supremum of the parametric family  $\{g_{l, \bar{y}}\}_{\bar{y} \in A}$ . In this context, it is stated in [57, Proposition 4.5.2] an exact formula for the subdifferential of the

maximum of convex functions. In order to apply this proposition, it is sufficient to verify the following statements:

- $(A, \|\cdot\|)$  is a compact Hausdorff space.  
This is obvious given our compactness assumption.
- For any  $\bar{y} \in A$ , the functional  $g_{l,\bar{y}}$  is convex and continuous at  $\bar{x}$ .  
Since  $A \subseteq F(\bar{x})$ , the statement follows directly from Lemma 3.7 (i).
- For every  $x \in X$ , the functional  $g_l(x, \cdot)$  is u.s.c at every point of  $A$ .  
Indeed, fix  $x \in X$  and take  $\bar{y} \in A, \alpha \in \mathbb{R}$  such that  $g_l(x, \bar{y}) < \alpha$ . This is equivalent to

$$\inf_{y \in F(x)} \Psi_e(y - \bar{y}) < \alpha,$$

and hence we can find  $y' \in F(x)$  such that  $\Psi_e(y' - \bar{y}) < \alpha$ . Because of the continuity of  $\Psi_e$ , we can find a neighborhood  $V(\bar{y})$  of  $\bar{y}$  such that for every  $z \in V(\bar{y})$ , the inequality  $\Psi_e(y' - z) < \alpha$  holds. This, together with the definition of  $g_l(x, \cdot)$ , gives us

$$\forall z \in V(\bar{y}) \cap A : g_l(x, z) \leq \Psi_e(y' - z) < \alpha,$$

as desired.

Applying now [57, Proposition 4.5.2], we obtain that

$$(4.4) \quad \partial f_{l,\bar{x}}(\bar{x}) = \overline{\text{conv}}^* \left( \bigcup_{\bar{y} \in \tilde{S}} \partial g_{l,\bar{y}}(\bar{x}) \right),$$

where

$$\tilde{S} = \{\bar{y} \in A : g_{l,\bar{y}}(\bar{x}) = f_{l,\bar{x}}(\bar{x})\}.$$

Recall that  $\text{WMin}(F(\bar{x}), K) \neq \emptyset$  according to (4.3). Then, by Proposition 3.5 (iii), we know that  $f_{l,\bar{x}}(\bar{x}) = 0$ . Hence,  $\bar{y} \in \tilde{S}$  if and only if  $g_{l,\bar{y}}(\bar{x}) = 0$ . Fix  $\bar{y} \in A$ . Note that, because of the monotonicity of  $\Psi_e$ , we have

$$\begin{aligned} g_{l,\bar{y}}(\bar{x}) &= \inf_{y \in F(\bar{x})} \Psi_e(y - \bar{y}) = \inf_{y \in F(\bar{x})+K} \Psi_e(y - \bar{y}) \\ &= \inf_{y \in A+K} \Psi_e(y - \bar{y}) = \inf_{y \in A} \Psi_e(y - \bar{y}). \end{aligned}$$

Then, following the same lines in the proof of Proposition 3.5 (i), we get

$$\inf_{y \in A} \Psi_e(y - \bar{y}) = 0 \iff \bar{y} \in \text{WMin}(A, K).$$

This shows that

$$(4.5) \quad \tilde{S} = \text{WMin}(A, K).$$

Now, since  $A$  is compact, we can apply [35, Proposition 9.3.7] to obtain that  $A$  satisfies the so called domination property, i.e,

$$(4.6) \quad A \subseteq \text{Min}(A, K) + K.$$

Hence, taking into account (4.4), (4.5) and Lemma 4.3, we obtain

$$(4.7) \quad \partial f_{l,\bar{x}}(\bar{x}) = \overline{\text{conv}}^* \left( \bigcup_{\bar{y} \in \text{WMin}(A, K)} D^* \mathcal{E}_F(\bar{x}, \bar{y}) [\partial \Psi_e(0)] \right).$$

By (4.6), for every  $\bar{y} \in \text{WMin}(A, K)$  there exists  $\bar{y}_1 \in \text{Min}(A, K)$  such that  $\bar{y}_1 \preceq_K \bar{y}$ . This, together with the fact that  $\partial \Psi_e(0) \subseteq K^*$ , allows us to apply Lemma 4.4 to obtain

$$(4.8) \quad D^* \mathcal{E}_F(\bar{x}, \bar{y}) [\partial \Psi_e(0)] \subseteq D^* \mathcal{E}_F(\bar{x}, \bar{y}_1) [\partial \Psi_e(0)].$$

Combining equations (4.7) and (4.8), we have

$$\begin{aligned} \partial f_{l,\bar{x}}(\bar{x}) &= \overline{\text{conv}}^* \left( \bigcup_{\bar{y} \in \text{WMin}(A, K)} D^* \mathcal{E}_F(\bar{x}, \bar{y}) [\partial \Psi_e(0)] \right) \\ &\subseteq \overline{\text{conv}}^* \left( \bigcup_{\bar{y}_1 \in \text{Min}(A, K)} D^* \mathcal{E}_F(\bar{x}, \bar{y}_1) [\partial \Psi_e(0)] \right). \end{aligned}$$

Since the reverse inclusion is obviously true, we obtain

$$\partial f_{l,\bar{x}}(\bar{x}) = \overline{\text{conv}}^* \left( \bigcup_{\bar{y} \in \text{Min}(A, K)} D^* \mathcal{E}_F(\bar{x}, \bar{y}) [\partial \Psi_e(0)] \right).$$

The desired result follows from (4.3).  $\square$

Next, we analyze the case on which  $F$  is locally Lipschitzian at  $\bar{x}$ . Similar to the convex case, we start by establishing an upper estimate of the subdifferential of the inner function.

**Lemma 4.7.** *Let Assumption 1 be fulfilled with  $X, Y$  being Asplund, and let  $\bar{y} \in \text{WMin}(F(\bar{x}), K)$ . Suppose also that:*

- (i)  $F$  is closed at  $\bar{x}$ ,
- (ii)  $S_F^{l,1}(x, y)$  is inner semicompact at  $(\bar{x}, \bar{y})$ ,
- (iii)  $\text{gph } F$  is locally closed around every point in the set  $\{\bar{x}\} \times F(\bar{x}) \cap (\bar{y} - \text{bd } K)$ .

Then,

$$(4.9) \quad \partial g_l(\bar{x}, \bar{y}) \subseteq \bigcup_{\substack{\bar{z} \in F(\bar{x}) \cap (\bar{y} - \text{bd } K) \\ z^* \in \partial \Psi_e(\bar{z} - \bar{y})}} D^* F(\bar{x}, \bar{z})(z^*) \times \{-z^*\}.$$

*Proof.* Consider the set-valued mapping  $\tilde{F} : X \times Y \rightrightarrows Y$  and the functional  $f : X \times Y \times Y \rightarrow \mathbb{R}$  defined as

$$\tilde{F}(x, y) := F(x), \quad f(x, y, z) := \Psi_e(z - y).$$

Thus, we have

$$g_l(x, y) = \inf_{z \in \tilde{F}(x, y)} f(x, y, z).$$

Now, we check that it is possible to apply Theorem 2.12. First, note that the associated solution map in this case is just  $S_F^{l,1}$ , from Definition 4.1. Next, observe that  $g_l(\bar{x}, \bar{y}) = 0$  by Proposition 3.5 (i). Hence, using the representability property of  $\Psi_e$  we get

$$(4.10) \quad S_F^{l,1}(\bar{x}, \bar{y}) = \{z \in F(\bar{x}) : \Psi_e(z - \bar{y}) = 0\} = F(\bar{x}) \cap (\bar{y} - \text{bd } K) \supseteq \{\bar{y}\} \neq \emptyset.$$

We proceed to check that the hypothesis of the theorem are fulfilled.

- $\tilde{F}$  is closed at  $(\bar{x}, \bar{y})$ .  
This is obvious given the definition of  $\tilde{F}$  and condition (i) above.
- $S_F^{l,1}$  is inner semicompact at  $(\bar{x}, \bar{y})$ .  
This is precisely condition (ii) in the lemma.
- There is a neighborhood  $U'$  of  $(\bar{x}, \bar{y})$  such that  $f$  is Lipschitzian on  $U' \times Y$ .  
This follows directly from the definition of  $f$  and Proposition 3.1 (ii).
- $\text{gph } \tilde{F}$  is locally closed around every point in the set  $\{(\bar{x}, \bar{y})\} \times S_F^{l,1}(\bar{x}, \bar{y})$ .  
Taking into account (4.10), the statement follows from condition (iii).

Applying now Theorem 2.12 we obtain

$$(4.11) \quad \partial g_l(\bar{x}, \bar{y}) \subseteq \bigcup_{\substack{\bar{z} \in F(\bar{x}) \cap (\bar{y} - \text{bd } K) \\ (x^*, y^*, z^*) \in \partial f(\bar{y}, \bar{y}, \bar{z})}} \left\{ (x^*, y^*) + D^* \tilde{F}(\bar{x}, \bar{y}, \bar{z})(z^*) \right\}.$$

We now simplify the above inclusion. The first step will be to examine  $D^* \tilde{F}(\bar{x}, \bar{y}, \bar{z})$ . Note that

$$\text{gph } \tilde{F} = \{(x, y, z) : z \in F(x)\}.$$

Hence, we obtain

$$N((\bar{x}, \bar{y}, \bar{z}), \text{gph } \tilde{F}) = \{(x^*, 0, z^*) \in X^* \times Y^* \times Y^* : (x^*, z^*) \in N((\bar{x}, \bar{z}), \text{gph } F)\}.$$

From this we deduce that

$$(4.12) \quad \begin{aligned} D^* \tilde{F}(\bar{x}, \bar{y}, \bar{z})(z^*) &= \{(x^*, 0) \in X^* \times Y^* : (x^*, -z^*) \in N((\bar{x}, \bar{z}), \text{gph } F)\} \\ &= D^* F(\bar{x}, \bar{z})(z^*) \times \{0\}. \end{aligned}$$



Next, we compute  $\partial f(\bar{x}, \bar{y}, \bar{z})$ . For this, we first note that  $f$  is convex and continuous at every point. Considering the operator  $T \in \mathcal{L}(X \times Y \times Y, Y)$  defined as  $T(x, y, z) := z - y$ , we get  $f = \Psi_e \circ T$ . By the classical chain rule in convex analysis [52, Proposition 3.28], we now obtain

$$\partial f(\bar{x}, \bar{y}, \bar{z}) = \partial f(\bar{x}, \bar{y}, \bar{z}) = T^*[\partial \Psi_e(\bar{z} - \bar{y})] = T^*[\partial \Psi_e(\bar{z} - \bar{y})],$$

where  $T^*$  denotes the adjoint operator of  $T$ . Moreover, it is easy to check that  $T^*(z^*) = (0, -z^*, z^*)$ . Hence, we get

$$(4.13) \quad \partial f(\bar{x}, \bar{y}, \bar{z}) = \{0\} \times \bigcup_{z^* \in \partial \Psi_e(\bar{z} - \bar{y})} (-z^*, z^*).$$

Substituting now (4.12) and (4.13) into (4.11), the desired estimate is obtained.  $\square$

**Theorem 4.8.** *In addition to Assumption 1, let  $X$  and  $Y$  be Asplund. Suppose also that:*

- (i)  $F$  is locally Lipschitzian at  $\bar{x}$ ,
- (ii)  $\text{WMin}(F(\bar{x}), K) \neq \emptyset$ ,
- (iii)  $F$  is closed at  $\bar{x}$ ,
- (iv)  $S_F^{l,1}$  is inner semicompact at every point of  $\{\bar{x}\} \times \text{WMin}(F(\bar{x}), K)$ ,
- (v)  $S_F^{l,2}$  is inner semicompact at  $\bar{x}$ ,
- (vi)  $\text{gph } F$  is locally closed around every point in the set  $\{\bar{x}\} \times \text{WMin}(F(\bar{x}), K)$ .

Then,

$$(4.14) \quad \partial f_{l,\bar{x}}(\bar{x}) \subseteq \overline{\text{conv}}^* \left( \bigcup_{\bar{y} \in \text{WMin}(F(\bar{x}), K)} \left\{ x^* \in X^* : \exists y^* \in N(\bar{y}, F(\bar{x})) : (x^*, y^*) \in G_{(\bar{x}, \bar{y})} \right\} \right),$$

where

$$G_{(\bar{x}, \bar{y})} = \overline{\text{conv}}^* \left( \bigcup_{\substack{\bar{z} \in F(\bar{x}) \cap (\bar{y} - \text{bd } K) \\ z^* \in \partial \Psi_e(\bar{z} - \bar{y})}} D^* F(\bar{x}, \bar{z})(z^*) \times \{-z^*\} \right).$$

*Proof.* Consider the (constant) set-valued mapping  $\tilde{F} : X \rightrightarrows Y$  defined as  $\tilde{F}(x) := F(\bar{x})$  for every  $x \in X$ . Then, we can write

$$f_{l,\bar{x}}(x) = \sup_{y \in \tilde{F}(x)} g_l(x, y).$$

Next, note that the solution map in this case is  $S_F^{l,2}$ . Furthermore, as a consequence of (ii) and Proposition 3.5 (iii), we obtain  $f_{l,\bar{x}}(\bar{x}) = 0$ . The

definition of  $\tilde{F}$  and  $g_l$  allow us then to apply Proposition 3.5 (i) to obtain that

$$S_F^{l,2}(\bar{x}) = \text{WMin}(F(\bar{x}), K).$$

We now check that it is possible to apply Theorem 2.12 to obtain an upper estimate of Mordukhovich's subdifferential of  $f_{l,\bar{x}}$  at  $\bar{x}$ .

- $\tilde{F}$  is closed at  $\bar{x}$ .  
It is easy to see that the closedness of  $\tilde{F}$  at  $\bar{x}$  is equivalent to the closedness of the set  $F(\bar{x})$ . The statement follows from condition (iii).
- $S_F^{l,2}$  is inner semicompact at  $\bar{x}$ .  
This is precisely condition (v).
- There is a neighborhood  $U$  of  $\bar{x}$  such that  $g_l$  is Lipschitzian on  $U \times Y$ .  
This follows from conditions (i), (ii) and Lemma 3.10 (i).
- $\text{gph } \tilde{F}$  is locally closed around every point of the set  $\{\bar{x}\} \times S_F^{l,2}(\bar{x})$ .  
Again, this is deduced from the fact that  $F(\bar{x})$  is a closed set, which is implied by (iii).

Hence, taking into account the Lipschitzianity of  $f_{l,\bar{x}}$  from Theorem 3.11 (i), we obtain:

$$\begin{aligned}
 \partial f_{l,\bar{x}}(\bar{x}) &= \partial \left( - \inf_{y \in \tilde{F}(\cdot)} -g_l(\cdot, y) \right) (\bar{x}) \\
 &\stackrel{\text{(Remark 2.9)}}{\subseteq} -\overline{\text{conv}}^* \left( \partial \left( \inf_{y \in \tilde{F}(\cdot)} -g_l(\cdot, y) \right) (\bar{x}) \right) \\
 (4.15) \quad &\stackrel{\text{(Theorem 2.12)}}{\subseteq} -\overline{\text{conv}}^* \left( \bigcup_{\substack{\bar{y} \in S_F^{l,2}(\bar{x}) \\ (x^*, y^*) \in \partial(-g_l)(\bar{x}, \bar{y})}} \left[ x^* + D^* \tilde{F}(\bar{x}, \bar{y})(y^*) \right] \right).
 \end{aligned}$$

Now, we examine  $D^* \tilde{F}(\bar{x}, \bar{y})$  for any  $(\bar{x}, \bar{y}) \in X \times Y$ . Since  $\text{gph } \tilde{F} = X \times F(\bar{x})$ , we get in this case  $N((\bar{x}, \bar{y}), \text{gph } \tilde{F}) = \{0\} \times N(\bar{y}, F(\bar{x}))$ . From this, we deduce that

$$D^* \tilde{F}(\bar{x}, \bar{y})(y^*) = \begin{cases} \{0\}, & \text{if } y^* \in -N(\bar{y}, F(\bar{x})), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Plugging this back into (4.15) and taking into account that  $S_F^{l,2}(\bar{x}) = \text{WMin}(F(\bar{x}), K)$ , we obtain

$$(4.16) \quad \partial f_{l,\bar{x}}(\bar{x}) \subseteq \overline{\text{conv}}^* \left( \bigcup_{\bar{y} \in \text{WMin}(F(\bar{x}), K)} \left\{ x^* \in X^* : \exists y^* \in N(\bar{y}, F(\bar{x})) : -(x^*, y^*) \in \partial(-g_l)(\bar{x}, \bar{y}) \right\} \right).$$

On the other hand, taking into account the Lipschitzianity of  $g_l$  from Lemma 3.10 (i), for every  $\bar{y} \in \text{WMin}(F(\bar{x}), K)$  we also have:

$$(4.17) \quad \begin{aligned} \partial(-g_l)(\bar{x}, \bar{y}) &\stackrel{\text{(Remark 2.9)}}{\subseteq} -\overline{\text{conv}}^*(\partial g_l(\bar{x}, \bar{y})) \\ &\stackrel{\text{(Lemma 4.3)}}{\subseteq} -\overline{\text{conv}}^* \left( \bigcup_{\substack{\bar{z} \in F(\bar{x}) \cap (\bar{y} - \text{bd } K) \\ z^* \in \partial \Psi_e(\bar{z} - \bar{y})}} D^* F(\bar{x}, \bar{z})(z^*) \times \{-z^*\} \right). \end{aligned}$$

Finally, by putting (4.17) back into (4.16), we obtain our desired estimate.  $\square$

**Remark 4.9.** According to Remark 3.6, the scalarizing functional  $f_{l, \bar{x}}$  would remain unchanged if we substitute  $F$  by a set-valued mapping  $\tilde{F} : X \rightrightarrows Y$  of the form  $\tilde{F}(x) = F(x) + A$ , with  $A \subseteq K$  and  $0 \in A$ . Hence, in Theorem 4.8 we can substitute  $F$  by any other set-valued mapping  $\tilde{F}$  of the above form. By doing this, we can obtain different (maybe sharper) upper estimates of  $\partial f_{l, \bar{x}}(\bar{x})$ . This is worth keeping in mind when obtaining optimality conditions for set optimization problems, as these are based on the subdifferential of  $f_{l, \bar{x}}(\bar{x})$ , see Section 6.

**Remark 4.10.** Note that, since the upper estimate of  $\partial f_{l, \bar{x}}(\bar{x})$  obtained in (4.14) is convex, it also constitutes an upper estimate of  $\partial^\circ f_{l, \bar{x}}(\bar{x})$  according to [46, Theorem 3.57]. However, as we will see in Example 6.8, when applying this result to optimality conditions for set optimization problems, the convexity of the upper estimate can not be removed very easily.

The following corollary shows that if  $Y$  is finite dimensional our assumptions in Theorem 4.8 are natural.

**Corollary 4.11.** Let Assumption 1 be fulfilled with  $X$  being Asplund. Suppose that  $Y$  is finite dimensional and that  $\text{gph } F$  is closed. Furthermore, assume that  $F$  is locally Lipschitzian and locally bounded at  $\bar{x}$ . Then, inclusion (4.14) holds.

*Proof.* Since  $\text{gph } F$  is closed, in particular we have that  $F$  is closed valued. This, together with the local boundedness at  $\bar{x}$  and the finite dimensionality of  $Y$ , gives us the compactness of  $F(\bar{x})$ . Hence, according to [29, Theorem 6.3], we have  $\text{WMin}(F(\bar{x}), K) \neq \emptyset$ . Furthermore, the local boundedness of  $F$  at  $\bar{x}$  also implies that of the set-valued mappings  $S_F^{l,1}$  and  $S_F^{l,2}$  in the statement of Theorem 4.8. This, together with the fact that  $Y$  is finite dimensional gives us the inner semicompactness of  $S_F^{l,1}$  and  $S_F^{l,2}$ . Thus, all the conditions of Theorem 4.8 are satisfied. The statement follows.  $\square$

5. SUBDIFFERENTIAL OF THE SCALARIZING FUNCTIONAL ASSOCIATED TO THE UPPER LESS RELATION

In this section, we compute an approximation of the subdifferential of the functional  $f_{u,\bar{x}}$  given in Definition 3.4 at the point  $\bar{x}$ . We start again by defining two useful solution maps.

**Definition 5.1.** *Let Assumption 1 be fulfilled.*

(i) *The upper inner solution map  $S_F^{u,1} : Y \rightrightarrows Y$  is defined as*

$$S_F^{u,1}(y) := \{z \in F(\bar{x}) : g_{u,\bar{x}}(y) = \Psi_e(y - z)\}.$$

(ii) *The upper outer solution map  $S_F^{u,2} : Y \rightrightarrows Y$  is defined as*

$$S_F^{u,2}(y) := \{y \in F(x) : f_{u,\bar{x}}(x) = g_{u,\bar{x}}(y)\}.$$

In the next lemma, we obtain upper estimates for the subdifferentials of the inner function in both the convex and Lipschitzian cases.

**Lemma 5.2.** *Let Assumption 1 be fulfilled. The following statements hold:*

(i) *Let  $\bar{y} \in \text{WMax}(\mathcal{H}_F(\bar{x}), K)$  and suppose that  $\mathcal{H}_F(\bar{x})$  is a convex and  $K$ -upper bounded set. Then,  $g_{u,\bar{x}}$  is convex, continuous at  $\bar{x}$  and*

$$(5.1) \quad \partial g_{u,\bar{x}}(\bar{y}) = \partial \Psi_e(0) \cap N(\bar{y}, \mathcal{H}_F(\bar{x})).$$

(ii) *Let  $X$  and  $Y$  be Asplund and fix  $\bar{y} \in \text{WMax}(F(\bar{x}), K)$ . Suppose that:*

- (a)  $F(\bar{x})$  is closed,
- (b)  $S_F^{u,1}$  is inner semicompact at  $\bar{y}$ .

*Then,*

$$(5.2) \quad \partial g_{u,\bar{x}}(\bar{y}) \subseteq \bigcup_{\bar{z} \in F(\bar{x}) \cap (\bar{y} + \text{bd } K)} \partial \Psi_e(\bar{y} - \bar{z}) \cap N(\bar{z}, F(\bar{x})).$$

*Proof.* Our statements will follow from Theorem 2.11 and Theorem 2.12 respectively. In order to see this, we consider  $T \in \mathcal{L}(Y \times Y, Y)$  and  $f : Y \times Y \rightarrow \mathbb{R}$  defined respectively as

$$T(y, z) := y - z, \quad f(y, z) := (\Psi_e \circ T)(y, z).$$

Furthermore, we define the set-valued maps  $\tilde{F}, \hat{F} : Y \rightrightarrows Y$  respectively as  $\tilde{F}(y) = \mathcal{H}_F(\bar{x})$  and  $\hat{F}(y) = F(\bar{x})$  for every  $y \in Y$ . We can then write

$$g_{u,\bar{x}}(y) = \inf_{z \in \tilde{F}(y)} f(y, z),$$

with corresponding solution map  $S_{\mathcal{H}_F}^{u,1}$ , and

$$g_{u,\bar{x}}(y) = \inf_{z \in \hat{F}(y)} f(y, z),$$

with corresponding solution map  $S_F^{u,1}$ . By Proposition 3.5 (ii) and Proposition 3.1 (vi), we get

$$\begin{aligned} S_{\mathcal{H}_F}^{u,1}(\bar{y}) &= \{z \in \mathcal{H}_F(\bar{x}) \mid \Psi_e(\bar{y} - z) = g_{u,\bar{x}}(\bar{y})\} \\ &= \{z \in \mathcal{H}_F(\bar{x}) \mid \Psi_e(\bar{y} - z) = 0\} \\ &= \mathcal{H}_F(\bar{x}) \cap (\bar{y} + \text{bd } K). \end{aligned}$$

In particular, we deduce that  $\bar{y} \in S_{\mathcal{H}_F}^{u,1}(\bar{x})$ . Similarly, we obtain

$$(5.3) \quad S_F^{u,1}(\bar{y}) = F(\bar{x}) \cap (\bar{y} + \text{bd } K).$$

On the other hand, it is obvious that  $f$  is convex and continuous. Moreover, for any  $\bar{z} \in Y$ , the chain rule of convex analysis [52, Proposition 3.28] implies

$$\partial f(\bar{y}, \bar{z}) = T^*[\partial \Psi_e(T(\bar{y}, \bar{z}))] = T^*[\partial \Psi_e(\bar{y} - \bar{z})],$$

where  $T^* \in \mathcal{L}(Y^*, Y^* \times Y^*)$  is the adjoint operator of  $T$ . It is easy to verify that in this case  $T^*(y^*) = (y^*, -y^*)$ . Hence, we get

$$(5.4) \quad \partial f(\bar{y}, \bar{z}) = \bigcup_{y^* \in \partial \Psi_e(\bar{y} - \bar{z})} (y^*, -y^*).$$

We proceed now to analyze each case separately.

(i) The convexity and continuity follows from Lemma 3.7 (ii). The subdifferential formula will be a simple application of Theorem 2.11 and to do so, we check that the hypothesis are fulfilled. Indeed, by assumption,  $\mathcal{H}_F(\bar{x})$  is a convex set and hence  $\tilde{F}$  is a convex set-valued mapping. Moreover, from Proposition 3.1 (i), (ii) it follows that  $f$  is a proper convex function that is continuous at any point of  $\text{gph } \tilde{F}$  and hence, in particular, the regularity condition (ii) in Theorem 2.11 is satisfied. As a consequence of Proposition 3.5 (ii), we also have that  $\bar{y} \in \text{dom } g_{u,\bar{x}}$  and  $\text{dom } g_{u,\bar{x}}(\bar{y}) = 0 < +\infty$ .

Since  $\bar{y} \in S_{\mathcal{H}_F}^{u,1}(\bar{x})$ , we can apply now Theorem 2.11 to obtain

$$(5.5) \quad \partial g_{u,\bar{x}}(\bar{y}) = \bigcup_{(y^*, z^*) \in \partial f(\bar{y}, \bar{y})} \left[ y^* + D^* \tilde{F}(\bar{y}, \bar{y})(z^*) \right].$$

Next, we examine the term  $D^* \tilde{F}(\bar{y}, \bar{y})(z^*)$  in the above formula. Note that  $\text{gph } \tilde{F} = Y \times \mathcal{H}_F(\bar{x})$ . Hence, we get  $N((\bar{y}, \bar{y}), \text{gph } \tilde{F}) = \{0\} \times N(\bar{y}, \mathcal{H}_F(\bar{x}))$  and from this it follows that, for any  $y^* \in Y^*$ :

$$\begin{aligned}
D^* \tilde{F}(\bar{y}, \bar{y})(-y^*) &= \{z^* \in Y^* \mid (z^*, y^*) \in N(\bar{y}, \mathcal{H}_F(\bar{x}))\} \\
&= \{z^* \in Y^* \mid (z^*, y^*) \in \{0\} \times N(\bar{y}, \mathcal{H}_F(\bar{x}))\} \\
&= \begin{cases} \{0\}, & \text{if } y^* \in N(\bar{y}, \mathcal{H}_F(\bar{x})), \\ \emptyset, & \text{otherwise.} \end{cases}
\end{aligned}$$

Taking this into account together with (5.4), we obtain the following in (5.5):

$$\begin{aligned}
\partial g_{u, \bar{x}}(\bar{y}) &= \bigcup_{y^* \in \partial \Psi_e(0)} \left[ y^* + D^* \tilde{F}(\bar{y}, \bar{y})(-y^*) \right] \\
&= \bigcup_{y^* \in \partial \Psi_e(0)} \left[ y^* + \begin{cases} \{0\}, & \text{if } y^* \in N(\bar{y}, \mathcal{H}_F(\bar{x})), \\ \emptyset, & \text{otherwise} \end{cases} \right] \\
&= \partial \Psi_e(0) \cap N(\bar{y}, \mathcal{H}_F(\bar{x})),
\end{aligned}$$

as expected.

(ii) In this case, we will apply Theorem 2.12 to obtain an upper estimate of  $\partial g_{u, \bar{x}}(\bar{y})$ . We check that all the conditions of the theorem are fulfilled:

- $\hat{F}$  is closed at  $\bar{y}$ .  
This follows from condition (a).
- $S_F^{u,1}$  is inner semicompact at  $\bar{y}$ .  
This is just condition (b).
- There exists a neighborhood  $V$  of  $\bar{y}$  such that  $f$  is Lipschitzian on  $V \times Y$ .  
Follows directly from the Lipschitzianity of  $\Psi_e$  in Proposition 3.1 (ii).
- $\text{gph } \tilde{F}$  is locally closed around every point in the set  $\{\bar{y}\} \times S_F^{u,1}(\bar{y})$ .  
This is a consequence of (a).

Theorem 2.12 together with (5.3) gives us now

$$(5.6) \quad \partial g_{u, \bar{x}}(\bar{y}) \subseteq \bigcup_{\substack{\bar{z} \in F(\bar{x}) \cap (\bar{y} + \text{bd } K) \\ (y^*, z^*) \in \partial f(\bar{y}, \bar{z})}} \left[ y^* + D^* \hat{F}(\bar{y}, \bar{z})(z^*) \right].$$

Analogous to the proof of statement (i), we obtain

$$D^* \hat{F}(\bar{y}, \bar{z})(z^*) = \begin{cases} \{0\}, & \text{if } z^* \in -N(\bar{z}, F(\bar{x})), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Finally, by substituting this and (5.4) into (5.6), the desired estimate is obtained.  $\square$

Next, we state the main result of the section.

**Theorem 5.3.** *In addition to Assumption 1, let  $X$  and  $Y$  be Asplund. Suppose also that:*

- (i)  $F$  is locally Lipschitzian at  $\bar{x}$ ,
- (ii)  $\text{WMax}(F(\bar{x}), K) \neq \emptyset$ ,
- (iii)  $F$  is closed at  $\bar{x}$ ,
- (iv)  $S_F^{u,1}$  is inner semicompact at every point in the set  $\text{WMax}(F(\bar{x}), K)$ ,
- (v)  $S_F^{u,2}$  is inner semicompact at  $\bar{x}$ .
- (vi)  $\text{gph } F$  is locally closed around any point in the set  $\{\bar{x}\} \times \text{WMax}(F(\bar{x}), K)$ .

Then,

$$(5.7) \quad \partial f_{u,\bar{x}}(\bar{x}) \subseteq -\overline{\text{conv}}^* \left( \bigcup_{\bar{y} \in \text{WMax}(F(\bar{x}), K)} D^* F(\bar{x}, \bar{y}) [H_{(\bar{x}, \bar{y})}] \right),$$

where

$$H_{(\bar{x}, \bar{y})} := -\overline{\text{conv}}^* \left( \bigcup_{\bar{z} \in F(\bar{x}) \cap (\bar{y} + \text{bd } K)} \partial \Psi_e(\bar{y} - \bar{z}) \cap N(\bar{z}, F(\bar{x})) \right).$$

*Proof.* Consider the function  $f : X \times Y \rightarrow Y$  defined as  $f(x, y) = g_{u,\bar{x}}(y)$ . By definition, we have

$$f_{u,\bar{x}}(x) = \sup_{y \in F(x)} f(x, y).$$

We verify that we can apply Theorem 2.12. First, note that the solution map in this case is just  $S_F^{u,2}$ . Hence, Proposition 3.5 (iii) can be applied to obtain  $f_{u,\bar{x}}(\bar{x}) = 0$ . Then, from Proposition 3.5 (ii) we get

$$(5.8) \quad S_F^{u,2}(\bar{x}) = \text{WMax}(F(\bar{x}), K) \neq \emptyset.$$

We proceed to check the rest of the assumptions:

- $F$  is closed at  $\bar{x}$ ,  
This is just condition (iii) in the theorem.
- $S_F^{u,2}$  is inner semicompact at  $\bar{x}$ ,  
This is exactly condition (v) in our theorem.
- There is a neighborhood  $U$  of  $\bar{x}$  such that  $f$  is Lipschitzian on  $U \times Y$ .  
Follows directly from condition (i) and Lemma 3.10 (ii).
- $\text{gph } F$  is locally closed around every point in the set  $\{\bar{x}\} \times S_F^{u,2}(\bar{x})$ .  
This follows from (5.8) and condition (vi) in the theorem.

Hence, taking into account the Lipschitzianity of  $f_{u,\bar{x}}$  from Theorem 3.11 (ii), we obtain:

$$\begin{aligned}
\partial f_{u,\bar{x}}(\bar{x}) &= \partial \left( - \inf_{y \in F(\cdot)} -f(\cdot, y) \right) (\bar{x}) \\
&\stackrel{\text{(Remark 2.9)}}{\subseteq} -\overline{\text{conv}}^* \left( \partial \left( \inf_{y \in F(\cdot)} -f(\cdot, y) \right) (\bar{x}) \right) \\
(5.9) \quad &\stackrel{\text{(Theorem 2.12 + (5.8))}}{\subseteq} -\overline{\text{conv}}^* \left( \bigcup_{\substack{\bar{y} \in \text{WMax}(F(\bar{x}), K) \\ (x^*, y^*) \in \partial(-f)(\bar{x}, \bar{y})}} \left[ x^* + D^* F(\bar{x}, \bar{y})(y^*) \right] \right).
\end{aligned}$$

Note that  $f$  is independent of the argument in the space  $X$ . Furthermore, since  $F$  is closed at  $\bar{x}$ , we also have that  $F(\bar{x})$  is a closed set. Hence, together with condition (iv), it is easy to see that the assumptions of Lemma 5.2 are satisfied. Then, for any  $\bar{y} \in \text{WMax}(F(\bar{x}), K)$ , we get:

$$\begin{aligned}
\partial(-f)(\bar{x}, \bar{y}) &= \{0\} \times \partial(-g_{u,\bar{x}})(\bar{y}) \\
&\stackrel{\text{(Remark 2.9)}}{\subseteq} -\{0\} \times \overline{\text{conv}}^* (\partial g_{u,\bar{x}}(\bar{y})) \\
(5.10) \quad &\stackrel{\text{(Lemma 5.2 (ii))}}{\subseteq} -\{0\} \times \overline{\text{conv}}^* \left( \bigcup_{\bar{z} \in F(\bar{x}) \cap (\bar{y} + \text{bd } K)} \partial \Psi_e(\bar{y} - \bar{z}) \cap N(\bar{z}, F(\bar{x})) \right).
\end{aligned}$$

Substituting (5.10) into (5.9), we obtain the desired estimate.  $\square$

**Remark 5.4.** *Similar to Remark 4.9, the functional  $f_{u,\bar{x}}$  remains unchanged if we substitute  $F$  by  $\tilde{F} : X \rightrightarrows Y$  of the form  $\tilde{F}(x) = F(x) - A$ , with  $A \subseteq K$  and  $0 \in A$ . Hence, in Theorem 5.3 we can substitute  $F$  by any other set-valued mapping  $\tilde{F}$  of the above form. From this, we can obtain different (maybe sharper) upper estimates of  $\partial f_{u,\bar{x}}(\bar{x})$ , which can be translated into sharper optimality conditions set optimization problems, see Section 6.*

**Remark 5.5.** *Similarly to Remark 4.10, we mention that, although the upper estimate in (5.7) is convex (and hence we are also estimating  $\partial^\circ f_{u,\bar{x}}(\bar{x})$ ), Example 6.8 illustrates that convexity is necessary.*

The proof of the following corollary is similar to that of Corollary 4.11, and it is hence omitted.

**Corollary 5.6.** *Let Assumption 1 be fulfilled with  $X$  being Asplund. Suppose that  $Y$  is finite dimensional and that  $\text{gph } F$  is closed. Furthermore, assume that  $F$  is locally Lipschitzian and locally bounded at  $\bar{x}$ . Then, inclusion (5.7) holds.*

We conclude this section with a sharper result in the convex case.

**Theorem 5.7.** *In addition to Assumption 1, let  $X$  and  $Y$  be Asplund. Suppose also that*



- (i)  $F$  is  $\preceq_K^{(u)}$ -convex and locally  $u$ -upper bounded at  $\bar{x}$ ,
- (ii)  $\mathcal{H}_F$  is convex valued in a neighborhood of  $\bar{x}$ ,
- (iii)  $\mathcal{H}_F$  is closed at  $\bar{x}$ ,
- (iv)  $\text{WMax}(\mathcal{H}_F(\bar{x}), K) \neq \emptyset$ ,
- (v)  $S_{\mathcal{H}_F}^{u,2}(x)$  is inner semicompact at  $\bar{x}$ .
- (vi)  $\text{gph } \mathcal{H}_F$  is locally closed around any point in the set  $\{\bar{x}\} \times \text{WMax}(\mathcal{H}_F(\bar{x}), K)$ .

Then,

$$\partial f_{u,\bar{x}}(\bar{x}) \subseteq -\overline{\text{conv}}^* \left( \bigcup_{\bar{y} \in \text{WMax}(\mathcal{H}_F(\bar{x}), K)} D^* \mathcal{H}_F(\bar{x}, \bar{y}) [-\partial \Psi_e(0) \cap N(\bar{y}, \mathcal{H}_F(\bar{x}))] \right).$$

*Proof.* Because of conditions (i) and (ii), we can apply [61, Theorem 7.4.9] to obtain that  $\mathcal{H}_F$  is locally Lipschitzian at  $\bar{x}$ . Then, it is easy to see that assumptions (i) – (iii), (v) – (vi) of Theorem 5.3 are satisfied if we replace  $F$  by  $\mathcal{H}_F$ . Since these assumptions are the only ones needed to obtain (5.9), we can take into account Remark 5.4 to get in this case

$$(5.11) \quad \partial f_{u,\bar{x}}(\bar{x}) \subseteq -\overline{\text{conv}}^* \left( \bigcup_{\substack{\bar{y} \in \text{WMax}(\mathcal{H}_F(\bar{x}), K) \\ (x^*, y^*) \in \partial(-f)(\bar{x}, \bar{y})}} \left[ x^* + D^* \mathcal{H}_F(\bar{x}, \bar{y})(y^*) \right] \right),$$

where  $f$  is the same function defined in Theorem 5.3. Similar to (5.10), but applying Lemma 5.2 (i) instead, we obtain

$$(5.12) \quad \partial(-f)(\bar{x}, \bar{y}) \subseteq -\{0\} \times \left( \partial \Psi_e(0) \cap N(\bar{y}, \mathcal{H}_F(\bar{x})) \right).$$

The estimate is then obtained by replacing the term  $\partial(-f)(\bar{x}, \bar{y})$  in (5.11) by the upper estimate obtained in (5.12).  $\square$

## 6. OPTIMALITY CONDITIONS FOR SET OPTIMIZATION PROBLEMS

In this section we will obtain optimality conditions for set optimization problems based on our previous results. We start by formally defining the set optimization problem and the solution concepts that will be considered.

**Definition 6.1.** *Let Assumption 1 be fulfilled and let  $r \in \{l, u\}$ . The set optimization problem is defined as*

$$(\mathcal{SOP}) \quad \min_{x \in \Omega} F(x),$$

and its minimal solutions are understood in the following sense: we say that  $\bar{x} \in \Omega$  is a

- (i)  $\preceq_K^{(r)}$ -weakly minimal solution of (SOP) if

$$\nexists x \in \Omega \setminus \{\bar{x}\} : F(x) \prec_K^{(r)} F(\bar{x}).$$

(ii)  $\preceq_K^{(r)}$ -strictly minimal solution of  $(\mathcal{SOP})$  if

$$\nexists x \in \Omega \setminus \{\bar{x}\} : F(x) \preceq_K^{(r)} F(\bar{x}).$$

(iii) weakly minimal solution of  $(\mathcal{SOP})$  if

$$\exists \bar{y} \in F(\bar{x}) : F(\Omega) \cap (\bar{y} - \text{int } K) = \emptyset.$$

If in the above definition we replace  $\Omega$  by  $\Omega \cap U$ , with  $U$  being a neighborhood of  $\bar{x}$ , we say that  $\bar{x}$  is a local  $(\preceq_K^{(r)}$ -weakly,  $\preceq_K^{(r)}$ -,  $\preceq_K^{(r)}$ - strictly, weakly)minimal solution respectively.

**Remark 6.2.** It is easy to see that  $\preceq_K^{(r)}$ - strictly minimal solutions are  $\preceq_K^{(r)}$ -weakly minimal. In addition, the minimality concept in Definition 6.1 (iii) is the one used in the vector approach for set optimization problems [35]. It is known [24, Proposition 2.10] that weakly minimal solutions of  $\mathcal{SOP}$  are also  $\preceq_K^{(l)}$ - weakly minimal in a slightly different sense. A similar statement can be made about the set relation  $\preceq_K^{(u)}$ , see also [24, Remark 2.11] Conversely, it was proved in [40] that, if  $F(\bar{x})$  has a strongly minimal element and  $\bar{x}$  is a  $\preceq_K^{(l)}$ - weakly minimal solution of  $(\mathcal{SOP})$ , then  $\bar{x}$  is also a weakly minimal solution.

Of course, global solutions of  $(\mathcal{SOP})$  are also local solutions. Our next proposition confirms that, as in the scalar case, the converse holds under convexity.

**Proposition 6.3.** Let Assumption 1 be fulfilled and fix  $r \in \{l, u\}$ . Suppose that  $\Omega$  is convex, that  $F$  is  $\preceq_K^{(r)}$ -convex and that  $\bar{x}$  is a local  $\preceq_K^{(r)}$ -weakly minimal solution of  $(\mathcal{SOP})$ . The following statements are true:

- (i) If  $r = l$ , then  $\bar{x}$  is also a global  $\preceq_K^{(l)}$ -weakly minimal solution.
- (ii) If  $r = u$  and  $\mathcal{H}_F(\bar{x})$  is convex, then  $\bar{x}$  is also a global  $\preceq_K^{(u)}$ -weakly minimal solution.

*Proof.* Since the proofs are similar and resemble the one in the scalar case, we only show (ii). See also [21, Proposition 5] for a proof of (i) with a slightly different optimality concept. Let  $U$  be the neighborhood of  $\bar{x}$  such that

$$\forall x \in \Omega \cap U \setminus \{\bar{x}\} : F(x) \not\prec_K^{(r)} F(\bar{x})$$

and suppose that  $\bar{x}$  is not a global  $\preceq_K^{(u)}$ -weakly minimal solution of  $(\mathcal{SOP})$ . Then, we can find  $\tilde{x} \in \Omega \setminus \{\bar{x}\}$  such that  $F(\tilde{x}) \prec_K^{(u)} F(\bar{x})$ . Hence, we get the existence of  $\lambda \in (0, 1]$  such that  $x_\lambda := \lambda\tilde{x} + (1 - \lambda)\bar{x} \in \Omega \cap U \setminus \{\bar{x}\}$ . It follows that

$$\begin{aligned}
F(x_\lambda) &\subseteq F(x_\lambda) - K \\
&\stackrel{(F \text{ is } \preceq_K^{(u)}\text{-convex})}{\subseteq} \lambda F(\tilde{x}) + (1 - \lambda)F(\bar{x}) - K \\
&\subseteq \lambda \mathcal{H}_F(\tilde{x}) + (1 - \lambda)\mathcal{H}_F(\bar{x}) - K \\
&\stackrel{(\text{as } F(\tilde{x}) \prec_K^{(u)} F(\bar{x}))}{\subseteq} \lambda \mathcal{H}_F(\bar{x}) + (1 - \lambda)\mathcal{H}_F(\bar{x}) - \text{int } K \\
&\stackrel{(\mathcal{H}_F(\bar{x}) \text{ is convex})}{=} \mathcal{H}_F(\bar{x}) - \text{int } K \\
&= F(\bar{x}) - \text{int } K,
\end{aligned}$$

which is equivalent to  $F(x_\lambda) \prec_K^{(u)} F(\bar{x})$ . This contradicts to the local minimality of  $F$  at  $\bar{x}$ .  $\square$

In the following theorem we establish relationships between the set-valued problem and a corresponding scalar problem. We want to mention that a similar statement to (i) below have been established in [24, Corollary 4.11] for the case  $r = l$ .

**Theorem 6.4.** *Let Assumption 1 be fulfilled and, for  $r \in \{l, u\}$ , consider the functional  $f_{r,\bar{x}}$  in Definition 3.4 (iii). The following assertions are true:*

(i) *If  $\bar{x}$  is a local  $\preceq_K^{(r)}$ -weakly minimal solution of  $(\mathcal{SOP})$ , then  $\bar{x}$  is a local solution of the problem*

$$(P_r) \quad \min_{x \in \Omega} f_{r,\bar{x}}(x).$$

(ii) *Conversely, suppose that  $\bar{x}$  is a local strict solution of problem  $(P_r)$  and either  $r = l$  and  $\text{WMin}(F(\bar{x}), K) \neq \emptyset$ , or  $r = u$  and  $\text{WMax}(F(\bar{x}), K) \neq \emptyset$ . Then,  $\bar{x}$  is a local  $\preceq_K^{(r)}$ -strictly minimal solution of  $(\mathcal{SOP})$ .*

*Proof.* (i) Assume that  $\bar{x}$  is not a local solution of  $(P_r)$ . Then, for every neighborhood  $U$  of  $\bar{x}$  we can find  $\tilde{x} \in \Omega \cap U$  such that

$$(6.1) \quad f_{r,\bar{x}}(\tilde{x}) < f_{r,\bar{x}}(\bar{x}) \leq 0.$$

We just analyze the case  $r = u$  since the other one is similar. From the definition of  $f_{u,\bar{x}}$  and (6.1), we deduce that for every  $\tilde{y} \in F(\tilde{x})$ , the inequality  $g_{u,\bar{x}}(\tilde{y}) < 0$  holds. Equivalently, we obtain

$$\forall \tilde{y} \in F(\tilde{x}) \exists \bar{y} \in F(\bar{x}) : \Psi_e(\tilde{y} - \bar{y}) < 0.$$

Again, by Proposition 3.1 (vi), we obtain  $F(\tilde{x}) \prec_K^{(u)} F(\bar{x})$ , a contradiction.

(ii) By Proposition 3.5 (iii) we know that  $f_{r,\bar{x}}(\bar{x})$  is finite. Assume that  $\bar{x}$  is not a local  $\preceq_K^{(r)}$ -strictly minimal solution of  $(\mathcal{SOP})$ . Then, for any neighborhood  $U$  of  $\bar{x}$  we can find  $\tilde{x} \in (\Omega \cap U) \setminus \{\bar{x}\}$  such that

$$F(\tilde{x}) \preceq_K^{(r)} F(\bar{x}).$$

Hence, according to Theorem 3.3, we get

$$f_{r,\bar{x}}(\tilde{x}) \leq f_{r,\bar{x}}(\bar{x}).$$

This contradicts the fact that  $\bar{x}$  is a local strict solution of  $(P_r)$ .  $\square$

Necessary optimality conditions for  $(\mathcal{SOP})$  with respect to the relation  $\preceq_K^{(l)}$  are established in the next theorem.

**Theorem 6.5.** *Let Assumption 1 be fulfilled and suppose that  $\bar{x}$  is a local  $\preceq_K^{(l)}$ -weakly minimal solution of  $(\mathcal{SOP})$ . The following statements are true:*

- (i) *Suppose that  $\Omega$  is convex, that  $F$  is  $\preceq_K^{(l)}$ -convex and locally  $l$ -bounded at  $\bar{x}$ , and that  $F(\bar{x})$  is strongly  $K$ -compact. Then,*

$$(6.2) \quad 0 \in \overline{\text{conv}}^* \left( \bigcup_{\bar{y} \in \text{Min}(F(\bar{x}), K)} D^* \mathcal{E}_F(\bar{x}, \bar{y}) [\partial \Psi_e(0)] \right) + N(\bar{x}, \Omega).$$

*This condition is sufficient for optimality provided that, in addition,  $F$  is strongly  $K$ -compact valued in  $\Omega$ .*

- (ii) *Suppose that  $X$  and  $Y$  are Asplund spaces, that  $F$  is locally Lipschitzian at  $\bar{x}$ , and that the rest of the conditions in Theorem 4.8 are fulfilled. Then,*

$$(6.3) \quad 0 \in \overline{\text{conv}}^* \left( \bigcup_{\bar{y} \in \text{WMin}(F(\bar{x}), K)} \left\{ x^* \in X^* : \exists y^* \in N(\bar{y}, F(\bar{x})) : (x^*, y^*) \in G_{(\bar{x}, \bar{y})} \right\} \right) + N(\bar{x}, \Omega),$$

where

$$G_{(\bar{x}, \bar{y})} = \overline{\text{conv}}^* \left( \bigcup_{\substack{\bar{z} \in F(\bar{x}) \cap (\bar{y} - \text{bd } K) \\ z^* \in \partial \Psi_e(\bar{z} - \bar{y})}} D^* F(\bar{x}, \bar{z})(z^*) \times \{-z^*\} \right).$$

*Proof.* By Theorem 6.4, it follows that  $\bar{x}$  is a solution of

$$(P_l) \quad \min_{x \in \Omega} f_{l,\bar{x}}(x).$$

(i) Because of Theorem 3.8 (i), we know that  $f_{l,\bar{x}}$  is convex and continuous at  $\bar{x}$ . The classical necessary and sufficient condition for convex problems [57, Proposition 5.1.1] is now read as  $0 \in \partial f_{l,\bar{x}}(\bar{x}) + N(\bar{x}, \Omega)$ . Hence, the first part of the statement follows from Theorem 4.6.

Suppose now that  $F$  is strongly  $K$ -compact valued in  $\Omega$  and that  $\bar{x}$  is not a  $\preceq_K^{(l)}$ -weakly minimal solution of  $(\mathcal{SOP})$ . Then, without loss of generality we can assume that  $F$  is compact valued and that there exists  $\tilde{x} \in \Omega$  such that

$$(6.4) \quad F(\tilde{x}) \prec_K^{(l)} F(\bar{x}).$$

We claim that  $f_{l,\bar{x}}(\tilde{x}) < 0 = f_{l,\bar{x}}(\bar{x})$ , which contradicts (6.2). Indeed, note that because  $F(\tilde{x})$  is compact, the functional  $g_l(\tilde{x}, \cdot)$  is finite. It is also upper semicontinuous in  $Y$  because it is the infimum of continuous functionals. Since  $F(\bar{x})$  is compact, the classical Weierstrass's theorem tells us that the problem

$$\max_{z \in F(\bar{x})} g_l(\tilde{x}, z)$$

has a solution  $\bar{y}$ . According to (6.4), we can find  $\tilde{y} \in F(\tilde{x})$  such that  $\tilde{y} \prec_K \bar{y}$ . Hence, we get

$$f_{l,\bar{x}}(\tilde{x}) = g_l(\tilde{x}, \tilde{y}) \leq \Psi_e(\tilde{y} - \bar{y}) < 0,$$

as desired.

(ii) Similarly to the previous case, by Theorem 3.11 (i) we obtain that  $f_{l,\bar{x}}$  is locally Lipschitzian at  $\bar{x}$ . Hence, all the assumptions for the necessary optimality conditions in [47, Proposition 5.3] are satisfied. From this we get  $0 \in \partial f_{l,\bar{x}}(\bar{x}) + N(\bar{x}, \Omega)$ . The result follows then from Theorem 4.8.  $\square$

With a similar argument to the one in the previous theorem, we can obtain the optimality conditions for problems with the relation  $\preceq_K^{(u)}$ . The proof is hence omitted.

**Theorem 6.6.** *In addition to Assumption 1, suppose that  $X$  and  $Y$  are Asplund spaces and that  $\bar{x}$  is a local  $\preceq_K^{(u)}$ -weakly minimal solution of (SOP). The following statements are true:*

(i) *Suppose that  $F$  is  $\preceq_K^{(u)}$ -convex and locally  $u$ -upper bounded at  $\bar{x}$ , and that the conditions in Theorem 5.7 are fulfilled. Then,*

$$0 \in -\overline{\text{conv}}^* \left( \bigcup_{\bar{y} \in \text{WMax}(\mathcal{H}_F(\bar{x}), K)} D^* \mathcal{H}_F(\bar{x}, \bar{y}) [-\partial \Psi_e(0) \cap N(\bar{y}, \mathcal{H}_F(\bar{x}))] \right) + N(\bar{x}, \Omega).$$

(ii) *Suppose that the  $F$  is locally Lipschitzian at  $\bar{x}$  and that the conditions in Theorem 5.3 are fulfilled. Then,*

$$(6.5) \quad 0 \in -\overline{\text{conv}}^* \left( \bigcup_{\bar{y} \in \text{WMax}(F(\bar{x}), K)} D^* F(\bar{x}, \bar{y}) [H_{(\bar{x}, \bar{y})}] \right) + N(\bar{x}, \Omega),$$

where

$$H_{(\bar{x}, \bar{y})} := -\overline{\text{conv}}^* \left( \bigcup_{\bar{z} \in F(\bar{x}) \cap (\bar{y} + \text{bd } K)} \partial \Psi_e(\bar{y} - \bar{z}) \cap N(\bar{z}, F(\bar{x})) \right).$$

Theorem 6.5 and Theorem 6.6 motivates the following definition.

**Definition 6.7.** *Let Assumption 1 be fulfilled. We say that  $\bar{x}$  is a*

- (i)  $\preceq_K^{(l)}$ - stationary point of  $(\mathcal{SOP})$ , if (6.3) is fulfilled,
- (ii)  $\preceq_K^{(u)}$ - stationary point of  $(\mathcal{SOP})$ , if (6.5) is fulfilled.

We conclude this section with the following example, that illustrate our results and compare them with other results obtained for the vector approach.

**Example 6.8.** *Let  $X = \Omega = \mathbb{R}$ ,  $Y = \mathbb{R}^2$ ,  $K = \mathbb{R}_+^2$ ,  $e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $\bar{x} = 0$ . Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}^2$  and the set-valued mapping  $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$  defined respectively as*

$$f(x) := \begin{pmatrix} x+1 \\ x-1 \end{pmatrix}, \quad F(x) := \{f(x), -f(x)\}.$$

*In particular, we have  $\nabla f(\bar{x}) = (1 \ 1)$ . Then:*

- (i)  $F$  is locally Lipschitzian at  $\bar{x}$ .
- (ii)  $\bar{x}$  is both a local  $\preceq_K^{(l)}$ ,  $\preceq_K^{(u)}$ - weakly minimal solution of  $(\mathcal{SOP})$ :  
Indeed, it is easy to verify that, choosing  $U = (-1, 1)$  :

$$\forall x \in U : F(x) \not\prec_K^{(l)} F(\bar{x}), \quad F(x) \not\prec_K^{(u)} F(\bar{x}).$$

- (iii)  $\bar{x}$  is not a local weakly minimal nor a local weakly maximal solution with the vector approach:

*Indeed, note that in any neighborhood  $U$  of  $\bar{x}$  we can find  $x \in U \setminus \{\bar{x}\}$  such that  $-x \in U$ . Then, it is easy to check that*

$$F(\bar{x}) \subset (F(x) + \text{int } K) \cup (F(-x) + \text{int } K),$$

$$F(\bar{x}) \subset (F(x) - \text{int } K) \cup (F(-x) - \text{int } K).$$

- (iv)  $\bar{x}$  is not a stationary point in the sense of the vector approach:

*Since  $f(\bar{x}) \neq -f(\bar{x})$  and  $\text{gph } F = \text{gph } f \cup \text{gph}(-f)$ , we have that  $\text{gph } F = \text{gph } f$  and  $\text{gph } F = \text{gph}(-f)$  around  $(\bar{x}, f(\bar{x}))$  and  $(\bar{x}, -f(\bar{x}))$  respectively. By the differentiability of  $f$  and Remark 2.10, we obtain*

$$(6.6) \quad \forall z^* \in \mathbb{R}^2 : D^*F(\bar{x}, f(\bar{x}))(z^*) = \{\nabla f(\bar{x})z^*\} = \{z_1^* + z_2^*\},$$

$$(6.7) \quad \forall z^* \in \mathbb{R}^2 : D^*F(\bar{x}, -f(\bar{x}))(z^*) = \{-\nabla f(\bar{x})z^*\} = \{-(z_1^* + z_2^*)\}.$$

*Now, we recall that  $\bar{x}$  is a stationary point of  $F$  in the sense of the vector approach (see [6, Theorem 5.1] and [12, Theorem 3.11] ) if there exists  $\bar{y} \in F(\bar{x})$  and  $y^* \in K^* \setminus \{0\}$  such that*

$$0 \in D^*F(\bar{x}, \bar{y})(y^*).$$

Since  $K^* = K$  in our context, it is then easy to check that

$$0 \in D^*F(\bar{x}, f(\bar{x}))(y^*), y^* \in K^* \iff y^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Similarly, we obtain that

$$0 \in D^*F(\bar{x}, -f(\bar{x}))(y^*), y^* \in K^* \iff y^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It follows that  $\bar{x}$  is not a stationary point in the sense of the vector approach.

(v)  $\bar{x}$  is both  $\preceq_K^{(l)}$ - and  $\preceq_K^{(u)}$ -stationary:

Of course, this is a direct consequence of Theorem 6.5 and Theorem 6.6, but we show the calculus for completeness. First, we note that  $\text{WMin}(F(\bar{x}), K) = \text{WMax}(F(\bar{x}), K) = F(\bar{x})$ . Because  $F(\bar{x})$  consists of isolated points, we obtain

$$(6.8) \quad N(f(\bar{x}), F(\bar{x})) = N(-f(\bar{x}), F(\bar{x})) = \mathbb{R}^2.$$

On the other hand, from Proposition 3.1 (v) we have

$$(6.9) \quad \partial\Psi_e(0) = \{k^* \in \mathbb{R}_+^2 : k_1^* + k_2^* = 1\}.$$

The  $\preceq_K^{(l)}$ -stationarity of  $\bar{x}$  is now equivalent to  $0 \in \overline{\text{conv}}(A_1 \cup A_2)$ , where

$$(6.10) \quad A_1 := \{x^* \in \mathbb{R} : \exists y^* \in \mathbb{R}^2 : (x^*, y^*)^T \in G_{(\bar{x}, f(\bar{x}))}\},$$

$$(6.11) \quad A_2 := \{x^* \in \mathbb{R} : \exists y^* \in \mathbb{R}^2 : (x^*, y^*)^T \in G_{(\bar{x}, -f(\bar{x}))}\}.$$

We have

$$\begin{aligned} G_{(\bar{x}, f(\bar{x}))} &\stackrel{(6.6)}{=} \overline{\text{conv}} \left( \bigcup_{z^* \in \partial\Psi_e(0)} \{z_1^* + z_2^*\} \times \{-z^*\} \right) \\ &\stackrel{(6.9)}{=} \overline{\text{conv}} \left( \bigcup_{z^* \in \partial\Psi_e(0)} \{1\} \times \{-z^*\} \right) \\ &= \{1\} \times (-\partial\Psi_e(0)). \end{aligned}$$

From this, we deduce that  $A_1 = \{1\}$ . Using a similar argument we can obtain  $G_{(\bar{x}, -f(\bar{x}))} = \{-1\} \times (-\partial\Psi_e(0))$ , from which we obtain  $A_2 = \{-1\}$ . Hence, we have

$$0 \in [-1, 1] = \overline{\text{conv}}(A_1 \cup A_2),$$

and the  $\preceq_K^{(l)}$ -stationarity of  $\bar{x}$  follows.

Next, we show that  $\bar{x}$  is also  $\preceq_K^{(u)}$ -stationary. This is equivalent to  $0 \in \overline{\text{conv}}(B_1 \cup B_2)$ , where

$$(6.12) \quad B_1 := -D^*F(\bar{x}, f(\bar{x})) [H_{(\bar{x}, f(\bar{x}))}],$$

$$(6.13) \quad B_2 := -D^*F(\bar{x}, f(\bar{x})) [H_{(\bar{x}, -f(\bar{x}))}].$$

In this case we have

$$\begin{aligned} H_{(\bar{x}, f(\bar{x}))} &= -\overline{\text{conv}}(\partial\Psi_e(0) \cap N(f(\bar{x}), F(\bar{x}))) \\ &\stackrel{(6.8)}{=} -\partial\Psi_e(0). \end{aligned}$$

From this, we deduce that

$$B_1 \stackrel{(6.12)}{=} -D^*F(\bar{x}, f(\bar{x}))[-\partial\Psi_e(0)] \stackrel{(6.6)}{=} \{1\}.$$

Similarly, we can obtain  $H_{(\bar{x}, -f(\bar{x}))} = -\partial\Psi_e(0)$ , from which we get

$$B_2 \stackrel{(6.13)}{=} -D^*F(\bar{x}, -f(\bar{x}))[-\partial\Psi_e(0)] \stackrel{(6.7)}{=} \{-1\}.$$

Hence, we have  $0 \in [-1, 1] = \overline{\text{conv}}(B_1 \cup B_2)$ , and  $\bar{x}$  is  $\preceq_K^{(u)}$ -stationary.

## 7. CONCLUSIONS

In this paper, we considered the set optimization problem with respect to the lower and upper less relations. The main contributions are the optimality conditions in Theorem 6.5 and Theorem 6.6, that are derived under the Lipschitzianity of the set-valued objective mapping and other natural assumptions. Perhaps the most attractive feature of our necessary conditions is that we do not require neither convexity nor compactness of the images of  $F$ , nor the existence of a strongly minimal element in the optimal set, which are some of the drawbacks of the other approaches in the literature [1, 2, 11, 20, 21, 26, 30, 32, 40, 41, 51, 54, 55].

The results obtained also open several ideas for further research. In particular, the scheme employed could be easily extended to other set relations, like those described in [28, 33, 42, 44] and that were not mentioned here. In addition, it is also of interest to relax the Lipschitzian assumption, maybe replacing it by some type of lower semicontinuity property, and therefore obtaining stronger results. Finally, we believe that our optimality conditions are the first step towards deriving algorithms for set optimization problems that converge to stationary points.

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(G. Bouza) FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF HAVANA, 10400, HAVANA, CUBA  
*Email address:* `gema@matcom.uh.cu`

(E. Quintana) FACULTY OF NATURAL SCIENCES II, INSTITUTE OF MATHEMATICS, MARTIN-LUTHER-UNIVERSITY HALLE-WITTENBERG, 06099 HALLE, GERMANY  
*Email address:* `ernest.quintana@mathematik.uni-halle.de`

(C. Tammer) FACULTY OF NATURAL SCIENCES II, INSTITUTE OF MATHEMATICS, MARTIN-LUTHER-UNIVERSITY HALLE-WITTENBERG, 06099 HALLE, GERMANY  
*Email address:* `christiane.tammer@mathematik.uni-halle.de`

(V. A. Tuan) FACULTY OF NATURAL SCIENCES II, INSTITUTE OF MATHEMATICS, MARTIN-LUTHER-UNIVERSITY HALLE-WITTENBERG, 06099 HALLE, GERMANY  
*Email address:* `anh.vu@mathematik.uni-halle.de`