Noname manuscript No.
(will be inserted by the editor)

Upper and Lower Bounds for Large Scale Multistage Stochastic Optimization Problems: Decomposition Methods

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December 20, 2019

Abstract We consider a large scale multistage stochastic optimization problem involving multiple units. Each unit is a (small) control system. Static constraints couple units at each stage. To tackle such large scale problems, we propose two decomposition methods, whether handling the coupling constraints by prices or by resources. We introduce the sequence (one per stage) of global Bellman functions, depending on the collection of local states of all units. We show that every Bellman function is bounded above by a sum of local resource-decomposed value functions, and below by a sum of local price-decomposed value functions — each local decomposed function having for arguments the corresponding local unit state variables. We provide conditions under which these local value functions can be computed by Dynamic Programming. These conditions are established assuming a centralized information structure, that is, when the information available for each unit consists of the collection of noises affecting all the units. We finally study the case where each unit only observes its own local noise (decentralized information structure).

Keywords Stochastic Programming · Discrete time stochastic optimal control · Decomposition methods · Dynamic programming

Mathematics Subject Classification (2000) 93A15 · 93E20 · 49M27 · 49L20

1 Introduction

Multistage stochastic optimization problems are, by essence, complex because their solutions are indexed both by stages (time) and by uncertainties (scenarios). Hence, their large scale nature makes decomposition methods appealing.

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We refer to [1] and [2] for a generic description of decomposition methods in stochastic programming problems. We sketch decomposition methods along three dimensions: *temporal decomposition* methods like Dynamic Programming [3,4] break the multistage problem into a sequence of interconnected static subproblems; *scenario decomposition* methods [5] split large scale stochastic optimization problems scenario by scenario, yielding deterministic subproblems; *spatial decomposition* methods break the spatial coupling of the global problem to obtain local decoupled subproblems. Dynamic Programming methods, and their extensions, have been used on a wide panel of problems, for example in dam management [6]. Scenario decomposition is a well-established method and has been successfully applied to the resolution of unit-commitment problems [7], among others. Such methods gain a lot of interests recently, with extension to mixed-integer problems [8,9] and network formulations [10]. Spatial decomposition of large-scale optimization problems was first studied in [11], and extended to open-loop stochastic optimization problems [12]. Recent developments have mixed spatial decomposition methods with Dynamic Programming to solve large scale multistage stochastic optimization problems. This work led to the introduction of the Dual Approximate Dynamic Programming (DADP) algorithm, which was first applied to unit-commitment problems with a single central coupling constraint linking different stocks altogether [13], and later applied to dams management problems [14]. Once the global problem decomposed by DADP, it is possible to solve each subproblem locally by temporal decomposition, that is, by Dynamic Programming (DP).

This article moves one step further by considering altogether two types of decompositions when dealing with general coupling constraints among units. Such constraints often arise from flows conservation on a graph. Optimization problems on graphs (monotropic optimization) have been studied since long [15,16]. The motivation of this paper comes from electrical microgrid management. In particular, we are interested in multistage stochastic problems corresponding to a district microgrid with different prosumers exchanging energy via a local network. The global problem can naturally be formulated as a sum of local multistage stochastic optimization subproblems coupled together via a global network constraint. Such a problem is analyzed in a companion paper [17], which presents a case study consisting of a large scale district microgrid coupling up to 48 small scale units together.

The paper is organized as follows. In Sect. 2, we introduce a generic stochastic multistage problem with different subsystems linked together via a set of coupling constraints. For this problem, we present price and resource decomposition schemes, that make use of so-called admissible coordination processes. We show how to bound the global Bellman functions above by a sum of local resource-decomposed value functions, and below by a sum of local price-decomposed value functions. In Sect. 3, we study the special case of deterministic coordination processes. First, we show that the local price and resource decomposed value functions satisfy recursive Dynamic Programming equations. Second, we outline how to improve the bounds obtained by the decomposition algorithms. Third, we provide an analysis of the decentralized information structure, that is, when the controls of a given subsystem only depend on the past observations of the noise in that system. Finally, we show how to use the decomposed Bellman functions to devise admissible policies for the global problem.

2 Upper and lower bounds by spatial decomposition

We focus in §2.1 on a generic decomposable optimization problem and present price and resource decomposition schemes. In §2.2, we apply these two methods to a multistage stochastic optimization problem, by decomposing a global static coupling constraint by means of so-called price and resource coordination processes. For such problems, we define the notions of centralized and decentralized information structures.

2.1 Bounds for an optimization problem under coupling constraints via decomposition

We first introduce a generic optimization problem under coupling constraints in §2.1.1 and show in §2.1.2 that, by decomposition, we are able to bound its optimal value.

2.1.1 Global optimization problem formulation

We describe a generic optimization problem coupling different local units. We borrow here the abstract duality formalism of [18].

Let $\mathcal{Z}^1, \dots, \mathcal{Z}^N$ be N sets and $J^i : \mathcal{Z}^i \to (-\infty, +\infty]$, $i \in [\![1, N]\!]$, be local criteria taking values in the extended reals and supposed proper ($J^i \not\equiv +\infty$), where $[\![1, N]\!] = \{1, 2, \cdots, N-1, N\}$ denotes the set of integers between 1 and N. Let $\mathcal{R}^1, \dots, \mathcal{R}^N$ be N vector spaces and $\vartheta^i : \mathcal{Z}^i \to \mathcal{R}^i$, $i \in [\![1, N]\!]$, be mappings that model local constraints.

From these *local* data, we formulate a *global* minimization problem under constraints. We define the product set $\mathcal{Z} = \mathcal{Z}^1 \times \cdots \times \mathcal{Z}^N$ and the product space $\mathcal{R} = \mathcal{R}^1 \times \cdots \times \mathcal{R}^N$. Finally, we introduce a subset $S \subset \mathcal{R}$ that captures the coupling constraints between the N units.

We define the *global optimization* problem as

$$V^\sharp = \inf_{(z^1, \cdots, z^N) \in \mathcal{Z}} \sum_{i=1}^{N} J^i(z^i) , \tag{1a}$$

under the *global coupling constraint*

$$\big(\vartheta^1(z^1), \cdots, \vartheta^N(z^N)\big) \in -S . \tag{1b}$$

The set S is called the *primal admissible set*, and an element $(r^1, \cdots, r^N) \in -S$ is called an *admissible resource vector*. We note that, without Constraint (1b), Problem (1) would decompose into N independent subproblems in a straightforward manner.

We moreover assume that the spaces $\mathcal{R}^1, \ldots, \mathcal{R}^N$ (resources) are paired with spaces $\mathcal{P}^1, \ldots, \mathcal{P}^N$ (prices) by the bilinear forms $\langle \cdot\,, \cdot \rangle \;:\; \mathcal{P}^i \times \mathcal{R}^i \to \mathbb{R}$ (duality pairings). We define the product space $\mathcal{P} = \mathcal{P}^1 \times \cdots \times \mathcal{P}^N$, so that $\mathcal{R}$ and $\mathcal{P}$ are paired by the duality pairing $\langle p\,, r \rangle = \sum_{i=1}^N \langle p^i\,, r^i \rangle$ (see [18] for further details; a typical example of paired spaces is a Hilbert space and its topological dual space).

2.1.2 Upper and lower bounds from price and resource value functions

Consider the global optimization problem (1). For each $i \in [\![1, N]\!]$, we introduce *local price value functions* $\underline{V}^i : \mathcal{P}^i \to [-\infty, +\infty]$ by

$$\underline{V}^i[p^i] = \inf_{z^i \in \mathcal{Z}^i} J^i(z^i) + \langle p^i\,, \vartheta^i(z^i) \rangle \;, \tag{2}$$

and *local resource value functions* $\overline{V}^i : \mathcal{R}^i \to [-\infty, +\infty]$ by

$$\overline{V}^i[r^i] = \inf_{z^i \in \mathcal{Z}^i} J^i(z^i) \quad \text{s.t.} \quad \vartheta^i(z^i) = r^i \;. \tag{3}$$

We denote by $S^\star \subset \mathcal{P}$ the dual cone associated with the constraint set S defined by

$$S^\star = \{ p \in \mathcal{P} \mid \langle p\,, r \rangle \geq 0 \;,\; \forall r \in S \} \;. \tag{4}$$

The cone $S^\star$ is called the *dual admissible set*, and an element $(p^1, \cdots, p^N) \in S^\star$ is called an *admissible price vector*.

The next proposition states that lower and upper bounds are available for Problem (1), and that they can be computed in a decomposed way, that is, unit by unit.

Proposition 2.1 *For any admissible price vector $p = (p^1, \cdots, p^N) \in S^\star$ and for any admissible resource vector $r = (r^1, \cdots, r^N) \in -S$, we have the following lower and upper decomposed estimates of the global minimum $V^\sharp$ of Problem* (1)*:*

$$\sum_{i=1}^N \underline{V}^i[p^i] \;\leq\; V^\sharp \;\leq\; \sum_{i=1}^N \overline{V}^i[r^i] \;. \tag{5}$$

Proof. For a given $p = (p^1, \cdots, p^N) \in S^\star$, we have

$$
\begin{aligned}
\sum_{i=1}^{N} \underline{V}^i[p^i] &= \sum_{i=1}^{N} \inf_{z^i \in \mathcal{Z}^i} \; J^i(z^i) + \big\langle p^i \,, \vartheta^i(z^i) \big\rangle \;, \\
&= \inf_{z \in \mathcal{Z}} \; \sum_{i=1}^{N} J^i(z^i) + \big\langle p \,, \big(\vartheta^1(z^1), \dots, \vartheta^N(z^N)\big) \big\rangle \;, \\
&\leq \inf_{z \in \mathcal{Z}} \; \sum_{i=1}^{N} J^i(z^i) + \big\langle p \,, \big(\vartheta^1(z^1), \dots, \vartheta^N(z^N)\big) \big\rangle \\
&\qquad \text{s.t. } \big(\vartheta^1(z^1), \dots, \vartheta^N(z^N)\big) \in -S \\
&\qquad\qquad\qquad\qquad\qquad \text{(minimizing on a smaller set)} \\
&\leq \inf_{z \in \mathcal{Z}} \; \sum_{i=1}^{N} J^i(z^i) \\
&\qquad \text{s.t. } \big(\vartheta^1(z^1), \dots, \vartheta^N(z^N)\big) \in -S \;, \\
&\qquad\qquad\qquad\qquad \text{(negative pairing between } -S \text{ and } S^\star)
\end{aligned}
$$

which gives the lower bound inequality. The upper bound arises directly, as the optimal value $V^\sharp$ of Problem (1) is given by $\inf_{r \in -S} \sum_{i=1}^{N} \overline{V}^i[r^i] \leq \sum_{i=1}^{N} \overline{V}^i[r^i]$ for any $r \in -S$. □

2.2 The special case of multistage stochastic optimization problems

Now, we turn to the case where Problem (1) corresponds to a multistage stochastic optimization problem elaborated from local data (local states, local controls, and local noises), with global coupling constraints at each time step.

We consider a time span $\{0, \dots, T\}$ where $T \in \mathbb{N}^\star$ is a finite horizon, and a number $N \in \mathbb{N}^\star$ of local units.

2.2.1 Local data for local stochastic control problems

We detail the *local* data describing each unit. Let $\{\mathbb{X}_t^i\}_{t \in [\![0,T]\!]}$, $\{\mathbb{U}_t^i\}_{t \in [\![0,T-1]\!]}$ and $\{\mathbb{W}_t^i\}_{t \in [\![1,T]\!]}$ be sequences of measurable spaces for $i \in [\![1,N]\!]$. We consider two other sequences of measurable vector spaces $\{\mathcal{R}_t^i\}_{t \in [\![0,T-1]\!]}$ and $\{\mathcal{P}_t^i\}_{t \in [\![0,T-1]\!]}$ such that for all t, $\mathcal{R}_t^i$ and $\mathcal{P}_t^i$ are paired spaces, equipped with a bilinear form $\langle p^i \,, r^i \rangle$ for all $p^i \in \mathcal{P}_t^i$ and $r^i \in \mathcal{R}_t^i$. We also introduce, for all $i \in [\![1,N]\!]$ and for all $t \in [\![0,T-1]\!]$,

- measurable *local dynamics* $g_t^i : \mathbb{X}_t^i \times \mathbb{U}_t^i \times \mathbb{W}_{t+1}^i \to \mathbb{X}_{t+1}^i$,
- measurable *local coupling functions* $\Theta_t^i : \mathbb{X}_t^i \times \mathbb{U}_t^i \to \mathcal{R}_t^i$,
- measurable *local instantaneous costs* $L_t^i : \mathbb{X}_t^i \times \mathbb{U}_t^i \times \mathbb{W}_{t+1}^i \to (-\infty, +\infty]$,

and a measurable *local final cost* $K^i : \mathbb{X}^i_T \to (-\infty, +\infty]$. We incorporate possible local constraints (for instance constraints coupling the control with the state) directly in the instantaneous costs L^i_t and the final cost K^i, since they are extended real valued functions which can possibly take the value $+\infty$.

2.2.2 Global data and information structures

From local data given above, we define the global state, control, noise, resource and price spaces at time t as products (over units) of the local spaces as

$$\mathbb{X}_t = \prod_{i=1}^N \mathbb{X}^i_t \ , \ \ \mathbb{U}_t = \prod_{i=1}^N \mathbb{U}^i_t \ , \ \ \mathbb{W}_t = \prod_{i=1}^N \mathbb{W}^i_t \ , \ \ \mathcal{R}_t = \prod_{i=1}^N \mathcal{R}^i_t \ , \ \ \mathcal{P}_t = \prod_{i=1}^N \mathcal{P}^i_t \ ,$$

and we introduce the *global constraint set* $S_t \subset \mathcal{R}_t$ at time t. The global coupling constraint at time t is a combination of the local couplings terms Θ^i_t:

$$\big(\Theta^1_t(x^1_t, u^1_t), \cdots, \Theta^N_t(x^N_t, u^N_t)\big) \in -S_t \ .$$

We also define the global resource and price spaces $\mathcal{R}$ and $\mathcal{P}$, as well as the global set $S \subset \mathcal{R}$, as

$$\mathcal{R} = \prod_{t=0}^{T-1} \mathcal{R}_t \ , \ \ \mathcal{P} = \prod_{t=0}^{T-1} \mathcal{P}_t \ , \ \ S = \prod_{t=0}^{T-1} S_t \ ,$$

and we denote by $S^\star \subset \mathcal{P}$ the dual cone of S (see Equation (4)).

We introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For all $i \in [\![1, N]\!]$, we introduce *local exogenous noise processes* $\boldsymbol{W}^i = \{\boldsymbol{W}^i_t\}_{t\in[\![1,T]\!]}$, where each $\boldsymbol{W}^i_t : \Omega \to \mathbb{W}^i_t$ is a random variable.[1] We denote by $\boldsymbol{W} = (\boldsymbol{W}_1, \cdots, \boldsymbol{W}_T)$ the *global noise process*, where

$$\boldsymbol{W}_t = (\boldsymbol{W}^1_t, \cdots, \boldsymbol{W}^N_t) \ . \tag{6}$$

We consider two possible *information structures* [19, Chap. 3] for decision making in stochastic optimization problems.

- The *centralized* information structure is associated with the global noise process $\boldsymbol{W}$, which materializes at any time $t \in [\![0, T]\!]$ through the σ-field $\mathcal{F}_t$ generated by all noises up to time t:

$$\mathcal{F}_t = \sigma(\boldsymbol{W}_1, \cdots, \boldsymbol{W}_t) \ , \tag{7a}$$

  with the convention $\mathcal{F}_0 = \{\emptyset, \Omega\}$. The σ-field $\mathcal{F}_t$ captures the information provided by the uncertainties *in all units* at time t. We introduce the *filtration* $\mathcal{F} = (\mathcal{F}_t)_{t\in[\![0,T]\!]}$.

[1] Random variables are denoted using bold letters.

- We also consider a *decentralized* information structure. For any $t \in [\![0, T]\!]$ and any $i \in [\![1, N]\!]$, we denote by $\mathcal{F}_t^i$ the i-th *local* σ-field which captures the information provided by the uncertainties *in unit i only* up to time t:

$$\mathcal{F}_t^i = \sigma(\boldsymbol{W}_1^i, \cdots, \boldsymbol{W}_t^i) , \tag{7b}$$

with $\mathcal{F}_0^i = \{\emptyset, \Omega\}$. For all $i \in [\![1, N]\!]$, we have that $\mathcal{F}_t^i \subset \mathcal{F}_t = \bigvee_{i=1}^N \mathcal{F}_t^i$. We also introduce the filtrations $\mathcal{F}^i = (\mathcal{F}_t^i)_{t \in [\![0,T]\!]}$ for all $i \in [\![1, N]\!]$.

In the sequel, for a given filtration $\mathcal{G}$ and a given measurable space $\mathbb{Y}$, we denote by $\mathbb{L}^0(\Omega, \mathcal{G}, \mathbb{P}; \mathbb{Y})$ the space of *$\mathcal{G}$-adapted processes taking values in the space* $\mathbb{Y}$.

2.2.3 Global stochastic control problem

With the data detailed in §2.2.1 and §2.2.2, we formulate the *global optimization problem*[2]

$$\begin{aligned}
V_0(x_0) = \min_{\boldsymbol{X}, \boldsymbol{U}} \mathbb{E}\Bigg[\sum_{i=1}^{N} \sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}^i) + K^i(\boldsymbol{X}_T^i)\Bigg] , \qquad & \text{(8a)} \\
\text{s.t.} , \ \forall t \in [\![0, T-1]\!] , \qquad & \\
\boldsymbol{X}_{t+1}^i = g_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}^i) , \ \ \boldsymbol{X}_0^i = x_0^i , \qquad & \text{(8b)} \\
\sigma(\boldsymbol{U}_t^i) \subset \mathcal{G}_t^i , \qquad & \text{(8c)} \\
\big(\Theta_t^1(\boldsymbol{X}_t^1, \boldsymbol{U}_t^1), \cdots, \Theta_t^N(\boldsymbol{X}_t^N, \boldsymbol{U}_t^N)\big) \in -S_t , \qquad & \text{(8d)}
\end{aligned}$$

where $x_0 = (x_0^1, \cdots, x_0^N) \in \mathbb{X}_0$ is the initial state, where $\sigma(\boldsymbol{U}_t^i)$ is the σ-field generated by the random variable $\boldsymbol{U}_t^i$, and where the σ-field $\mathcal{G}_t^i$ is either equal to $\mathcal{F}_t$ (centralized information structure) or to $\mathcal{F}_t^i$ (decentralized information structure), as detailed in §2.2.2.

Constraints (8c) express the fact that each decision $\boldsymbol{U}_t^i$ is $\mathcal{G}_t^i$-measurable, that is, measurable either with respect to the global information available at time t (see Equation (7a)) or to the local information available at time t for unit i (see Equation (7b)). Note that Constraints (8d) have to be taken in the $\mathbb{P}$-almost sure sense.

We denote by $\boldsymbol{X}_t = (\boldsymbol{X}_t^1, \cdots, \boldsymbol{X}_t^N)$ and $\boldsymbol{U}_t = (\boldsymbol{U}_t^1, \cdots, \boldsymbol{U}_t^N)$ the global state and global control at time t. The stochastic processes $\boldsymbol{X} = (\boldsymbol{X}_0, \cdots, \boldsymbol{X}_T)$ and $\boldsymbol{U} = (\boldsymbol{U}_0, \cdots, \boldsymbol{U}_{T-1})$ are called *global state* and *global control* processes. The stochastic processes $\boldsymbol{X}^i = (\boldsymbol{X}_0^i, \cdots, \boldsymbol{X}_T^i)$ and $\boldsymbol{U}^i = (\boldsymbol{U}_0^i, \cdots, \boldsymbol{U}_{T-1}^i)$ are called *local state* and *local control processes.*

[2] We suppose that measurability and integrability assumptions hold true, so that the expected value in (8) is well defined.

At each time $t \in [\![0, T]\!]$, the *global value function* $V_t : \prod_{i=1}^N \mathbb{X}_t^i \to (-\infty, +\infty]$ is defined by, for all $(x_t^1, \cdots, x_t^N) \in \mathbb{X}_t^1 \times \cdots \times \mathbb{X}_t^N$,

$$V_t(x_t^1, \cdots, x_t^N) = \min_{\boldsymbol{X},\boldsymbol{U}} \mathbb{E}\Big[\sum_{i=1}^{N} \sum_{s=t}^{T-1} L_s^i(\boldsymbol{X}_s^i, \boldsymbol{U}_s^i, \boldsymbol{W}_{s+1}^i) + K^i(\boldsymbol{X}_T^i) \Big] , \tag{9a}$$

$$\text{s.t. }, \ \forall s \in [\![t, T-1]\!] ,$$

$$\boldsymbol{X}_{s+1}^i = g_s^i(\boldsymbol{X}_s^i, \boldsymbol{U}_s^i, \boldsymbol{W}_{s+1}^i) \ , \ \ \boldsymbol{X}_t^i = x_t^i \ , \tag{9b}$$

$$\sigma(\boldsymbol{U}_s^i) \subset \mathcal{G}_s^i \ , \tag{9c}$$

$$\big(\Theta_s^1(\boldsymbol{X}_s^1, \boldsymbol{U}_s^1), \ldots, \Theta_s^N(\boldsymbol{X}_s^N, \boldsymbol{U}_s^N)\big) \in -S_s \ , \tag{9d}$$

with the convention $V_T = \sum_{i=1}^N K^i$. Of course, the value function defined by (9) for $t = 0$ is the same as the function V_0 defined by (8). In the global value function (9), the expected value is taken w.r.t. (with respect to) the global uncertainty process $(\boldsymbol{W}_{t+1}, \cdots, \boldsymbol{W}_T)$. We assume that the expected values in problems (8) and (9) are well defined. In order to have optimal policies, we also assume that the sets defined by the optimal solutions of Problems (9) are nonempty.

2.2.4 Local price and resource value functions

As in §2.1.2, we are able to define local price and local resource value functions for the global multistage stochastic problem (8).

Let $i \in [\![1, N]\!]$ be a local unit, and $\boldsymbol{P}^i = (\boldsymbol{P}_0^i, \cdots, \boldsymbol{P}_{T-1}^i) \in \mathbb{L}^0(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{P}^i)$ be a *local price process*, that is, adapted to the global filtration $\mathcal{F}$ in (7a) generated by the global noises (note that we do not assume that it is adapted to the local filtration $\mathcal{F}^i$ in (7b) generated by the local noises). When specialized to the context of Problem (8), the *local price value function* defined in Equation (2) is written

$$\underline{V}_0^i[\boldsymbol{P}^i](x_0^i) = \min_{\boldsymbol{X}^i,\boldsymbol{U}^i} \mathbb{E}\Big[\sum_{t=0}^{T-1} \Big(L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}^i) + \big\langle \boldsymbol{P}_t^i \ , \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \big\rangle \Big) + K^i(\boldsymbol{X}_T^i) \Big] , \tag{10a}$$

$$\text{s.t. }, \ \forall t \in [\![0, T-1]\!] ,$$

$$\boldsymbol{X}_{t+1}^i = g_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}^i) \ , \ \ \boldsymbol{X}_0^i = x_0^i \ , \tag{10b}$$

$$\sigma(\boldsymbol{U}_t^i) \subset \mathcal{G}_t^i \ . \tag{10c}$$

We introduce paired spaces $\widetilde{\mathbb{L}}(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{R}^i)$ and $\widetilde{\mathbb{L}}^\star(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{P}^i)$ such that the duality product terms $\mathbb{E}\big[\sum_{t=0}^{T-1} \big\langle \boldsymbol{P}_t^i \ , \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \big\rangle\big]$ make sense. This ensures that Problem (10) is well-posed. An example is to consider the case of square integrable random variables, that is, $\Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathcal{R}_t^i)$ and $\boldsymbol{P}_t^i \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathcal{P}_t^i)$. Another possibility is that $\Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) \in \mathbb{L}^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathcal{R}_t^i)$ is a bounded random variable, and that $\boldsymbol{P}_t^i \in \mathbb{L}^1(\Omega, \mathcal{F}_t, \mathbb{P}; \mathcal{P}_t^i)$. We refer to [20] for a discussion about the duality in the (L^∞, L^1) pairing case.

At each time $t \in [\![0, T]\!]$, we also introduce the *local price value functions* $\underline{V}_t^i[\boldsymbol{P}^i] : \mathbb{X}_t^i \to (-\infty, +\infty]$ as, for all $x_t^i \in \mathbb{X}_t^i$,

$$\underline{V}_t^i[\boldsymbol{P}^i](x_t^i) = \min_{\boldsymbol{X}^i, \boldsymbol{U}^i} \mathbb{E}\bigg[\sum_{s=t}^{T-1} \Big(L_s^i(\boldsymbol{X}_s^i, \boldsymbol{U}_s^i, \boldsymbol{W}_{s+1}^i) + \big\langle \boldsymbol{P}_s^i \,, \Theta_s^i(\boldsymbol{X}_s^i, \boldsymbol{U}_s^i) \big\rangle \Big) + K^i(\boldsymbol{X}_T^i) \bigg] \,, \tag{11a}$$

$$\text{s.t.}\,,\ \forall s \in [\![t, T-1]\!]\,,$$

$$\boldsymbol{X}_{s+1}^i = g_s^i(\boldsymbol{X}_s^i, \boldsymbol{U}_s^i, \boldsymbol{W}_{s+1}^i)\,, \ \ \boldsymbol{X}_t^i = x_t^i\,, \tag{11b}$$

$$\sigma(\boldsymbol{U}_s^i) \subset \mathcal{G}_s^i\,, \tag{11c}$$

with the convention $\underline{V}_T^i[\boldsymbol{P}^i] = K^i$. We define the *global price value function* at time $t \in [\![0, T]\!]$ as the sum of the corresponding local price value functions:

$$\underline{V}_t[\boldsymbol{P}](x_t) = \sum_{i=1}^N \underline{V}_t^i[\boldsymbol{P}^i](x_t^i)\,. \tag{12}$$

In the same vein, let $\boldsymbol{R}^i = (\boldsymbol{R}_0^i, \cdots, \boldsymbol{R}_{T-1}^i) \in \widetilde{\mathbb{L}}(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{R}^i)$ be a *local resource process*. The *local resource value function*, defined in Equation (3) is written here

$$\overline{V}_0^i[\boldsymbol{R}^i](x_0^i) = \min_{\boldsymbol{X}^i, \boldsymbol{U}^i} \mathbb{E}\bigg[\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}^i) + K^i(\boldsymbol{X}_T^i) \bigg] \,, \tag{13a}$$

$$\text{s.t.}\,,\ \forall t \in [\![0, T-1]\!]\,,$$

$$\boldsymbol{X}_{t+1}^i = g_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}^i)\,, \ \ \boldsymbol{X}_0^i = x_0^i\,, \tag{13b}$$

$$\sigma(\boldsymbol{U}_t^i) \subset \mathcal{G}_t^i\,, \tag{13c}$$

$$\Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = \boldsymbol{R}_t^i\,. \tag{13d}$$

We introduce the *local resource value functions* $\underline{V}_t^i[\boldsymbol{R}^i] : \mathbb{X}_t^i \to (-\infty, +\infty]$ at each time $t \in [\![0, T]\!]$ as, for all $x_t^i \in \mathbb{X}_t^i$,

$$\overline{V}_t^i[\boldsymbol{R}^i](x_t^i) = \min_{\boldsymbol{X}^i, \boldsymbol{U}^i} \mathbb{E}\bigg[\sum_{s=t}^{T-1} L_s^i(\boldsymbol{X}_s^i, \boldsymbol{U}_s^i, \boldsymbol{W}_{s+1}^i) + K^i(\boldsymbol{X}_T^i) \bigg] \,, \tag{14a}$$

$$\text{s.t.}\,,\ \forall s \in [\![t, T-1]\!]\,,$$

$$\boldsymbol{X}_{s+1}^i = g_s^i(\boldsymbol{X}_s^i, \boldsymbol{U}_s^i, \boldsymbol{W}_{s+1}^i)\,, \ \ \boldsymbol{X}_t^i = x_t^i\,, \tag{14b}$$

$$\sigma(\boldsymbol{U}_s^i) \subset \mathcal{G}_s^i\,, \tag{14c}$$

$$\Theta_s^i(\boldsymbol{X}_s^i, \boldsymbol{U}_s^i) = \boldsymbol{R}_s^i\,, \tag{14d}$$

with the convention $\overline{V}^i_T[\boldsymbol{R}^i] = K^i$. We define the *global resource value function* at time $t \in [\![0, T]\!]$ as the sum of the local resource value functions as

$$\overline{V}_t[\boldsymbol{R}](x_t) = \sum_{i=1}^{N} \overline{V}^i_t[\boldsymbol{R}^i](x^i_t) . \tag{15}$$

We call the global processes $\boldsymbol{R} \in \widetilde{\mathbb{L}}(\Omega, \mathcal{F}; \mathbb{P}, \mathcal{R})$ and $\boldsymbol{P} \in \widetilde{\mathbb{L}}^\star(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{P})$ *coordination processes.*

2.2.5 Global upper and lower bounds

Applying Proposition 2.1 to the local price value functions (10) and resource value functions (13) makes it possible to bound the global problem (8). For this purpose, we first define the notion of *admissible* price and resource coordination processes.

We introduce the primal admissible set $\boldsymbol{\mathcal{S}}$ of stochastic processes associated to the almost sure constraints (8d):

$$\boldsymbol{\mathcal{S}} = \big\{ \boldsymbol{Y} = (\boldsymbol{Y}_0, \cdots, \boldsymbol{Y}_{T-1}) \in \widetilde{\mathbb{L}}(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{R}) \\ \text{s.t. } \boldsymbol{Y}_t \in S_t \ \mathbb{P}\text{-a.s.} , \ \forall t \in [\![0, T-1]\!] \big\} . \tag{16a}$$

We also define the dual admissible cone of $\boldsymbol{\mathcal{S}}$ as

$$\boldsymbol{\mathcal{S}}^\star = \{ \boldsymbol{Z} = (\boldsymbol{Z}_0, \cdots, \boldsymbol{Z}_{T-1}) \in \widetilde{\mathbb{L}}^\star(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{P}) \\ \text{s.t. } \mathbb{E}\big[\big\langle \boldsymbol{Y}_t \, , \boldsymbol{Z}_t \big\rangle\big] \geq 0 \ , \ \forall \, \boldsymbol{Y} \in \boldsymbol{\mathcal{S}} \ , \ \forall t \in [\![0, T-1]\!] \} . \tag{16b}$$

We say that $\boldsymbol{P} \in \widetilde{\mathbb{L}}^\star(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{P})$ is an *admissible coordination price process* if $\boldsymbol{P} \in \boldsymbol{\mathcal{S}}^\star$. In a similar manner, we say that $\boldsymbol{R} \in \widetilde{\mathbb{L}}(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{R})$ is an *admissible coordination resource process* if $\boldsymbol{R} \in -\boldsymbol{\mathcal{S}}$.

By considering admissible price and resource coordination processes, we are able to bound up and down Problem (8) and all the global value functions (9) with the local value functions (11) and (14).

Proposition 2.2 *Let $\boldsymbol{P} = (\boldsymbol{P}^1, \ldots, \boldsymbol{P}^N) \in \boldsymbol{\mathcal{S}}^\star$ be any admissible coordination price process and let $\boldsymbol{R} = (\boldsymbol{R}^1, \cdots, \boldsymbol{R}^N) \in -\boldsymbol{\mathcal{S}}$ be any admissible coordination resource process. Then, for any $t \in [\![0, T]\!]$ and for all $x_t = (x^1_t, \cdots, x^N_t) \in \mathbb{X}_t$, we have*

$$\sum_{i=1}^{N} \underline{V}^i_t[\boldsymbol{P}^i](x^i_t) \leq V_t(x_t) \leq \sum_{i=1}^{N} \overline{V}^i_t[\boldsymbol{R}^i](x^i_t) . \tag{17}$$

Proof. For $t = 0$, the proof of the following proposition is a direct application of Proposition 2.1 to Problem (8).

For $t \in [\![1, T-1]\!]$, from the definitions (16) of $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{S}}^\star$, the assumption that $(\boldsymbol{R}_0, \cdots, \boldsymbol{R}_{T-1})$ (resp. $(\boldsymbol{P}_0, \cdots, \boldsymbol{P}_{T-1})$) is an admissible process implies that the reduced process $(\boldsymbol{R}_t, \cdots, \boldsymbol{R}_{T-1})$ (resp. $(\boldsymbol{P}_t, \cdots, \boldsymbol{P}_{T-1})$) is also admissible on the reduced time interval $[\![t, T-1]\!]$, hence the result by applying Proposition 2.1. □

3 Decomposition of local value functions by Dynamic Programming

We have seen in §2.2 that we are able to obtain upper and lower bounds of optimization problems by *spatial* decomposition. We now show that spatial decomposition schemes can be made compatible with time decomposition. We will thus obtain a mix of spatial and time decompositions. In §3.1, we show that the local price value functions (10) and the local resource value functions (13) can be computed by Dynamic Programming if price and resource processes are chosen deterministic. In §3.2, we detail methods to obtain tighter bounds by appropriately choosing the deterministic price and resource processes. In §3.3, we analyze the case of a *decentralized information structure*. Eventually, in §3.4, we show how to use the local price and resource value functions as a surrogate for the global Bellman value functions in order to compute global admissible policies.

In the sequel, we make the following key assumption.

Assumption 1. *The global uncertainty process* $(\boldsymbol{W}_1, \cdots, \boldsymbol{W}_T)$ *in* (6) *consists of stagewise independent random variables.*

Under Assumption 1, and in the case where $\mathcal{G}_t^i = \mathcal{F}_t$ for all t and all i (centralized information structure), Problem (8) can be solved by Dynamic Programming, and the global value functions (9) satisfy the Dynamic Programming equation (see [19] for further details):

$$V_T(x_T) = \sum_{i=1}^{N} K^i(x_T^i) \ , \tag{18a}$$

and, for $t = T-1, \ldots, 0$,

$$V_t(x_t) = \min_{u_t \in \mathbb{U}_t} \mathbb{E}\Big[\sum_{i=1}^{N} L_t^i(x_t^i, u_t^i, \boldsymbol{W}_{t+1}^i) + V_{t+1}\big(\boldsymbol{X}_{t+1}^1, \cdots, \boldsymbol{X}_{t+1}^N\big) \Big] \tag{18b}$$

$$\text{s.t.} \ \ \boldsymbol{X}_{t+1}^i = g_t^i(x_t^i, u_t^i, \boldsymbol{W}_{t+1}^i) \ , \tag{18c}$$

$$\big(\Theta_t^1(x_t^1, u_t^1), \cdots, \Theta_t^N(x_t^N, u_t^N)\big) \in -S_t \ . \tag{18d}$$

In the case where $\mathcal{G}_t^i = \mathcal{F}_t^i$ for all t and all i (decentralized information structure), the common assumptions under which the global value functions (9) satisfy a Dynamic Programming equation are not met.

3.1 Decomposed value functions by means of deterministic coordination processes

We prove now that, if the coordination price process $\boldsymbol{P}$ and the coordination resource process $\boldsymbol{R}$ are deterministic, the local problems (10) and (13) satisfy local Dynamic Programming equations, provided that Assumption 1 holds true. We first study the local price value function (10).

Proposition 3.1 *Let $p^i = (p^i_0, \cdots, p^i_{T-1}) \in \mathcal{P}^i$ be a deterministic price process. Then, be it for the centralized or the decentralized information structure (see §2.2.2), the local price value functions* (11) *satisfy the recursive Dynamic Programming equation:*

$$\underline{V}^i_T[p^i](x^i_T) = K^i(x^i_T) , \tag{19a}$$

and, for $t = T-1, \dots, 0$,

$$\underline{V}^i_t[p^i](x^i_t) = \min_{u^i_t \in \mathbb{U}^i_t} \mathbb{E}\Big[L^i_t(x^i_t, u^i_t, \boldsymbol{W}^i_{t+1}) + \big\langle p^i_t \,, \Theta^i_t(x^i_t, u^i_t) \big\rangle + \underline{V}^i_{t+1}[p^i]\big(g^i_t(x^i_t, u^i_t, \boldsymbol{W}^i_{t+1}) \big) \Big] . \tag{19b}$$

Proof. Let $p^i = (p^i_0, \cdots, p^i_{T-1}) \in \mathcal{P}^i$ be a deterministic price vector. Then, the price value function (10) has the following expression:

$$\underline{V}^i_0[p^i](x^i_0) = \min_{\boldsymbol{X}^i, \boldsymbol{U}^i} \mathbb{E}\bigg[\sum_{t=0}^{T-1} L^i_t(\boldsymbol{X}^i_t, \boldsymbol{U}^i_t, \boldsymbol{W}^i_{t+1}) + \big\langle p^i_t \,, \Theta^i_t(\boldsymbol{X}^i_t, \boldsymbol{U}^i_t) \big\rangle + K^i(\boldsymbol{X}^i_T) \bigg] , \tag{20a}$$

$$\text{s.t.} , \ \forall t \in [\![0, T-1]\!] ,$$

$$\boldsymbol{X}^i_{t+1} = g^i_t(\boldsymbol{X}^i_t, \boldsymbol{U}^i_t, \boldsymbol{W}^i_{t+1}) , \ \ \boldsymbol{X}^i_0 = x^i_0 , \tag{20b}$$

$$\sigma(\boldsymbol{U}^i_t) \subset \mathcal{G}^i_t . \tag{20c}$$

In the case where $\mathcal{G}^i_t = \mathcal{F}_t$, and provided that Assumption 1 holds true, the optimal value of Problem (20) can be obtained by the recursive Dynamic Programming equation (19). Consider now the case $\mathcal{G}^i_t = \mathcal{F}^i_t$. Since the i-th local value function and local dynamics in (20) only depend on the local noise process $\boldsymbol{W}^i$, there is no loss of optimality to replace the constraint $\sigma(\boldsymbol{U}^i_t) \subset \mathcal{F}_t$ by $\sigma(\boldsymbol{U}^i_t) \subset \mathcal{F}^i_t$. Moreover, Assumption 1 implies that the local uncertainty process $(\boldsymbol{W}^i_1, \dots, \boldsymbol{W}^i_T)$ consists of stagewise independent random variables, so that the solution of Problem (20) can be obtained by the recursive Dynamic Programming equation (19). Finally, replacing the *global* σ-field $\mathcal{F}_t$ by the *local* σ-field $\mathcal{F}^i_t$ (see Equation (7)) does not change anything in Problem (20). □

A similar result holds for the local resource value function (13). The proof of the following proposition is left to the reader.

Proposition 3.2 *Let $r^i = (r^i_0, \cdots, r^i_{T-1}) \in \mathcal{R}^i$ be a deterministic resource process. Then, be it for the centralized or the decentralized information structure, the local resource value functions* (14) *satisfy the recursive Dynamic Programming equations:*

$$\overline{V}^i_T[r^i](x^i_T) = K^i(x^i_T) , \tag{21a}$$

and, for $t = T-1, \dots, 0$,

$$\overline{V}^i_t[r^i](x^i_t) = \min_{u^i_t \in \mathbb{U}^i_t} \mathbb{E}\Big[L^i_t(x^i_t, u^i_t, \boldsymbol{W}^i_{t+1}) + \overline{V}^i_{t+1}[r^i]\big(g^i_t(x^i_t, u^i_t, \boldsymbol{W}^i_{t+1})\big)\Big] \ , \quad (21b)$$

$$s.t. \quad \Theta^i_t(x^i_t, u^i_t) = r^i_t \ . \quad (21c)$$

In the context of deterministic *admissible* coordination price and resource processes, the double inequality (17) in Proposition 2.2 becomes

$$\sum_{i=1}^{N} \underline{V}^i_t[p^i](x^i_t) \leq V_t(x_t) \leq \sum_{i=1}^{N} \overline{V}^i_t[r^i](x^i_t) \ . \quad (22)$$

- Both in the lower bound and the upper bound of V_t in (22), the sum over indices i materializes the spatial decomposition for the computation of the bounds. For each of the bound, this decomposition leads to N independent optimization subproblems that can be processed in parallel.
- For a given index i, as stated in Propositions 3.1 and 3.2, the computation of the local value functions $\underline{V}^i_t[p^i]$ and $\overline{V}^i_t[r^i]$ for $t \in [\![0, T]\!]$ can be performed by Dynamic Programming. The corresponding loop in backward time materializes the temporal decomposition, processed sequentially.

Figure 1 illustrates this double decomposition.

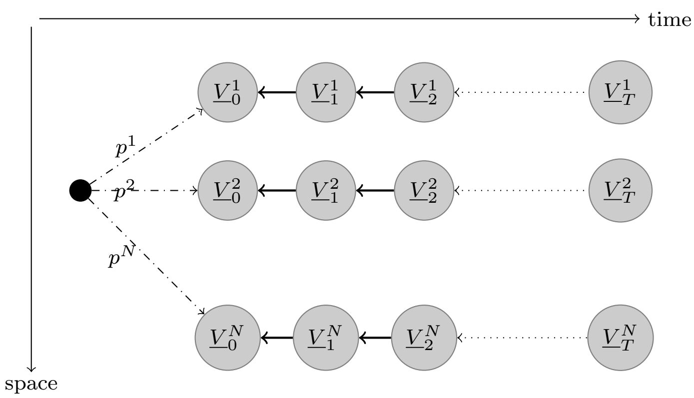

Fig. 1 Mix of spatial and temporal decompositions

Remark 3.1 The results obtained in §3.1 when considering *deterministic* coordination processes p and r can be extended to the case of *Markovian* coordination processes $\boldsymbol{P}$ and $\boldsymbol{R}$. The reader is referred to [21, Chap. 7] for further details. ◊

3.2 Improving bounds

We have seen that given deterministic admissible price and resource coordination processes we are able to obtain upper and lower bounds for the global Bellman functions (9). By choosing the best possible coordination processes, we will obtain tighter bounds in Equation (22). Moreover, that allows to interpret the problems corresponding to the class of deterministic coordination processes in terms of a relaxation (for price) or a restriction (for resource) of the original primal problem (8).

3.2.1 Selecting a deterministic global price process

By Propositions 2.2 and 3.1, for any deterministic $p = (p_0, \cdots, p_{T-1}) \in S^\star$, we have

$$\sum_{i=1}^{N} \underline{V}_0^i[p^i](x_0^i) \ \leq\ V_0(x_0) \ .$$

As a consequence, solving the following optimization problem

$$\sup_{p\in S^\star} \min_{\boldsymbol{X},\boldsymbol{U}} \mathbb{E}\bigg[\sum_{i=1}^{N}\sum_{t=0}^{T-1}\Big(L_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_{t+1}^i) + \big\langle p_t^i\,, \Theta_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i)\big\rangle\Big) + K^i(\boldsymbol{X}_T^i)\bigg] \ , \tag{23a}$$

$$\text{s.t.}\ ,\ \forall t\in [\![0,T-1]\!]\ ,$$

$$\boldsymbol{X}_{t+1}^i = g_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_{t+1}^i)\ ,\ \ \boldsymbol{X}_0^i = x_0^i\ , \tag{23b}$$

$$\sigma(\boldsymbol{U}_t^i) \subset \mathcal{G}_t^i\ , \tag{23c}$$

gives the greatest possible lower bound in the class of deterministic price coordination processes. Since $\mathbb{E}\big[\big\langle p_t^i\,, \Theta_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i)\big\rangle\big] = \big\langle p_t^i, \mathbb{E}\big[\Theta_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i)\big]\big\rangle$ for all $t \in [\![0, T-1]\!]$, Problem (23) can be interpreted as the dual problem of

$$\min_{\boldsymbol{X},\boldsymbol{U}} \mathbb{E}\bigg[\sum_{i=1}^{N}\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_{t+1}^i) + K^i(\boldsymbol{X}_T^i)\bigg]\ , \tag{24a}$$

$$\text{s.t.}\ ,\ \forall t\in [\![0,T-1]\!]\ ,$$

$$\boldsymbol{X}_{t+1}^i = g_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_{t+1}^i)\ ,\ \ \boldsymbol{X}_0^i = x_0^i\ , \tag{24b}$$

$$\sigma(\boldsymbol{U}_t^i) \subset \mathcal{G}_t^i\ , \tag{24c}$$

$$\Big(\mathbb{E}\big[\Theta_t^1(\boldsymbol{X}_t^1,\boldsymbol{U}_t^1)\big], \cdots, \mathbb{E}\big[\Theta_t^N(\boldsymbol{X}_t^N,\boldsymbol{U}_t^N)\big]\Big) \in -S_t\ . \tag{24d}$$

Problem (24) is a relaxation of Problem (8), in the sense that the almost sure constraint (8d) is replaced by the constraint in expectation (24d). We refer to [20, Chapter 8] for considerations on the duality between Problem (24) and Problem (23).

3.2.2 Selecting a deterministic global resource process

By Propositions 2.2 and 3.2, for any deterministic $r = (r_0, \cdots, r_{T-1}) \in -S$, we have

$$V_0(x_0^1, \cdots, x_0^N) \;\le\; \sum_{i=1}^{N} \overline{V}_0^i[r^i](x_0^i) \; .$$

Solving the following optimization problem

$$\inf_{r \in -S} \min_{\boldsymbol{X}, \boldsymbol{U}} \mathbb{E}\bigg[\sum_{i=1}^{N} \sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}^i) + K^i(\boldsymbol{X}_T^i) \bigg] \; , \tag{25a}$$

$$\text{s.t.}\,,\ \forall t \in [\![0, T-1]\!] \; ,$$

$$\boldsymbol{X}_{t+1}^i = g_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i, \boldsymbol{W}_{t+1}^i) \; , \;\; \boldsymbol{X}_0^i = x_0^i \; , \tag{25b}$$

$$\sigma(\boldsymbol{U}_t^i) \subset \mathcal{F}_t \; , \tag{25c}$$

$$\Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = r_t^i \; , \tag{25d}$$

thus gives the lowest possible upper bound in the set of deterministic resource coordination processes. Furthermore, imposing by Constraint (25d) that each term $\Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i)$ has to be deterministic amounts to hardening the constraints of Problem (8), so that Problem (25) is a restriction of Problem (8).

3.3 Analysis of the decentralized information structure

An interesting consequence of Propositions 3.1 and 3.2 is that the local price and resource value functions $\underline{V}_t^i[p^i]$ and $\overline{V}_t^i[r^i]$ remain the same when choosing either the centralized information structure or the decentralized one. In contrast, the global value functions V_t depend on that choice. Let us denote by V_t^{C} (resp. V_t^{D}) the value functions (9) in the centralized (resp. decentralized) case. Since the admissible set induced by Constraint (9c) in the centralized case is larger than the one in the decentralized case (because $\mathcal{F}_t^i \subset \mathcal{F}_t$ by (7)), we deduce that the lower bound is tighter for the centralized problem, and the upper bound tighter for the decentralized problem: for all $x_t = (x_t^1, \cdots, x_t^N) \in \mathbb{X}_t$,

$$\sum_{i=1}^{N} \underline{V}_t^i[p^i](x_t^i) \le V_t^{\mathrm{C}}(x_t) \le V_t^{\mathrm{D}}(x_t) \le \sum_{i=1}^{N} \overline{V}_t^i[r^i](x_t^i) \; . \tag{26}$$

In some specific cases (but often encountered in practical applications), we can even show that the best upper bound in (26) is equal to the optimal value $V_t^{\mathrm{D}}(x_t)$ of the decentralized problem. We have the following proposition.

Proposition 3.3 *Assume that, for all $t \in [\![0, T-1]\!]$, the global coupling constraint $\big(\Theta_t^1(\boldsymbol{X}_t^1, \boldsymbol{U}_t^1), \ldots, \Theta_t^N(\boldsymbol{X}_t^N, \boldsymbol{U}_t^N)\big) \in -S_t$ is equivalent to*

$$\exists (r_t^1, \ldots, r_t^N) \in -S_t \; , \;\; \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = r_t^i \quad \forall i \in [\![1, N]\!] \; . \tag{27}$$

Then,

$$V_0^{\mathrm{D}}(x_0) = \inf_{(r^1,\cdots,r^N)\in -S} \sum_{i=1}^{N} \overline{V}_0^i[r^i](x_0^i) \ . \tag{28}$$

Proof. Using Assumption (27), Problem (8) can be written as

$$\begin{aligned}
V_0^{\mathrm{D}}(x_0) = \inf_{(r^1,\cdots,r^N)\in -S} & \left(\sum_{i=1}^{N} \min_{\boldsymbol{X}^i,\boldsymbol{U}^i} \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_{t+1}^i) + K^i(\boldsymbol{X}_T^i) \Big] \right) , \\
\text{s.t.} & \ , \ \forall t \in [\![0, T-1]\!] \ , \\
& \boldsymbol{X}_{t+1}^i = g_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i,\boldsymbol{W}_{t+1}^i) \ , \ \ \boldsymbol{X}_0^i = x_0^i \ , \\
& \sigma(\boldsymbol{U}_t^i) \subset \mathcal{F}_t^i \ , \\
& \Theta_t^i(\boldsymbol{X}_t^i,\boldsymbol{U}_t^i) = r_t^i \ , \\
= \inf_{(r^1,\cdots,r^N)\in -S} & \sum_{i=1}^{N} \overline{V}_0^i[r^i](x_0^i) \ ,
\end{aligned}$$

the last equality arising from the definition of $\overline{V}_0^i[r^i]$ in (13). □

As an application of the previous proposition, we consider the case of a decentralized information structure, and we assume moreover that

- the global uncertainty process $(\boldsymbol{W}^1, \cdots, \boldsymbol{W}^N)$ consists of independent random processes, that is, the processes $\boldsymbol{W}^1, \cdots, \boldsymbol{W}^N$ are (spatially) independent to each others.
- the coupling constraints (8d) are of the form $\sum_{i=1}^N \Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) = 0$.

Note that we add here an *independence assumption in space*, whereas Assumption 1 was an *independence assumption in time*.

A well-known result is that, if a sum of independent random variables is zero, then every random variable in the sum is constant (deterministic). From the dynamic constraint (8b) and from the measurability constraint (8c), we conclude that each term $\Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i)$ is $\mathcal{F}_t^i$-measurable in the decentralized information structure case. From the space independence assumption, the random variables $\Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i)$ are independent, and hence *constant*. By introducing new variables $(r_t^1, \ldots, r_t^N)$, Constraints (8d) is written equivalently $\Theta_t^i(\boldsymbol{X}_t^i, \boldsymbol{U}_t^i) - r_t^i = 0 \ \forall\, i \in [\![1, N]\!]$ and $\sum_{i=1}^N r_t^i = 0$. From Proposition 3.3, we deduce that the optimal value of the problem with a decentralized information structure is, in that case, equal to the upper bound obtained by resource decomposition provided that the deterministic resource process $(r^1, \cdots, r^N)$ is chosen at best:

$$V_0^{\mathrm{D}}(x_0) = \inf_{(r^1,\cdots,r^N)\in -S} \sum_{i=1}^{N} \overline{V}_0^i[r^i](x_0^i) \ .$$

In this specific example, resource decomposition allows to compute the optimal value of the global problem (8) when using a decentralized information structure.

3.4 Producing admissible policies

We again restrict ourselves to deterministic coordination processes. We moreover assume that the information structure corresponds to the centralized one ($\mathcal{G}_t^i = \mathcal{F}_t$ in (7)), in order to be able to compute Dynamic Programming based policies (see however Remark 3.2 below for the case of a decentralized information structure).

Here we suppose that we have at our disposal pre-computed *local* value functions $\{\underline{V}_t^i\}_{t\in[\![0,T]\!]}$ and $\{\overline{V}_t^i\}_{t\in[\![0,T]\!]}$ solving Equations (19) for the price value functions and Equations (21) for the resource value functions. Using the sum of these local value functions as a surrogate for a global Bellman value function, we can compute *global* admissible policies. More precisely, two admissible policies are built as follows:
1) a *global price policy* $\underline{\gamma} = (\underline{\gamma}_0, \cdots, \underline{\gamma}_{T-1})$, with, for any $t = 0, \dots, T-1$, $\underline{\gamma}_t : \mathbb{X}_t \to \mathbb{U}_t$ defined for all $x_t = (x_t^1, \cdots, x_t^N) \in \mathbb{X}_t$ by

$$\underline{\gamma}_t(x_t) \in \operatorname*{arg\,min}_{u_t^1,\cdots,u_t^N} \mathbb{E}\bigg[\sum_{i=1}^N L_t^i(x_t^i, u_t^i, \boldsymbol{W}_{t+1}^i) + \underline{V}_{t+1}^i\big(g_t^i(x_t^i, u_t^i, \boldsymbol{W}_{t+1}^i)\big)\bigg] \ , \tag{29a}$$

$$\text{s.t.} \ \ \big(\Theta_t^1(x_t^1, u_t^1), \cdots, \Theta_t^N(x_t^N, u_t^N)\big) \in -S_t \ , \tag{29b}$$

2) a *global resource policy* $\overline{\gamma} = (\overline{\gamma}_0, \cdots, \overline{\gamma}_{T-1})$, with, for any $t = 0, \dots, T-1$, $\overline{\gamma}_t : \mathbb{X}_t \to \mathbb{U}_t$ defined for all $x_t = (x_t^1, \cdots, x_t^N) \in \mathbb{X}_t$ by

$$\overline{\gamma}_t(x_t) \in \operatorname*{arg\,min}_{u_t^1,\cdots,u_t^N} \mathbb{E}\bigg[\sum_{i=1}^N L_t^i(x_t^i, u_t^i, \boldsymbol{W}_{t+1}^i) + \overline{V}_{t+1}^i\big(g_t^i(x_t^i, u_t^i, \boldsymbol{W}_{t+1}^i)\big)\bigg] \ , \tag{30a}$$

$$\text{s.t.} \ \ \big(\Theta_t^1(x_t^1, u_t^1), \cdots, \Theta_t^N(x_t^N, u_t^N)\big) \in -S_t \ . \tag{30b}$$

Given a policy $\gamma = (\gamma_0, \cdots, \gamma_{T-1})$, and any time $t \in [\![0, T]\!]$, the expected cost of policy γ starting from state x_t at time t is equal to

$$V_t^\gamma(x_t) = \mathbb{E}\bigg[\sum_{i=1}^N \sum_{s=t}^{T-1} L_s^i(\boldsymbol{X}_s^i, \gamma_s^i(\boldsymbol{X}_s), \boldsymbol{W}_{t+1}^i) + K^i(\boldsymbol{X}_T^i)\bigg] \ , \tag{31a}$$

$$\text{s.t.} \ \ , \ \forall s \in [\![t, T-1]\!] \ , \quad \boldsymbol{X}_{s+1}^i = g_s^i(\boldsymbol{X}_s^i, \gamma_s^i(\boldsymbol{X}_s), \boldsymbol{W}_{s+1}^i) \ , \ \ \boldsymbol{X}_t^i = x_t^i \ . \tag{31b}$$

We prove hereafter that we are able to bound the performance of the global resource policy defined in (30).

Proposition 3.4 *Let $t \in [\![0, T]\!]$ and $x_t = (x_t^1, \cdots, x_t^N) \in \mathbb{X}_t$ be a given state. Then, we have the following upper bound on the expected value of the global resource policy* (30)

$$V_t^{\overline{\gamma}}(x_t^1, \cdots, x_t^N) \leq \sum_{i=1}^N \overline{V}_t^i(x_t^i) \ . \tag{32}$$

Proof. We prove the result by backward induction. At time $t = T$, the result is straightforward as $\overline{V}^i_t = K^i$ for all $i \in [\![1, N]\!]$.

Let $t \in [\![0, T-1]\!]$ such that (32) holds true at time $t+1$. Then, for all $x_t \in \mathbb{X}_t$,

$$V_t^{\overline{\gamma}}(x_t) = \mathbb{E}\bigg[\sum_{i=1}^N \big(L^i_t(x^i_t, \overline{\gamma}^i_t(x_t), \boldsymbol{W}^i_{t+1})\big) + V^{\overline{\gamma}}_{t+1}(\boldsymbol{X}_{t+1})\bigg] ,$$

since Dynamic Programming applies to (31). Now, using the induction assumption, we deduce that

$$V_t^{\overline{\gamma}}(x_t) \leq \mathbb{E}\bigg[\sum_{i=1}^N L^i_t(x^i_t, \overline{\gamma}^i_t(x_t), \boldsymbol{W}^i_{t+1}) + \overline{V}^i_{t+1}(\boldsymbol{X}^i_{t+1})\bigg] .$$

From the very definition (30) of the global resource policy, $\overline{\gamma}$, we obtain

$$\begin{aligned} V_t^{\overline{\gamma}}(x_t) \leq \min_{u^1_t, \cdots, u^N_t} \; & \mathbb{E}\bigg[\sum_{i=1}^N L^i_t(x^i_t, u^i_t, \boldsymbol{W}^i_{t+1}) + \overline{V}^i_{t+1}(\boldsymbol{X}^i_{t+1})\bigg] , \\ & \text{s.t.} \;\; \big(\Theta^1_t(x^1_t, u^1_t), \cdots, \Theta^N_t(x^N_t, u^N_t)\big) \in -S_t \; . \end{aligned}$$

Introducing a deterministic admissible resource process $(r^1_t, \cdots, r^N_t) \in -S_t$ and restraining the constraint to it reinforces the inequality

$$V_t^{\overline{\gamma}}(x_t) \leq \min_{u^1_t, \cdots, u^N_t} \; \mathbb{E}\bigg[\sum_{i=1}^N L^i_t(x^i_t, u^i_t, \boldsymbol{W}^i_{t+1}) + \overline{V}^i_{t+1}(\boldsymbol{X}^i_{t+1})\bigg] \tag{33a}$$

$$\text{s.t.} \;\; \Theta^1_t(x^1_t, u^1_t) = r^1_t \; , \;\; \cdots \; , \;\; \Theta^N_t(x^N_t, u^N_t) = r^N_t \; , \tag{33b}$$

so that

$$V_t^{\overline{\gamma}}(x_t) \leq \sum_{i=1}^N \bigg(\min_{u^i_t} \mathbb{E}\big[L^i_t(x^i_t, u^i_t, \boldsymbol{W}^i_{t+1}) + \overline{V}^i_{t+1}(\boldsymbol{X}^i_{t+1})\big] \;\; \text{s.t.} \;\; \Theta^i_t(x^i_t, u^i_t) = r^i_t \bigg) ,$$

as we do not have any coupling left in (33). By Equation (21), we deduce that

$$V_t^{\overline{\gamma}}(x_t) \leq \sum_{i=1}^N \overline{V}^i_t(x^i_t) ,$$

hence the result at time t. □

Furthermore, for any admissible policy γ, we have $V_t(x_t) \leq V^\gamma_t(x_t)$ as the global Bellman function gives the minimal cost starting at any point $x_t \in \mathbb{X}_t$. We therefore obtain the following bounds

$$\sum_{i=1}^N \underline{V}^i_t(x^i_t) \leq V_t(x_t) \leq V_t^{\overline{\gamma}}(x_t) \leq \sum_{i=1}^N \overline{V}^i_t(x^i_t) , \tag{34a}$$

and

$$V_t(x_t) \leq \min\big\{V_t^{\underline{\gamma}}(x_t), V_t^{\overline{\gamma}}(x_t)\big\} . \tag{34b}$$

Remark 3.2 In the case of a decentralized information structure (7b), it seems difficult to produce Bellman-based online policies. Indeed, the *global* Dynamic Programming principle does not apply: neither the global price policy in (29) nor the global resource policy in (30) are implementable since both policies require the knowledge of the global state $(x_t^1, \cdots, x_t^N)$ for each unit i, which is incompatible with the information constraint (7b). Nevertheless, one can use the results given by resource decomposition to compute an online policy. Knowing a deterministic admissible resource process $r \in -S$, solving at time t and for each $i \in [\![1, N]\!]$ the subproblem

$$\begin{aligned}\overline{\gamma}_t^i(x_t^i) \in \arg\min_{u_t^i} \ & \mathbb{E}\Big[L_t^i(x_t^i, u_t^i, \boldsymbol{W}_{t+1}^i) + \overline{V}_{t+1}^i\big(\boldsymbol{X}_{t+1}^i\big)\Big] \ , \\ \text{s.t.} \ & \boldsymbol{X}_{t+1}^i = g_t^i(x_t^i, u_t^i, \boldsymbol{W}_{t+1}^i) \ , \\ & \Theta_t^i(x_t^i, u_t^i) = r_t^i \ ,\end{aligned}$$

generates a global admissible policy $\big(\overline{\gamma}_t^1(x_t^1), \ldots, \overline{\gamma}_t^N(x_t^N)\big)$ compatible with the decentralized information structure (7b). ♢

4 Conclusion

In this article, we have presented a formalism for joint temporal and spatial decomposition. More precisely, we have decomposed multistage stochastic optimization problems — made of interconnected subsystems — first by prices and by resources, and then by Dynamic Programming. We have proved that, under proper assumptions, we could bound the Bellman value functions of the original problem up and down by summing decomposed local Bellman functions, for all time (Sect. 2). Moreover, we have obtained tighter bounds by optimizing the choice of deterministic price and resource processes. As we consider multistage stochastic problems, we have stressed the key role played by information structures in the design of the decomposition schemes.

An application of this methodology is presented in the companion paper [17]. We use the decomposition algorithms on a specific case study, the management of a district microgrid with different prosumers located at the nodes of the grid and exchanging energy altogether. Numerical results show the effectiveness of the approach: the decomposition algorithms beat the reference Stochastic Dual Dynamic Programming (SDDP) for large-scale problems with more than 12 nodes (more than 16 state variables); on problems with up to 48 nodes (up to 64 state variables), we observe that their performance scale well as the number of nodes grows.

A natural extension is the following. In this paper, we have only considered deterministic price and resource coordination processes. Using more complex stochastic processes such as *Markovian* coordination processes would make it possible to improve the performance of the algorithms (see Remark 3.1). However, one would need to analyze how to obtain a good trade-off between accuracy and numerical performance.

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