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On set-valued perturbation stability of metric regularity*

Yiran He[1] · **Wending Xu**[1,2]

Abstract Two stability results of the metric regularity under set-valued perturbation are established. As the perturbation is of addition type, the first result assumes that the graph of the sum mapping is locally closed, the perturbation mapping is locally Lipschitz in the sense of Hausdorff metric with the diameter of its image at the reference point being bounded above by a given scalar, while the second result replaces the local closedness of the graph of the sum mapping by both mappings being of closed graph. Compared to a known result, we allow the diameter of the image of the perturbation mapping at the reference point to vary in a larger area and we apply the Ekeland variational principle only once while the known result using twice. We present an example to which our results can be applied but not the known results.

Keywords Metric regularity · Perturbation stability · Diameter · Ekeland's variational principle

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Yiran He
E-mail: yrhe@sicnu.edu.cn

✉ Wending Xu
E-mail: wd-xu@hotmail.com

1. Department of Mathematics, Sichuan Normal University, Chengdu, Sichuan, China

2. Department of Tourism and Culture Industry, Sichuan Tourism University, Chengdu, Sichuan, China

1 Introduction

Let X, Y be Banach spaces with norms denoted by $\|\cdot\|$, $d(x, A) := \inf\{\|x - y\| \ |y \in A\}$ be the distance from a point x towards a set A and $\operatorname{diam} A := \sup\{\|x - x'\| \mid x, x' \in A\}$ be the diameter of a set A. For two sets C and D in a same space, $e(C, D) := \sup\{d(x, D)|x \in C\}$ represents the excess from C towards D. For a set-valued mapping $F : X \rightrightarrows Y$, we use the notations $\operatorname{gph} F := \{(x, y)|y \in F(x)\}$ and $\operatorname{dom} F := \{x \in X|F(x) \neq \emptyset\}$ for the graph and the domain of F, respectively. The inverse of F, denoted by F^{-1}, is defined by $F^{-1}(y) := \{x \in X|y \in F(x)\}$. The closed ball with center x and radius $r > 0$ is denoted by $B(x, r)$.

A set C is said to be locally closed at $x \in C$ if there exists a neighborhood U of x such that $C \cap U$ is closed.

It is a classical result that, for a continuous linear mapping $A : X \to Y$, the following properties are equivalent:

(a) A is open;
(b) A is surjective;
(c) There exists $c > 0$ such that $d(x, A^{-1}(y)) \leq c\|y - Ax\|$ for any $(x, y) \in X \times Y$.

Property (c) is called the metric regularity of A on $X \times Y$. In general, for a set-valued mapping $F : X \rightrightarrows Y$ and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, we say that F is **metrically regular** at $\bar{x}$ for $\bar{y}$ with constant c and neighborhood $U \times V$ if there exists $c > 0$ and neighborhood U of $\bar{x}$ and V of $\bar{y}$ such that

$$d(x, F^{-1}(y)) \leq cd(y, F(x)), \text{ for all } (x, y) \in U \times V. \tag{1}$$

The following proposition gives some equivalent descriptions for the metric regularity (see [9] and [10]).

Proposition 1 *For a set-valued mapping $F : X \rightrightarrows Y$ and constants $c, r > 0$, the following properties are equivalent:*

(i) F is metrically regular at $\bar{x}$ for $\bar{y}$ with constant c and neighborhood $U \times V$;
(ii) There exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that (1) holds for any $(x, y) \in U \times V$ with $d(y, F(x)) < r$;
(iii) For any $\varepsilon > 0$, the mapping $F_0 : \operatorname{gph} F \to Y$ defined by $F_0(x, y) = y$ is metrically regular at $(\bar{x}, \bar{y})$ for $\bar{y}$ with constant $c + \varepsilon$ on

$$W = \{((x', v'), y') \mid v' \in F(x'), (x', y') \in U \times V\},$$

where d_ε is the $\varepsilon-$metric on $X \times Y$ defined by

$$d_\varepsilon((x, y), (x', y')) := \|x - x'\| + \varepsilon\|y - y'\|. \tag{2}$$

Based on a result given by Graves in [8], Dontchev observed in [3] that a nonlinear single-valued mapping $f : X \to Y$ is metrically regular at $\bar{x}$ for $f(\bar{x})$ if it is a small perturbation of a linear mapping A which is metrically regular at $\bar{x}$ for $A(\bar{x})$. The small perturbation is in the sense that $f - A$ is locally Lipschitz continuous around $\bar{x}$, that is, there exists $0 < k < c^{-1}$ (where c is the regularity constant of A at $\bar{x}$ for $A(\bar{x})$) and $\delta > 0$ such that for any $x', x \in B(\bar{x}, \delta)$, we have

$$\|(f-A)(x') - (f-A)(x)\| \leq k\|x' - x\|.$$

With some slight updates, the result can be stated as follows:

Theorem 1 *Let $f : X \to Y$ be a single-valued mapping, $A : X \to Y$ be a continuous linear mapping, and $\bar{x} \in dom f$ be a point. Let positive constants c, k with $ck < 1$ be such that*

(i) A is metrically regular at $\bar{x}$ for $A(\bar{x})$ with constant c;
(ii) $f - A$ is locally Lipschitz continuous around $\bar{x}$ with constant k.

Then f is metrically regular at $\bar{x}$ for $f(\bar{x})$ with constant $\frac{c}{1-ck}$.

Theorem 1 can be viewed as a stability result of metric regularity for linear mappings with nonlinear single-valued perturbations. If f is strictly differentiable at $\bar{x}$, then by taking $A := f'(\bar{x})$ in Theorem 1, one can obtain the following implication.

$$f'(\bar{x}) \text{ is surjective} \quad \Rightarrow \quad f \text{ is metrically regular at } \bar{x} \text{ for } f(\bar{x}).$$

Actually, the inverse of the implication above is also true and can be derived easily by the following result, named Extended Lyusternik-Graves Theorem in [9], which generalizes Theorem 1 to set-valued mappings with nonlinear single-valued perturbations (see [4],[5] and [9]).

Theorem 2 *(Extended Lyusternik-Graves Theorem) Consider a set-valued mapping $F : X \rightrightarrows Y$ and a point $(\bar{x}, \bar{y})$ at which $\mathrm{gph}\, F$ is locally closed. Let $g : X \to Y$ be a single-valued mapping. Assume that a, b, c, k be positive constants with $ck < 1$ such that*

$$d(x, F^{-1}(y)) \leq cd(y, F(x)), \text{ for all } (x, y) \in B(\bar{x}, a) \times B(\bar{y}, b)$$

and

$$\|g(x') - g(x)\| \leq k\|x' - x\|, \text{ for all } x', x \in B(\bar{x}, a).$$

Then the mapping $H := F + g$ is metrically regular at $\bar{x}$ for $\bar{y} + g(\bar{x})$ with constant $\frac{c}{1-ck}$.

In Theorem 2, the perturbation mapping is of single-valued. Stability results of metric regularity under single-valued perturbations can be seen in [2–6, 9] and references therein. When the perturbation becomes a set-valued mapping, a counter-example (see Example 5I.1 in [5]) shows that the metric regularity fails to be preserved. However, by adding a restriction on the diameter of the image of the perturbation mapping at the reference point $\bar{x}$, [1] proved the following result, which to the best of our knowledge, is the first stability result of the metric regularity under set-valued perturbations of addition type.

Theorem 3 *(Theorem 3.2 in [1]) Let $F, G : X \rightrightarrows Y$ be two set-valued mappings with closed graph, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and $(\bar{x}, \bar{z}) \in \operatorname{gph} G$ be two points. Suppose that a, b, c, k be positive constants with $ck < 1$ such that*

(i)

$$d(x, F^{-1}(y)) \leq cd(y, F(x)), \text{ for all } (x, y) \in B(\bar{x}, a) \times B(\bar{y}, b).$$

(ii)

$$e(G(x), G(x')) \leq k\|x - x'\|, \text{ for all } x', x \in B(\bar{x}, a). \tag{3}$$

If

$$\operatorname{diam} G(\bar{x}) < \frac{1}{2} \min\{Ma, b\}, \tag{4}$$

where $M := \max\{1, \frac{1}{c}\}$, then the set-valued mapping $F+G$ is metrically regular at $\bar{x}$ for $\bar{y} + \bar{z}$ with constant $\frac{c}{1-ck}$.

In the aspect of the proof method, when the perturbation mappings are of single-valued, the stability result can be proved through different arguments such as the Picard type iteration, the generalized Banach fixed point theorem, and the Ekeland variational principle. When the perturbation mappings are of set-valued, it seems that the Ekeland variational principle is the best way to obtain the perturbation stability results of metric regularity. Indeed, Theorem 3 is proved by the Ekeland variational principle, however, the variational principle was actually applied twice in the proof. In this paper, we use the the variational principle only once and prove a set-valued perturbation stability result for metric regularity. Besides using a new argument, the restriction (4) is relaxed in our results. An example is given in which our result can be applied while Theorem 3 fails.

2 Stability of local metric regularity under set-valued perturbations

In this section, we present the main results of this paper. Let $F, G : X \rightrightarrows Y$ be two set-valued mappings, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and $(\bar{x}, \bar{z}) \in \operatorname{gph} G$ be two points. Set $H := F + G$. In the first result, we assume that $\operatorname{gph} H$ is locally closed, while in the second result, we replace the local closedness of $\operatorname{gph} H$ by assuming that both $\operatorname{gph} F$ and $\operatorname{gph} G$ are closed.

Theorem 4 *Assume that $\operatorname{gph} H$ is locally closed at $(\bar{x}, \bar{y} + \bar{z})$. Suppose that a, b, c and k are positive constants with $kc < 1$ such that*

(i)

$$d(x, F^{-1}(y)) \leq cd(y, F(x)), \text{ for all } (x, y) \in B(\bar{x}, a) \times B(\bar{y}, b). \tag{5}$$

(ii)

$$e(G(x), G(x')) \leq k\|x - x'\|, \text{ for all } x, x' \in B(\bar{x}, a). \tag{6}$$

If

$$\operatorname{diam} G(\bar{x}) < b, \tag{7}$$

then the mapping H is metrically regular at $\bar{x}$ for $\bar{y}+\bar{z}$ with constant $\tau := \frac{c}{1-kc}$.

Proof Take $r > 0$ such that

$$r < \min\{\frac{a}{c+2\tau+2}, \frac{a}{\tau(c+1)k}, \frac{b-\operatorname{diam} G(\bar{x})}{(2\tau+1)k+1}\}. \tag{8}$$

From Proposition 1, it suffices to prove that for any $(x,y) \in B(\bar{x},r)\times B(\bar{y}+\bar{z},r)$ with $d(y,H(x)) < r$, we have

$$d(x,H^{-1}(y)) \le \tau(d(y,H(x))). \tag{9}$$

Fix any $(x,y) \in B(\bar{x},r) \times B(\bar{y}+\bar{z},r)$ with $d(y,H(x)) < r$. Pick a sufficiently small $\varepsilon \in (0,1)$ such that $d(y,H(x)) + \varepsilon < r$. Then we can find $z \in H(x)$ satisfying

$$\|y-z\| < d(y,H(x)) + \varepsilon < r. \tag{10}$$

Define a norm on $X \times Y$ by

$$\|(x_1,x_2),(y_1,y_2)\|_1 := \max\{\|x_1-x_2\|, \tau_1\|y_1-y_2\|\}, \tag{11}$$

where $\tau_1 = \frac{\tau(1-kc)}{1+kc}$. Set

$$E := \{(x',y')|(x',y') \in \operatorname{gph} H - (0,y), \|(x',y'),(\bar{x},0)\|_1 \le a\}. \tag{12}$$

Since $\operatorname{gph} H$ is locally closed at $(\bar{x},\bar{y}+\bar{z})$, we can without loss of generality assume that

$$(\operatorname{gph} H - (0,\bar{y}+\bar{z})) \cap \{(x',y') \mid \|(x',y'),(\bar{x},0)\|_1 \le a\}$$

is closed. Since y can be chosen sufficiently close to $\bar{y}+\bar{z}$, we know that

$$(\operatorname{gph} H - (0,y)) \cap \{(x',y') \mid \|(x',y'),(\bar{x},0)\|_1 \le a\}$$

is closed and hence E is a complete metric space. Then by (10) we have $(x,z-y) \in E$. Applying Ekeland's variational principle (see [5], [7]) to the function

$$(x',y') \mapsto \|y'\|$$

over E with $(x,z-y)$ as the reference point, one has for any $\lambda \in (\frac{1}{2\tau}, \frac{1}{\tau})$, there exists $(p,q) \in E$ such that

$$\|q\| + \lambda\|(x,z-y),(p,q)\|_1 \le \|z-y\|, \tag{13}$$

and

$$\|q\| \le \lambda\|(p,q),(u,v)\|_1 + \|v\|, \quad \text{for all } (u,v) \in E. \tag{14}$$

From the definition of E and the fact that $(p,q) \in E$, we know that

$$y+q \in H(p). \tag{15}$$

Now we prove that $q = 0$ for any $\lambda \in (\frac{1}{2\tau}, \frac{1}{\tau})$. If we are done, then (15) implies $p \in H^{-1}(y)$ and hence from (10) and (13) we have

$$d(x, H^{-1}(y)) \leq \|x - p\| \leq \frac{1}{\lambda}\|z - y\| \leq \frac{1}{\lambda}(d(y, H(x)) + \varepsilon). \tag{16}$$

Letting $\lambda \to \tau^{-1}$ and $\varepsilon \to 0^+$ in (16), we obtain (9).

Assume on the contrary that $q \neq 0$. From (10) and (13) we know

$$0 < \|q\| \leq \|z - y\| < r. \tag{17}$$

Since $y + q \in H(p) = F(p) + G(p)$, we can find $q_1 \in G(p)$ such that

$$q + y - q_1 \in F(p). \tag{18}$$

It follows from (8), (10) and (13) that

$$\|p - \bar{x}\| \leq \|p - x\| + \|x - \bar{x}\| \leq (\frac{1}{\lambda} + 1)r < (2\tau + 1)r < a, \tag{19}$$

and hence, by (6), we have

$$d(q_1, G(\bar{x})) \leq e(G(p), G(\bar{x})) \leq k\|p - \bar{x}\| < (2\tau + 1)kr. \tag{20}$$

Thus, there exists $q_2 \in G(\bar{x})$ such that $\|q_1 - q_2\| \leq (2\tau + 1)kr$. From (8), we have

$$\begin{aligned}\|y - q_1 - \bar{y}\| &\leq \|y - \bar{z} - \bar{y}\| + \|q_1 - q_2\| + \|q_2 - \bar{z}\| \\ &< [(2\tau + 1)k + 1]r + \mathrm{diam}(G(\bar{x})) \leq b.\end{aligned}$$

It follows from (5) and (18) that

$$\begin{aligned}d(p, F^{-1}(y - q_1)) &\leq cd(y - q_1, F(p)) < cd(y - q_1, F(p)) + \varepsilon\|q\| \\ &\leq c\|q\| + \varepsilon\|q\| = (c + \varepsilon)\|q\| < (c + \varepsilon)r.\end{aligned}$$

Thus, there exists $u' \in F^{-1}(y - q_1)$ such that

$$\|p - u'\| < (c + \varepsilon)\|q\| < (c + \varepsilon)r.$$

Note that

$$\begin{aligned}\|u' - \bar{x}\| &\leq \|u' - p\| + \|p - \bar{x}\| \\ &\leq (c + \varepsilon + 2\tau + 1)r < (c + 2\tau + 2)r \leq a,\end{aligned}$$

where the second inequality follows from (19). Hence, it follows from (6) and (17) that

$$d(q_1, G(u')) \leq e(G(p), G(u')) \leq k\|p - u'\| < (c + \varepsilon)k\|q\| < (c + \varepsilon)kr.$$

Thus we can find $q_3 \in G(u')$ which satisfies

$$\|q_1 - q_3\| < (c + \varepsilon)k\|q\| \leq (c + \varepsilon)kr.$$

Observing that $\tau_1 < \tau$ and $\varepsilon < 1$, we have

$$\tau_1\|q_3 - q_1\| \leq \tau_1(c+\varepsilon)kr < \tau(c+1)kr \leq a,$$

and hence $(u', q_3 - q_1) \in E$.
Substituting (u, v) by $(u', q_3 - q_1)$ in (14), we obtain

$$\begin{aligned}\|q\| &\leq \lambda \max\{\|u' - p\|, \tau_1\|q_3 - q_1 - q\|\} + \|q_3 - q_1\| \\ &\leq \lambda \max\{\|u' - p\|, \tau_1[\|q_3 - q_1\| + \|q\|]\} + \|q_3 - q_1\| \\ &\leq \lambda \max\{(c+\varepsilon), \tau_1((c+\varepsilon)k + 1)\}\|q\| + (c+\varepsilon)k\|q\|. \qquad (21)\end{aligned}$$

Since $\|q\| > 0$, dividing $\|q\|$ on both sides of (21), we obtain

$$1 \leq \lambda \max\{(c+\varepsilon), \tau_1((c+\varepsilon)k + 1)\} + (c+\varepsilon)k. \qquad (22)$$

Observing that $\tau_1(1 + kc) = c$ and $kc < 1$ and letting $\varepsilon \to 0^+$ in (22), we have

$$\begin{aligned}1 &\leq \lambda \max\{c, \tau_1(kc+1)\} + kc \\ &< \frac{1}{\tau} \cdot \tau_1(1 + kc) + kc = \frac{1}{\tau} \cdot \frac{\tau(1-kc)}{1+kc}(1+kc) + kc = 1\end{aligned}$$

which yields a contradiction. Thus $q = 0$ must hold. □

Theorem 4 assumes that $\operatorname{gph} H$ is locally closed. Without this assumption, the same perturbation stability result holds if the graphs $\operatorname{gph} F$ and $\operatorname{gph} G$ are both closed.

Theorem 5 *Assume that* $\operatorname{gph} F$ *and* $\operatorname{gph} G$ *are closed. Suppose that* a, b, c *and* k *are positive constants with* $kc < 1$ *such that* (5)*,* (6) *and* (7) *hold. Then the mapping* H *is metrically regular at* $\bar{x}$ *for* $\bar{y} + \bar{z}$ *with constant* $\tau := \frac{c}{1-kc}$.

Proof Take $r > 0$ as (8). Fix any $(x, y) \in B(\bar{x}, \frac{r}{2}) \times B(\bar{y} + \bar{z}, \frac{r}{2})$. Since $H = F + G$, for any $t > 0$, we can find $v \in F(x)$ such that

$$d(y, v + G(x)) \leq d(y, H(x)) + t. \qquad (23)$$

Pick any sufficiently small $\varepsilon > 0$ satisfying

$$(c+\varepsilon)k < 1 \quad \text{and} \quad \varepsilon\|v - \bar{y}\| \leq \frac{r}{2}, \qquad (24)$$

and define the $\varepsilon - metric$ on $X \times Y$ by (2). Then from (5) and the implication (i)⇒(iii) in Proposition 1 we know that the mapping $F_0 : \operatorname{gph} F \to Y$ defined by $F_0(x, y) = y$ is metrically regular at $(\bar{x}, \bar{y})$ for $\bar{y}$ with constant $c+\varepsilon$ on $\{((x', v'), y') \mid v' \in F(x'), (x', v') \in B_\varepsilon((\bar{x}, \bar{y}), a), y' \in B(\bar{y}, b)\}$, where B_ε denotes the closed ball under the $\varepsilon - metric$.

From (2), (6) and (7) we know that the mapping $G_0 : \operatorname{gph} F \rightrightarrows Y$ defined by $G_0(x, y) = G(x)$ is locally Lipschitz on $B_\varepsilon((\bar{x}, \bar{y}), a) \cap \operatorname{gph} F$ with constant k and that $\operatorname{diam} G_0(\bar{x}, \bar{y}) = G(\bar{x}) < b$.

Set $\Psi := F_0 + G_0$. Now we prove that $\operatorname{gph} \Psi$ is closed. In fact, let $\{((x_n, v_n), y_n)\}$ be an arbitrary sequence in $\operatorname{gph} \Psi$ converging to $((\hat{x}, \hat{v}), \hat{y})$. Since $((x_n, v_n), y_n) \in$

gph Ψ, we have $(x_n, v_n) \in \operatorname{gph} F$ and $(x_n, y_n - v_n) \in \operatorname{gph} G$. It follows from the closedness of gph F and gph G that $(\hat{x}, \hat{v}) \in \operatorname{gph} F$ and $(\hat{x}, \hat{y} - \hat{v}) \in \operatorname{gph} G$, and hence

$$\hat{y} \in \hat{v} + G(\hat{x}) = F_0(\hat{x}, \hat{v}) + G_0(\hat{x}, \hat{v}) = \Psi(\hat{x}, \hat{v}),$$

which yields $((\hat{x}, \hat{v}), \hat{y}) \in \operatorname{gph} \Psi$.

Thus, by applying Theorem 4 with $X := X \times Y$, $Y := Y$, $F := F_0$ and $G := G_0$, we obtain that the mapping $\Psi := F_0 + G_0$ is metrically regular at $(\bar{x}, \bar{y})$ for $\bar{y} + \bar{z}$ with constant $\frac{c+\varepsilon}{1-(c+\varepsilon)k}$ on

$$W := \{((x', v'), y') \mid v' \in F(x'), (x', v') \in B_\varepsilon((\bar{x}, \bar{y}), r), y' \in B(\bar{y} + \bar{z}, r)\}.$$

That is, for any $((x', v'), y') \in W$, we have

$$d_\varepsilon((x', v'), \Psi^{-1}(y')) \leq \frac{c+\varepsilon}{1-(c+\varepsilon)k} d(y', \Psi(x', v')). \tag{25}$$

It follows from (24) that

$$d_\varepsilon((x, v), (\bar{x}, \bar{y})) = \|x - \bar{x}\| + \varepsilon \|v - \bar{y}\| \leq \frac{r}{2} + \frac{r}{2} = r.$$

Thus, substituting $((x', v'), y')$ by $((x, v), y)$, we obtain

$$d_\varepsilon((x, v), \Psi^{-1}(y)) \leq \frac{c+\varepsilon}{1-(c+\varepsilon)k} d(y, \Psi(x, v)) = \frac{c+\varepsilon}{1-(c+\varepsilon)k} d(y, v + G(x)),$$

which yields that for any $\eta > 0$, there exists $(u, w) \in \Psi^{-1}(y)$ such that

$$d_\varepsilon((x, v), (u, w)) \leq \frac{c+\varepsilon}{1-(c+\varepsilon)k} d(y, v + G(x)) + \eta. \tag{26}$$

Since $(u, w) \in \Psi^{-1}(y) \subset \operatorname{gph} F$, we have

$$y \in \Psi(u, w) = w + G(u) \subset F(u) + G(u) = H(u)$$

and hence $u \in H^{-1}(y)$. Therefore, it follows from (23) and (26) that

$$\begin{aligned} d(x, H^{-1}(y)) &\leq \|x - u\| \leq d_\varepsilon((x, v), (u, w)) \\ &\leq \frac{c+\varepsilon}{1-(c+\varepsilon)k} (d(y, H(x)) + t) + \eta. \end{aligned} \tag{27}$$

Then, it holds by letting $\varepsilon \to 0^+$ and $\eta \to 0^+$ in (27) that

$$d(x, H^{-1}(y)) \leq \frac{c}{1-ck} (d(y, H(x)) + t). \tag{28}$$

Finally, letting $t \to 0^+$ in (28), we obtain

$$d(x, H^{-1}(y)) \leq \frac{c}{1-ck} d(y, H(x)),$$

which completes the proof. $\square$

It is obvious that the condition (7) is weaker than (4). Example below shows that our result can be applied to obtain the perturbation stability of metric regularity while Theorem 3 cannot be applied.

Example 1 Let set-valued mappings $F : R \rightrightarrows R$ and $G : R \rightrightarrows R$ be defined by

$$F(x) = \{x^2 - 2x, 1\} \quad \text{and} \quad G(x) = \{sinx, \tfrac{1}{5}\}.$$

For $(0, 0) \in \operatorname{gph} F$, we claim that F is metrically regular at 0 for 0 with constant $\frac{3}{5}$ and neighborhood $B(0, \frac{1}{6}) \times B(0, \frac{1}{4})$.

In fact, fix any $(x, y) \in B(0, \frac{1}{6}) \times B(0, \frac{1}{4})$, we have

$$d(x, F^{-1}(y)) = |x - 1 + \sqrt{1 + y}|, \tag{29}$$

and

$$\begin{aligned} d(y, F(x)) = |y - x^2 + 2x| &= |x - 1 + \sqrt{1 + y}| \cdot |x - 1 - \sqrt{1 + y}| \\ &= |x - 1 - \sqrt{1 + y}| \cdot d(x, F^{-1}(y)). \end{aligned}$$

It can be checked that

$$|x - 1 - \sqrt{y + 1}| \geq \frac{5}{3}.$$

Therefore, it holds that

$$d(x, F^{-1}(y)) \leq \frac{3}{5} d(y, F(x)).$$

It is clear that $0 \in G(0)$, G is Lipschitz around 0 with constant 1 and that $\operatorname{diam} G(0) = \frac{1}{5} < \frac{1}{4}$. Then from Theorem 5 we obtain that

$$H := F + G = \{x^2 - 2x + sinx,\ x^2 - 2x + \frac{1}{5},\ 1 + sinx,\ \frac{6}{5}\}$$

is metrically regular at 0 for 0 with constant $\frac{3}{2}$.

However, Theorem 3 is not applicable since the condition (4) is not satisfied.

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