The maximum k-colorable subgraph problem and related problems

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Abstract

The maximum k-colorable subgraph (MkCS) problem is to find an induced k-colorable subgraph with maximum cardinality in a given graph. This paper is an in-depth analysis of the MkCS problem that considers various semidefinite programming relaxations including their theoretical and numerical comparisons. To simplify these relaxations we exploit the symmetry arising from permuting the colors, as well as the symmetry of the given graphs when applicable. We also show how to exploit invariance under permutations of the subsets for other partition problems and how to use the MkCS problem to derive bounds on the chromatic number of a graph.

Our numerical results verify that the proposed relaxations provide strong bounds for the MkCS problem, and that those outperform existing bounds for most of the test instances.

Keywords: k-colorable subgraph problem; stable set; chromatic number of a graph; generalized theta number; semidefinite programming; Johnson graphs; Hamming graphs.

1 Introduction

The maximum k-colorable subgraph (MkCS) problem is to find the largest induced subgraph in a given graph that can be colored in k colors such that no two adjacent vertices have the same color. The MkCS problem is also known as the maximum k-partite induced subgraph problem since the k-coloring corresponds to a k-partition of the subgraph. The MkCS problem for $k = 2$ is also known as the maximum bipartite subgraph problem.

In the literature, the name "maximum k-colorable subgraph problem" is sometimes used for the maximum k-cut problem [19, 53]. In the latter problem one searches for the partition of the graph into k subsets such that the number of edges crossing the subsets is maximized. If one colors vertices in the resulting subsets in different colors, the crossing edges are properly colored, that is, the endpoints of these edges have different colors. However, the MkCS and the maximum k-cut are different problems, and we do not consider the latter problem in this paper. We refer interested readers to [10, 19, 26, 54, 61, 64] for more information on the maximum k-cut problem and related semidefinite programming (SDP) relaxations.

The MkCS problem falls into the class of NP-complete problems considered by Lewis and Yannakakis [41]. Moreover, even approximating this problem is NP-hard [45]. For $k = 1$ the MkCS problem reduces to the famous maximum stable set problem, which was shown to be NP-complete by Karp [39]. Another well-known problem from the list of Karp [39] related to the MkCS problem is the chromatic number problem. The chromatic number problem is to determine whether the

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vertices of a given graph can be colored in k colors. If one can solve the MkCS problem for any given number of colors, then one can also solve the maximum stable set and the chromatic number problems. However, efficient algorithms for the latter two problems do not necessarily result in efficient algorithms for the MkCS problem. For instance, the chromatic number and the stability number on perfect graphs can be computed in polynomial time while the MkCS problem is NP-complete on chordal graphs which is a subfamily of the perfect graphs [66]. However, there are special classes of graphs for which the MkCS problem is polynomial-time solvable. Some examples are graphs where every odd cycle has two non-crossing chords for any k [1], clique-separable graphs for $k = 2$ [1], chordal graphs for fixed k [66], interval graphs for any k [66], circular-arc graphs and tolerance graphs for $k = 2$ [49]. It is known that the size of the maximum stable set of the Kneser graphs $K(v, d)$, where $v \geq 2d$, equals $\binom{v-1}{d-1}$, see [14]. For the Kneser graphs $K(v, 2)$, the size of the maximum k-colorable subgraph is also known, see e.g., [20].

The MkCS problem for $k = 2$ has been studied, among others, by Bresar and Valencia-Pabon [4], Fouilhoux and Mahjoub [17], Godsil and Royle [25], Hüffner [33], Lee et al. [40]. However, the MkCS problem for $k > 2$ is rarely considered in the literature. Januschowski and Pfetsch [34] notice that such a lack of attention might be related to the connection of the MkCS problem to the earlier mentioned prominent problems. One of the few significant sources of information of the MkCS problem for $k > 2$, but also $k = 2$, are Narasimhan [49] and Narasimhan and Manber [50], where the authors introduce an upper bound on the optimal value of the MkCS problem called the generalized ϑ-number. Namely, they generalize the concept of the famous ϑ-number by Lovász [44], which is an upper bound on the size of the maximum stable set of a graph. Alizadeh [2] formulated the generalized ϑ-number problem using SDP. Mohar and Poljak [48] present the bound by Narasimhan and Manber [50] among important applications of eigenvalues of graphs in combinatorial optimization. To our knowledge, the quality of the generalized ϑ-number has never been evaluated.

Other related work considers integer programming (IP) formulations of the MkCS problem, see e.g. [8, 34, 35]. Januschowski and Pfetsch [34], and Campêlo and Corrêa [8] provide computational results for $k \geq 2$. In particular, Januschowski and Pfetsch [34] consider graphs with symmetry, which enables them to provide numerical results for large graphs up to 1085 vertices. Hertz et al. [32] analyze the performance of various existing online algorithms for the MkCS problem, Bresar and Valencia-Pabon [4] provide theoretical lower and upper bounds for the MkCS problem for $k \geq 2$ on the Kneser graphs.

It is worth mentioning that the MkCS problem has a number of applications, such as channel assignment in spectrum sharing networks (e.g., Wi-Fi or cellular) [3, 30, 31, 62], VLSI design [18, 47] and human genetic research [18, 42].

Outline and main results

This paper is an in-depth analysis of the the MkCS problem that includes various semidefinite programming relaxations and their theoretical and numerical comparisons. This analysis also extends results for the MkCS problem to other graph partition problems, but also relates the MkCS results with the stable set and the chromatic number problems.

We begin our study with the eigenvalue bound by Narasimhan and Manber [50], that is also known as the generalized ϑ-number. We present the generalized ϑ-number as the optimal solution of an SDP relaxation, see also Alizadeh [2]. In order to strengthen the mentioned SDP relaxation we add non-negativity constraints to the matrix variable, and call the solution of the resulting SDP relaxation the generalized ϑ'-number. This number can be seen as the generalization of the Schrijver ϑ'-number [58]. Both, the generalized ϑ-number and ϑ'-number, require solving an SDP relaxation with one matrix variable of the order n, where n is the number of vertices in the graph.

Next, we derive vector lifting and matrix lifting based SDP relaxations. The sizes of matrix variables in the resulting relaxations depend on n and k. We reduce the sizes of the SDP relaxations by exploiting the invariance of the MkCS problem under permutations of the colors. In particular, we exploit the fact that all constraints in the relaxations are satisfied for any color labeling, and the objective does not change if the labeling changes. This property is inherited by our SDP relaxations from the IP formulations of the MkCS problem. By exploiting color invariance, our matrix lifting SDP relaxation reduces to a model with one SDP constraint of the order $n+1$, and our strongest SDP relaxation reduces to a model with two SDP constraints of the order $n+1$ and n, respectively, for a graph with n vertices. Thus, it turns out that matrix sizes in the vector and matrix lifting relaxations are independent of k.

To strengthen our relaxations, we add inequalities from the boolean quadric polytope. We also show how to further reduce our SDP relaxations for highly symmetric graphs. This reduction results in a linear programming program arising from the generalized ϑ'-number, or in programs with linear objective, one second order cone constraint, and many linear constraints.

Since the k-colorable subgraph problem is also a graph partition problem, we are able to apply our symmetry reduction approach based on the invariance under permutations of the subsets, to other partition problems. In particular, we prove in an elegant way that the vector and matrix lifting relaxations for the k-equipartition problem are equivalent. We obtain a similar result for the max-k-cut problem.

Finally, we evaluate the quality of all here presented SDP upper bounds on instances from [8, 34]. We also propose two heuristic approaches to compute lower bounds for the MkCS problem. Our computational results show that our lower and upper bounds for the the MkCS problem are strong and can be computed fast for dense graphs or highly symmetric graphs. Our lower bounds on the chromatic number of a graph are competitive with existing bounds in the literature.

This paper is organized as follows. We present several equivalent integer programming formulations of the MkCS problem in Section 2. In Section 3 we present first the generalized ϑ-number by Narasimhan and Manber [50], and then the related strengthened bound that we call the generalized ϑ'-number. In Section 4 we first propose two vector lifting SDP relaxations and compare them, and then we show how to apply symmetry reduction on colors in order to reduce their sizes. In order to further tighten our SDP relaxations we consider adding inequalities from the boolean quadric polytope in Section 4.2. In Section 4.3 we show that the MkCS problem is the stable set problem on the Cartesian product of the complete graph on k vertices and the graph under the consideration. A matrix lifting SDP relaxation and its symmetry reduced version are given in Section 5. In Section 6 we show how to further simplify and reduce our SDP relaxations by exploiting symmetry of graphs. The application of symmetry reduction on colors is extended to other graph partition problems in Section 7. Section 8 contains numerical results. We summarize the results in Section 9.

Notation. The space of $n \times n$ symmetric matrices is denoted by $\mathbb{S}^n$. For $A, B \in \mathbb{S}^n$, the trace inner product of A and B is denoted by $\langle A, B \rangle := \text{trace}(AB)$. For two matrices $A, B \in \mathbb{R}^{n \times n}$, $A \geq B$, means $a_{ij} \geq b_{ij}$, for all i, j.

We use the notation I_n (resp., J_n) for the identity matrix (resp., the matrix of all ones) of order n. If the dimension of the matrices is clear from the context, we omit the subsctipt. We use u_i to denote the i-th standard basis vector, and e to denote the vector of all ones. Here, we use $[m]$ to denote the set $\{1, \dots, m\}$.

The 'vec' operator stacks the columns of a matrix, while the 'diag' operator maps an $n \times n$

matrix to the n-vector given by its diagonal. The adjoint operator of 'diag' is denoted by 'Diag'. The Kronecker product $A \otimes B$ of matrices $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{r\times s}$ is defined as the $pr \times qs$ matrix composed of pq blocks of size $r \times s$, with block ij given by $A_{i,j}B$, $i = 1,\dots,p$, $j = 1,\dots,q$.

2 Problem formulation

Let $G = (V, E)$ be a simple undirected graph with the vertex set V and the edge set E. Let $|V| = n$, and let k be a given integer such that $1 \le k \le n-1$. We say that G is k-colorable if one can assign to each vertex in G one of the k colors such that adjacent vertices do not have the same color. A graph $G' = (V', E')$ is called an induced subgraph of a given graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ is the set of all edges in E connecting the vertices in V'.

The maximum k-colorable subgraph problem is to find an induced k-colorable subgraph with maximum cardinality in the given graph G. We denote by $\alpha_k(G)$ the number of vertices in the maximum k-colorable subgraph of G. When $k = 1$, the MkCS problem corresponds to the stable set problem, i.e., $\alpha_1(G) = \alpha(G)$, where $\alpha(G)$ denotes the stability number of G (the maximum cardinality of a stable set in G).

For any $k \ge 1$, the MkCS problem can be formulated as an integer programming problem. Let $X \in \{0,1\}^{n\times k}$ be the matrix with one in the entry (i, r) if vertex $i \in [n]$ is colored with color $r \in [k]$ and zero otherwise. An IP formulation for the MkCS problem is given by (1a)–(1c):

$$\alpha_k(G) = \max_{X\in\{0,1\}^{n\times k}} \sum_{i\in[n],r\in[k]} X_{ir} \tag{1a}$$

$$\text{s. t.} \quad X_{ir}X_{jr} = 0, \quad \text{for all } \{ij\} \in E, r \in [k] \tag{1b}$$

$$\sum_{r\in[k]} X_{ir} \le 1, \quad \text{for all } i \in [n]. \tag{1c}$$

Here, constraints (1b) ensure that two adjacent vertices are not colored with the same color, and constraints (1c) that each vertex is colored with at most one color.

Alternative IP formulation for the MkCS problem can be obtained by adding a binary slack variable to each inequality constraint in (1c), i.e., by replacing those constraints with the following ones:

$$\sum_{r\in[k+1]} X_{ir} = 1, \qquad \text{for all } i \in [n]. \tag{2}$$

It is known that a model with equality constraints may provide stronger relaxations than alternative one, see e.g., [7, 55]. In Section 4 we use the IP model with equality constraints (2) to derive our strongest SDP relaxation for the MkCS problem.

Another IP model for the MkCS problem is obtained by replacing (1b) constraints with the following ones:

$$X_{ir} + X_{jr} \le 1, \quad \{ij\} \in E,\ r \in [k]. \tag{3}$$

The IP formulation (1a), (1c), (3) is exploited in [34, 35].

3 The generalized ϑ- and ϑ'-number

In this section we present the generalized ϑ-number by Narasimhan and Manber [50], which is an upper bound for the MkCS problem. This eigenvalue upper bound was reformulated as an SDP relaxation in [2]. Here, we strengthen the mentioned SDP relaxation by adding non-negativity constraints, and introduce the generalized ϑ'-number.

3.1 Eigenvalue and SDP formulations of the generalized ϑ-number

Narasimhan and Manber [50] introduce $\vartheta_k(G)$, the generalized ϑ-number, as an upper bound for $\alpha_k(G)$:

$$\alpha_k(G) \le \vartheta_k(G) = \min_{A\in\mathbb{S}^n} \left\{ \sum_{i=1}^{k} \lambda_i(A) : \ A_{ij} = 1 \text{ for } \{ij\} \notin E \text{ or } i = j \right\}. \tag{4}$$

The minimum in (4) is attained, see Theorem 12 in Narasimhan [49]. To show that $\vartheta_k(G)$ is an upper bound on $\alpha_k(G)$, we follow the reasoning of Mohar and Poljak [48] who use Fan's theorem.

Theorem 1 (Fan [16])**.** *Let A be a symmetric matrix with eigenvalues $\lambda_1(A) \ge \lambda_2(A) \ge \cdots \ge \lambda_n(A)$. Then*

$$\sum_{i=1}^{k} \lambda_i(A) = \max_{X\in\mathbb{R}^{n\times k}} \left\{ \langle A, XX^\top \rangle \ \text{ s. t. } \ X^\top X = I_k \right\}. \tag{5}$$

Let $X^* \in \{0,1\}^{n\times k}$ be an optimal solution to the IP problem (1). That is, for every $r \in [k]$ the r^{th} column of X^* is the incidence vector of the stable set colored in color r. Let $\hat{X}$ be the matrix whose columns are the columns of X^* normalized to one. Thus, $\hat{X}^\top \hat{X} = I_k$ by construction. Now, using Theorem 1, for any matrix A feasible for problem (4) we have

$$\alpha_k(G) = \langle A, \hat{X}\hat{X}^\top \rangle \le \max_{X\in\mathbb{R}^{n\times k}} \left\{ \langle A, XX^\top \rangle \ \text{ s. t. } \ X^\top X = I_k \right\} = \sum_{i=1}^{k} \lambda_i(A).$$

Hence $\alpha_k(G) \le \vartheta_k(G)$. For $k = 1$, $\vartheta_k(G)$ is the eigenvalue formulation of the ϑ-number by Lovász [44]. In the original paper by Narasimhan and Manber [50], the generalized ϑ-number is introduced for the clique number of G, which implies that the generalized ϑ-number there is defined for the complement of G. Here, we are interested in the maximum number of vertices in a k-partite subgraph of G and therefore define $\vartheta_k(G)$ for G.

It is known that problem (5) can be formulated as an SDP relaxation, see Alizadeh [2]. This implies that $\vartheta_k(G)$ can be obtained as the optimal solution of the following SDP relaxation:

$$\begin{aligned} \vartheta_k(G) = \min_{Y\in\mathbb{S}^n, \mu, \{x_{ij}\}_{\{ij\}\in E}} \quad & \langle I, Y \rangle + \mu k \\ \text{s. t.} \quad & \sum_{\{ij\}\in E} \mathcal{E}^{ij} x_{ij} - J + \mu I + Y \succeq 0, \ Y \succeq 0, \end{aligned} \tag{6}$$

where $\mathcal{E}^{ij} = u_i u_j^\top + u_j u_i^\top$ for $i, j \in [n]$ such that $i \neq j$. The dual of the above problem is

$$\begin{aligned} \vartheta_k(G) = \max_{Z\in\mathbb{S}^n} \ & \langle J, Z \rangle \\ \text{s. t.} \quad & Z_{ij} = 0 \text{ for } \{ij\} \in E, \\ & \langle I, Z \rangle = k, \\ & Z \succeq 0, \ I - Z \succeq 0. \end{aligned} \tag{7}$$

Note that $Z = \frac{k}{n} I$ is a strictly feasible point for the SDP relaxation (7), in particular strong duality holds for the primal-dual pair (6)–(7).

Alternatively, the SDP relaxation (7) can be obtained directly from the IP model (1). As before, let $X \in \{0,1\}^{n\times k}$ be an optimal solution to problem (1), and let $\hat{X}$ be the matrix whose columns are the columns of X normalized to one. Then the matrix $Z := \hat{X}\hat{X}^\top$ is feasible for the relaxation (7) with the objective value $\alpha_k(G)$. In particular, this follows since for any $r = 1, \ldots, k$ we have $\left(\hat{X}\hat{X}^\top\right)_{ij} = \frac{1}{c_r}$ if vertices i and j are colored with color r, where c_r is the total number of vertices colored in color r; $\left(\hat{X}\hat{X}^\top\right)_{ij} = 0$ otherwise. The first constraint in (7) is satisfied by construction of $\hat{X}$ and clearly $\hat{X}\hat{X} \succeq 0$. The second and the last constraints in (7) are satisfied since the columns of $\hat{X}$ are normalized to one, $\hat{X}\hat{X}^\top$ has k eigenvalues equal to one, and $n-k$ eigenvalues equal to zero.

For $k = 1$ constraint $I - Z \succeq 0$ in (7) becomes redundant since Z is positive semidefinite and its eigenvalues sum up to one. In this case the SDP relaxation (7) reduces to a formulation of the ϑ-number by Lovász [44], i.e.,

$$
\begin{aligned}
\vartheta(G) = \max_{Z\in\mathbb{S}^n}\ & \langle J, Z\rangle \\
\text{s.t. }\ & Z_{ij} = 0 \text{ for } \{ij\} \in E, \\
& \langle I, Z\rangle = 1, \\
& Z \succeq 0.
\end{aligned} \tag{8}
$$

3.2 Strengthening the generalized ϑ-number

Note that all entries of an optimal solution to the integer programming problem (1) are non-negative. Therefore, we can add non-negativity constraints to the matrix variable in (7) to strengthen the relaxation. This leads to the following SDP relaxation:

$$
\begin{aligned}
\vartheta_k'(G) = \max_{Z\in\mathbb{S}^n}\ & \langle J, Z\rangle \\
\text{s.t. }\ & Z_{ij} = 0 \text{ for } \{ij\} \in E \\
& \langle I, Z\rangle = k \\
& Z \succeq 0,\ I - Z \succeq 0, \\
& Z \geq 0.
\end{aligned} \tag{9}
$$

We refer to the solution of the above SDP relaxation as the generalized ϑ'-number. Note that for $k = 1$, $\vartheta_k'(G)$ equals the $\vartheta'(G)$ upper bound on $\alpha(G)$ by Schrijver [58].

4 Vector lifting SDP relaxations

In this section we derive two new SDP relaxations for the MkCS problem. To derive relaxations we use the vector lifting approach, see e.g., [12, 61, 65]. In Section 4.1, we reduce the SDP models by exploiting the fact that the MkCS problem is invariant under any permutation of the colors. To strengthened the relaxations, we add inequalities from the boolean quadric polytope in Section 4.2. In Section 4.3 we relate the stable set problem on the Cartesian product of the complete graph on k vertices and G, with the MkCS problem. In particular, we show that our weaker vector lifting relaxation is equivalent to the SDP model for the Schrijver ϑ'-number on the Cartesian product of two mentioned graphs.

To derive our first SDP relaxation in this section, we consider the IP model for the MkCS problem with all equality constraints. Thus, assume that $X \in \{0,1\}^{n\times(k+1)}$ is feasible for the IP model (1a), (1b) and (2), i.e., X is the matrix with one in the entry (i,r) if vertex i is colored with color r and zero otherwise, where 'color' $k+1$ represents the uncolored vertices. Let $x = \mathrm{vec}(X)$ and $Y = xx^\top$. As $x \in \{0,1\}^{n(k+1)}$, it follows that $Y = \mathrm{diag}(Y)\mathrm{diag}(Y)^\top$, which can be relaxed to $Y - \mathrm{diag}(Y)\mathrm{diag}(Y)^\top \succeq 0$ which is equivalent to the following convex constraint:

$$\begin{bmatrix} 1 & \mathrm{diag}(Y)^\top \\ \mathrm{diag}(Y) & Y \end{bmatrix} \succeq 0. \tag{10}$$

Matrix Y consists of $(k+1)^2$ blocks of the size $n \times n$. We denote by Y^{rl} the $n \times n$ block of Y located in position $(r,l) \in [k+1] \times [k+1]$. From here on we use the subscripts i, j to indicate vertices and use superscripts r, l to indicate colors in matrix variables. From (1b), there follows

$$Y_{ij}^{rr} = 0, \quad \forall \{ij\} \in E, \ r \in [k],$$

from (2),

$$\sum_{r \in [k+1]} Y_{ii}^{rr} = 1, \qquad \text{for all } i \in [n].$$

Also, from (2) and the fact that X is binary, we have

$$Y_{ii}^{rl} = 0, \qquad \text{for all } i \in [n], \ r, l \in [k+1], \ r \neq l.$$

We collect all above listed constraints, add non-negativity constraints, and arrive to the following SDP relaxation for the MkCS problem:

$$\begin{aligned} \max_{Y \in \mathbb{S}^{n(k+1)}} \quad & \sum_{r \in [k]} \sum_{i \in [n]} Y_{ii}^{rr} \\ \text{s.t.} \quad & Y_{ij}^{rr} = 0, \ \text{ for all } \{ij\} \in E, \ r \in [k] \\ & \sum_{r \in [k+1]} Y_{ii}^{rr} = 1, \qquad \text{for all } i \in [n] \\ & Y_{ii}^{rl} = 0, \ \text{ for all } i \in [n], \ r, l \in [k+1], \ r \neq l \\ & Y \geq 0, \ \begin{bmatrix} 1 & \mathrm{diag}(Y)^\top \\ \mathrm{diag}(Y) & Y \end{bmatrix} \succeq 0. \end{aligned} \tag{11}$$

Similarly, we derive a vector lifting SDP relaxation from the IP problem (1). Namely, by exploiting $X \in \{0,1\}^{n \times k}$ that is feasible for (1), we obtain the following SDP relaxation:

$$\max_{Y \in \mathbb{S}^{nk}} \quad \sum_{r \in [k]} \sum_{i \in [n]} Y_{ii}^{rr} \tag{12a}$$

$$\text{s.t.} \quad Y_{ij}^{rr} = 0, \ \text{ for all } \{ij\} \in E, \ r \in [k] \tag{12b}$$

$$Y_{ii}^{rl} = 0, \ \text{ for all } i \in [n], \ r, l \in [k], \ r \neq l \tag{12c}$$

$$Y \geq 0, \ \begin{bmatrix} 1 & \mathrm{diag}(Y)^\top \\ \mathrm{diag}(Y) & Y \end{bmatrix} \succeq 0. \tag{12d}$$

Note that it is not necessary to include in (12) the constraints $\sum_{r \in [k]} Y_{ii}^{rr} \leq 1$ $(i \in [n])$ that arise naturally from (1c), as they are redundant. Namely, we have the following result.

Lemma 1. *For each $i \in [n]$, the constraint $\sum_{r \in [k]} Y_{ii}^{rr} \leq 1$ is redundant for the SDP relaxation (12).*

Proof. Let Y be feasible for (12) and $i \in [n]$. Consider the principal submatrix of Y, $Y(\alpha)$, of the size $(k+1)\times(k+1)$ whose rows and columns are indexed by $\alpha = \{1, i+1, n+i+1, \ldots, (k-1)n+i+1\}$. From $Y_{ii}^{rl} = 0$ for $r, l \in [k]$ such that $r \neq l$, it follows that

$$Y(\alpha) = \begin{bmatrix} 1 & y^\top \\ y & \mathrm{Diag}(y) \end{bmatrix}, \text{ where } y = [Y_{ii}^{11}, Y_{ii}^{22}, \ldots, Y_{ii}^{kk}].$$

Since $Y(\alpha) \succeq 0$, we have

$$e^\top \mathrm{Diag}(y)e - (e^\top y)(y^\top e) = \sum_{r=1}^{k} y_r - \left(\sum_{r=1}^{k} y_r\right)^2 \geq 0,$$

which implies $\sum_{r\in[k]} Y_{ii}^{rr} \leq 1$. □

It is not difficult to verify that the SDP relaxation (12) is dominated by the SDP relaxation (11). We show below that those two relaxations are equivalent after adding the following inequality constraints to the relaxation (12):

$$1 - \sum_{r\in[k]} Y_{ii}^{rr} - \sum_{r\in[k]} Y_{jj}^{rr} + \sum_{r\in[k]}\sum_{l\in[k]} Y_{ij}^{rl} \geq 0, \text{ for all } i > j \tag{13}$$

$$Y_{ii}^{ll} - \sum_{r\in[k]} Y_{ij}^{rl} \geq 0, \quad \text{for all } i \neq j,\ l, \tag{14}$$

where $i, j \in [n]$ and $l, r \in [k]$. These inequalities are based on the reformulation-linearization technique by Sherali and Adams [59]. In particular, inequalities (13) are linearizations of the products of pairs of constraints (1c). Inequalities (14) represent multiplication of elementwise non-negativity constraint on X with each individual constraint in (1c). Similar inequalities are used in [55] and [56] to improve SDP relaxations for the min-cut problem and the bandwidth problem, respectively.

Theorem 2. *The SDP relaxation* (12) *with additional constraints* (13), (14) *is equivalent to the SDP relaxation* (11)*.*

Proof. (See also [55], Section 5.2.) Let Z be feasible for problem (12) and assume it also satisfies (13), (14). We define $z := \mathrm{diag}(Z)$. For ease of presentation we write constraints (13) and (14) as matrix inequalities using the following transformation matrix:

$$M = \begin{bmatrix} 1 & -(m^1)^\top \\ \ldots & \ldots \\ 1 & -(m^n)^\top \end{bmatrix}, \text{ where } m_j^i = \begin{cases} 1, & j \in \{i, n+i, \ldots, n(k-1)+i\} \\ 0, & \text{otherwise.} \end{cases}$$

Then inequalities (13) can be written as follows:

$$M \begin{bmatrix} 1 & z^\top \\ z & Z \end{bmatrix} M^\top \geq 0,$$

and inequalities (14) as follows:

$$M \begin{bmatrix} z^\top \\ Z \end{bmatrix} \geq 0.$$

Now, it is not difficult to verify that

$$Y = \begin{bmatrix} I \\ M \end{bmatrix} \begin{bmatrix} 1 & z^\top \\ z & Z \end{bmatrix} \begin{bmatrix} I & M^\top \end{bmatrix} \succeq 0$$

is feasible for the SDP relaxation (11).

Conversely, let Y be feasible for the SDP relaxation (11). Then $Z := Y(1{:}kn, 1{:}kn)$ is feasible for the SDP relaxation (12) by construction of both relaxations. To show that Z also satisfies (13) and (14) note that that those inequalities correspond to the missing blocks $Y^{(k+1)r}$ for $r \in [k+1]$. □

To strengthen relaxations (11) and (12), one may tend to add the clique constraints i.e., $\sum_{i\in C} \operatorname{diag}(Y_{ii}^{rr}) \leq 1$ where $C \subseteq [n]$ denotes a set of indices corresponding to vertices in a clique, and $r \in [k]$. However, those constraints are redundant, see Remark 2, page 13 for an explanation.

4.1 Symmetry reduction on colors

In this section we exploit the fact that the MkCS problem is invariant under permutations of the colors, in order to reduce the sizes the vector lifting SDP relaxations from the previous section. Our vector lifting SDP relaxations are not strictly feasible, which hampers numerical stability of the interior point methods. However, we show that the symmetry-reduced SDP relaxations are strictly feasible.

We begin with a lemma related to strict feasibility.

Lemma 2. *Let $n \geq 1, k \geq 1$, and let $e \in \mathbb{R}^n$ be the vector of all ones. Then*

$$M := \begin{bmatrix} k & \frac{1}{(n+1)} e^\top \\ \frac{1}{(n+1)} e & \frac{1}{(n+1)} I \end{bmatrix} \succ 0.$$

Proof. Let $\left[\begin{smallmatrix} x_0 \\ x \end{smallmatrix}\right] \in \mathbb{R}^{n+1} \setminus \{0\}$ we have

$$\left[\begin{smallmatrix} x_0 \\ x \end{smallmatrix}\right]^\top M \left[\begin{smallmatrix} x_0 \\ x \end{smallmatrix}\right] = \frac{1}{n+1}\left(((n+1)k - n)x_0^2 + \|x_0 e + x\|^2\right) > 0$$

□

To reduce the size of the SDP relaxation (12) with additional constraints (13) and (14), we need the following result.

Lemma 3 (Lemma 2.8 in [29])**.** *Let $Y \in \mathbb{R}^{kn}$ be a block matrix that consists of k^2 blocks of the size $n \times n$. Let Y have a matrix $A \in \mathbb{S}^n$ as its diagonal blocks, and a matrix $B \in \mathbb{S}^n$ as its non-diagonal blocks, i.e.,*

$$Y = \underbrace{\begin{bmatrix} A & B & \dots & B \\ B & A & \dots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \dots & A \end{bmatrix}}_{k \text{ blocks}} = I \otimes A + (J - I) \otimes B.$$

Then $Y \succeq 0$ if and only if $A - B \succeq 0$ and $A + (k-1)B \succeq 0$.

Now, we are ready to prove our main result in this section.

Theorem 3. *The SDP relaxation* (11) *and the SDP relaxation* (12) *with additional constraints* (13)*,* (14) *are equivalent to the following SDP relaxation:*

$$\theta_k^1(G) = \max_{Z,X\in\mathbb{S}^n} \ \langle I, Z\rangle \tag{15a}$$

$$\begin{aligned}
\text{s.t.}\quad & Z_{ij} = 0, \quad \textit{for } \{ij\}\in E \\
& X_{ii} = 0, \quad \textit{for } i\in[n] && (15b)\\
& Z\geq 0,\ X\geq 0 && (15c)\\
& Z - X \succeq 0 && (15d)\\
& \begin{bmatrix} 1 & \operatorname{diag}(Z)^\top \\ \operatorname{diag}(Z) & Z+(k-1)X\end{bmatrix} \succeq 0. && (15e)\\
& 1-Z_{ii}-Z_{jj}+Z_{ij}+(k-1)X_{ij}\geq 0, \quad \textit{for } i,j\in[n], i>j && (15f)\\
& Z_{ii}-Z_{ij}-(k-1)X_{ij}\geq 0, \quad \textit{for } i,j\in[n], i\neq j, && (15g)
\end{aligned}$$

and the above SDP relaxation is strictly feasible.

Proof. First, the relaxations (11) and (12) with additional constraints (13), (14) are equivalent by Theorem 2. We show here that the SDP relaxation (12) with (13), (14) is equivalent to (15).

Let Y be a feasible solution to the SDP relaxation (12) with additional constraints (13), (14). If we permute the color labels, we permute the "columns" and "rows" of blocks in Y. For instance, permuting color r and color l results in permuting blocks Y^{pl} and Y^{pr}, for all $p\in[k]$, and then permuting blocks Y^{lp} and Y^{rp}, for all $p\in[k]$. In particular Y^{rr} and Y^{ll} are permuted.

Let $\bar{Y}$ be the average over all permutations of each pair of colors of Y. By construction problem (12) with constraints (13), (14) is convex and invariant under color permutations. Therefore, $\bar{Y}$ is feasible for (12) and satisfies the additional constraints (13), (14). Notice that $\bar{Y}$ has the form

$$\bar{Y} = \frac{1}{k}\underbrace{\begin{bmatrix} Z & X & \dots & X\\ X & Z & \dots & X\\ \vdots & \vdots & \ddots & \vdots \\ X & X & \dots & Z\end{bmatrix}}_{k \text{ blocks}} = \tfrac{1}{k}I_k\otimes Z + \tfrac{1}{k}(J_k - I_k)\otimes X, \tag{16}$$

where $Z=\sum_{r=1}^k Y^{rr}$ and $X=\frac{1}{k-1}\sum_{r=1}^k\sum_{l=1}^k Y^{rl}$.

The SDP relaxation (12) with constraints (13), (14) can be restricted without loss of generality to matrices of the form (16), which results in (15). Indeed, the objective and all linear constraints of the SDP relaxation (15) are obtained by rewriting the objective and the corresponding linear constraints of (12), (13) and (14) for matrices of the form (16). Now, consider the SDP constraint

$$\begin{bmatrix} 1 & \operatorname{diag}(Y)^\top \\ \operatorname{diag} Y & Y\end{bmatrix}\succeq 0,$$

where Y is of the form (16) which, by the Schur complement, is equivalent to $Y - \operatorname{diag}(Y)\operatorname{diag}(Y)^\top \succeq 0$. We have

$$Y - \operatorname{diag}(Y)\operatorname{diag}(Y)^\top = \tfrac{1}{k}I\otimes(Z-\tfrac{1}{k}\operatorname{diag}(Z)\operatorname{diag}(Z)^\top) + \tfrac{1}{k}(J-I)\otimes(X-\tfrac{1}{k}\operatorname{diag} Z(\operatorname{diag} Z)^\top).$$

Hence by Lemma 3, the positive semidefinite constraint holds if and only if

$$Z-X\succeq 0 \text{ and } \ Z+(k-1)X-\operatorname{diag}(Z)\operatorname{diag}(Z)^\top\succeq 0.$$

Now we show strict feasibility of (15). Let $A_{\bar{G}}$ be the adjacency matrix of the complement of G and $M \succ 0$ be given as in Lemma 2. Since also $\frac{1}{k(n+1)} I \succ 0$, there exists $0 < \varepsilon < \frac{1}{kn(n+1)}$ such that

$$\frac{1}{k} M + \varepsilon \begin{bmatrix} 0 & 0^\top \\ 0 & A_{\bar{G}} + (k-1)(J-I) \end{bmatrix} \succ 0 \;\text{ and }\; \frac{1}{k(n+1)} I + \varepsilon A_{\bar{G}} - \varepsilon(J-I) \succ 0.$$

Define $\bar{Z} := \frac{1}{k(n+1)} I + \varepsilon A_{\bar{G}}$, $\bar{X} := \varepsilon(J-I)$. By the choice of ε one can verify that $(\bar{Z}, \bar{X})$ strictly satisfies constraints (15c), (15f) and (15g). For example, we verify below that constraints (15f) are strictly satisfied for $(\bar{Z}, \bar{X})$:

$$1 - \bar{Z}_{ii} - \bar{Z}_{jj} + \bar{Z}_{ij} + (k-1)\bar{X}_{ij} \geq 1 - \frac{2}{k(n+1)} = \frac{kn+k-2}{k(n+1)} \geq \frac{1}{k(n+1)},$$

for all $i, j \in [n], i > j$. □

Remark 1. *We denote by $\theta_k^2(G)$ the optimal value of the SDP relaxation obtained from* (15) *where constraints* (15f) *and* (15g) *removed. Thus, $\theta_k^2(G)$ equals to the optimal value of the SDP relaxation* (12).

For the sake of completeness, we also present the symmetry-reduced version of the relaxation (11).

Corollary 1. *The SDP relaxation* (11) *is equivalent to the following relaxation:*

$$\begin{aligned}
\max_{Z,X,V,W \in \mathbb{S}^n} \quad & \langle I, Z \rangle \\
\text{s.t.} \quad & Z_{ij} = 0 \textit{ for } \{ij\} \in E \\
& X_{ii} = 0, \quad \textit{for } i \in [n] \\
& W_{ii} = 0, \quad \textit{for } i \in [n] \\
& Z_{ii} + V_{ii} = 1 \textit{ for } i \in [n] \\
& Z \geq 0, \; X \geq 0, \; V \geq 0, \; W \geq 0 \\
& Z - X \succeq 0 \\
& \begin{bmatrix} k & \sqrt{k}\, \mathrm{diag}(Z)^\top & \mathrm{diag}(V)^\top \\ \sqrt{k}\, \mathrm{diag}(Z) & Z + (k-1)X & W \\ \mathrm{diag}(V) & W & V \end{bmatrix} \succeq 0.
\end{aligned}$$

Proof. First, relaxations (11) and (15) are equivalent by Theorem 2 and Theorem 3. Now, consider a feasible solution Y to the SDP relaxation (11). The problem is invariant under permutations of k colors, which correspond to the first k^2 blocks of Y, i.e., Y^{ij}, $(i, j = 1, \dots, k)$. By invariance of the problem, it is enough to consider the solutions of the form

$$Y = \frac{1}{k} \begin{bmatrix} 1 & (e \otimes \mathrm{diag}(Z))^\top & \mathrm{diag}(V)^\top \\ e \otimes \mathrm{diag}(Z) & I \otimes Z + (J-I) \otimes X & e \otimes W \\ \mathrm{diag}(V) & (e \otimes W)^\top & V \end{bmatrix}.$$

Further we use the same approach as in the proof of Theorem 3, so the details are omitted. □

Note that the SDP relaxation from Corollary 1 has a matrix variable of order $3n$, while the SDP relaxation from Theorem 3 has a matrix variable of order $2n$. However, they are equivalent and the later one is strictly feasible, thus more appropriate for numerical experiments.

4.2 Boolean quadric polytope inequalities

To further strengthen our strongest SDP relaxation, one can add inequalities from the boolean quadric polytope (BQP), see e.g., Padberg [52]. Namely, let X be a feasible solution to the binary problem (1) and consider $Y = \operatorname{vec}(X)\operatorname{vec}(X)^\top$. Then for all $i, j, p \in [nk]$ the following BQP inequalities are valid for Y:

$$0 \le Y_{i,j} \le Y_{i,i} \tag{17}$$

$$Y_{i,i} + Y_{j,j} \le 1 + Y_{i,j} \tag{18}$$

$$Y_{i,p} + Y_{j,p} \le Y_{p,p} + Y_{i,j} \tag{19}$$

$$Y_{i,i} + Y_{j,j} + Y_{p,p} \le Y_{i,j} + Y_{i,p} + Y_{j,p} + 1. \tag{20}$$

Therefore, one can add those constraints to the SDP relaxation (12) with additional constraints (13), (14) in order to further strengthen it. In the view of the symmetry reduction on colors, we may consider only those feasible solutions Y to the relaxation (15) that can be written as in (16). Then, (17)–(20) reduce to

$$0 \le Z_{i,j} \le Z_{i,i}, \qquad 0 \le X_{i,j} \le Z_{i,i}, \tag{21}$$

$$Z_{i,i} + Z_{j,j} \le k + Z_{i,j}, \qquad Z_{i,i} + Z_{j,j} \le k + X_{i,j}, \tag{22}$$

$$\begin{aligned} &X_{i,p} + X_{j,p} \le Z_{p,p} + X_{i,j}, && Z_{i,p} + Z_{j,p} \le Z_{p,p} + Z_{i,j},\\ &X_{i,p} + X_{j,p} \le Z_{p,p} + Z_{i,j}, && X_{i,p} + Z_{j,p} \le Z_{p,p} + X_{i,j},\\ &Z_{i,p} + X_{j,p} \le Z_{p,p} + X_{i,j}, && \end{aligned} \tag{23}$$

$$\begin{aligned} &Z_{i,i} + Z_{j,j} + Z_{p,p} \le X_{i,j} + X_{i,p} + X_{j,p} + k,\\ &Z_{i,i} + Z_{j,j} + Z_{p,p} \le Z_{i,j} + Z_{i,p} + Z_{j,p} + k,\\ &Z_{i,i} + Z_{j,j} + Z_{p,p} \le Z_{i,j} + X_{i,p} + X_{j,p} + k,\\ &Z_{i,i} + Z_{j,j} + Z_{p,p} \le X_{i,j} + X_{i,p} + Z_{j,p} + k,\\ &Z_{i,i} + Z_{j,j} + Z_{p,p} \le X_{i,j} + Z_{i,p} + X_{j,p} + k, \end{aligned} \tag{24}$$

where $i, j, p \in [n], i \ne j \ne p$. Inequalities (21) correspond to (17), (22) correspond to (18), etc.

Some of the BQP constraints (21)–(24) are redundant for the relaxation (15). In particular, (21) follows from (15g). Also, constraints (22) are redundant since the 2-clique constraints are redundant for the SDP relaxations (11) and (12). Therefore, in numerical experiments we use inequalities of the type (23) and (24). Notice that the first inequality in each of the two sets (23) and (24) is only valid for $k \ge 3$.

One could also consider the triangle inequalities for Y in (12), i.e., $Y_{ij} + Y_{jp} - Y_{ip} \le 1$ for any $i, j, p \in [n]$. However, those inequalities follow from (19) and the non-negativity of Y.

4.3 The MkCS as the maximum stable set problem

The main result in this section is that the MkCS problem on a graph G can be considered as a stable set problem on the Cartesian product of the complete graph on k vertices and G. We also show that the Schrijver's ϑ'-number on the Cartesian product of mentioned two graphs equals the optimal value of the vector lifting relaxation (12).

We denote by $K_k = (V_k, E_k)$, where $V_k = [k]$, the complete graph on k vertices. The Cartesian product $K_k \square G$ of graphs K_k and $G = (V, E)$ is a graph with the vertex set $V_k \times V$ and the edge set

$E_\Box$ where two vertices (u,i) and (v,j) are adjacent if $u=v$ and $(i,j)\in E$ or $i=j$ and $(u,v)\in E_k$. The following result shows that the MkCS problem on G corresponds to the stable set problem on $K_k\Box G$.

Theorem 4. *Let $G=(V,E)$, and let K_k be the complete graph on k vertices. Then $\alpha_k(G)=\alpha(K_k\Box G)$.*

Proof. First, if $S_1,\dots,S_k$ are disjoint stable sets in G, then $\{1\}\times S_1,\dots,\{k\}\times S_k$ is a stable set in $K_k\Box G$. On the other hand, let S be a stable set in $K_k\Box G$ of the largest cardinality. Then S can be partitioned into $S_1,\dots,S_k$ such that $S_1=\{1\}\times\hat{S}_1,\dots,S_k=\{k\}\times\hat{S}_k$, $\hat{S}_1\subseteq V,\dots,\hat{S}_k\subseteq V$. Moreover, $\hat{S}_1,\dots,\hat{S}_k$ are disjoint since $u\in\hat{S}_l\cap\hat{S}_p$ for some $l,p\in[k]$ with $l\neq p$ implies that there is an edge between (l,u) and (p,u) that is also in the stable set S. Hence $\hat{S}_1,\dots,\hat{S}_k$ are disjoint stable sets in G. □

The Schrijver's ϑ'-number on $K_k\Box G$ is as follows

$$\begin{aligned}\vartheta'(K_k\Box G)=\max_{Y\in\mathbb{S}^{nk}}\ &\langle J,Y\rangle &&(25)\\ \text{s.t. }\ &Y_{ij}^{rr}=0,\quad \text{for all } \{ij\}\in E,\ r\in[k]\\ &Y_{ii}^{rl}=0,\quad \text{for all } i\in[n],\ r,l\in[k],\ r\neq l\\ &\langle I,Y\rangle=1,\\ &Y\succeq 0\\ &Y\geq 0,\end{aligned}$$

where Y is of the size $nk\times nk$. The above SDP relaxation follows directly from (9) and the definition of $K_k\Box G$. We show below the following interesting result:

$$\vartheta'(K_k\Box G)=\theta_k^2(G)\geq\alpha_k(G),$$

where $\theta_k^2(G)$ is the optimal value of the vector lifting relaxation (12). To see this, first, notice that any feasible solution to the vector lifting relaxation (12) provides a feasible solution to problem (25). In particular let Y be feasible for (12), then it readily follows that $\hat{Y}=\frac{1}{\langle I,Y\rangle}Y$ is feasible for (25), see also [28]. Moreover, the SDP constraint in the vector lifting relaxation (12) implies

$$\langle J,\hat{Y}\rangle=\tfrac{1}{\langle I,Y\rangle}\langle J,Y\rangle\geq\tfrac{1}{\langle I,Y\rangle}\langle J,\operatorname{diag}(Y)\operatorname{diag}(Y)^\top\rangle=\langle I,Y\rangle.$$

For the opposite direction, we assume that $\hat{Y}$ is feasible for (25). Now, we use Theorem 1 from Galli and Letchford [22], to define $Y\in\mathbb{S}^{kn}$ as follows:

$$Y_{ij}:=\hat{Y}_{ij}\frac{\sum_{m=1}^{kn}\hat{Y}_{im}\sum_{m=1}^{kn}\hat{Y}_{jm}}{\langle J,\hat{Y}\rangle\hat{Y}_{ii}\hat{Y}_{jj}}\ \text{for } i\neq j,\ \text{and } Y_{ii}:=\frac{\left(\sum_{j=1}^{kn}\hat{Y}_{ij}\right)^2}{\langle J,\hat{Y}\rangle\hat{Y}_{ii}},\ \text{for } i\in[kn].$$

Then, Y is feasible for (25) and satisfies $\langle I,Y\rangle\geq\langle J,\hat{Y}\rangle$. Note that the authors of [22] and [28] consider the SDP relaxations of the Lovász ϑ-number. Alternatively, one can obtain the above result by considering the equivalent SDP relaxation of the Lovász ϑ-number from [27].

Remark 2. *It is well known that the clique constraints are redundant for the Lovász ϑ-number, e.g., Chapter 9 of [27]. Therefore, the clique constraints are redundant for the SDP relaxation* (25)*, and consequently also for* (12) *and* (11)*.*

5 Matrix lifting SDP relaxations

In this section we derive a matrix lifting SDP relaxation for the MkCS problem. Relaxations obtained by the matrix lifting approach are known to have less variables and constraints than the corresponding relaxations obtained by the vector lifting approach. However, relaxations obtained by those two approaches may be equal, see Ding et al. [12]. We show here that our matrix lifting SDP relaxation is dominated by the vector lifting relaxations from the previous section. However, numerical results show (see Section 8) that the here derived relaxation is the best to compute for large graphs. We also apply symmetry reduction on colors to further reduce the here introduced relaxation.

Let $X \in \{0,1\}^{n\times k}$ be a solution to the IP problem (1) and consider

$$Y = \begin{bmatrix} I_k \\ X \end{bmatrix} \begin{bmatrix} I_k & X^\top \end{bmatrix} = \begin{bmatrix} I_k & X^\top \\ X & XX^\top \end{bmatrix}.$$

Linearizing the block $XX^\top$, we obtain the following matrix lifting SDP relaxation for (1):

$$\begin{aligned} \max_{Z\in\mathbb{S}^n, X\in\mathbb{R}^{n\times k}} \quad & \langle I, Z\rangle \\ \text{s.t.} \quad & Z_{ij} = 0 \text{ for } \{ij\} \in E \\ & Z_{ii} \leq 1 \text{ for } i \in [n] \\ & Z_{ii} = \sum_{r\in[k]} X_{ir} \text{ for } i \in [n] \\ & Z \geq 0,\ X \geq 0 \\ & \begin{bmatrix} I_k & X^\top \\ X & Z \end{bmatrix} \succeq 0. \end{aligned} \tag{26}$$

Here, the positive semidefinite constraint is imposed on a matrix variable of the size $kn \times kn$. The zero pattern and constraints on the diagonal follow directly from the construction. The above relaxation has no constraints that ensure that a vertex can be colored by only one color, while vector lifting relaxations have such constraints.

Notice that the constraint $\operatorname{diag}(Y) \leq e$ is redundant when $k = 1$, and that the resulting SDP relaxation corresponds to one of the SDP relaxations for the Schrijver number, see [27].

5.1 Symmetry reduction on colors

In this section we exploit the invariance of the MkCS problem to reduce the number of variables in the SDP relaxation (26).

Let (Z, X) be a feasible solution to (26), and $\bar{X}$ be the average over all column permutations of X. Since the SDP relaxation (26) is convex and invariant under the permutations of the colors, $(Z, \bar{X})$, is feasible for (26). By construction, all columns of $\bar{X}$ are equal to each other. Therefore it is sufficient to consider solutions (Z, X) for (26), where the columns of X are equal to each other. If we denote a column of X by x, then the constraint $Z_{ii} = \sum_{r\in[k]} X_{ir}$ for $i \in [n]$ reduces to

$$\operatorname{diag}(Z) = kx, \tag{27}$$

and the SDP constraint to

$$\begin{bmatrix} I_k & ex^\top \\ xe^\top & Z \end{bmatrix} \succeq 0,$$

where $e \in \mathbb{R}^k$. Now, we use the Schur complement and (27) to rewrite the SDP constraint:

$$Z \succeq 0,\ Z - xe^\top ex^\top = Z - kxx^\top = Z - \tfrac{1}{k}\mathrm{diag}Z(\mathrm{diag}Z)^\top \succeq 0.$$

The following result follows from the above discussion.

Theorem 5. *The matrix lifting SDP relaxation* (26) *is equivalent to the following relaxation:*

$$\begin{aligned} \theta_k^3(G) = \quad & \max_{Z \in \mathbb{S}^n} \quad \langle I, Z\rangle \\ & \text{s.t.} \quad Z_{ij} = 0 \textit{ for } \{ij\} \in E \\ & \qquad\ \ Z_{ii} \leq 1 \textit{ for } i \in [n] \\ & \qquad\ \ Z \geq 0 \\ & \qquad\ \ \begin{bmatrix} k & \mathrm{diag}(Z)^\top \\ \mathrm{diag}(Z) & Z \end{bmatrix} \succeq 0, \end{aligned} \tag{28}$$

and the latter problem is strictly feasible.

Proof. The first part follows from the construction. To show strict feasibility, consider $M \succ 0$ from Lemma 2. Let $A_{\bar{G}}$ be the adjacency matrix of the complement of G. Then there exists $\varepsilon > 0$ such that $M + \varepsilon\begin{bmatrix} 0 & 0^\top \\ 0 & A_{\bar{G}} \end{bmatrix} \succ 0$. Therefore matrix $Z = \frac{1}{(n+1)}I + \varepsilon A_{\bar{G}}$ is a strictly feasible solution of (28) by construction. □

Let us relate our matrix and vector lifting relaxations for the MkCS problem. Note that the matrix lifting relaxation does not impose orthogonality constraints that correspond to the incidence vectors of different colors. Further, note that $\mathrm{diag}(Z) \leq e$ is redundant for (15) when the orthogonality constraints are imposed, see Lemma 1. Now, when we put those observations together, we arrive to the following result.

Theorem 6. *The SDP relaxation* (28) *is equivalent to the following vector lifting relaxation*

$$\begin{aligned} & \max_{Z,X \in \mathbb{S}^n} \quad \langle I, Z\rangle \\ & \text{s.t.} \quad Z_{ij} = 0, \quad \textit{for } \{ij\} \in E \\ & \qquad\ \ Z_{ii} \leq 1, \quad \textit{for } i \in [n] \\ & \qquad\ \ Z \geq 0,\ X \geq 0 \\ & \qquad\ \ Z - X \succeq 0 \\ & \qquad\ \ \begin{bmatrix} 1 & \mathrm{diag}(Z)^\top \\ \mathrm{diag}(Z) & Z + (k-1)X \end{bmatrix} \succeq 0. \end{aligned} \tag{29}$$

Proof. First, let (Z, X) be a feasible solution for the SDP relaxation from the theorem. We claim that Z is feasible for problem (28), and the corresponding objective values are equal. The linear constraints in (28) are readily satisfied. To verify that the SDP constraint in (28) is also satisfied, we proceed as follows

$$Z - \tfrac{1}{k}\mathrm{diag}(Z)\mathrm{diag}(Z)^\top \succeq Z - \tfrac{1}{k}Z - \tfrac{k-1}{k}X = \tfrac{k-1}{k}\big(Z - X\big) \succeq 0.$$

Here we exploit two positive semidefinite constraints from (29).

Now, let Z be a feasible solution to (28). We claim that $(Z, X) := (Z, \frac{1}{k}\mathrm{diag}(Z)\mathrm{diag}(Z)^\top)$ is feasible for the SDP relaxation (29), and the corresponding objective values are equal. The linear constraints of the SDP relaxation (29) are clearly satisfied for previously defined (Z, X). The SDP constraint of (28) implies $Z \succeq 0$ and

$$Z - X = Z - \tfrac{1}{k}\text{diag}(Z)\text{diag}(Z)^\top \succeq 0,$$

by the Schur complement. Therefore the first SDP constraint in (29) is satisfied. Finally,

$$Z + (k-1)X = Z + \tfrac{(k-1)}{k}\text{diag}(Z)\text{diag}(Z)^\top \succeq 0$$

and

$$Z + (k-1)X - \text{diag}(Z)\text{diag}(Z)^\top = Z + (k-1)X - kX = Z - X \succeq 0,$$

which implies the second SDP constraint in (29). □

Note that the SDP relaxation from Theorem 6 is weakened SDP relaxation for $\theta_k^2(G)$. The following result follows directly from the previous theorem and Theorem 2:

$$\theta_k^1(G) \le \theta_k^2(G) \le \theta_k^3(G),$$

where $\theta_k^1(G)$ is the optimal solution of the SDP relaxation (15), $\theta_k^2(G)$ is the optimal solution of (15) without constraints (15f), (15g), and $\theta_k^3(G)$ is the optimal solution of (28).

In the previous section we concluded that $\theta_k^3(G) = \vartheta'(G)$ when $k = 1$, where $\vartheta'(G)$ is the Schrijver number. We were not able to establish a relation between $\theta_k^3(G)$ and $\vartheta'(G)$ when $k > 1$. However, our numerical results in Section 8 suggest the following result.

Conjecture 1. *For $k \ge 2$, the upper bound $\theta_k^3(G)$, see* (28)*, is at least as good as the upper bound $\vartheta'_k(G)$, see* (9)*.*

We conclude this section listing some cases when our bounds are tight.

Lemma 4. *For a given k, let G be a graph such that $\alpha_k(G) = k\vartheta(G)$, then*

$$\vartheta_k(G) = \vartheta'_k(G) = \theta_k^1(G) = \theta_k^2(G) = \theta_k^3(G) = k\alpha(G) = \alpha_k(G).$$

Proof. Let G be any graph. We have $\alpha_k(G) \le k\alpha(G) \le k\vartheta(G)$. We have already shown $\alpha_k(G) \le \theta_k^1(G) \le \theta_k^2(G) \le \theta_k^3(G)$. We claim that $\theta_k^3(G) \le k\vartheta(G)$. To prove this, given Z a feasible solution to (28) let $\hat{Z} = \frac{1}{\langle I, Z\rangle} Z$. Then $\hat{Z}$ is feasible for (8). It is enough to show then that $k\langle J, \hat{Z}\rangle \ge \langle I, Z\rangle$ that is, $k\langle J, Z\rangle \ge \langle I, Z\rangle^2$. But, using the Schur complement, the last constraint from (28) is equivalent to $kZ - \text{diag}Z(\text{diag}Z)^\top \succeq 0$ which implies $k\langle J, Z\rangle - \langle I, Z\rangle^2 = ke^T Ze - (e^T \text{diag}Z)^2 = e^T(kZ - \text{diag}Z(\text{diag}Z)^\top)e \ge 0$.

Also, $\alpha_k(G) \le \vartheta'_k(G) \le \vartheta_k(G)$. We claim that $\vartheta_k(G) \le k\vartheta(G)$. To prove this notice that if Z is a feasible solution to (7) then $\frac{1}{k}Z$ is feasible for (8).

Now, if G is such that $\alpha_k(G) = k\vartheta(G)$, all previous inequalities become equalities. □

Notice that $\alpha_k(G) \le k\alpha(G) \le k\vartheta(G)$ and thus the assumption $\alpha_k(G) = k\vartheta(G)$ in Lemma 4 is equivalent to $\alpha_k(G) = k\alpha(G)$ and $\alpha(G) = \vartheta(G)$. Several families of graphs satisfy these conditions. For instance if G is a perfect graph with at least K non-intersecting independent sets of size $\alpha(G)$, then $\alpha_k(G) = k\alpha(G) = k\vartheta(G)$ for all $k \le K$. In Proposition 1, we characterize a family of (not necessarily perfect) graphs satisfying the conditions of Lemma 4. The condition is given in terms of the chromatic number of the complement graph of G, $\chi(\bar{G})$. Notice that $\chi(\bar{G})$ is equal to the clique cover number of G.

Proposition 1. *Let G be a graph such that $|V(G)| = \chi(\bar{G})\chi(G)$. Then Lemma* (4) *holds for all $k \leq \chi(G)$.*

Proof. For any graph $\chi(\bar{G}) \geq \alpha(G) \geq |V(G)|/\chi(G)$. Thus $|V(G)| = \chi(\bar{G})\chi(G)$ implies $\chi(\bar{G}) = \alpha(G)$ and $\alpha_{\chi(G)}(G) = \chi(G)\alpha(G)$. Using the Lovász sandwich theorem [44] $\alpha(G) \leq \vartheta(G) \leq \chi(\bar{G})$ we obtain that $\alpha(G) = \vartheta(G)$. Then the statement follows from Lemma 4. □

The set of vertex transitive graphs contains a number of non-trivial examples for Proposition 1, for instance see Table 1. In the table $H(v,d) := H(v,d,1)$ denotes the Hamming graph, while $J(v,d) := J(v,d,d-1)$ denotes the Johnson graph. For definitions of this graphs see Section 6.

Table 1: Graphs for which all our SDP upper bounds are tight.

Graph	$\|V(G)\|$	$\chi(G)$	$\chi(\bar{G})$	
$H(v,d)$	d^v	d	d^{v-1}	[57]
$J(v,2)$, v even	$\binom{v}{2}$	$v-1$	$\frac{v}{2}$	[15]
$J(v,3)$, $v \equiv 1$ or $3 \bmod 6$	$\binom{v}{3}$	$v-2$	$\frac{v(v-1)}{6}$	[6],[15]

6 Reductions using graph symmetry

In this section we first prove that several inequalities in the strongest vector lifting relaxation (15) are redundant for vertex-transitive graphs. Then, we present reduced SDP relaxations for different classes of highly symmetric graphs.

Proposition 2. *Let G be a vertex transitive graph. Then constraints* (15g) *are redundant for the SDP relaxation* (15a)–(15e).

Proof. Let (Z,X) be an optimal solution for the SDP relaxation (15a)–(15e). By averaging and vertex transitivity of the graph we obtain an optimal solution $(\bar{Z},\bar{X})$ such that $\bar{Z}_{ii} = \bar{Z}_{jj}$ for all i,j. Given i and j, let $z = \bar{Z}_{ii} = \bar{Z}_{jj}$ and $y = \bar{Z}_{ij} + (k-1)\bar{X}_{ij}$. Looking at the principal sub-matrix of constraint (15e) indexed by i and j we obtain

$$\begin{bmatrix} z & y \\ y & z \end{bmatrix} \succeq 0.$$

Which implies $z^2 \geq y^2$, that is equivalent to (15g) from the non-negativity of z and y. □

In general, constraints (15f) are not redundant for the SDP relaxation (15) and vertex transitive graphs. For example, for the Petersen graph and $k=2$ we have that $\theta_k^1(G)$ equals 7.5, while $\theta_k^2(G)$ is equal to 8.

Below we consider symmetry reduction for the Hamming and Johnson graphs. We apply the general theory of symmetry reduction to our SDP relaxations, see e.g., [9, 23, 58, 63], and therefore omit details. While in the mentioned and other papers in the literature, symmetry reduced relaxations are linear programming relaxations, our simplified relaxations have also second order cone constraints.

Let us now define the Hamming graphs. The vertex set V is the set of d-tuples of letters from an alphabet of size q, so $n := |V| = q^d$. The adjacency matrices $H(d,q,j)$ $(j = 0,\ldots,d)$ of the Hamming association scheme are defined by the number of positions in which two d-tuples differ.

In particular, $H(d,q,j)_{x,y} = 1$ if x and y differ in j positions, for $x, y \in V$ $(j = 0, \ldots, d)$, i.e., if their Hamming distance $d(x,y) = j$. $H(d,q,1)$ is the adjacency matrix of the well-known Hamming graph, which can also be obtained as the Cartesian product of d copies of the complete graph K_q. Further, we denote by $H^-(d,q,j)$ the graph whose Hamming distance $d(x,y) \leq j$. The matrices of the Hamming association scheme can be simultaneously diagonalized. The eigenvalues (character table) of the Hamming scheme can be expressed in terms of Krawtchouk polynomials:

$$K_i(u) := \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} \binom{u}{j} \binom{d-u}{i-j}, \quad i, u = 0, \ldots, d.$$

In particular, eigenvalues of $B_i := H(d,q,i)$ $(i = 0, 1, \ldots, d)$ are $K_i(j)$ for $j = 0, 1, \ldots, d$.

Now, let us consider the SDP relaxation (9). Since the relaxation is invariant under the permutation group of the Hamming graph, we can restrict optimization of the SDP relaxation to feasible points in the Bose-Mesner algebra, see e.g., [9, 23]. Therefore, we assume $Z = \sum_{i=0}^{d} z_i B_i$ in (9). Further we consider the case in which the adjacency matrix corresponds to $H(d,q,1)$. Relaxations are similar for any other $H(d,q,j)$ or $H^-(d,q,j)$ $(j = 2 \ldots, d)$. After the substitution, the SDP relaxation (9) reduces to:

$$\begin{aligned} \vartheta'_k(G) = \max_{z \in \mathbb{R}^{d+1}} \ & k + \sum_{i=2}^{d} z_i \langle J, B_i \rangle \\ \text{s.t.} \ & \tfrac{k}{n} + \sum_{i=2}^{d} z_i K_i(j) \geq 0, \ \text{for } j \in \{0, 1, \ldots, d\} \\ & 1 - \tfrac{k}{n} - \sum_{i=2}^{d} z_i K_i(j) \geq 0, \ \text{for } j \in \{0, 1, \ldots, d\} \\ & z_0 = \tfrac{k}{n}, \ z_1 = 0, \ z_i \geq 0, \ \text{for } i \in \{2, \ldots, d\}. \end{aligned} \tag{30}$$

Note that (30) is a linear program. Next, we reduce our matrix lifting SDP relaxation (28) by using similar arguments as before. The resulting θ^3-bound is as follows:

$$\begin{aligned} \theta^3(G) = \max_{z \in \mathbb{R}^{d+1}} \ & n \cdot z_0 \\ \text{s.t.} \ & \sum_{i=0}^{d} z_i K_i(0) - \tfrac{n}{k} z_0^2 \geq 0 \\ & \sum_{i=0}^{d} z_i K_i(j) \geq 0, \ \text{for } j \in \{1, \ldots, d\} \\ & z_0 \leq 1, \ z_1 = 0, \ z_i \geq 0, \ \text{for } i \in \{0, 2, 3, \ldots, d\}. \end{aligned} \tag{31}$$

Note that (31) is the optimization problem with linear objective, $2d+1$ linear inequalities and one convex quadratic constraint. Finally, we simplify the SDP relaxation whose optimal value is denoted by $\theta^2(G)$, i.e., the vector lifting SDP relaxation (15a) without (15f) and (15g). Here, we also may restrict $X = \sum_{i=0}^{d} x_i B_i$.

$$\begin{aligned}
\theta^2(G) = \max_{z,x \in \mathbb{R}^{d+1}} \ & n \cdot z_0 && (32) \\
\text{s. t.} \ & \sum_{i=0}^{d} (z_i - x_i) K_i(j) \geq 0, \ \text{for } j \in \{0, 1, \ldots, d\} \\
& \sum_{i=0}^{d} z_i K_i(0) + (k-1) \sum_{i=0}^{d} x_i K_i(0) - n z_0^2 \geq 0 \\
& \sum_{i=0}^{d} z_i K_i(j) + (k-1) \sum_{i=0}^{d} x_i K_i(j) \geq 0, \ \text{for } j \in \{1, \ldots, d\} \\
& x_0 = 0, \ x_i \geq 0, \ \text{for } i \in \{1, 2, 3, \ldots, d\} \\
& z_0 \leq 1, \ z_1 = 0, \ z_i \geq 0, \ \text{for } i \in \{0, 2, 3, \ldots, d\}.
\end{aligned}$$

Note that the optimization problem (32) has a linear objective, one second order cone constraint, and several linear constraints.

Finally, to compute $\theta^1(G)$, we add to (32) the symmetry reduced inequalities (15f), i.e.,

$$1 - 2z_0 + z_i + (k-1)x_i \geq 0, \qquad i = 1, \ldots, d.$$

One can similarly derive simplified SDP relaxations for graphs whose corresponding algebra is diagonalizable, such as for the Johnson graph $J(v, d, q)$. The Johnson graph is defined as follows. Let Ω be a fixed set of size v and let d be an integer such that $1 \leq d \leq v/2$. The vertices of the Johnson graph $J(v, d, q)$ are the subsets of Ω with size d. Two vertices are connected if the corresponding sets have q elements in common. In the literature, the graph $J(v, d, d-1)$ is known as the Johnson graph $J(v, d)$, while $J(v, d, 0)$ is known as the Kneser graph $K(v, d)$. Matrices corresponding to $J(v, d, q)$, $q = 0, 1, \ldots, d$ can be simultaneously diagonalized. The eigenvalues (character table) of the Johnson scheme can be expressed in terms of Eberlein polynomials:

$$E_i(u) := \sum_{j=0}^{i} (-1)^j \binom{u}{j} \binom{d-u}{i-j} \binom{v-d-u}{i-j}, \quad i, u = 0, \ldots, d.$$

Eigenvalues of $J(v, d, i)$ $(i = 0, 1, \ldots, d)$ are $E_i(j)$ for $j = 0, 1, \ldots, d$. Now, one can proceed similarly as with the Hamming graphs in order to obtain simplified relaxations for the Johnson graphs. The resulting relaxations for the Johnson graphs differ from the relaxations for the Hamming graphs in the type of polynomials.

7 Symmetry reductions for other partition problems

Notice that a k-colorable subgraph of a graph corresponds to a partition of the graph's vertices into k+1 subsets, i.e., k independent sets and the rest of the vertices. Therefore one can consider the k-colorable subgraph problem as a graph partition problem, which is invariant under permutations of the subsets. It is not difficult to verify that other graph partition problems such as the max-k-cut problem and the k-equipartition problem are also invariant under permutations of the subsets. The max-k-cut problem is the problem of partitioning the vertex set of a graph into k sets such that the total weight of edges joining different sets is maximized. For the problem formulation and related SDP relaxations see e.g., [10, 11, 54]. The k-equipartition problem is the problem of partitioning the vertex set of a graph into k sets of equal cardinality such that the total weight of edges joining different sets is minimized. For the problem formulation and related SDP relaxations see e.g., [38, 60, 63, 65].

It is known that vector and matrix lifting SDP relaxations for the max-k-cut and k-equipartition problems are equivalent. In particular, De Klerk et al. [10] prove the equivalence of the relaxations for the max-k-cut problem, by exploiting the invariance of the max-k-cut problem under permutations of the subsets. Sotirov [60] prove the equivalence of three different SDP relaxations for the k-equipartition problem; a matrix lifting relaxation, a vector lifting relaxation, and an SDP relaxation for the k-equipartition problem derived as a special case of the quadratic assignment problem.

Here, we prove the same results by using the approach from Section 4.1 and 5.1. We remark that the proof here is more elegant than the one from [60].

We denote the optimal value of the vector lifting SDP relaxation for the max-k-cut (resp., k-equipartition) problem on graph G by $MkC_v(G)$ (resp., $Ek_v(G)$). The vector lifting relaxations of both problems are particular cases of the relaxation for the general graph partition problem by Wolkowicz and Zhao [65]. Let L be the Laplacian matrix of G, then the symmetry-reduced versions of vector lifting relaxations are given as follows:

$$
\begin{aligned}
MkC_v(G) = \max_{Z,X\in\mathbb{S}^n} \ & \tfrac{1}{2}\langle L, Z\rangle \\
\text{s. t.} \ & X_{ii} = 0, \quad \text{for } i \in [n] \\
& Z_{ii} = 1, \quad \text{for } i \in [n] \\
& Z \geq 0, \ X \geq 0 \\
& Z - X \succeq 0 \\
& Z + (k-1)X - J \succeq 0.
\end{aligned}
\qquad
\begin{aligned}
Ek_v(G) = \min_{Z,X\in\mathbb{S}^n} \ & \tfrac{1}{2}\langle L, Z\rangle \\
\text{s. t.} \ & X_{ii} = 0, \quad \text{for } i \in [n] \\
& Z_{ii} = 1, \quad \text{for } i \in [n] \\
& Z \geq 0, \ X \geq 0 \\
& Z - X \succeq 0 \\
& Z + (k-1)X - J \succeq 0 \\
& Ze = \tfrac{n}{k}e.
\end{aligned}
$$

Next, we look at the matrix lifting relaxations for the two problems. For the max-k-cut problem, reducing the matrix lifting SDP relaxation results in the relaxation by van Dam and Sotirov [64], which is equivalent to the relaxation by Frieze and Jerrum [19]. Similarly, for the k-equipartition problem, reducing the matrix lifting SDP relaxation results in a well-known SDP relaxation by Karisch and Rendl [38], which is equivalent to the relaxation by Sotirov [61]. Both relaxations are presented below, and notation is analogous to the notation in the vector lifting case.

$$
\begin{aligned}
MkC_m(G) = \max_{Z\in\mathbb{S}^n} \ & \tfrac{1}{2}\langle L, Z\rangle \\
\text{s. t.} \ & Z_{ii} = 1, \quad \text{for } i \in [n] \\
& Z \geq 0, \\
& Z - \tfrac{1}{k}J \succeq 0.
\end{aligned}
\qquad
\begin{aligned}
Ek_m(G) = \min_{Z\in\mathbb{S}^n} \ & \tfrac{1}{2}\langle L, Z\rangle \\
\text{s. t.} \ & Z_{ii} = 1, \quad \text{for } i \in [n] \\
& Z \geq 0 \\
& Z - \tfrac{1}{k}J \succeq 0 \\
& Ze = \tfrac{n}{k}e.
\end{aligned}
$$

To show that the vector and matrix lifting relaxations are equivalent, observe that for both the k-equipartition and max-k-cut problem one can construct feasible solutions to the matrix lifting relaxation from the vector lifting relaxation with the same objective value, and the other way round. First, from a feasible solution Z for each of the symmetry-reduced matrix lifting relaxations we obtain a feasible solution $(Z, \frac{1}{k-1}(J-Z))$ for the corresponding symmetry-reduced vector lifting relaxation. For the opposite direction, a feasible solution (Z, X) to each of the symmetry-reduced vector lifting relaxations provides a feasible solution Z to the corresponding matrix lifting relaxation, as in the case of the MkCS problem, see also the proof of Theorem 6. Hence the vector and matrix lifting relaxations are equivalent.

8 Numerical results

In Section 8.1 we compare our upper and lower bounds with the bounds for the MkCS problem on graphs from the literature. In Section 8.2 we present upper and lower bounds for highly symmetric graphs of larger sizes by exploiting the results from Section 6. In Section 8.3, we exploit bounds for the MkCS problem to compute bounds on the chromatic number $\chi(G)$ of a graph G.

All computations are done in MATLAB R2018b with Yalmip [43] on a computer with two processors Intel® Xeon® Gold 6126 CPU @ 2.60 GHz and 512 GB of RAM. Semidefinite programs are solved with MOSEK, Version 8.0.0.80.

Lower bounds. To obtain lower bounds for the MkCS problem, we use two heuristics. Our first heuristic is based on the MATLAB heuristic algorithm for finding maximum independent set [51]. Namely, we transform the MkCS to the stable set problem as described in Section 4.3, and then use the mentioned algorithm. The heuristic first finds the vertices with the minimum degree. If there is only one such vertex, it is added to the stable set, and its neighbors are excluded from the stable set. If there are several vertices with the minimum degree, then the heuristic looks at the support of each such vertex, i.e., the sum of degrees of this vertex's neighbors. The vertex with the largest support is added to the stable set, and its neighbors are excluded from it. If there are several vertices with the largest support, they are chosen according to the priority specified by the user in advance. The procedure is repeated until all vertices are considered.

Our second heuristic is a tabu search algorithm. A greedy heuristic is used to find the initial solution. Our greedy algorithm sorts vertices of a graph in order of ascending degree, and going down this list, tries to color as many vertices as possible with the same color. When the end of the list is reached, the colored nodes are removed from the list and the algorithm starts again at the top of the list with the next color. After the initial solution is found, a tabu list is used to obtain a better solution. In each iteration of the main algorithm one of the colored vertices that has the most edges in common with non-colored vertices is uncolored, and the other uncolored nodes are checked if they can be colored. For those that get a color, we prevent them from being uncolored for a certain number of iterations. The tabu search algorithm outperforms the other heuristic on dense graphs.

8.1 Bounds for the MkCS problem for general graphs

In this section, we compute upper bounds on $\alpha_k(G)$ for benchmark graphs from the literature. In particular we compare our bounds with the bounds from Campêlo and Corrêa [8], Januschowski and Pfetsch [34]. In the former paper, the authors present an IP formulation of the MkCS and implement a parallel subgradient algorithm. In the latter paper, the authors propose a branch-and-cut method that accounts for both, the symmetry with respect to color permutations and the inner graph symmetry.

Graphs. Graphs used in the mentioned two papers are from two different sources. We first describe graphs from the Second DIMACS Implementation Challenge for the max-clique problem [36]. Graphs "brockx_y" are random graphs with x vertices and depth of clique hiding y, see [5]. Graphs "gen$x$_py-z" are artificially generated with x vertices, edge density y and known embedded clique of the size z. "san$x$_py_z" and "sanr$x$_py are randomly generated graph instances where parameters x, y and z have the same meaning as for "gen" instances. "C$x.y$" are random graphs on x vertices with edge probability y. "p-hat" graphs are generated with the p-hat generator, which generalizes the classical uniform random graph generator and the resulting graphs have larger cliques than uniform graphs, see [24]. Graphs "keller$x$" are based on Keller's conjecture on tilings using hypercubes. "c-fat$x$-y" are graphs with x vertices based on fault diagnosis problems.

The second group of graphs are from the COLOR02 symposium [37]. "Myciel" are graphs based on the Mycielski transformation. They are triangle free, but the coloring number increases in problem size so that the graphs can have arbitrary large gaps between their clique and chromatic numbers. The "FullIns" and "Insertions" graphs are generalizations of the Mycielski graphs. "DSJC$n.p$" are standard random graphs with n vertices where an edge between two vertices appears with probability p. "DSJR$n.p$" are graphs with n vertices randomly distributed in the unit square with an edge between two vertices if the length of the line between them is less than $p/10$, "queen$n$-n" graph is a graph with vertices that correspond to the squares of the $n\times n$ chess board and are connected by an edge if the corresponding squares are in the same row, column, or diagonals (according to the queen move rule at the chess game). "MANN_ax" is the graph from the clique formulation of the set covering problem for the Steiner triple system on a set with x elements see, e.g., [46] for the problem description.

Graphs in Tables 2 to 9 that are marked by a superscript ^{02} were used as benchmarks in the COLOR02 symposium. Other graphs are benchmarks from the Second DIMACS Challenge [36] or their complements[1]. All graphs in the tables that end with a superscript ^{c} are complements of the original graphs.

Table 2 (resp., Table 4) compares our bounds with the results from Campêlo and Corrêa [8] (resp., Januschowski and Pfetsch [34]) for graphs up to 200 vertices. The first column in Table 2 lists graphs, while the second, third, and fourth columns list the number of vertices, number of edges, and density of the graphs, respectively. The fifth column in Table 2 specifies the number of colors k, and sixth upper bounds on $\alpha_k(G)$ from Campêlo and Corrêa [8]. In the remaining columns we list the following upper bounds: $\vartheta_k(G)$ see (7), $\vartheta'_k(G)$ see (9), $\theta_k^3(G)$ see (28), $\theta_k^2(G)$ that is the optimal solution of (15) without constraints (15f), (15g), and $\theta_k^1(G)$ see (15). We also compute $\theta_k^1(G)$ with BQP inequalities (23) and (24). In particular, we use a cutting plane method that adds up to $2n$ most violated BQP inequalities for at most four iterations. In the last column of Table 2 we list lower bounds obtained by one of our heuristics. We report only the best lower bound. Table 4 is organized similarly to Table 2. However, the sixth column of Table 4 lists the optimal solution for $\alpha_k(G)$ computed by Januschowski and Pfetsch [34].

We highlight by boldface the best upper bounds for a given graph and k. All upper bounds are rounded to the nearest second digit. We omit in this section results for the Johnson and Hamming graphs used in [8, 34] since we devote the whole next section to highly symmetric graphs. Table 3 and 5 report MOSEK running times in seconds.

Our numerical results in Table 2 show that for all compared graphs, except "c-fat200-5^c ", our upper bounds dominate the upper bounds from [8]. In particular, for eight out of ten graphs and all tested k our best SDP bounds improve the upper bounds from [8], and for "c-fat200-2^c " already θ_k^3 bound is tight for both k. Our lower bounds significantly improve the lower bounds from [8] for most of the instances. Table 4 shows that our upper bounds provide optimal values for nine out of

[1]The description in [8] does not mention taking the complements of the graphs, but the edge densities and bounds in the paper correspond to the complements of the DIMACS graphs.

twenty one instances.

Our best upper bound, as expected, is $\theta_k^1(G)$ with BQP inequalities (23) and (24). However, in many cases several SDP relaxations provide the same bound as the strongest one, while the computational times for the vector lifting relaxations are substantially larger than the computational times for other relaxations. Table 3 and 5 also show that it takes less computational time to solve SDP relaxations for dense graphs. Note that several of the graphs considered here have additional symmetry (see Januschowski and Pfetsch [34] for details) that can be exploited to reduce computational effort. However, we do not exploit symmetry for those graphs here.

For all graphs the bound $\theta_k^3(G)$ is at least as good as the bound $\vartheta'_k(G)$ that dominates $\vartheta_k(G)$. However, computational times for all three bounds are similar.

Table 2: Results for graphs with up to 200 vertices considered by Campêlo and Corrêa [8].

Graph	n	$\|E\|$	ρ,%	k	UB [8]	ϑ_k	ϑ'_k	θ_k^3	θ_k^2	θ_k^1	θ_k^1 + BQP	LB
brock200_2^c	200	10024	50	2	30.10	28.45	**28.26**	**28.26**	**28.26**	**28.26**	**28.26**	19
				3	44.90	42.68	**42.39**	**42.39**	**42.39**	**42.39**	**42.39**	28
brock200_4^c	200	6811	34	2	51.80	42.59	**42.24**	**42.24**	**42.24**	**42.24**	**42.24**	30
				3	76.60	63.88	**63.36**	**63.36**	**63.36**	**63.36**	**63.36**	42
C125.9^c	125	787	10	2	79.40	75.61	75.09	74.63	74.41	74.11	**74.10**	64
				3	115.60	112.86	112.18	107.27	106.96	105.90	**105.31**	89
keller4^c	171	5100	35	2	27.90	28.02	**26.93**	**26.93**	**26.93**	**26.93**	**26.93**	22
				3	41.90	42.04	**40.40**	**40.40**	**40.40**	**40.40**	**40.40**	31
c-fat200-2^c	200	16665	84	2	**46.00**	46.33	46.33	**46.00**	**46.00**	**46.00**	**46.00**	46
				3	**68.00**	68.33	68.33	**68.00**	**68.00**	**68.00**	**68.00**	68
c-fat200-5^c	200	11427	57	2	**116.00**	120.69	120.69	120.69	120.69	120.69	119.42	116
				3	**172.00**	181.04	181.04	181.03	181.01	175.99	175.24	172
gen200_p0.9_44^c	200	1990	10	2	88.00	88.00	88.00	88.00	88.00	88.00	**87.99**	81
				3	132.00	132.00	132.00	131.94	131.90	131.84	**131.66**	114
gen200_p0.9_55^c	200	1990	10	2	109.00	103.23	102.79	100.84	100.52	**100.36**	**100.36**	93
				3	161.60	150.52	149.70	146.22	146.01	145.61	**145.34**	128
san200_0.7_2^c	200	5970	30	2	36.00	36.00	35.68	35.63	35.62	35.60	**35.60**	31
				3	54.00	53.98	53.34	53.24	53.24	53.24	**53.22**	44
san200_0.9_2^c	200	1990	10	2	117.00	109.05	108.68	106.61	106.24	**106.03**	**106.03**	99
				3	184.00	157.29	156.59	152.84	152.26	151.44	**151.16**	137

Table 3: Running times for upper bounds of graphs with up to 200 vertices from [8], in seconds.

Graph	n	$\|E\|$	ρ,%	k	[8]	ϑ_k	ϑ'_k	θ_k^3	θ_k^2	θ_k^1	θ_k^1 + BQP
brock200_2^c	200	10024	50	2	162	71	99	60	1333	1267	1291
				3	242	71	104	61	1088	1147	1168
brock200_4^c	200	6811	34	2	143	144	193	115	1290	1671	1697
				3	221	143	195	117	1742	1575	1636
C125.9^c	125	787	10	2	70	29	51	29	228	295	1130
				3	125	32	54	25	322	342	1806
keller4^c	171	5100	35	2	27	59	114	62	963	1090	1217
				3	30	68	125	68	975	1152	1190
c-fat200-2^c	200	16665	84	2	<1	7	9	5	940	879	3627
				3	<1	7	8	5	726	616	639
c-fat200-5^c	200	11427	57	2	<1	36	61	35	902	1199	7649
				3	<1	40	52	33	1018	1148	6955
gen200_p0.9_44^c	200	1990	10	2	378	330	507	258	2176	2976	13570
				3	364	330	460	258	3013	3528	16242
gen200_p0.9_55^c	200	1990	10	2	46	284	500	338	2542	3512	14187
				3	71	311	475	312	3268	3532	10893
san200_0.7_2^c	200	5970	30	2	32	243	415	220	2540	3003	12066
				3	28	199	373	190	2593	2988	5508
san200_0.9_2^c	200	1990	10	2	1,800	282	478	353	2954	3645	13590
				3	46	316	515	328	3555	3899	11889

Table 4: Results for graphs with up to 200 vertices considered by Januschowski and Pfetsch [34].

Graph	n	$\|E\|$	ρ,%	k	α_k [34]	ϑ_k	ϑ'_k	θ^3_k	θ^2_k	θ^1_k	θ^1_k + BQP
1-FullIns_4[02]	93	593	14	3	87	93.00	93.00	92.59	92.57	92.43	**91.33**
4-FullIns_3[02]	114	541	8	3	106	114.00	114.00	107.40	107.31	107.30	**107.25**
5-FullIns_3[02]	154	792	7	3	144	154.00	154.00	145.33	145.25	145.25	**145.23**
1-Insertions_4[02]	67	232	10	3	63	**67.00**	**67.00**	**67.00**	**67.00**	**67.00**	**67.00**
c-fat200-1[c]	200	18366	92	6	72	**72.00**	**72.00**	**72.00**	**72.00**	**72.00**	**72.00**
				7	84	**84.00**	**84.00**	**84.00**	**84.00**	**84.00**	**84.00**
c-fat200-1	200	1534	8	10	180	184.67	184.67	184.67	184.67	**184.65**	**184.65**
c-fat200-2[c]	200	16665	84	7	156	156.33	156.33	**156.00**	**156.00**	**156.00**	**156.00**
				8	178	178.33	178.33	**178.00**	**178.00**	**178.00**	**178.00**
DSJC125.9[02]	125	6961	90	4	16	**16.00**	**16.00**	**16.00**	**16.00**	**16.00**	**16.00**
				5	20	**20.00**	**20.00**	**20.00**	**20.00**	**20.00**	**20.00**
				6	23	23.95	23.95	**23.73**	**23.73**	**23.73**	**23.73**
gen200_p0.9_44	200	17910	90	4	20	**20.00**	**20.00**	**20.00**	**20.00**	**20.00**	**20.00**
gen200_p0.9_55	200	17910	90	4	17	18.22	18.20	**18.15**	**18.15**	**18.15**	**18.15**
myciel5[02]	47	236	22	4	44	**47.00**	**47.00**	**47.00**	**47.00**	**47.00**	**47.00**
				5	46	**47.00**	**47.00**	**47.00**	**47.00**	**47.00**	**47.00**
myciel6[02]	95	755	17	3	83	95.00	95.00	95.00	95.00	95.00	**93.32**
queen6_6[02]	36	580	92	6	32	35.97	35.97	35.84	35.84	**35.81**	**35.81**
san200_0.9_1	200	17910	90	4	16	16.10	16.08	**16.07**	**16.07**	**16.07**	**16.07**
san200_0.9_2	200	17910	90	4	16	17.23	17.21	17.21	17.21	17.21	**17.20**
sanr200_0.9	200	17863	90	4	16	**17.91**	**17.91**	**17.91**	**17.91**	**17.91**	**17.91**

Table 5: Running times for graphs with up to 200 vertices from [34], seconds.

Graph	n	$\|E\|$	ρ,%	k	ϑ_k	ϑ'_k	θ_k^3	θ_k^2	θ_k^1	θ_k^1 + BQP
1-FullIns_4[02]	93	593	14	3	6	8	11	107	103	554
4-FullIns_3[02]	114	541	8	3	18	22	20	230	196	901
5-FullIns_3[02]	154	792	7	3	85	100	89	1,047	713	3,551
1-Insertions_4[02]	67	232	10	3	2	2	1	14	10	34
c-fat200-1[c]	200	18366	92	6	2	2	1	577	776	806
				7	2	3	1	739	778	804
c-fat200-1	200	1534	8	10	300	742	300	3,612	5,184	5,485
c-fat200-2[c]	200	16665	84	7	7	10	5	615	620	638
				8	6	10	4	537	644	673
DSJC125.9[02]	125	6961	90	4	<1	<1	<1	103	104	115
				5	<1	<1	<1	92	106	118
				6	<1	1	<1	145	141	147
gen200_p0.9_44	200	17910	90	4	3	3	2	955	847	1,002
gen200_p0.9_55	200	17910	90	4	5	6	4	1,238	1,365	1,386
myciel5[02]	47	236	22	4	<1	<1	<1	1	1	4
				5	<1	<1	<1	2	2	7
myciel6[02]	95	755	17	3	5	6	4	43	37	366
queen6_6[02]	36	580	92	6	<1	<1	<1	<1	<1	3
san200_0.9_1	200	17910	90	4	4	4	3	1,103	1,282	2,897
san200_0.9_2	200	17910	90	4	4	5	3	963	1,136	2,036
sanr200_0.9	200	17863	90	4	4	4	2	871	888	918

Computing $\theta_k^2(G)$ and $\theta_k^1(G)$ for graphs with more than 200 vertices is computationally very demanding. This is due to the presence of two SDP constraints and a large number of non-negativity constraints in each of the related SDP relaxations. Therefore we do not compute $\theta_k^2(G)$ and $\theta_k^1(G)$ for graphs with more than 200 vertices. In Table 6 and 7 (resp., Table 8 and 9) we present numerical results for graphs with more than 200 vertices and compare them with results from [8] (resp., [34]). Table 6 and Table 8 are organized similarly to Table 2 and Table 4, respectively. If a graph with more than 200 vertices from [8] and [34] is not present in our computations, that means we were not able to compute the corresponding bounds due to a memory issue related to the solver. We exclude computations for the Hamming graphs here.

Table 6 shows that our upper bounds dominate bounds from [8] also for large graphs. Our lower bounds also significantly improve lower bounds from the same paper. Table 8 shows that our SDP bounds are tight for most of the instances. In Table 8 we omitted results for DSJR500.1[c] instance since there is discrepancy between our results and the results from [34]. Namely, for DSJR500.1[c] we get $\theta_3^3 = 36$ and also 36 for the lower bound, while [34] reports $\alpha_3 = 37$. Similarly, we get for the same graph that $\theta_4^3 = 47.24$ (resp., $\theta_5^3 = 58.24$) for the lower bound 47 (resp., 58), while [34] reports $\alpha_4 = 48$ (resp., 59). Note that computational times in Tables 7 and 9 indicate that we are able to compute strong bounds for large dense graphs in reasonable time even without exploiting graph symmetry.

Table 6: Results for graphs with more than 200 vertices considered by Campêlo and Corrêa [8].

Graph	n	$\|E\|$	ρ,%	k	UB [8]	ϑ_k	ϑ'_k	θ^3_k	LB
C250.9^c	250	3141	10	2	134.50	112.48	**111.63**	**111.63**	83
				3	201.60	168.72	167.45	**167.15**	123
c-fat500-2^c	500	115611	92	2	**52.00**	**52.00**	**52.00**	**52.00**	52
				3	**78.00**	**78.00**	**78.00**	**78.00**	78
p_hat300-1^c	300	33917	76	2	20.90	20.14	**20.04**	**20.04**	14
				3	31.00	30.20	**30.06**	**30.06**	21
p_hat300-2^c	300	22922	51	2	62.10	53.93	53.43	**52.96**	43
				3	91.00	80.80	80.03	**77.28**	61

Table 7: Running times for upper bounds of graphs with more than 200 vertices from [8], in seconds.

Graph	n	$\|E\|$	ρ,%	k	[8]	ϑ_k	ϑ'_k	θ^3_k
C250.9^c	250	3141	10	2	804	1,717	3,245	1,952
				3	798	1,382	3,490	1,861
c-fat500-2^c	500	115611	92	2	<1	70	115	52
				3	<1	71	116	47
p_hat300-1^c	300	33917	76	2	57	133	174	107
				3	79	119	183	109
p_hat300-2^c	300	22922	51	2	1,212	629	1,098	626
				3	1,592	636	955	718

Table 8: Results for graphs with more than 200 vertices considered by Januschowski and Pfetsch [34].

Graph	n	$\|E\|$	ρ,%	k	α_k [34]	ϑ_k	ϑ'_k	θ^3_k
2-FullIns_4^{02}	212	1621	7	3	202	212.00	212.00	**207.65**
c-fat500-1^c	500	120291	96	4	56	**56.00**	**56.00**	**56.00**
				5	70	**70.00**	**70.00**	**70.00**
				6	84	**84.00**	**84.00**	**84.00**
DSJC250.9^{02}	250	27897	90	3	14	14.93	14.93	**14.86**
				4	18	19.81	19.80	**19.72**
MANN_a27	378	70551	99	15	45	**45.00**	**45.00**	**45.00**
				17	51	**51.00**	**51.00**	**51.00**
				20	60	**60.00**	**60.00**	**60.00**
				22	66	**66.00**	**66.00**	**66.00**
				25	75	**75.00**	**75.00**	**75.00**

Table 9: Running times for graphs with more than 200 vertices from [34], seconds.

Graph	n	$\|E\|$	ρ,%	k	ϑ_k	ϑ'_k	θ^3_k
2-FullIns_4	212	1621	7	3	512	576	993
c-fat500-1^c	500	120291	96	4	32	39	20
				5	28	31	18
				6	23	31	16
DSJC250.9	250	27897	90	3	13	17	16
				4	18	25	15
MANN_a27	378	70551	99	15	5	6	6
				17	6	6	8
				20	7	8	8
				22	7	7	8
				25	7	7	7

8.2 Bounds for the MkCS problem for highly symmetric graphs

In this section we present results for highly symmetric graphs. In particular, we consider Johnson $J(v,d,q)$ and Hamming graphs $H(d,q,j)$. For definition and simplified relaxations see Section 6.

We first compare our upper bounds for the Johnson graphs $J(v,d,q)$ for which $\alpha_k(G)$ is known. In particular, for the Kneser graph $K(v,d) = J(v,d,0)$ we have the following lower bound on the size of the MkCS problem:

$$\alpha_k(K(v,d)) \geq \sum_{i=1}^{k} \binom{v-i}{d-1}. \tag{33}$$

The lower bound above is tight for $d = 2$ and $v \geq k+3$, see [4]. Therefore we use those parameters in our first experiment to demonstrate the quality of our upper bounds for the Kneser graphs.

Table 10 presents results for $d = 2$, $v = 10, 15, 20$, and increasing k. We keep increasing k as long as our bounds are nontrivial, i.e., smaller than the number of vertices in the graph. The table reads similarly to the tables in the previous section. For all listed examples except for $J(15,2,0)$ and $k = 7$, bounds $\vartheta_k(G)$, $\vartheta'_k(G)$, $\theta^3_k(G)$, $\theta^2_k(G)$, $\theta^1_k(G)$ are equal to each other. However we have $\vartheta_k(J(15,2,0)) = \vartheta'_k(J(15,2,0)) = \theta^3_k(J(15,2,0)) = \theta^2_k(J(15,2,0)) = 98$ and $\theta^1_k(J(15,2,0)) = 97.5$. Our bounds do not provide optimal values, but the ratio of $\theta^1_k(G)$ to α_k does not exceed 1.3. The table also indicates that the quality of the upper bound deteriorate when k increases. We do not present computational times required to solve the relaxations since they small, i.e., only a few seconds.

Table 10: Results for the Kneser graphs $K(v,2)$.

Graph	n	$\|E\|$	ρ,%	k	α_k	θ_k^1	θ_k^1/α_k
J(10,2,0)	45	630	64	1	9	9.00	1.00
				2	17	18.00	1.06
				3	24	27.00	1.13
				4	30	36.00	1.20
J(15,2,0)	105	4095	75	1	14	14.00	1.00
				2	27	28.00	1.04
				3	39	42.00	1.08
				4	50	56.00	1.12
				5	60	70.00	1.17
				6	69	84.00	1.22
				7	77	97.50	1.27
J(20,2,0)	190	14535	81	1	19	19.00	1.00
				2	37	38.00	1.03
				3	54	57.00	1.06
				4	70	76.00	1.09
				5	85	95.00	1.12
				6	99	114.00	1.15
				7	112	133.00	1.19
				8	124	152.00	1.23
				9	135	171.00	1.27

For $k = 2$ and $v \geq \frac{1}{2}(3+\sqrt{5})d$, there exists the following upper bound on $\alpha_2(K(v,d))$, see [21]:

$$\alpha_2(K(v,d)) \leq \binom{v-1}{d-1} + \binom{v-2}{d-1}. \tag{34}$$

Therefore, in our next experiment we first compute bounds (33) and (34) for $k = 2$ and various (v,d) s.t., $v \geq \frac{1}{2}(3+\sqrt{5})d$ in order to find graphs for which these two bounds coincide. Table 11 presents examples of such graphs. Although for each graph in Table 11 all our upper bounds are equal, there are existing Kneser graphs for which our relaxations do not provide the same bounds. For example, for the Kneser graph $J(14,5,0)$ and $k = 2$ we have $\vartheta_k(J(14,5,0)) = \theta_k^2(J(14,5,0)) = 1430$ and $\theta_k^1(J(14,5,0)) = 1386$ while $\alpha_k(J(14,5,0)) = 1210$.

Table 11: Results for the Kneser graphs $K(v, d)$, $d \geq 3$ and $k = 2$.

Graph	n	$\|E\|$	ρ,%	α_k	θ_k^1	θ_k^1/α_k
J(15,3,0)	455	50050	48	169	182.00	1.08
J(16,3,0)	560	80080	51	196	210.00	1.07
J(17,3,0)	680	123760	54	225	240.00	1.07
J(18,3,0)	816	185640	56	256	272.00	1.06
J(16,4,0)	1820	450450	27	819	910.00	1.11
J(17,4,0)	2380	850850	30	1015	1120.00	1.10
J(18,4,0)	3060	1531530	33	1240	1360.00	1.10
J(16,5,0)	4368	1009008	11	2366	2730.00	1.15
J(17,5,0)	6188	2450448	13	3185	3640.00	1.14
J(18,5,0)	8568	5513508	15	4200	4760.00	1.13

For $k = 2$ and the same v and d as those used in Table 11, we compute upper and lower bounds for $J(v, d, d-1)$ graphs, see Table 12. Those graphs are more sparse than the corresponding Kneser graphs. The table shows that $\alpha_2(J(15, 3, 2)) = 70$, see also Table 1. For each graph in the table all our upper bounds are equal. Gaps between upper and lower bounds for graphs $J(v, d, d-1)$ are larger on average than gaps between upper bounds and $\alpha_k(G)$ for $J(v, d, 0)$. Table 12 might be used as a benchmark for highly symmetric graphs.

Table 12: Results for the Johnson graphs $J(v, d, d-1)$, $d \geq 3$ and $k = 2$.

Graph	n	$\|E\|$	ρ,%	θ_k^1	LB	θ_k^1/LB
$J(15, 3, 2)$	455	8190	8	70.00	70	1.00
$J(16, 3, 2)$	560	10920	7	80.00	73	1.09
$J(17, 3, 2)$	680	14280	6	90.67	88	1.03
$J(18, 3, 2)$	816	18360	6	102.00	94	1.09
$J(16, 4, 3)$	1820	43680	3	280.00	269	1.04
$J(17, 4, 3)$	2380	61880	2	340.00	291	1.17
$J(18, 4, 3)$	3060	85680	2	408.00	367	1.11
$J(16, 5, 4)$	4368	120120	1	728.00	565	1.29
$J(17, 5, 4)$	6188	185640	1	952.00	771	1.23
$J(18, 5, 4)$	8568	278460	<1	1224.00	974	1.26

8.3 Lower bounds on the chromatic number

In this section we further exploit upper bounds on the MkCS problem for deriving lower bounds on the chromatic number of a graph $G = (E, V)$. In particular, if an upper bound on $\alpha_k(G)$ is smaller than the number of vertices in the graph, then G is not k-colorable. Hence the lower bound on $\chi(G)$ is at least $k + 1$. Using this principle, we obtain a lower bound on $\chi(G)$ as

$$\Psi(G) = \max\{\ k : \text{upper bound on } \alpha_k < |V|\} + 1. \tag{35}$$

We test below our lower bound on the chromatic number (35) for several graphs. We first consider vertex transitive graphs from Djukanovic and Rendl [13], Table 10. Table 13 presents results for two out of four Johnson graphs and several Hamming graphs from [13]. 'n.a.' in the table means that we could't compute bounds. For all graphs in Table 13, we have that $\vartheta_k(G)$ differs from $\theta_k^1(G)$. Table 13 shows that $\Psi(J(12, 7, 3)) = 7$ and $\Psi(J(14, 7, 3)) = 12$, which corresponds to the lower bounds on the chromatic number of the corresponding graphs from [13]. These lower bounds are obtained from (35) with $\theta_k^1(G)$ as an upper bound. Moreover, we obtain the same lower bounds on $\chi(G)$ as the authors from [13] for all Johnson and Hamming graphs from Table 10, in [13].

It is interesting to note that the lower bound (35) with $\vartheta_k(G)$ as an upper bound, provides the same lower bounds on the chromatic number of G as $\Theta(G)$, and thus also $\Theta^-(G)$, for all graphs from Table 10, in [13]. Table 13 also shows that for $J(12, 7, 3)$ and $k = 1, 2, 3$ we compute optimal solutions for the MkCS problem, and similar conclusion follows for $H(6, 2, 4)$ and $H^-(12, 2, 7)$.

Table 13: Results for the Johnson and Hamming graphs from [13], Table 10.

Graph	n	$\|E\|$	ρ, %	k	ϑ_k	θ_k^1	LB	θ_k^1/LB
$J(12,7,3)$	792	69,300	22	1	214.50	120.00	120	1.00
				2	429.00	240.00	240	1.00
				3	643.50	360.00	360	1.00
				4	792.00	480.00	437	1.09
				5	792.00	600.00	522	1.15
				6	792.00	720.00	584	1.23
				7	792.00	792.00	640	1.23
$J(14,7,3)$	3432	2,102,100	36	7	3003.00	2032.80	1262	1.61
				8	3432.00	2323.20	1428	1.63
				11	3432.00	3194.40	n.a.	n.a.
				12	3432.00	3432.00	n.a.	n.a.
H(6,2,4)	64	480	23	1	16.00	12.00	12	1.00
				2	32.00	24.00	24	1.00
				3	48.00	36.00	36	1.00
				4	64.00	48.00	48	1.00
				5	64.00	60.00	52	1.15
				6	64.00	64.00	60	1.07
H(10,2,8)	1024	23,040	4	2	768.00	640.00	520	1.23
				3	1024.00	960.00	700	1.37
				4	1024.00	1024.00	880	1.16
$H^-(12,2,7)$	4096	6,760,448	81	1	7.70	4.00	4	1.00
				2	15.41	8.00	8	1.00
				3	23.11	12.00	12	1.00
				4	30.81	16.00	16	1.00
				5	38.52	20.00	20	1.00
				531	4090.70	2124.00	n.a.	n.a.
				532	4096.00	2128.00	n.a.	n.a.
				1023	4096.00	4092.00	n.a.	n.a.
				1024	4096.00	4096.00	n.a.	n.a.

Further, we consider graphs from the COLOR02 symposium [37] whose lower bounds on $\chi(G)$ were also computed in [13]. We compute bounds on the chromatic number for all mentioned graphs except for 4-Insertions-4 that has 475 vertices. For all graphs except myciel7, we obtain the same lower bounds on the chromatic number of a graph as those reported in [13]. Moreover, for many graphs it is sufficient to compute $\theta_k^3(G)$ to obtain the best lower bound.

For myciel7 graph we improve a lower bound on the chromatic number. Namely, it follows from Table 8.3 that our lower bound on χ(myciel7) is four while the lower bound from [13] is three. It is also known that the the chromatic number for myciel7 is eight.

Table 14: Results for myciel7 graph.

Graph	n	$\|E\|$	ρ, %	k	θ_k^1	θ_k^1 with BQP
myciel7	191	2360	13	3	191	186.84
				4	191	191

Notice that we are able to improve the lower bound for χ(myciel7) from [13] only when using our strongest relaxation that includes BQP inequalities. However, for other graphs from the COLOR02 symposium [37] even that relaxation does not help to improve the results. Finally, we also compute $\theta_k^1(G)$ with BQP inequalities for $H(6,2,4)$ from Table 13, but the lower bound on $\chi(H(6,2,4))$ remains six.

The result in this section show that one can compute strong lower bounds on the chromatic number of a graph by exploiting upper bounds for the MkCS problem.

9 Conclusion

This paper combines several modelling approaches to derive strong bounds for the maximum k-colorable subgraph problem and related problems.

We first analyze the existing upper bound for the MkCS problem known as the generalized ϑ-number, see (7). Then, we strengthen it by adding non-negativity constraints to the corresponding SDP relaxation. We call the resulting upper bound the generalized ϑ'-number, see (9). Then, we propose several new SDP relaxations for the MkCS problem with increasing complexities. The sizes of our new SDP relaxations initially depend on the number of colors k and the number of the vertices in the graph, see (11), (12) and (26). To reduce the sizes of those three SDP relaxations, we exploit the fact that the MkCS problem is invariant with respect to color permutations. The reduction results in the SDP relaxations with at most two SDP constraints of order at most $(n+1)$ for any k and any graph type, see Theorem 3, Corollary 1 and Theorem 5. The resulting relaxations provide the following upper bounds for the MkCS problem: θ_k^1 see (15), θ_k^2 see (15) without constraints (15f), (15g), and θ_k^3 see (28). In Proposition 1 we characterise a family of graphs for which those bounds are tight. To improve our strongest relaxation we add non-redundant, symmetry-reduced, boolean quadric polytope inequalities, see (23)–(24).

We further reduce relaxations for several classes of highly symmetric graphs including the Johnson and Hamming graphs, see Section 6. The resulting relaxations are linear programs or linear programs with one convex quadratic constraint. Finally, we show that the vector and matrix lifting relaxations for the max-k-cut problem and the k-equipartition problem are equivalent by exploiting the invariance of the problems under permutations of the subsets, see Section 7.

We compute upper and lower bounds for graphs considered in [8, 34] with up to 500 vertices. We also compute bounds for the MkCS problem for highly symmetric graphs with up to $6,760,448$ edges. We solve the problem for several graphs to optimality and obtain stronger bounds than in [8] for all but one tested graphs. Our lower bounds on the chromatic number of a graph are competitive with bounds from the literature.

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